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For the ideal Fermi gas that fills a quantum well confined by two parallel planes, there are calculated the thermodynamic characteristics in general form for arbitrary temperatures, namely: the thermodynamic potential, energy, entropy, equations of state, heat capacities and compressibilities. The distance between planes is considered as an additional thermodynamic variable. Owing to the anisotropy, the pressure of the Fermi gas along and transverse to the planes is different, so that the system is characterized by two equations of state and a set of different heat capacities. Limiting cases of low and high temperatures are considered. The temperature dependencies of the entropy and heat capacities at low temperatures remain linear, just as in the volume case, and their dependencies on the chemical potential and density undergo jumps at the beginning of the filling of new discrete levels. It is shown that the behavior of thermodynamic quantities with the distance between plates can be either oscillating or monotonic, depending on what quantity is assumed to be fixed: the volume or surface density. For high temperatures the corrections to thermodynamic quantities are obtained, which are proportional to the ratio of the thermal de Broglie wavelength to the distance between planes.

**KEY WORDS:** Fermi particle, quantum well, thermodynamic functions, low-dimensional systems, equation of state, heat capacity, compressibility

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Для ідеального фермі-газу, що заповнює квантову яму обмежену двома паралельними площинами, у загальному вигляді для довільних температур обчислені термодинамічні характеристики, а саме: термодинамічний потенціал, енергія, ентропія, рівняння стану, теплоємності та стисливості. Відстань між площинами розглядається як додаткова термодинамічна змінна. Через анізотропію тиск фермі-газу уздовж і перпендикулярно площинам відмінний, так що система характеризується двома рівняннями стану і набором різних теплоємностей. Розглянуті граничні випадки низьких і високих температур. Температурні залежності ентропії і теплоємностей при низьких температурах залишаються лінійними, як і в об'ємному випадку, а їх залежності від хімічного потенціалу і густини зазнають стрибки при початку заповнення нового дискретного рівня. Показано, що зміна термодинамічних величин з відстанню між пластинами може мати або осциляційний, або монотонний характер залежно від того, що передбачається фіксованим, об'ємна або поверхнева густина. В області високих температур знайдені поправки до термодинамічних величин, пропорційні відношенню теплової хвилі де Бройля до відстані між площинами.

**КЛЮЧОВІ СЛОВА:** фермі-частинка, квантова яма, термодинамічні функції, низькорозмірні системи, рівняння стану, теплоємність, стисливість

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Для идеального ферми-газа, заполняющего квантовую яму ограниченную двумя параллельными плоскостями, в общем виде для произвольных температур вычислены термодинамические характеристики, а именно: термодинамический потенциал, энергия, энтропия, уравнения состояния, теплоемкости и сжимаемости. Расстояние между плоскостями рассматривается как дополнительная термодинамическая переменная. В силу анизотропии давление ферми-газа вдоль и перпендикулярно плоскостям различно, так что система характеризуется двумя уравнениями состояния и набором различных теплоемностей. Рассмотрены предельные случаи низких и высоких температур. Температурные зависимости энтропии и теплоемностей при низких температурах остаются линейными, как и в объемном случае, а их зависимости от химического потенциала и плотности испытывают скачки при начале заполнения нового дискретного уровня. Показано, что изменение термодинамических величин с расстоянием между пластинами может иметь либо осцилляционный, либо монотонный характер в зависимости от того, что предполагается фиксированным, объемная или поверхностная плотность. В области высоких температур найдены поправки к термодинамическим величинам, пропорциональные отношению тепловой волны де Бройля к расстоянию между плоскостями.

**КЛЮЧЕВЫЕ СЛОВА:** ферми-частица, квантовая яма, термодинамические функции, низкоразмерные системы, уравнение состояния, теплоемкость, сжимаемость

The model of the ideal Fermi gas is the basis for understanding the properties of electron and other many-fermion systems. In many cases it is also possible to describe with reasonable accuracy the behavior of systems of interacting fermi-particles within the approximation of an ideal gas of quasiparticles whose dispersion law differs from the dispersion law of free particles. It is essential that thermodynamic characteristics of the ideal Fermi gas at arbitrary

temperatures in the volume case can be expressed through the special Fermi functions and, therefore, it is possible to obtain and verify all relations of the phenomenological thermodynamics on the basis of the quantum microscopic model.

In recent time, much attention has been paid to investigation of low-dimensional systems, in particular to properties of the two-dimensional Fermi gas in quantum wells, because apart from purely scientific interest the study of such objects is rather promising for the solid-state electronics [1-5]. Thermodynamics relations for the Fermi gas in the confined geometry have been studied much less than in the volume case [6] and require further investigation. A detailed understanding of the properties of such systems must serve as a basis for the study of low-dimensional systems of interacting particles.

It is usual to consider that strongly correlated Fermi systems, to which also two-dimensional Fermi liquids are attributed, in many respects essentially differ from the usual Fermi systems and often show “non-Fermi-liquid” behavior [7]. At that the properties of quasi-two-dimensional and quasi-one-dimensional systems are compared with the theory of bulk Fermi liquid [8,9]. However, as seen even on the example of the quasi-two-dimensional system of noninteracting particles which is considered in detail in the present work, its properties can substantially differ, especially at low temperatures, from the properties of the bulk system owing to the quantum size effect. Therefore, the theory of Fermi liquid itself in conditions of the confined geometry must, generally speaking, be formulated differently than in the volume case. Note that the Migdal’s theory of finite Fermi systems [10] does not essentially differ in this respect from the Fermi liquid theory of uniform systems.

The consideration of low-dimensional models of interacting Fermi particles leads to a conclusion about, in many cases, unique properties of such systems [11,12]. It should be kept in mind, however, that real systems are always three-dimensional and their low dimensionality manifests itself only in the boundedness of motion of particles in one, two or three coordinates. In considering statistical properties of the three-dimensional many-particle systems one usually passes to the thermodynamic limit, setting in final formulas the volume and number of particles to infinity at a fixed density. It is of general theoretical interest to study the statistical properties of many-particle systems occupying a volume, one or two dimensions of which remain fixed, and the thermodynamic limiting transition is carried out only over remaining coordinates. In this case the coordinates, over which the thermodynamic limiting transition is not performed, should be considered as additional thermodynamic variables. The model of the ideal Fermi gas allows to build the thermodynamics of such systems on the basis of the statistical treatment.

