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Riemann-Hilbert approach for the integrable nonlocal nonlinear Schrödinger equation with step-like initial data

Ya. Rybalko[1], D. Shepelsky[2]

[1] *B. Verkin Institute for Low Temperature Physics and Engineering, Ukraine*
[2] *V.Karazin Kharkiv National University, 4, Svobody Sq., Kharkiv, 61022, Ukraine*
rybalkoyan@gmail.com, shepelsky@yahoo.com

We study the Cauchy problem for the integrable nonlocal nonlinear Schrödinger (NNLS) equation

$$iq_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)\bar{q}(-x,t) = 0$$

with a step-like initial data: $q(x,0) = o(1)$ as $x \to -\infty$ and $q(x,0) = A + o(1)$ as $x \to \infty$, where $A > 0$ is an arbitrary constant. We develop the inverse scattering transform method for this problem in the form of the Riemann-Hilbert approach and obtain the representation of the solution of the Cauchy problem in terms of the solution of an associated Riemann-Hilbert-type analytic factorization problem, which can be efficiently used for further studying the properties of the solution, including the large time asymptotic behavior.
Keywords: nonlocal nonlinear Schrödinger equation, inverse scattering transform method, Riemann-Hilbert problem.

Рибалко Я., Шепельський Д. **Метод задачі Рімана-Гільберта для інтегровного нелокального нелінійного рівняння Шредінгера з початковими даними типу сходинки.** Ми розглядаємо задачу Коші для інтегровного нелокального нелінійного рівняння Шредінгера (ННШ)

$$iq_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)\bar{q}(-x,t) = 0$$

з початковими даними типу сходинки: $q(x,0) = o(1)$, при $x \to -\infty$, $q(x,0) = A + o(1)$, при $x \to \infty$, де $A > 0$ – будь-яка константа. Ми розробляємо метод оберненої задачі розсіяння для цієї задачі у вигляді методу задачі Рімана-Гільберта, та отримуємо зображення для розв'язку вихідної задачі у термінах розв'язку відповідної задачі аналітичної факторизації типу Рімана-Гільберта, яке може бути ефективно використано для подальшого дослідження властивостей розв'язку, зокрема, його асимптотики за великим часом.
Ключові слова: нелокальне нелінійне рівняння Шредінгера, метод оберненої задачі розсіяння, задача Рімана-Гільберта.

Рыбалко Я., Шепельский Д. **Метод задачи Римана-Гильберта для интегрируемого нелокального нелинейного уравнения Шредингера с начальными данными типа ступеньки.** Мы рассматриваем задачу Коши для интегрируемого нелокального нелинейного уравнения Шредингера

$$iq_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)\bar{q}(-x,t) = 0$$

с начальными данными типа ступеньки: $q(x,0) = o(1)$ при $x \to -\infty$, $q(x,0) = A + o(1)$ при $x \to \infty$, где $A > 0$ некоторая константа. Мы разрабатываем метод обратной задачи рассеяния для этой проблемы в виде метода задачи Римана-Гильберта, и получаем представление для решения исходной задачи в терминах решения соответствующей задачи аналитической факторизации типа Римана-Гильберта, которое может быть эффективно использовано для дальнейшего изучения свойств решения, в частности его асимптотики при больших временах
Ключевые слова: нелокальное нелинейное уравнение Шредингера, метод обратной задачи рассеяния, задача Римана-Гильберта.

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1. Introduction

In this letter we consider the following Cauchy problem for the focusing nonlocal nonlinear Schrödinger (NNLS) equation with a step-like initial data:

$$iq_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)\bar{q}(-x,t) = 0, \qquad x \in \mathbb{R},\, t > 0, \tag{1a}$$

$$q(x,0) = q_0(x), \tag{1b}$$

where $\bar{q}$ denotes the complex conjugate of q and

$$q_0(x) \to 0 \ \text{ as } x \to -\infty \ \text{ and } \ q_0(x) \to A \text{ as } x \to \infty \tag{1c}$$

sufficiently fast, with some $A > 0$.

The nonlocal nonlinear Schrödinger equation (1a) was introduced by M. Ablowitz and Z. Musslimani in [2]. This equation is a reduction of a member of the AKNS hierarchy [1]

$$iq_t + q_{xx} + 2q^2 r = 0, \tag{2a}$$

$$-ir_t + r_{xx} + 2r^2 q = 0, \tag{2b}$$

with $r(x,t) = \bar{q}(-x,t)$, which introduces a remarkably simple nonlocality to the above system and reduces it to equation (1a). The NNLS equation has been widely considered recently due to its interesting physical and mathematical features. First, this equation is invariant under the joint transformations $x \to -x$, $t \to -t$, and complex conjugation. Therefore, it is parity-time (PT) symmetric and thus is related to this state-of-art research area of modern physics [4]. Moreover, in [9] it was shown, that NNLS is gauge-equivalent to the unconventional system of coupled Landau-Lifshitz (CLL) equations and consequently can be used in the physics of nanomagnetic artificial materials. Finally, the focusing NNLS equation (1a), in contrast to the conventional (local) NLS equation, can simultaneously support both bright and dark soliton solutions [11].

