Вісник Харківського національного університету імені В.Н. Каразіна
Серія "Математика, прикладна математика і механіка"
Том 85, 2017, с.62–68

Visnyk of V.N.Karazin Kharkiv National University
Ser. "Mathematics, Applied Mathematics and Mechanics"
Vol. 85, 2017, p. 62–68
DOI: 10.26565/2221-5646-2017-85-05

УДК 517.9

To the generalization of the Newton-Kantorovich theorem

S. M. Chuiko

Donbass State Pedagogical University,
Slavyansk, 84116, Generala-Batuka str., 19, Ukraine
E-mail: chujko-slav@inbox.ru

Constructive conditions for solvability are obtained, as well as an iterative scheme for finding solutions of the nonlinear equation that generalize the well-known Newton-Kantorovich theorem. The case of a nonlinear equation whose dimension does not coincide with the dimension of the unknown has been researched.
Keywords: Newton-Kantorovich method; iterative scheme; nonlinear equation; pseudoinverse matrices.

Чуйко С. М. **Про узагальнення теореми Ньютона-Канторовича.** Отримано конструктивні умови розв'язності, а також ітераційну схему, для знаходження розв'язків нелінійного рівняння, які узагальнюють відому теорему Ньютона-Канторовича. Досліджено випадок нелінійного рівняння, розмірність якого, не збігається з розмірністю невідомої.
Ключові слова: метод Ньютона-Канторовича; ітераційна схема; нелінійне рівняння; псевдообернена матриця.

Чуйко С. М. **К обобщению теоремы Ньютона-Канторовича.** Получены конструктивные условия разрешимости, а также итерационная схема, применимая для нахождения решений нелинейного уравнения, обобщающие известную теорему Ньютона-Канторовича. Исследован случай нелинейного уравнения, размерность которого, не совпадает с размерностью неизвестной.
Ключевые слова: метод Ньютона-Канторовича; итерационная схема; нелинейное уравнение; псевдообратная матрица.

2000 Mathematics Subject Classification: 15A24; 34B15; 34C25.

1. Formulation of the problem

We investigate the problem of finding the solution $z \in \mathbb{R}^n$ of the nonlinear equation

$$\varphi(z) = 0. \tag{1}$$

We assume that the function

$$\varphi(z) : \mathbb{R}^n \to \mathbb{R}^m, \quad m \neq n$$

is twice continuously differentiable with respect to z in some domain $\Omega \subseteq \mathbb{R}^n$. To construct an iteration scheme $\{z_k\}$, that converges to the solution $\tilde{z} \in \mathbb{R}^n$, we use the Newton method [1, 2, 3].

Interest in the use of the Newton method is associated with its effective application in solving nonlinear equations, as well as in the theory of nonlinear oscillations [1, 2, 3, 4], including in the theory of non-linear Noetherian boundary value problems [5, 6, 7, 8].

2. The main result

Suppose an approximation z_k is found that is sufficiently close to an exact solution $\tilde{z}$ of the equation (1). We expand the function $\varphi(z)$ in a neighborhood of the exact solution

$$\varphi(\tilde{z}) = \varphi(z_k) + \varphi'(z_k, \varepsilon)\Big(\tilde{z} - z_k\Big) + R(\xi_k,\, \tilde{z} - z_k), \tag{2}$$

where

$$R(\xi_k,\, \tilde{z} - z_k) := \int_0^1 (1-s)\, d^2\varphi(\xi_k\,; \tilde{z} - z_k)\, ds.$$

Here ξ_k is a point lying between the points $\tilde{z}$ and z_k. In a small neighborhood of the exact solution we have the approximate equality

$$\varphi(z_k) + \varphi'(z_k)\Big(\tilde{z} - z_k\Big) \approx 0,$$

therefore, in order to find the next approximation of z_{k+1} to the exact solution, it is natural to put