The idea of taking account of the spatial quantization when calculating the electron heat capacity of small particles was for the first time used by Fröhlich [13]. Thermodynamic properties of the Fermi gas at low temperatures in the confined geometry within the quasiclassical approach were considered in the works of I.M. Lifshits and A.M. Kosevich [14,15]. Since the fermions possess a quasidiscrete spectrum in the confined geometry, in a similar way as it takes place for electrons in the magnetic field [6], the authors of these works showed that the thermodynamic potential contains under such conditions a component that oscillates with varying the chemical potential. However, as seen from the results of the given work, the presence of the oscillating component in the thermodynamic potential does not yet guarantee that the full thermodynamic potential and thermodynamic quantities are oscillating.

It should be noted that in experiments the quantum oscillation phenomena in thin metallic, semimetallic, semiconducting films and nanostructures were observed for the kinetic coefficients such as the conductivity, the mobility, the Hall coefficient and others [2,5], but not for the thermodynamic quantities.

The aim of the proposed work is obtaining general thermodynamic relations for the Fermi gas in conditions of the confined geometry on the basis of a microscopic treatment. In the work there are derived exact formulas for the thermodynamic potential, energy, entropy, pressures, heat capacities and compressibilities which allow to analyze the equilibrium properties of the Fermi system at arbitrary temperatures and geometric dimensions. Any thermodynamic quantity proves to be expressed through some standard functions and their derivatives that depend only on the dimensionless temperature and the dimensionless chemical potential. It is shown that, owing to the anisotropy of the system under consideration, the pressure of the Fermi gas along and transverse to planes is different, so that the system is characterized by two equations of state and a set of different heat capacities. The cases of low and high temperatures are studied. At low temperatures the dependencies of the entropy and heat capacities on temperature remain linear, just as in the volume case, and their dependencies on the chemical potential and density undergo jumps at the beginning of the filling of new discrete levels. It is shown that the behavior of thermodynamic quantities with the distance between plates can be qualitatively different, depending on what quantity is assumed to be fixed: the volume or surface density. Thus, at a fixed surface density the chemical potential and pressure vary monotonically with the thickness and at a fixed volume density these dependencies have an oscillating character. In the area of high temperatures the corrections to thermodynamic quantities are obtained, proportional to the ratio of the thermal de Broglie wavelength to the distance between planes.

#### FERMI GAS IN A VOLUME

Before proceeding to consideration of the Fermi gas of particles with mass  $m$  in conditions of the confined geometry, here we give the basic formulas for the three-dimensional Fermi gas enclosed in the rectangular parallelepiped of volume  $V = L_x L_y L_z$ , lengths of all sides of which are large. In the case when  $L_i \gg 2\pi/k_i$  ( $\mathbf{k}$  is a

wave vector), the wave functions of particles have the form of plane waves  $\varphi_{\mathbf{k}}(\mathbf{r}) = V^{-1/2} e^{i\mathbf{k}\mathbf{r}}$  and the energy of particles  $\varepsilon_{\mathbf{k}} = \hbar^2 k^2 / 2m$ . Generally, in books authors restrict themselves to considering the two limiting cases: either the case of high temperatures in which a description of motion of gas particles based on the laws of the classical mechanics is valid, or the case of very low temperatures when the Fermi gas is degenerate [6]. Meanwhile, all thermodynamic functions of the Fermi gas at arbitrary temperature can be exactly expressed through the special functions

$$\Phi_s(t) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{z^{s-1} dz}{e^{z-t} + 1}, \quad (1)$$

where  $s$  is an integer or half-integer positive number,  $\Gamma(s)$  is the gamma function. For calculation of the bulk properties of the Fermi gas it is sufficient to know the functions (1) with half-integer indices  $s = 1/2, 3/2, 5/2$ .

The thermodynamic potential  $\Omega$ , energy  $E$ , entropy  $S$ , particle number density  $n$  and pressure  $p$  of the Fermi gas of particles with the spin 1/2 in a volume, expressed through the functions (1), are given by the formulas:

$$\begin{aligned} \Omega &= -\frac{2TV}{\Lambda^3} \Phi_{5/2}(t), & E &= \frac{3TV}{\Lambda^3} \Phi_{5/2}(t), & S &= \frac{2V}{\Lambda^3} \left[ \frac{5}{2} \Phi_{5/2}(t) - t \Phi_{3/2}(t) \right], \\ n &= \frac{N}{V} = \frac{2}{\Lambda^3} \Phi_{3/2}(t), & p &= \frac{2T}{\Lambda^3} \Phi_{5/2}(t). \end{aligned} \quad (2)$$

Here  $t \equiv \mu/T$ ,  $\mu$  is the chemical potential. The thermal de Broglie wavelength enters into the formulas (2):

$$\Lambda \equiv \left( \frac{2\pi\hbar^2}{mT} \right)^{1/2}. \quad (3)$$

In a bulk gas in addition to the thermal wavelength of a particle (3) there exists one more characteristic length  $l = n^{-1/3}$  which defines an average distance between particles. The ratio of these lengths

$$q_\Lambda \equiv \frac{\Lambda}{l} = \Lambda n^{1/3} = [2\Phi_{3/2}(t)]^{1/3} \quad (4)$$

characterizes the extent of proximity of the gas to the degenerate state, so to say the measure of its ‘‘quantumness’’. Depending on the density of number of particles and temperature the quantum mechanical properties of the gas will manifest themselves to a greater or lesser extent. If  $q_\Lambda$  is small, the system can be well described by the classical mechanics. With decreasing temperature at a fixed density the thermal wavelength increases and, therefore, the ‘‘quantumness’’ of the gas increases in this case, and besides, as we see, the parameter  $q_\Lambda$  depends on the single parameter  $t$ .

Here we also give the formulas for the heat capacities at a constant volume and a constant pressure:

$$\begin{aligned} C_V &= \frac{15}{2} \frac{V}{\Lambda^3} \left[ \Phi_{5/2}(t) - \frac{3}{5} \frac{\Phi_{3/2}^2(t)}{\Phi_{1/2}(t)} \right], \\ C_p &= \frac{25}{2} \frac{V}{\Lambda^3} \Phi_{5/2}(t) \left[ \frac{\Phi_{1/2}(t) \Phi_{3/2}(t)}{\Phi_{3/2}^2(t)} - \frac{3}{5} \right]. \end{aligned} \quad (5)$$

It should be noted that the heat capacities per one particle  $C_V/N$  and  $C_p/N$  also depend on the single parameter  $t$ . The properties of Fermi systems in the presence of the discrete levels in magnetic field were studied with the help of the functions (1) in work [16].