In [3] the authors presented the Inverse Scattering Transform (IST) method to the study of the Cauchy problem for equation (1a), based on a variant of the Riemann-Hilbert approach, in the case of decaying initial data and obtained the one- and two-soliton solutions. In the present paper we assume that the solution $q(x,t)$ of problem (1a-1b) satisfies the following boundary conditions for all $t>0$:

$$q(x,t)=o(1), \qquad x\to-\infty, \tag{3a}$$
$$q(x,t)=A+o(1), \qquad x\to+\infty, \tag{3b}$$

This choice of initial data and boundary values is inspired by the shock problems for the classical (local) NLS equation, which has been considered since 1980s (see e.g. [5], [6]). Particularly, in [6] the authors study the asymptotics of the Cauchy problem for the NLS equation with the step-like initial data.

The present paper aims at the development of the Riemann-Hilbert approach to the initial value problem (1) with boundary values (3). It is organized as follows. In Section 2 we construct the dedicated solutions of the Lax pair equations fixing their large x behavior (Jost solution). In Section 3 we discuss the properties of the associated spectral functions. Finally, in Section 4, we gives the representation of the solution of the Cauchy problem in terms of a Riemann-Hilbert problem. Notice that this RH problem has a form suitable for further large time asymptotic analysis by using an appropriate adaptation of the nonlinear steepest descent method [8, 7].

2. Eigenfunctions of the Lax pair equations with step-like boundary conditions

The focusing NNLS equation (1a) is a compatibility condition of the following two linear equations (Lax pair)

$$\begin{cases} \Phi_x+ik\sigma_3\Phi=U(x,t)\Phi \\ \Phi_t+2ik^2\sigma_3\Phi=V(x,t,k)\Phi \end{cases} \tag{4}$$

where $\sigma_3=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$, $\Phi(x,t,k)$ is a 2×2 matrix-valued function, $k\in\mathbb{C}$ is an auxiliary (spectral) parameter, and the matrix coefficients $U(x,t)$ and $V(x,t,k)$ are given in terms of $q(x,t)$:

$$U(x,t)=\begin{pmatrix}0 & q(x,t)\\ -\bar q(-x,t) & 0\end{pmatrix}, \qquad V=\begin{pmatrix}V_{11} & V_{12}\\ V_{21} & V_{22}\end{pmatrix}, \tag{5}$$

where $V_{11}=-V_{22}=iq(x,t)\bar q(-x,t)$, $V_{12}=2kq(x,t)+iq_x(x,t)$, and $V_{21}=-2k\bar q(-x,t)+i(\bar q(-x,t))_x$.

Introduce the notations

$$U_+ = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \; U_- = \begin{pmatrix} 0 & 0 \\ -A & 0 \end{pmatrix}, \; V_+ = \begin{pmatrix} 0 & 2kA \\ 0 & 0 \end{pmatrix}, \; V_- = \begin{pmatrix} 0 & 0 \\ -2kA & 0 \end{pmatrix}. \tag{6}$$

Then, assuming that there exists $q(x,t)$ satisfying (1) and (3), it follows that

$$U(x,t) \to U_\pm \text{ and } V(x,t,k) \to V_\pm(k) \quad \text{as } x \to \pm\infty. \tag{7}$$

It is easy to see that that the system (4) is compatible with U, V replaced by U_+, V_+ or U_-, V_-. Particularly, these equations are satisfied by $\Phi_\pm(x,t,k)$ defined as follows:

$$\Phi_\pm(x,t,k) = N_\pm(k) e^{-(ikx+2ik^2t)\sigma_3}, \tag{8}$$

where $N_+(k) = \begin{pmatrix} 1 & \frac{A}{2ik} \\ 0 & 1 \end{pmatrix}$, $N_-(k) = \begin{pmatrix} 1 & 0 \\ \frac{A}{2ik} & 1 \end{pmatrix}$. Notice that $\det \Phi_\pm \equiv 1$. On the other hand, the singularities of $N_\pm(k)$ at $k = 0$ will significantly affect the analysis that follows. Namely, the solution of the basic RH problem has a singularity as $k \to 0$, i.e. at a point on the contour of the RH problem (see (38) and (39) below).