$$\varphi(z_k) + \varphi'(z_k)\Big(z_{k+1} - z_k\Big) = 0, \tag{3}$$

whence under the condition

$$P_{J_k^*} = 0, \quad J_k := \varphi'(z_k) \in \mathbb{R}^{m\times n} \tag{4}$$

we find

$$z_{k+1} = z_k - J_k^+\, \varphi(z_k). \tag{5}$$

Here $P_{J_k^*}\,:\ \mathbb{R}^m \to \mathbb{N}(J_k^*)$ is an orthogonal projector of the matrix $J_k^* \in \mathbb{R}^{n\times m}$ and J_k^+ is the pseudoinverse Moore-Penrose matrix [5, 9]. Note that condition (4) is equivalent to the requirement of completeness of the rank matrix J_k and is possible only in case $m \leq n$. We show that the iteration scheme (5) converges to the exact solution $\tilde{z}$. Suppose that in the neighborhood of the exact solution $\tilde{z}$ there are inequalities

$$\left|\left|J_k^+\right|\right| \le \sigma_1(k), \quad \left|\left|d^2\varphi(\xi_k\,;\tilde{z}-z_k)\right|\right| \le \sigma_2(k)\cdot ||\tilde{z}-z_k||^2$$

and note that it follows from the equalities (2) and (3) that

$$\varphi'(z_k,\varepsilon)\Big(\tilde{z}-z_k\Big) = -R(\xi_k,\,\tilde{z}-z_k),$$

so

$$||\tilde{z}-z_{k+1}|| \le \left|\left|J_k^+\right|\right|\cdot\left|\left|R(\xi_k,\,\tilde{z}-z_k)\right|\right| \le \frac{\sigma_1(k)\sigma_2(k)}{2}\cdot||\tilde{z}-z_k||^2.$$

Let there be a constant

$$\theta := \sup_{k\in N}\left\{\frac{\sigma_1(k)\sigma_2(k)}{2}\right\}.$$

In this case, there is an estimate

$$|\tilde{z}-z_{k+1}| \le \theta\cdot|\tilde{z}-z_k|^2,$$

which holds that if the iteration scheme (5) converges to the exact solution $\tilde{z}$ of the equation (1), then this convergence is quadratic. Let us find the condition for the convergence of the iteration scheme (5) to the exact solution $\tilde{z}$ of the equation (1). To do this, we make estimates

$$|\tilde{z}-z_1| \le \theta\cdot|\tilde{z}-z_0|^2,$$

$$|\tilde{z}-z_2| \le \theta\cdot|\tilde{z}-z_1|^2 \le \theta^{1+2}\cdot|\tilde{z}-z_0|^{2^2},$$

$$|\tilde{z}-z_3| \le \theta\cdot|\tilde{z}-z_2|^2 \le \theta^{1+2+2^2}\cdot|\tilde{z}-z_0|^{2^3},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots ,$$

$$|\tilde{z}-z_k| \le \theta\cdot|\tilde{z}-z_{k-1}|^2 \le \theta^{1+2+2^2+\,\dots\,+2^{k-1}}\cdot|\tilde{z}-z_0|^{2^k},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots .$$

So there's an inequality [3]

$$|\tilde{z}-z_k| \le \theta^{\frac{2^k-1}{2-1}}\cdot|\tilde{z}-z_0|^{2^k} = \frac{1}{\theta}\cdot\Big(\theta\cdot|\tilde{z}-z_0|\Big)^{2^k},$$

indicating the convergence of the iterative process (5) to an exact solution $\tilde{z}$ of the equation (1) under condition

$$\theta\cdot|\tilde{z}-z_0| < 1. \tag{6}$$

In practice, the last inequality can be replaced by the following one:

$$\theta\cdot|z_k-z_0| < 1.$$

Theorem 0.1 *Suppose that for the equation (1) the following conditions are satisfied:*

1. *A non-linear vector-function* $f(z) : \mathbb{R}^n \to \mathbb{R}^m$*, twice continuously differentiable with respect to* z *in some region* $\Omega \subseteq \mathbb{R}^n$*, in a neighborhood of the point* z_0 *has a root* z^**.*

2. *In the neighborhood of the zeroth approximation* $z_0 \in \Omega \subseteq \mathbb{R}^n$ *there are inequalities*

$$\left|\left|J_k^+\right|\right| \leq \sigma_1(k), \left|\left|d^2\varphi(\xi_k\,; \tilde{z} - z_k)\right|\right| \leq \sigma_2(k) \cdot ||\tilde{z} - z_k||. \tag{7}$$