### THERMODYNAMICS OF THE FERMION GAS IN A RECTANGULAR QUANTUM WELL

The model of the ideal Fermi gas is the basis for studying the bulk properties of the Fermi systems for particles of different nature. In the two-dimensional case an analogous role is played by the ideal Fermi gas contained between two parallel planes, therefore a detailed study of such the system is also of general physical interest. In particular, it is important to obtain exact formulas for thermodynamic quantities of the Fermi gas contained between two parallel planes  $z = L/2$ ,  $z = -L/2$  and to make analysis of its thermodynamic properties. It is assumed everywhere that the spin of the Fermi particle is equal to 1/2. The lengths  $L_x, L_y$  are as before considered to be macroscopic, where  $A = L_x L_y$  is the area in the  $(x, y)$  plane, but no restrictions are imposed on the length of the third side  $L_z = L$  and it can be small, that corresponds to transition to the quasi-two-dimensional case. The total volume occupied by the Fermi gas  $V = AL$ . The case when  $L_x \gg L$ ,  $L_y \gg L$  is of the most interest and the main attention will be paid to it. Note that in work [15] this case was not considered in detail. Let us assume that the potential barrier at the points  $z = L/2$  and  $z = -L/2$  is infinite, so that the wave function of a particle turns into zero at boundaries. In this case solutions of the Schrödinger equation have the form

$$\begin{aligned}\varphi_{\mathbf{k},n}^{(+)}(x,y,z) &= \sqrt{\frac{2}{AL}} e^{i\mathbf{k}\mathbf{r}} \cos(2n+1)\frac{\pi z}{L}, & (n=0,1,2,\dots), \\ \varphi_{\mathbf{k},n}^{(-)}(x,y,z) &= \sqrt{\frac{2}{AL}} e^{i\mathbf{k}\mathbf{r}} \sin 2n\frac{\pi z}{L}, & (n=1,2,\dots).\end{aligned}\quad (6)$$

The first of these functions is even and the second is odd with respect to the transformation  $z \rightarrow -z$ . Here  $\mathbf{k} \equiv (k_x, k_y)$  and  $\mathbf{r} \equiv (x, y)$  are two-dimensional vectors. The energy of a particle:

$$\varepsilon_{kn} = \frac{\hbar^2 k^2}{2m} + \varepsilon_L n^2, \quad (7)$$

$n=1,2,\dots$  and  $k^2 = k_x^2 + k_y^2$ . For odd  $n$  the levels correspond to even wave functions, and for even  $n$  – to odd wave functions. The energy

$$\varepsilon_L \equiv \frac{\pi^2 \hbar^2}{2mL^2} \quad (8)$$

is conditioned by localization of a quantum particle between planes and increases with decreasing the distance  $L$  between them. This characteristic energy of a problem (we assume the particle is an electron), expressed in Rydbergs, can be written in the form  $\tilde{\varepsilon}_L = \pi^2 / \tilde{L}^2$ , where  $\tilde{L} = L/a_0$  is the distance in the Bohr radiuses. Considering that  $1 \text{ Ry} = 13.6 \text{ eV} = 15.8 \cdot 10^4 \text{ K}$ , we obtain: 1) at  $\tilde{L} = 10^2$  –  $\tilde{\varepsilon}_L = 156 \text{ K}$ ; 2) at  $\tilde{L} = 10^3$  –  $\tilde{\varepsilon}_L = 1.56 \text{ K}$ ; 3) at  $\tilde{L} = 10^4$  –  $\tilde{\varepsilon}_L = 0.016 \text{ K}$ ; 4) at  $\tilde{L} = 10^6$  –  $\tilde{\varepsilon}_L = 1.6 \cdot 10^{-6} \text{ K}$ . Thus, quantum effects connected with the presence of the energy (8), as it had to be expected, must manifest themselves essentially at low temperatures and small distances between planes. The distribution function in this case has the form

$$f_{kn} = [\exp(\varepsilon_{kn} - \mu)/T + 1]^{-1}. \quad (9)$$

After integration over momenta with the function (9), the thermodynamical potential

$$\Omega = -2T \sum_{k,n} \ln [1 + e^{-(\varepsilon_{kn} - \mu)/T}] \quad (10)$$

will be determined by the formula

$$\Omega = -\frac{2TA}{\Lambda^2} \Psi_2(\tau, \eta). \quad (11)$$

Instead of the parameter  $t = \mu/T$ , that was used in formulas for the volume case, it is convenient to introduce the dimensionless chemical potential  $\eta \equiv \mu/\varepsilon_L$  and the dimensionless temperature  $\tau \equiv T/\varepsilon_L$ , then  $t = \eta/\tau$ . We define the function

$$\Psi_s(\tau, \eta) \equiv \sum_{n=1}^{\infty} \Phi_s[\tau^{-1}(\eta - n^2)]. \quad (12)$$

The details of calculation of such functions are given in Appendix. The thermodynamic potential (11) is a function of the temperature, chemical potential, area and distance between plates:  $\Omega = \Omega(T, \mu, A, L)$ . In contrast to the volume case when  $\Omega$  is proportional to the volume  $V$ , in this case it is proportional to the area  $A$  and depends in a complicated manner on the distance  $L$ . This circumstance is conditioned by the evident anisotropy of the system under consideration, since here the motions in the  $(x, y)$  plane and in the direction of the  $z$  axis are qualitatively different. In statistical mechanics it is customary to pass in the final formulas to the thermodynamic limit  $V \rightarrow \infty, N \rightarrow \infty$  at  $n = N/V = \text{const}$ . In the present case it is more accurate to write down the thermodynamic limit somewhat differently, namely

$$A \rightarrow \infty, N \rightarrow \infty \quad \text{at} \quad n_A \equiv N/A = \text{const}. \quad (13)$$

It is thereby stressed that the transition to infinite volume occurs only owing to increasing the area, at a fixed distance  $L$ .