Now define the 2×2-valued functions $\Psi_j(x,t,k)$, $j = 1,2$, $-\infty < x < \infty$, $0 \le t < \infty$ as the solutions of the Volterra integral equations:

$$\Psi_1(x,t,k) = N_-(k) + \int_{-\infty}^{x} G_-(x,y,t,k) \left(U(y,t) - U_-\right) \Psi_1(y,t,k) e^{ik(x-y)\sigma_3}\, dy, \tag{9a}$$

$$\Psi_2(x,t,k) = N_+(k) + \int_{\infty}^{x} G_+(x,y,t,k) \left(U(y,t) - U_+\right) \Psi_2(y,t,k) e^{ik(x-y)\sigma_3}\, dy, \tag{9b}$$

where $G_\pm(x,y,t,k) = \Phi_\pm(x,t,k)[\Phi_\pm(y,t,k)]^{-1}$. The functions $\Psi_j(x,t,k)$, $j = 1,2$ are the main ingredients of the basic RH problem (see (31) below). The main properties of the matrices $\Psi_j(x,t,k)$ are summarized in Proposition 1, where we denote by $\Psi_j^{(i)}(x,t,k)$ the i-th column of $\Psi_j(x,t,k)$, $\mathbb{C}^\pm = \{k \in \mathbb{C} \,|\, \pm \Im k > 0\}$, and $\overline{\mathbb{C}^\pm} = \{k \in \mathbb{C} \,|\, \pm \Im k \ge 0\}$.

Proposition 1 *The matrices $\Psi_1(x,t,k)$ and $\Psi_2(x,t,k)$ have the following properties*

1. *The columns $\Psi_1^{(1)}(x,t,k)$ and $\Psi_2^{(2)}(x,t,k)$ are well-defined and analytic in $k \in \mathbb{C}^+$ and continuous in $\overline{\mathbb{C}^+} \setminus \{0\}$; moreover,*

   $$\Psi_1^{(1)}(x,t,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O\Big(\frac{1}{k}\Big) \;\; \text{and} \;\; \Psi_2^{(2)}(x,t,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O\Big(\frac{1}{k}\Big),\; k \to \infty$$

   for $k \in \mathbb{C}^+$.

2. *The columns $\Psi_1^{(2)}(x,t,k)$ and $\Psi_2^{(1)}(x,t,k)$ are well-defined and analytic in $k \in \mathbb{C}^-$ and continuous in $\overline{\mathbb{C}^-}$; moreover,*

   $$\Psi_1^{(2)}(x,t,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O\Big(\frac{1}{k}\Big) \;\; \text{and} \;\; \Psi_2^{(1)}(x,t,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O\Big(\frac{1}{k}\Big),\; k \to \infty$$

   for $k \in \mathbb{C}^-$.

3. *The functions* $\Phi_j(x,t,k)$, $j=1,2$ *defined by*

$$\Phi_j(x,t,k)=\Psi_j(x,t,k)e^{-(ikx+2ik^2t)\sigma_3},\qquad k\in\mathbb{R}\setminus\{0\},\qquad j=1,2,\quad (10)$$

are the Jost solutions of the Lax pair equations (4) satisfying the boundary conditions

$$\Phi_1(x,t,k)\to\Phi_-(x,t,k),\qquad x\to-\infty,\quad (11a)$$

$$\Phi_2(x,t,k)\to\Phi_+(x,t,k),\qquad x\to\infty.\quad (11b)$$

4. $\det\Psi_j(x,t,k)=1,\qquad x\in\mathbb{R},\ t\geq 0,\ k\in\mathbb{R},\qquad j=1,2.$

5. *The following symmetry relation holds:*

$$\Lambda\overline{\Psi_1(-x,t,-k)}\Lambda^{-1}=\Psi_2(x,t,k),\ \ k\in\mathbb{R}\setminus\{0\},\quad (12)$$

where $\Lambda=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$.

6. *As* $k\to 0$,

$$\Psi_1^{(1)}(x,t,k)=\frac{1}{k}\begin{pmatrix}v_1(x,t)\\v_2(x,t)\end{pmatrix}+O(1),\quad (13a)$$

$$\Psi_1^{(2)}(x,t,k)=\frac{2i}{A}\begin{pmatrix}v_1(x,t)\\v_2(x,t)\end{pmatrix}+O(k),\quad (13b)$$

$$\Psi_2^{(1)}(x,t,k)=-\frac{2i}{A}\begin{pmatrix}\overline{v_2}(-x,t)\\\overline{v_1}(-x,t)\end{pmatrix}+O(k),\quad (13c)$$

$$\Psi_2^{(2)}(x,t,k)=-\frac{1}{k}\begin{pmatrix}\overline{v_2}(-x,t)\\\overline{v_1}(-x,t)\end{pmatrix}+O(1),\quad (13d)$$

where $v_j(x,t)$, *j=1,2 solve the following system of linear Volterra integral equations:*

$$\begin{cases}v_1(x,t)=\int_{-\infty}^{x}q(y,t)v_2(y,t)\,dy,\\ v_2(x,t)=-i\frac{A}{2}-\int_{-\infty}^{x}\overline{q(-y,t)}v_1(y,t)\,dy.\end{cases}\quad (14)$$