3. *The folowing constant exists*

$$\theta := \sup_{k \in N} \left\{ \frac{\sigma_1(k)\sigma_2(k)}{2} \right\}.$$

Then, under conditions (4) and (6), to find the solution z^* *of equation (1) the iteration scheme (5) is applicable, and the rate of convergence of the sequence* $\{z_k\}$ *to the solution* z^* *of equation (1) is quadratic.*

Example 0.1 *The iterative scheme (5) is approximate for finding the solution of the non-linear equation (1), where the vector-function is as follows:*

$$\varphi(u) := \begin{pmatrix} x + \sin y + \cos z \\ y + \sin z + \cos x \end{pmatrix}, \quad u := \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

This vector-function $\varphi(u) : \mathbb{R}^3 \to \mathbb{R}^2$ is defined in any open domain $D \subset \mathbb{R}^3$ and is twice continuously differentiable with respect to z in the neighborhood $\Omega \subseteq D \subset \mathbb{R}^3$. We set

$$u_0 := \begin{pmatrix} -0,45 & -0,45 & -0,45 \end{pmatrix},$$

wherein

$$\operatorname{rank}\left[\varphi'(u_0)\right] = 2,$$

besides

$$u_1 \approx \begin{pmatrix} -0,455\ 961 & -0,457\ 894 & -0,455\ 547 \end{pmatrix}^*,$$

and

$$\operatorname{rank}\left[\varphi'(u_1)\right] = 2,$$

Then

$$\left|\left|\ \left[\varphi'(u_1)\right]^+\right|\right|_\infty := \sigma_1(1) \approx 2,09\ 903, \qquad \left|\left|d^2\varphi(u_1)\right|\right|_\infty := \sigma_2(1) \approx 0,897\ 838.$$

In this case, the weakened condition (6)

$$\theta_1 \cdot ||u_1 - u_0||_\infty \approx 0,00743\ 856 \ll 1, \quad \theta_1 := \frac{\sigma_1(1)\sigma_2(1)}{2} \approx 0,942\ 293$$

is satisfied. Since the condition (6) is satisfied for the first step of the iteration scheme (5), we find

$$u_2 \approx \begin{pmatrix} -0,455\ 968\ 239\ 769\ 595 \\ -0,457\ 889\ 951\ 795\ 185 \\ -0,455\ 537\ 594\ 550\ 856 \end{pmatrix}.$$

Then

$$\text{rank}\left[\varphi'(u_2)\right] = 2,$$

besides

$$\left|\left|\,\left[\varphi'(u_2)\right]^+\,\right|\right|_\infty := \sigma_1(2) \approx 2,099, \quad \left|\left|d^2\varphi(u_2)\right|\right|_\infty := \sigma_2(2) \approx 0,897\ 835.$$

In this case, the weakened condition (6)

$$\theta_2 \cdot ||u_2 - u_0||_\infty \approx 0,00743\ 453 \ll 1,$$

is satisfied, where

$$\theta_2 := \frac{\sigma_1(2)\,\sigma_2(2)}{2} \approx 0,00743\ 453.$$

For the second step of the iteration scheme (5) the discrepancy of the obtained approximation

$$||\varphi(u_2)||_\infty \approx 3,69\ 679 \times 10^{-11}$$

is sufficiently big, so we find

$$u_3 \approx \begin{pmatrix} 0,455\ 968\ 239\ 730\ 150 \\ 0,457\ 889\ 951\ 789\ 936 \\ 0,455\ 537\ 594\ 568\ 580 \end{pmatrix}.$$

Then

$$\text{rank}\left[\varphi'(u_3)\right] = 2,$$

besides

$$\left|\left|\,\left[\varphi'(u_3)\right]^+\,\right|\right|_\infty := \sigma_1(3) \approx 2,099, \quad \left|\left|d^2\varphi(u_3)\right|\right|_\infty := \sigma_2(3) \approx 0,897\ 835.$$

In this case, the weakened condition (6)

$$\theta_3 \cdot ||u_3 - u_0||_\infty \approx 0,00743\ 453 \ll 1$$

is satisfied, where

$$\theta_3 := \frac{\sigma_1(3)\,\sigma_2(3)}{2} \approx 0,942\ 278.$$

For the third step of the iteration scheme (5) the discrepancy of the obtained approximation is

$$||\varphi(u_3)||_\infty \approx 0,$$

so it's natural to confine with this approximation.