The differential of the thermodynamic potential (11) has the form

$$d\Omega = -\frac{2A}{\Lambda^2} \left( 2\Psi_2 + \tau \frac{\partial \Psi_2}{\partial \tau} \right) dT - \frac{2A}{\Lambda^2} \Psi_1 d\mu + \frac{\Omega}{A} dA - \frac{4AT}{\Lambda^2 L} \left( \tau \frac{\partial \Psi_2}{\partial \tau} + \eta \frac{\partial \Psi_2}{\partial \eta} \right) dL. \quad (14)$$

It was taken into account that  $\partial \Psi_2 / \partial \eta = \tau^{-1} \Psi_1$  and  $d\varepsilon_L = -(2\varepsilon_L/L)dL$ . Since  $S = -(\partial \Omega / \partial T)_{\mu, A, L}$  and  $N = -(\partial \Omega / \partial \mu)_{T, A, L}$ , from (14) there follow expressions for the entropy and number of particles:

$$S = \frac{2A}{\Lambda^2} \left( 2\Psi_2 + \tau \frac{\partial \Psi_2}{\partial \tau} \right), \quad (15)$$

$$N = \frac{2A}{\Lambda^2} \Psi_1. \quad (16)$$

The volume and surface densities of number of particles are defined by obvious relations:  $n \equiv N/AL$ ,  $n_A \equiv N/A$ . The same formulas for the entropy and number of particles can be derived, of course, straight by means of the distribution function (9). The energy is determined by the formula:

$$E = \frac{2AT}{\Lambda^2} \left( \Psi_2 + \frac{\eta}{\tau} \Psi_1 + \tau \frac{\partial \Psi_2}{\partial \tau} \right). \quad (17)$$

Naturally, the relation holds  $\Omega = E - TS - \mu N$ .

### PRESSURES

In a bulk system the pressure is connected with the thermodynamical potential by the known formula  $p = -\Omega/V$ . In the considered case the system is anisotropic, since the character of motion of particles in the directions parallel and perpendicular to planes is different, and, therefore, the usual formula for the pressure is invalid. The force exerted by the gas on the wall perpendicular to the  $z$  axis is different from the force exerted on the side walls perpendicular to the  $x$  and  $y$  axes. These forces can be calculated in the same way as in the volume case [6]. The pressures in directions parallel to the  $(x, y)$  plane and on the planes perpendicular to the  $z$  axis are given by the formulas

$$p_{\parallel} = -\frac{1}{L} \left( \frac{\partial E}{\partial A} \right)_{S,L}, \quad p_{\perp} = -\frac{1}{A} \left( \frac{\partial E}{\partial L} \right)_{S,A}. \quad (18)$$

Since

$$\left( \frac{\partial E}{\partial A} \right)_{S,N,L} = \left( \frac{\partial \Omega}{\partial A} \right)_{T,\mu,L}, \quad \left( \frac{\partial E}{\partial L} \right)_{S,N,A} = \left( \frac{\partial \Omega}{\partial L} \right)_{T,\mu,A}, \quad (19)$$

it is then more convenient to calculate the pressures (18) using the formulas

$$p_{\parallel} = -\frac{1}{L} \left( \frac{\partial \Omega}{\partial A} \right)_{T,\mu,L}, \quad p_{\perp} = -\frac{1}{A} \left( \frac{\partial \Omega}{\partial L} \right)_{T,\mu,A}. \quad (20)$$

The differential of the thermodynamic potential (14) can be represented in the form

$$d\Omega = -SdT - Nd\mu - p_{\parallel}LdA - p_{\perp}AdL. \quad (21)$$

Considering the form of the thermodynamic potential (11), we obtain the formulas determining the pressures through the functions (12):

$$p_{\parallel} = \frac{2T}{\Lambda^2 L} \Psi_2, \quad p_{\perp} = \frac{4T}{\Lambda^2 L} \left( \tau \frac{\partial \Psi_2}{\partial \tau} + \frac{\eta}{\tau} \Psi_1 \right). \quad (22)$$

The quantity  $-p_{\parallel}L$  is an analog of the surface tension in the theory of surfaces [6]. The energy (17) is connected with the pressures (22) by the relation

$$E = AL \left( p_{\parallel} + \frac{1}{2} p_{\perp} \right), \quad (23)$$

which in the volume limit  $p_{\parallel} = p_{\perp} = p$  turns into the known relation  $pV = (2/3)E$  for the Fermi gas [6].

### REDUCED FORM OF THERMODYNAMIC QUANTITIES

It is convenient to introduce dimensionless quantities, which we will call "reduced" and designate them by a tilde on top, for the entropy, energy, pressures, volume and surface densities:

$$\begin{aligned} \tilde{S} &\equiv \frac{2L^2}{\pi A} S, & \tilde{E} &\equiv \frac{2mL^4}{\pi^3 \hbar^2 A} E, & \tilde{p}_{\parallel} &\equiv \frac{2mL^5}{\pi^3 \hbar^2} p_{\parallel}, & \tilde{p}_{\perp} &\equiv \frac{2mL^5}{\pi^3 \hbar^2} p_{\perp}, \\ \tilde{n} &\equiv \frac{2L^3}{\pi} n, & \tilde{n}_A &\equiv \frac{2L^2}{\pi} n_A. \end{aligned} \quad (24)$$

The reduced quantities are functions of only two independent dimensionless variables – the temperature  $\tau$  and chemical potential  $\eta$ :

$$\begin{aligned} \tilde{S} &= \tau \left[ 2\Psi_2 + \tau \frac{\partial \Psi_2}{\partial \tau} \right], & \tilde{E} &= \frac{\tau^2}{2} \left( \Psi_2 + \frac{\eta}{\tau} \Psi_1 + \tau \frac{\partial \Psi_2}{\partial \tau} \right), & \tilde{n} &= \tilde{n}_A = \tau \Psi_1, \\ \tilde{p}_{\parallel} &= \frac{\tau^2}{2} \Psi_2, & \tilde{p}_{\perp} &= \tau^2 \left( \frac{\eta}{\tau} \Psi_1 + \tau \frac{\partial \Psi_2}{\partial \tau} \right). \end{aligned} \quad (25)$$

The use of the reduces quantities is convenient owing to the fact that they do not contain explicitly geometric dimensions of the system.