Proof. Properties 1–5 follows directly from the construction (9) of Ψ_j. Particularly, property 5 follows from the corresponding symmetry of U and V. Now let us prove property 6. From (9) and the structure of singularity of $N_\pm$ at $k=0$ it follows that, as $k\to 0$,

$$\Psi_1^{(1)}(x,t,k)=\frac{1}{k}\begin{pmatrix}v_1(x,t)\\v_2(x,t)\end{pmatrix}+O(1),\qquad\Psi_1^{(2)}(x,t,k)=\begin{pmatrix}\tilde v_1(x,t)\\\tilde v_2(x,t)\end{pmatrix}+O(k),\quad (15a)$$

$$\Psi_2^{(1)}(x,t,k)=\begin{pmatrix}\tilde w_1(x,t)\\\tilde w_2(x,t)\end{pmatrix}+O(k),\qquad\Psi_2^{(2)}(x,t,k)=\frac{1}{k}\begin{pmatrix}w_1(x,t)\\w_2(x,t)\end{pmatrix}+O(1).\quad (15b)$$

Then the symmetry relation (12) implies

$$\begin{pmatrix} w_1(x,t) \\ w_2(x,t) \end{pmatrix} = \begin{pmatrix} -\overline{v_2}(-x,t) \\ -\overline{v_1}(-x,t) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{w}_1(x,t) \\ \tilde{w}_2(x,t) \end{pmatrix} = \begin{pmatrix} \overline{\tilde{v}_2}(-x,t) \\ \overline{\tilde{v}_1}(-x,t) \end{pmatrix}. \tag{16}$$

Substituting (15) into (9) we conclude that $v_j(x,t)$, $j=1,2$ satisfy (14) whereas $\tilde{v}_j(x,t)$, $j=1,2$ solve the following system of equations

$$\begin{cases} \tilde{v}_1(x,t) = \int_{-\infty}^{x} q(y,t)\tilde{v}_2(y,t)\,dy, \\ \tilde{v}_2(x,t) = 1 - \int_{-\infty}^{x} \overline{q(-y,t)}\tilde{v}_1(y,t)\,dy. \end{cases} \tag{17}$$

Comparing this with (14) implies that

$$\begin{pmatrix} \tilde{v}_1(x,t) \\ \tilde{v}_2(x,t) \end{pmatrix} = \frac{2i}{A} \begin{pmatrix} v_1(x,t) \\ v_2(x,t) \end{pmatrix}. \tag{18}$$

3. Scattering matrix and spectral functions

Since $\Phi_1(x,t,k)$ and $\Phi_2(x,t,k)$ satisfy both equations in the Lax pair (4), the following dependence relation holds

$$\Phi_1(x,t,k) = \Phi_2(x,t,k)S(k), \;\; k \in \mathbb{R}\setminus\{0\}, \tag{19}$$

or, in terms of Ψ_j,

$$\Psi_1(x,t,k) = \Psi_2(x,t,k)e^{-(ikx+2ik^2t)\sigma_3}S(k)e^{(ikx+2ik^2t)\sigma_3}, \;\; k \in \mathbb{R}\setminus\{0\}, \tag{20}$$

where $S(k)$ is called the scattering matrix. The symmetry relation (12) implies the same relation for the Jost solutions:

$$\Lambda\overline{\Phi_1(-x,t,-\bar{k})}\Lambda^{-1} = \Phi_2(x,t,k), \qquad k \in \mathbb{R}\setminus\{0\}. \tag{21}$$

This implies that the scattering matrix $S(k)$ can be written as follows (cf. [3, 10])

$$S(k) = \begin{pmatrix} a_1(k) & -\overline{b(-\bar{k})} \\ b(k) & a_2(k) \end{pmatrix}, \qquad k \in \mathbb{R}\setminus\{0\}, \tag{22}$$

with some $b(k)$, $a_1(k)$, and $a_2(k)$; moreover, $a_1(k)$, and $a_2(k)$ are well defined in respectively $\overline{\mathbb{C}^+}\setminus\{0\}$ and $\overline{\mathbb{C}^-}\setminus\{0\}$, where they satisfy the symmetry relations

$$\overline{a_1(-\bar{k})} = a_1(k), \qquad \overline{a_2(-\bar{k})} = a_2(k). \tag{23}$$