The theorem just proved generalizes the corresponding results [2, 3, 4, 6, 7, 8] to the case of matrix J_k irreversibility and can be used in the theory of non-linear Noetherian boundary-value problems [5, 6, 7, 8], in the theory of stability of motion [10, 11], in the theory of matrix boundary-value problems [12], and also in the theory of matrix linear differential-algebraic boundary value problem [13, 14, 15, 16].

Acknowledgement. The work is done with the financial support of the State Fund for Fundamental Research. Number of state registration is 0115U003182.

REFERENCES

1. Bogolyubov N.N., Mitropolsky J.A., Samoilenko A.M. The method of accelerated convergence in nonlinear mechanics. — Kiev: Scientific thought, 1969. — 248 pp.

2. Kantorovich L.V., Akilov G.P. Functional analysis. — Moscow: Nauka. — 1977. — 744 pp.

3. Dennis J. Schnabel R. Numerical methods of unconditional optimization and solving nonlinear equations. — Moscow: Mir. — 1988. — 440 pp.

4. Polyak B.T. The Newton method and its role in optimization and computational mathematics // Trudy ICA RAN. — 2006. — **28**. — P. 48 — 66.

5. Boichuk A.A., Samoilenko A.M. Generalized inverse operators and Fredholm boundary-value problems (2-th edition). — Berlin; Boston: De Gruyter, 2016. — 298 pp.

6. Chuiko S.M., Boichuk I.A. An autonomous Noetherian boundary value problem in the critical case // Nonlinear Oscillations (N.Y.) — **12**. — 2009. № 3, P. 405 — 416.

7. Chuiko S.M., Boichuk I.A., Pirus O.E. On the approximate solution of an autonomous boundary-value problem the Newton - Kantorovich method // Journal of Mathematical Sciences — 2013. — **189**, № 5. — P. 867 — 881.

8. Chuiko S.M., Pirus O.E. On the approximate solution of autonomous boundary-value problems by the Newton method // Journal of Mathematical Sciences — 2013. — **191**, № 3. — P. 449 — 464.

9. Gantmakher F.R. Matrix theory. — Moscow: Nauka. — 1988. — 552 pp.

10. Korobov V.I. Bebiya M.O. Stabilization of one class of nonlinear systems // Avtomat. i Telemekh. — 2017. — № 1. — P. 3 — 18.

11. Bebiya M.O. Stabilization of systems with power nonlinearity // Visnyk of V.N.Karazin Kharkiv National University. Ser. Mathematics, Applied Mathematics and Mechanics. — 2014, № 1120. — Issue 69. — P. 75 — 84.

12. Chuiko S. Weakly nonlinear boundary value problem for a matrix differential equation // Miskolc Mathematical Notes. — 2016. — **17**, № 1. — P. 139 — 150.

13. Campbell S.L. Singular Systems of differential equations. — San Francisco – London – Melbourne: Pitman Advanced Publishing Program. — 1980. — 178 p.

14. Chuiko S.M. The Green's operator of a generalized matrix linear differential-algebraic boundary value problem // Siberian Mathematical Journal. — 2015. —**56**, № 4. — P. 752 — 760.

15. Chuiko S.M. A generalized matrix differential-algebraic equation // Journal of Mathematical Sciences (N.Y.). – 2015. – **210**, № 1. – P. 9 – 21.

16. Chuiko S.M. To the issue of a generalization of the matrix differential-algebraic boundary-value problem // Journal of Mathematical Sciences. — 2017. — **227**, № 1. — P. 16 — 32.

Article history: Received: 30 August 2017; Final form: 18 November 2017;
Accepted: 21 November 2017.