### HEAT CAPACITIES

An important directly observable thermodynamic quantity is the heat capacity. In the geometry under consideration heat capacities can be defined under various conditions different from that which take place in the volume case. In order to determine heat capacities, it is necessary to calculate the quantity  $C = T(dS/dT)$ . For this purpose, it is convenient to express the differential of the entropy through the reduced quantities:

$$dS = \frac{\pi A}{2L^2} \left( -\frac{\tilde{S}}{\tilde{n}_A} d\tilde{n}_A + d\tilde{S} \right) = \frac{\pi A}{2L^2} \left[ \left( \frac{\partial \tilde{S}}{\partial \eta} - \frac{\tilde{S}}{\tilde{n}_A} \frac{\partial \tilde{n}_A}{\partial \eta} \right) d\eta + \left( \frac{\partial \tilde{S}}{\partial \tau} - \frac{\tilde{S}}{\tilde{n}_A} \frac{\partial \tilde{n}_A}{\partial \tau} \right) d\tau \right]. \quad (26)$$

In the volume case at a fixed number of particles, which is assumed here, the equation of state is:  $p = p(T, V)$ . If the chemical potential is used as an independent variable, then the equation of state is defined parametrically by the equations  $p = p(T, V, \mu)$  and  $N = N(T, V, \mu)$ . To obtain the heat capacity as a function of only temperature, one constraint should be imposed between the pressure and the volume. In the simplest case, it is possible to fix either the volume or the pressure, thus determining the heat capacities  $C_V$  and  $C_p$ .

Under given conditions, owing to anisotropy of the system, there are two equations of state (22) for two pressures  $p_{\parallel} = p_{\parallel}(T, A, L, \mu)$  and  $p_{\perp} = p_{\perp}(T, A, L, \mu)$ , which at a fixed number of particles should be considered together with the equation (16)  $N = N(T, A, L, \mu)$ . To obtain the heat capacity as a function of only temperature, two additional constraints should be set between the pressures  $p_{\parallel}, p_{\perp}$  and the dimensions of the system  $A, L$ , namely  $F_1(p_{\parallel}, p_{\perp}, A, L) = 0$  and  $F_2(p_{\parallel}, p_{\perp}, A, L) = 0$ . In the simplest case, two of four quantities  $p_{\parallel}, p_{\perp}, A, L$  can be fixed. Then the heat capacity as a function of temperature can be considered under fixation of one of the following pairs of quantities:  $(A, L)$ ,  $(p_{\parallel}, p_{\perp})$ ,  $(A, p_{\parallel})$ ,  $(A, p_{\perp})$ ,  $(L, p_{\parallel})$ ,  $(L, p_{\perp})$ . Fixation of the first of pairs  $(A, L)$  corresponds to the volume case of the heat capacity at a constant volume, and of the second  $(p_{\parallel}, p_{\perp})$  – at a constant pressure.

With account of the fixation of a number of particles, we have

$$d\tilde{n}_A = \tilde{n}_A \left( 2 \frac{dL}{L} - \frac{dA}{A} \right). \quad (27)$$

Also it should be taken into account that

$$\frac{d\tau}{dT} = \frac{1}{\varepsilon_L} + \frac{2\tau}{L} \frac{dL}{dT}. \quad (28)$$

Finally, we obtain formulas for the reduced heat capacities  $\tilde{C} \equiv \frac{2L^2}{\pi A} C$  under different conditions for arbitrary temperatures:

$$\tilde{C}_{AL} = \tau \left[ \frac{\partial \tilde{S}}{\partial \tau} - \frac{\partial \tilde{S}}{\partial \eta} \left( \frac{\partial \tilde{n}_A}{\partial \tau} / \frac{\partial \tilde{n}_A}{\partial \eta} \right) \right], \quad (29)$$

$$\tilde{C}_{p_{\parallel} p_{\perp}} = \tau \frac{\left\{ \left( \frac{\partial \tilde{S}}{\partial \eta} - \frac{\tilde{S}}{\tilde{n}_A} \frac{\partial \tilde{n}_A}{\partial \eta} \right) \left( \tilde{p}_{\perp} \frac{\partial \tilde{p}_{\parallel}}{\partial \tau} - \tilde{p}_{\parallel} \frac{\partial \tilde{p}_{\perp}}{\partial \tau} \right) + \left( \frac{\partial \tilde{S}}{\partial \tau} - \frac{\tilde{S}}{\tilde{n}_A} \frac{\partial \tilde{n}_A}{\partial \tau} \right) \left( \tilde{p}_{\parallel} \frac{\partial \tilde{p}_{\perp}}{\partial \eta} - \tilde{p}_{\perp} \frac{\partial \tilde{p}_{\parallel}}{\partial \eta} \right) \right\}}{\left[ \tilde{p}_{\parallel} \frac{\partial \tilde{p}_{\perp}}{\partial \eta} - \tilde{p}_{\perp} \frac{\partial \tilde{p}_{\parallel}}{\partial \eta} + \frac{2}{5} \tau \left( \frac{\partial \tilde{p}_{\parallel}}{\partial \eta} \frac{\partial \tilde{p}_{\perp}}{\partial \tau} - \frac{\partial \tilde{p}_{\perp}}{\partial \tau} \frac{\partial \tilde{p}_{\parallel}}{\partial \eta} \right) \right]}, \quad (30)$$

$$\tilde{C}_{L p_{\parallel}} = \tau \left\{ \left( \frac{\partial \tilde{S}}{\partial \tau} - \frac{\tilde{S}}{\tilde{n}_A} \frac{\partial \tilde{n}_A}{\partial \tau} \right) - \left( \frac{\partial \tilde{S}}{\partial \eta} - \frac{\tilde{S}}{\tilde{n}_A} \frac{\partial \tilde{n}_A}{\partial \eta} \right) \left( \frac{\partial \tilde{p}_{\parallel}}{\partial \tau} / \frac{\partial \tilde{p}_{\parallel}}{\partial \eta} \right) \right\}, \quad (31)$$

$$\tilde{C}_{A p_{\parallel}} = \tau \frac{\left\{ \tilde{n}_A \frac{\partial \tilde{p}_{\parallel}}{\partial \eta} \left( \frac{\partial \tilde{S}}{\partial \tau} - \frac{\tilde{S}}{\tilde{n}_A} \frac{\partial \tilde{n}_A}{\partial \tau} \right) - \tilde{n}_A \frac{\partial \tilde{p}_{\parallel}}{\partial \tau} \left( \frac{\partial \tilde{S}}{\partial \eta} - \frac{\tilde{S}}{\tilde{n}_A} \frac{\partial \tilde{n}_A}{\partial \eta} \right) + \frac{5}{2} \tilde{p}_{\parallel} \left( \frac{\partial \tilde{S}}{\partial \eta} \frac{\partial \tilde{n}_A}{\partial \tau} - \frac{\partial \tilde{S}}{\partial \tau} \frac{\partial \tilde{n}_A}{\partial \eta} \right) \right\}}{\tilde{n}_A \frac{\partial \tilde{p}_{\parallel}}{\partial \eta} - \frac{5}{2} \tilde{p}_{\parallel} \frac{\partial \tilde{n}_A}{\partial \eta} + \tau \left( \frac{\partial \tilde{n}_A}{\partial \eta} \frac{\partial \tilde{p}_{\parallel}}{\partial \tau} - \frac{\partial \tilde{n}_A}{\partial \tau} \frac{\partial \tilde{p}_{\parallel}}{\partial \eta} \right)}. \quad (32)$$