The scattering matrix $S(k)$ is uniquely determined by the initial data $q_0(x)$. Indeed, introducing the notations $\psi_1(x,k) = (\Psi_1)_{11}(x,0,k)$, $\psi_2(x,k) = (\Psi_1)_{12}(x,0,k)$, $\psi_3(x,k) = (\Psi_1)_{21}(x,0,k)$ and $\psi_4(x,k) = (\Psi_1)_{22}(x,0,k)$, equations (9a) reduce to the systems of Volterra integral equations for ψ_1, ψ_3 and ψ_2, ψ_4 respectively:

$$\begin{cases} \psi_1(x,k) = & 1 + \int_{-\infty}^{x} q_0(y)\psi_3(y,k)\,dy, \\ \psi_3(x,k) = & \frac{A}{2ik} + \int_{-\infty}^{x} e^{2ik(x-y)}\left(A - \overline{q_0(-y)}\right)\psi_1(y,k)\,dy \\ & +\frac{A}{2ik}\int_{-\infty}^{x} q_0(y)\left(1 - e^{2ik(x-y)}\right)\psi_3(y,k)\,dy. \end{cases} \tag{24}$$

$$\begin{cases}\psi_2(x,k) = & \int_{-\infty}^{x} e^{-2ik(x-y)} q_0(y)\psi_4(y,k)\,dy,\\ \psi_4(x,k) = & 1+\int_{-\infty}^{x}\left(A-\overline{q_0(-y)}\right)\psi_2(y,k)\,dy\\ & +\frac{A}{2ik}\int_{-\infty}^{x} q_0(y)\left(e^{-2ik(x-y)}-1\right)\psi_4(y,k)\,dy.\end{cases} \tag{25}$$

Having these equations solved, the entries of $S(k)$ are given by

$$a_1(k) = \lim_{x\to\infty}\left(\psi_1(x,k) - \frac{A}{2ik}\psi_3(x,k)\right), \tag{26a}$$

$$b(k) = \lim_{x\to\infty} e^{-2ikx}\psi_3(x,k), \tag{26b}$$

$$a_2(k) = \lim_{x\to\infty} \psi_4(x,k). \tag{26c}$$

Alternatively, the spectral functions (the entries of the scattering matrix) can be written it terms of the determinant relations:

$$a_1(k) = \det\left(\Psi_1^{(1)}(0,0,k), \Psi_2^{(2)}(0,0,k)\right), \quad k\in\mathbb{C}^+, \tag{27a}$$

$$a_2(k) = \det\left(\Psi_2^{(1)}(0,0,k), \Psi_1^{(2)}(0,0,k)\right), \quad k\in\mathbb{C}^-, \tag{27b}$$

$$b(k) = \det\left(\Psi_2^{(1)}(0,0,k), \Psi_1^{(1)}(0,0,k)\right), \quad k\in\mathbb{R}. \tag{27c}$$

The properties of the spectral functions, which follow from Proposition 1, are summarized in

Proposition 2 *The spectral functions $a_j(k)$, j=1,2, and $b(k)$ have the following properties*

1. *$a_1(k)$ is analytic in $k\in\mathbb{C}^+$ and continuous in $\overline{\mathbb{C}^+}\setminus\{0\}$; $a_2(k)$ is analytic in $k\in\mathbb{C}^-$ and continuous in $\overline{\mathbb{C}^-}$.*
2. *$a_j(k) = 1 + O\left(\frac{1}{k}\right)$, $j=1,2$ and $b(k) = O\left(\frac{1}{k}\right)$ as $k\to\infty$ (the latter holds for $k\in\mathbb{R}$).*
3. *$\overline{a_1(-\bar{k})} = a_1(k)$, $k\in\overline{\mathbb{C}^+}\setminus\{0\}$; $\overline{a_2(-\bar{k})} = a_2(k)$, $k\in\overline{\mathbb{C}^-}$.*
4. *$a_1(k)a_2(k) + b(k)\overline{b(-\bar{k})} = 1$, $k\in\mathbb{R}\setminus\{0\}$ (follows from $\det S(k) = 1$).*
5. *As $k\to 0$, $a_1(k) = \frac{A^2 a_2(0)}{4k^2} + O\left(\frac{1}{k}\right)$ and $b(k) = \frac{Aa_2(0)}{2ik} + O(1)$.*

Remark 1. Concerning the last item of Proposition 2, we notice that substituting (13) into (26) yields, as $k\to 0$,

$$a_1(k) = \frac{1}{k^2}(|v_2(0,0)|^2 - |v_1(0,0)|^2) + O\left(\frac{1}{k}\right), \tag{28a}$$

$$a_2(k) = \frac{4}{A^2}(|v_2(0,0)|^2 - |v_1(0,0)|^2) + O(k), \tag{28b}$$

$$b(k) = -\frac{2i}{kA}(|v_2(0,0)|^2 - |v_1(0,0)|^2) + O(1), \tag{28c}$$

from which item 5 follows. Since (27) holds for any (x,t), the "conservation law" holds for the Jost solutions:

$$v_2(x,t)\bar{v}_2(-x,t) - v_1(x,t)\bar{v}_1(-x,t) = const.$$

Remark 2. In the case of the pure-step initial data:

$$q_0(x) = \begin{cases} 0, & x < 0 \\ A, & x > 0 \end{cases} \tag{29}$$

the scattering matrix $S(k)$ is as follows:

$$S(k) = [\Phi_2(0,0,k)]^{-1}\Phi_1(0,0,k) = N_+^{-1}(k)N_-(k) = \begin{pmatrix} 1 + \frac{A^2}{4k^2} & -\frac{A}{2ik} \\ \frac{A}{2ik} & 1 \end{pmatrix}. \tag{30}$$

4. The basic Riemann-Hilbert problem

The Riemann–Hilbert formalism of the IST is based on constructing (using the Jost soultions) a piece-wise meromorphic, 2×2-valued function in the k-complex plane, whose "lack of analyticity", i.e., the jump across a contour and, if appropriate, some conditions at singular points, can be fully characterized in terms of the spectral data (spectral functions and a discrete set of data related to the poles) uniquely determined by the initial data.