The heat capacities  $\tilde{C}_{L p_{\perp}}$  and  $\tilde{C}_{A p_{\perp}}$  are determined by the formulas (31), (32) with account of the replacement  $p_{\parallel} \rightarrow p_{\perp}$ .

**COMPRESSIBILITIES**

Another directly observable quantities are compressibilities. We define “parallel” and “perpendicular” compressibilities by the relations

$$\gamma_{\parallel} = \frac{1}{n} \left( \frac{\partial n}{\partial p_{\parallel}} \right)_A, \quad \gamma_{\perp} = \frac{1}{n} \left( \frac{\partial n}{\partial p_{\perp}} \right)_A. \tag{33}$$

Compressibilities can be calculated under condition of constant temperature (isothermal) and constant entropy (adiabatic). For compressibilities in isothermal conditions, defined by the relations (33), we obtain:

$$\tilde{\gamma}_{\parallel T} \equiv \frac{\pi^3 \hbar^2}{4mL^5} \gamma_{\parallel T} = \frac{(\partial \tilde{n} / \partial \eta)}{2\tilde{n} (\partial \tilde{p}_{\parallel} / \partial \eta)}, \tag{34}$$

$$\tilde{\gamma}_{\perp T} \equiv \frac{\pi^3 \hbar^2}{4mL^5} \gamma_{\perp T} = \frac{(\partial \tilde{n} / \partial \eta)}{4 \left[ \left( \frac{5}{2} \tilde{p}_{\perp} - \tau \frac{\partial \tilde{p}_{\perp}}{\partial \tau} \right) \frac{\partial \tilde{n}}{\partial \eta} - \left( \tilde{n} - \tau \frac{\partial \tilde{n}}{\partial \tau} \right) \frac{\partial \tilde{p}_{\perp}}{\partial \eta} \right]}. \tag{35}$$

The adiabaticity condition consists in the invariance of the entropy per one particle (and therefore of the total entropy in the system with a fixed number of particles). In the volume case in adiabatic processes the parameter  $t = \mu/T$  is constant. It is easy to verify, using the formulas (2), that this condition leads to the known equations of the adiabat:  $n/T^{3/2} = C_1$ ,  $p/n^{5/3} = C_2$ ,  $p/T^{5/2} = C_3$ , where  $C_i$  are constants.

In the considered case the adiabaticity condition has the form:

$$\sigma \equiv \frac{S}{N} = \frac{1}{\Psi_1} \left( 2\Psi_2 + \tau \frac{\partial \Psi_2}{\partial \tau} \right) \equiv \Theta(\tau, \eta) = \text{const}. \tag{36}$$

Together with the equation for the number of particles (16) the equation (36) determines relationships between the density, temperature and pressures in adiabatic processes. The adiabatic compressibilities are given by the formulas:

$$\tilde{\gamma}_{\parallel \sigma} = \frac{\frac{\partial \Theta}{\partial \eta} \frac{\partial \tilde{n}}{\partial \tau} - \frac{\partial \Theta}{\partial \tau} \frac{\partial \tilde{n}}{\partial \eta}}{2\tilde{n} \left[ \frac{\partial \Theta}{\partial \eta} \frac{\partial \tilde{p}_{\parallel}}{\partial \tau} - \frac{\partial \Theta}{\partial \tau} \frac{\partial \tilde{p}_{\parallel}}{\partial \eta} \right]}, \tag{37}$$

$$\tilde{\gamma}_{\perp \sigma} = \frac{\frac{\partial \Theta}{\partial \eta} \frac{\partial \tilde{n}}{\partial \tau} - \frac{\partial \Theta}{\partial \tau} \frac{\partial \tilde{n}}{\partial \eta}}{4 \left[ \frac{5}{2} \tilde{p}_{\perp} \left( \frac{\partial \Theta}{\partial \eta} \frac{\partial \tilde{n}}{\partial \tau} - \frac{\partial \Theta}{\partial \tau} \frac{\partial \tilde{n}}{\partial \eta} \right) - \tilde{n} \left( \frac{\partial \Theta}{\partial \eta} \frac{\partial \tilde{p}_{\perp}}{\partial \tau} - \frac{\partial \Theta}{\partial \tau} \frac{\partial \tilde{p}_{\perp}}{\partial \eta} \right) \right]}. \tag{38}$$

Certainly, at zero temperature the isothermal and adiabatic compressibilities coincide.

**ANALYSIS OF FUNCTIONS  $\Psi_1(\tau, \eta)$  AND  $\Psi_2(\tau, \eta)$**

As shown above, all thermodynamic quantities are expressed through the functions  $\Psi_1(\tau, \eta)$ ,  $\Psi_2(\tau, \eta)$  and their derivatives. In this section we study the properties of these functions. The details of calculations are given in Appendix. Note that when studying oscillations in the Fermi gas with quantized levels, usually the Poisson formula is used for the extraction of an oscillating part [6,14,15]. But a detailed analysis undertaken by the authors shows that it is more convenient to calculate the standard functions (12), by which thermodynamic quantities are expressed, without use of the Poisson formula. This, in particular, is connected with the fact that the possibility of extraction of an oscillating part in some function does not at all mean that the total function is oscillating, and the contribution of non-oscillating part should be analyzed as well. As a simple example let us consider the function  $f(x, a) = (a/2)x^2 + \sin 2x$ . Despite this function contains an oscillating term, its behavior depends on the value of its non-oscillating part, that is, the value of the parameter  $a$ . The form of this function and its derivative at some values of  $a$  is shown in Fig. 1. At  $a = 0.1$  both the function and its derivative oscillate (curves 1). At  $a = 2$  the function itself already proves to be monotonically increasing, while its derivative remains oscillating (curves 2). And at  $a = 4$  both the function and its derivative monotonically increase (curves 3). As it will be seen, a similar situation takes place as well for the functions considered in the present work. Also it should be noted that for establishing correct thermodynamic relations, the total thermodynamic potential should be considered, with account of contributions of both oscillating and non-oscillating parts.