Define the 2×2-valued function $M(x,t,k)$, piece-wise meromorphic relative to $\mathbb{R}$, as follows:

$$M(x,t,k) = \begin{cases} \left(\frac{\Psi_1^{(1)}(x,t,k)}{a_1(k)}, \Psi_2^{(2)}(x,t,k) \right), \ k \in \mathbb{C}^+ \setminus \{0\}, \\ \left(\Psi_2^{(1)}(x,t,k), \frac{\Psi_1^{(2)}(x,t,k)}{a_2(k)} \right), \ k \in \mathbb{C}^- \setminus \{0\}. \end{cases} \tag{31}$$

Then the scattering relation (20) implies that the boundary values $M_\pm(x,t,k) = \lim\limits_{k' \to k, k' \in \mathbb{C}^\pm} M(x,t,k')$, $k \in \mathbb{R}$ satisfy the multiplicative jump condition

$$M_+(x,t,k) = M_-(x,t,k)J(x,t,k), \qquad k \in \mathbb{R} \setminus \{0\}, \tag{32}$$

where

$$J(x,t,k) = \begin{pmatrix} 1 + r_1(k)r_2(k) & r_2(k)e^{-2ikx-4ik^2t} \\ r_1(k)e^{2ikx+4ik^2t} & 1 \end{pmatrix} \tag{33}$$

with the reflection coefficients defined by

$$r_1(k) := \frac{b(k)}{a_1(k)}, \quad r_2(k) := \frac{\overline{b(-k)}}{a_2(k)}. \tag{34}$$

Moreover, we have the normalization

$$M(x,t,k) \to I, \qquad k \to \infty, \tag{35}$$

where I is the 2×2 identity matrix.

Observe that the symmetry conditions 3 (see Proposition 2) imply that

$$r_1(-k)r_2(-k)=\overline{r_1(k)}\,\overline{r_2(k)},\quad k\in\mathbb{R}\setminus\{0\}. \tag{36}$$

By the determinant property 4 in Proposition 2, we also have

$$1+r_1(k)r_2(k)=\frac{1}{a_1(k)a_2(k)},\qquad k\in\mathbb{R}\setminus\{0\}. \tag{37}$$

Now notice that in view of (28), the behavior of M as $k\to 0$ depends qualitatively on whether $a_2(0)\neq 0$ or $a_2(0)=0$. The former case, which is generic, contains the case of "pure-step initial data", where $a_1(k)$ has (in $\mathbb{C}^+$) a single, simple zero located on the imaginary axis, and $a_2(k)$ has no zeros in $\mathbb{C}^-$. Since small (in the L_1 sense) perturbations of the pure-step initial data preserve these properties, this motivates us to concentrate, in the present paper, on the following two cases:

Generic case: The spectral function $a_1(k)$ has one simple zero in $\overline{\mathbb{C}^+}$, say $k=ik_1$, $k_1>0$; $a_2(k)$ has no zeros in $\overline{\mathbb{C}^-}$.

Non-generic case: The spectral function $a_1(k)$ has one simple zero in $\overline{\mathbb{C}^+}$, say $k=ik_1$, $k_1>0$; $a_2(k)$ has one simple zero in $\overline{\mathbb{C}^-}$ at $k=0$ (and thus $\dot a_2(0)\neq 0$). Moreover, we assume that $\lim\limits_{k\to 0}ka_1(k)\neq 0$.

Remark 3. From the symmetry relations (23) it follows that $a_{11}:=\lim\limits_{k\to 0}ka_1(k)$ is purely imaginary. Moreover, if $a_1(k)$ has one simple zero, then $\Im a_{11}<0$ in the non-generic case.

Taking into account the singularities of $\Psi_j(x,t,k)$, $j=1,2$ and $a_1(k)$ at $k=0$ (see Proposition 1), the behavior of $M(x,t,k)$ at $k=0$ can be described as follows: in the *generic case*,

$$M_+(x,t,k)=\begin{pmatrix}\frac{4}{A^2a_2(0)}v_1(x,t) & -\overline{v_2}(-x,t)\\ \frac{4}{A^2a_2(0)}v_2(x,t) & -\overline{v_1}(-x,t)\end{pmatrix}(I+O(k))\begin{pmatrix}k & 0\\ 0 & \frac{1}{k}\end{pmatrix},\ k\to +i0, \tag{38a}$$

$$M_-(x,t,k)=\frac{2i}{A}\begin{pmatrix}-\overline{v_2}(-x,t) & \frac{v_1(x,t)}{a_2(0)}\\ -\overline{v_1}(-x,t) & \frac{v_2(x,t)}{a_2(0)}\end{pmatrix}+O(k),\ k\to -i0, \tag{38b}$$

and in the *non-generic case*,

$$M_+(x,t,k)=\begin{pmatrix}\frac{v_1(x,t)}{a_{11}} & -\overline{v_2}(-x,t)\\ \frac{v_2(x,t)}{a_{11}} & -\overline{v_1}(-x,t)\end{pmatrix}(I+O(k))\begin{pmatrix}1 & 0\\ 0 & \frac{1}{k}\end{pmatrix},\ k\to +i0, \tag{39a}$$

$$M_-(x,t,k)=\frac{2i}{A}\begin{pmatrix}-\overline{v_2}(-x,t) & \frac{v_1(x,t)}{\dot a_2(0)}\\ -\overline{v_1}(-x,t) & \frac{v_2(x,t)}{\dot a_2(0)}\end{pmatrix}(I+O(k))\begin{pmatrix}1 & 0\\ 0 & \frac{1}{k}\end{pmatrix},\ k\to -i0 \tag{39b}$$

(recall that a_{11} is determined by $a_1(k) = \frac{a_{11}}{k} + O(1)$ as $k \to 0$).

Additionally, if $a_1(ik_1) = 0$ (simple zero) with $k_1 > 0$, then $M(x,t,k)$ satisfies the residue condition

$$\operatorname*{Res}_{k=ik_1} M^{(1)}(x,t,k) = \frac{\gamma_1}{\dot{a}_1(ik_1)} e^{-2k_1x-4ik_1^2t} M^{(2)}(x,t,ik_1), \quad |\gamma_1| = 1, \tag{40}$$

where

$$\Psi_1^{(1)}(0,0,ik_1) = \gamma_1 \Psi_2^{(2)}(0,0,ik_1). \tag{41}$$

Notice that the symmetry relation (12) yields $\overline{\Psi}_1^{(1)}(0,0,ik_1) = \gamma_1^{-1}\overline{\Psi}_2^{(2)}(0,0,ik_1)$ and thus $|\gamma_1| = 1$ (cf. [3]).

Now we summarize the results of the analysis above in the representation Theorem that gives the solution of the Cauchy problem (1), (3) in terms of the solution of the associated Riemann-Hilbert problem.

Theorem 1 *Let $q_0(x)$, $x \in (-\infty,\infty)$ be given such that*

$$\int_{-\infty}^{0} |q_0(x)|dx + \int_0^{\infty} |q_0(x) - A|dx < \infty.$$

Let $a_1(k)$, $a_2(k)$, and $b(k)$ be constructed according to (24)–(26). Assume that (i) $a_1(k)$ has a single, simple, pure imaginary zero k_1 in $\overline{\mathbb{C}^+}$; (ii) $a_2(k)$ has no zeros in $\overline{\mathbb{C}^+} \setminus \{0\}$ and, if $a_2(0) = 0$, then 0 is the simple zero of $a_2(k)$; (iii) $\lim_{k\to 0}(ka_1(k)) \neq 0$. Determine γ_1 according to (41).

Consider the following Riemann-Hilbert problem: find the 2×2-valued function $M(x,t,k)$, piece-wise meromorphic in k relative to $\mathbb{R}$ and satisfying the following conditions:

(i) *Jump conditions. The boundary values $M_\pm(x,t,k) = M(x,t,k \pm i0)$, $k \in \mathbb{R}\setminus\{0\}$ satisfy the condition*

$$M_+(x,t,k) = M_-(x,t,k)J(x,t,k), \qquad k \in \mathbb{R}\setminus\{0\}, \tag{42}$$

where the jump matrix $J(x,t,k)$ is given by (33), with r_1 and r_2 given by (34).