At fixed particle number density and at high temperatures, the same as in the volume case, the chemical potential is negative. With decreasing temperature it increases and at some temperature  $T_0$  turns into zero ( $\eta = 0$ ), becoming further positive. There is one more characteristic temperature  $T_L$ , at which  $\mu = \varepsilon_L$  ( $\eta = 1$ ). The dependencies of the

dimensionless chemical potential  $\eta$  on the dimensionless temperature  $\tau$  are shown in Fig. 2. The characteristic temperatures  $\tau_0 = T_0/\varepsilon_L$  and  $\tau_L = T_L/\varepsilon_L$  are determined from the equations:

$$\tilde{n}_A = \tau_0 \Psi_1(\tau_0, 0), \quad \tilde{n}_A = \tau_L \Psi_1(\tau_L, 1). \tag{39}$$

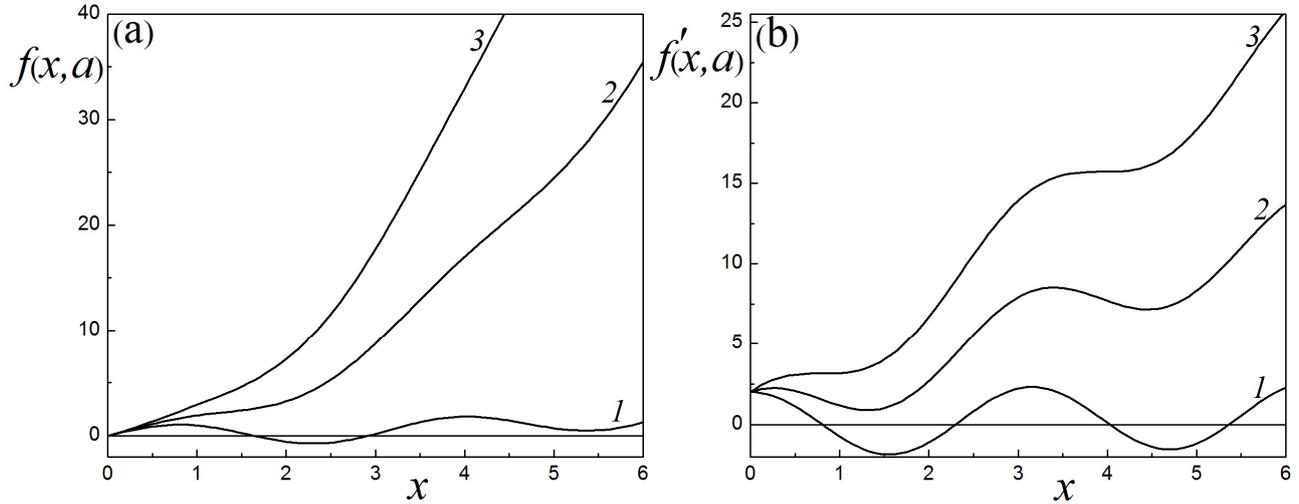


Fig. 1. Graphs of the functions  $f(x, a)$ ,  $f'(x, a)$  for some values of the parameter  $a$ .

(a) The function  $f(x, a) = \frac{a}{2}x^2 + \sin 2x$  for the values of the parameter  $a$ : (1) 0.1; (2) 2.0; (3) 4.0.

(b) The derivative  $f'(x, a) = ax + 2\cos 2x$  for the same values of the parameter  $a$ : (1) 0.1; (2) 2.0; (3) 4.0.

The region where  $\mu \leq \varepsilon_L$  ( $\eta \leq 1$ ) will be for convenience called the high temperature region, and the region  $\mu > \varepsilon_L$  ( $\eta > 1$ ) – the low temperature region. The functions  $\Psi_s(\tau, \eta)$  are calculated differently in these regions. At  $\eta \leq 1$  they can be calculated by the formula

$$\Psi_s(\tau, \eta) = \frac{1}{2} \sum_{l=1}^{\infty} \frac{(-1)^{l+1} e^{\frac{l\eta}{\tau}}}{l^s} \theta_3\left(0, \frac{l}{\pi^2 \tau}\right) - \frac{1}{2} \Phi_s\left(\frac{\eta}{\tau}\right), \tag{40}$$

where  $\theta_3(v, x) \equiv 1 + 2 \sum_{k=1}^{\infty} e^{-k^2 \pi^2 x} \cos 2\pi v x$  is the theta-function.

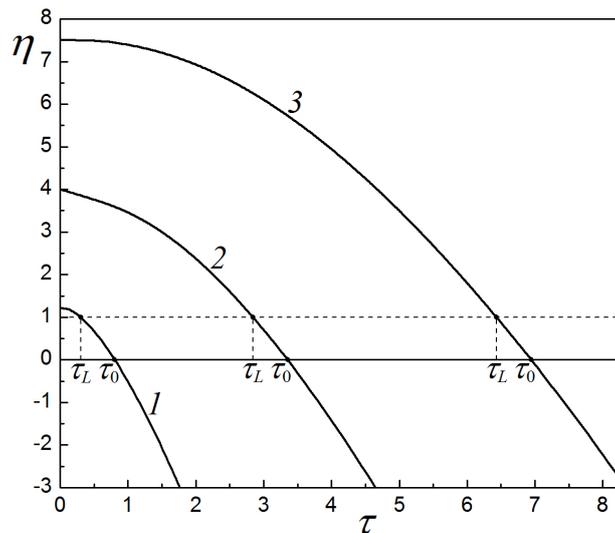


Fig. 2. The dependencies of the chemical potential on temperature  $\eta(\tau)$  at different values of the reduced density:

(1)  $\tilde{n}_A = 0.21$ ,  $\tau_L = 0.30$ ,  $\tau_0 = 0.81$ ; (2)  $\tilde{n}_A = 3.0$ ,  $\tau_L = 2.84$ ,  $\tau_0 = 3.35$ ; (3)  $\tilde{n}_A = 10.0$ ,  $\tau_L = 6.43$ ,  $\tau_0 = 6.95$ .