(ii) *Normalization at $k = \infty$:*

$$M(x,t,k) = I + O(k^{-1}), \qquad k \to \infty.$$

(iii) *Residue condition (40).*

(iv) *Structural conditions at $k = 0$: $M(x,t,k)$ satisfies (38) (generic case: $a_2(0) \neq 0$) or (39) (non-generic case: $a_2(0) = 0$), where $v_j(x,t)$, $j = 1,2$ are some (not prescribed) functions.*

Assume that the RH problem (i)–(iv) has a solution $M(x,t,k)$. Then the solution of the Cauchy problem (1), (3) is given in terms of the (12) and (21) entries of $M(x,t,k)$ as follows:

$$q(x,t) = 2i \lim_{k\to\infty} kM_{12}(x,t,k), \tag{43}$$

and

$$q(-x,t) = -2i \lim_{k\to\infty} k\overline{M_{21}(x,t,k)}. \tag{44}$$

The solution of the RH problem is unique, if exists. Indeed, if M and $\tilde{M}$ are two solutions, then conditions (38) or (39) provide the boundedness of $M(k)\tilde{M}^{-1}(k)$ at $k=0$. Moreover, $M(k)\tilde{M}^{-1}(k)$ has no jump across $\mathbb{R}$ and $M(k)\tilde{M}^{-1}(k)\to I$ as $k\to\infty$, which allows to deduce, using the Liouville theorem, that $M(k)\tilde{M}^{-1}(k)\equiv I$.

Remark 4. From (43) and (44) it follows that in order to present the solution of (1), (3) for all $x\in\mathbb{R}$, it is sufficient to have the solution of the RH problem for, say, $x\geq 0$ only.

Proposition 3 *The solution M of the Riemann–Hilbert problem* ***(i)-(iv)*** *satisfies the following symmetry condition (cf. (21)):*

$$M(x,t,k) = \begin{cases} \Lambda\overline{M(-x,t,-\bar{k})}\Lambda^{-1}\begin{pmatrix}\frac{1}{a_1(k)} & 0\\ 0 & a_1(k)\end{pmatrix}, & k\in\mathbb{C}^+\setminus\{0\},\\ \Lambda\overline{M(-x,t,-\bar{k})}\Lambda^{-1}\begin{pmatrix}a_2(k) & 0\\ 0 & \frac{1}{a_2(k)}\end{pmatrix}, & k\in\mathbb{C}^-\setminus\{0\}.\end{cases} \tag{45}$$

Proof. Follows from the symmetry of the jump matrix (33) in (42)

$$\Lambda\overline{J(-x,t,-k)}\Lambda^{-1} = \begin{pmatrix}a_2(k) & 0\\ 0 & \frac{1}{a_2(k)}\end{pmatrix} J(x,t,k)\begin{pmatrix}a_1(k) & 0\\ 0 & \frac{1}{a_1(k)}\end{pmatrix}, \qquad k\in\mathbb{R}\setminus\{0\}$$

(which, in turns, follows from (36) and (37)), and the fact that the structural conditions (38) and (39) and the residue condition (40) are consistent with (45).

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Рибалко Я., Шепельський Д. **Метод задачі Рімана-Гільберта для інтегровного нелокального нелінійного рівняння Шредінгера з початковими даними типу сходинки.** Досліджується задача Коші для інтегровного нелокального нелінійного рівняння Шредінгера (ННШ)

$$iq_t(x,t)+q_{xx}(x,t)+2q^2(x,t)\bar{q}(-x,t)=0$$

з початковими даними типу сходинки: припускаэться, що початкова функція $q(x,0)$ э такою, що $q(x,0)=o(1)$, коли $x\to-\infty$, та $q(x,0)=A+o(1)$, коли $x\to\infty$, де $A>0$ – довільний (фіксований) параметр, що відповідає ненульовому фону сходинки. Починаючи з 2013 року, коли рівняння ННШ було запропоновано як інтегровна модель, воно привертає значну увагу дослідників (як математиків, так і фізиків) у зв'язку з тим, що рівняння ННШ є симетричним у сенсі «парність-час»: воно інваріантне відносно спільного перетворення $x\to-x$, $t\to-t$, та комплексного спряження.

Зокрема, рівняння ННШ є калібрувально-еквівалентним до системи пов'язаних рівнянь Ландау-Ліфшиця та, відповідно, може знайти застосування у фізиці штучних наномагнітних матеріалів.

Для дослідження задачі Коші для рівняння ННШ, автори розробляють варіант методу оберненої задачі розсіяння у формі методу задачі Рімана-Гільберта. Метод базується на використанні розв'язків Йоста (у тому числі, на ненульовому фоні) для рівнянь відповідної пари Лакса та детальному аналізі їх аналітичних властивостей. У результаті отримано зображення для розв'язку вихідної задачі у термінах розв'язків відповідної задачі аналітичної факторизації типу Рімана-Гільберта. Це зображення має цілий ряд відмінностей від відповідного зображення у випадку класичного (локального) нелінійного рівняння Шредінгера. Зокрема, це стосується сингулярної поведінки у околі нуля спектрального параметра, особливістю якої є те, що ця сингулярність локалізована на контурі спряження для задачі Рімана-Гільберта. Отримане зображення може бути ефективно використано для подальшого дослідження властивостей розв'язків задачі Коші, зокрема, дослідження його асимптотичної поведінки за великим часом.

Ключові слова: нелокальне нелінійне рівняння Шредінгера, метод оберненої задачі розсіяння, розв'язки Йоста, задача Рімана-Гільберта.

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