More interesting is the case  $\eta > 1$ , which is realized at low temperatures. Then the considered functions can be represented in the form

$$\Psi_1(\tau, \eta) = \frac{\eta}{\tau} \Psi'_1(\eta) + \Psi''_1(\tau, \eta), \quad (41)$$

$$\Psi_2(\tau, \eta) = \frac{\eta^2}{2\tau^2} \Psi'_2(\eta) + \Psi''_2(\tau, \eta). \quad (42)$$

Here the functions

$$\Psi'_1(\eta) = [x_0] \left\{ 1 - \frac{1}{6\eta} ([x_0] + 1)(2[x_0] + 1) \right\}, \quad (43)$$

$$\Psi'_2(\eta) = [x_0] \left\{ 1 - \frac{1}{3\eta} ([x_0] + 1)(2[x_0] + 1) + \frac{1}{30\eta^2} ([x_0] + 1)(2[x_0] + 1)(3[x_0]^2 + 3[x_0] - 1) \right\} \quad (44)$$

determine the state of the system at zero temperature. For brevity here and in the following it is used the designation  $x_0 \equiv \sqrt{\eta}$ ,  $x_0 > 1$  at that, and  $[x_0]$  designates the whole part of a number  $x_0$ . Graphs of the functions (43), (44) and their derivatives are shown in Fig. 3a and 3b. The function  $\Psi'_1(\eta)$  is continuous and monotonically increasing, and its derivative at the specific points  $x_0 = [x_0]$  undergoes jumps  $\Delta(d\Psi'_1(\eta)/d\eta)|_{[x_0]} = 1/[x_0]^2$  (Fig. 3a). At  $\eta \gg 1$  it has the asymptote  $\Psi'_1(\eta) \approx 2\sqrt{\eta}/3$ . The function  $\Psi'_2(\eta)$  is also continuous and monotonically increasing, with the asymptote  $\Psi'_2(\eta) \approx 8\sqrt{\eta}/15$  at  $\eta \gg 1$ . The derivative of the function  $\Psi'_2(\eta)$  is an oscillating function with a varying amplitude (Fig. 3b).

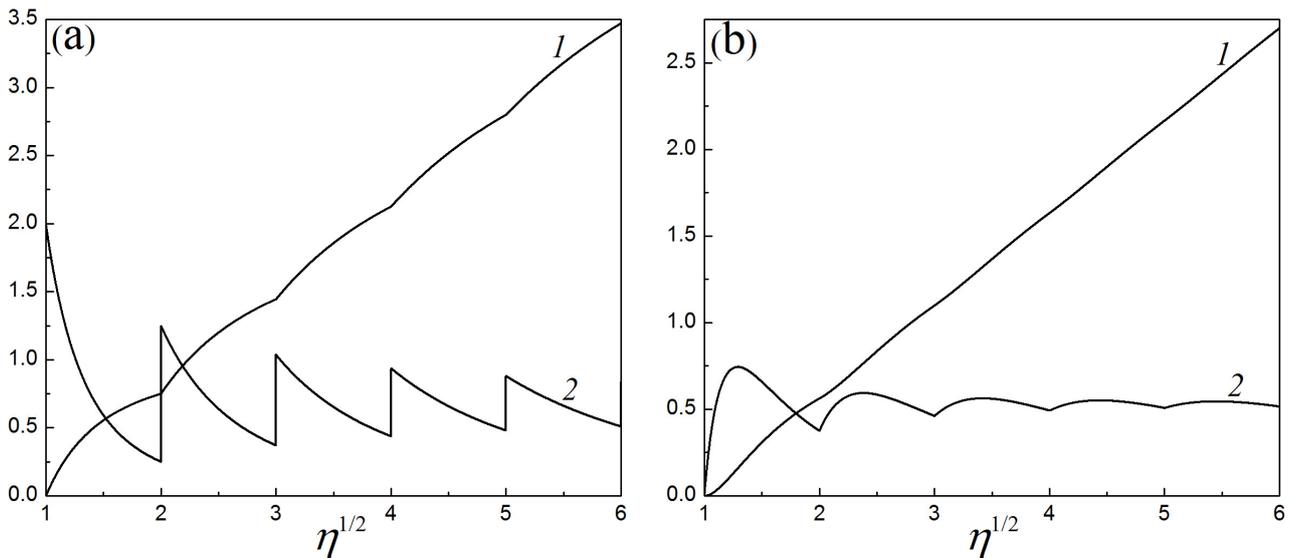


Fig. 3. Graphs of the functions  $\Psi'_1(\eta)$ ,  $\Psi'_2(\eta)$  and their derivatives.

(a) The functions  $\Psi'_1(\eta)$  (1) and  $d\Psi'_1(\eta)/d\sqrt{\eta}$  (2); (b) The functions  $\Psi'_2(\eta)$  (1) and  $d\Psi'_2(\eta)/d\sqrt{\eta}$  (2).

The functions

$$\Psi''_1(\tau, \eta) = \Phi_1 \left[ \tau^{-1} ([x_0]^2 - \eta) \right] + \Phi_1 \left[ \tau^{-1} (\eta - ([x_0] + 1)^2) \right] + \bar{\Psi}_1^{\text{exp}}(\tau, \eta), \quad (45)$$

$$\Psi''_2(\tau, \eta) = \frac{\pi^2}{6} [x_0] - \Phi_2 \left[ \tau^{-1} ([x_0]^2 - \eta) \right] + \Phi_2 \left[ \tau^{-1} (\eta - ([x_0] + 1)^2) \right] + \bar{\Psi}_2^{\text{exp}}(\tau, \eta) \quad (46)$$

describe the temperature dependencies of thermodynamic quantities at low temperatures. The form of the exponentially small at  $\tau \ll 1$  functions  $\bar{\Psi}_s^{\text{exp}}(\tau, \eta)$  is given in Appendix (the formula (11)). The dependencies of the functions (45), (46) and their derivatives on the chemical potential are shown in Fig. 4a and 4b. The function  $\Psi''_1(\tau, \eta)$  has an oscillating character and its derivative undergoes jumps at  $x_0 = [x_0]$  (Fig. 4a). The function  $\Psi''_2(\tau, \eta)$  is monotonically increasing and its derivative has oscillations (Fig. 4b). The dependencies of the total functions (41), (42) and their derivatives on the chemical potential are shown in Fig. 5a and 5b.

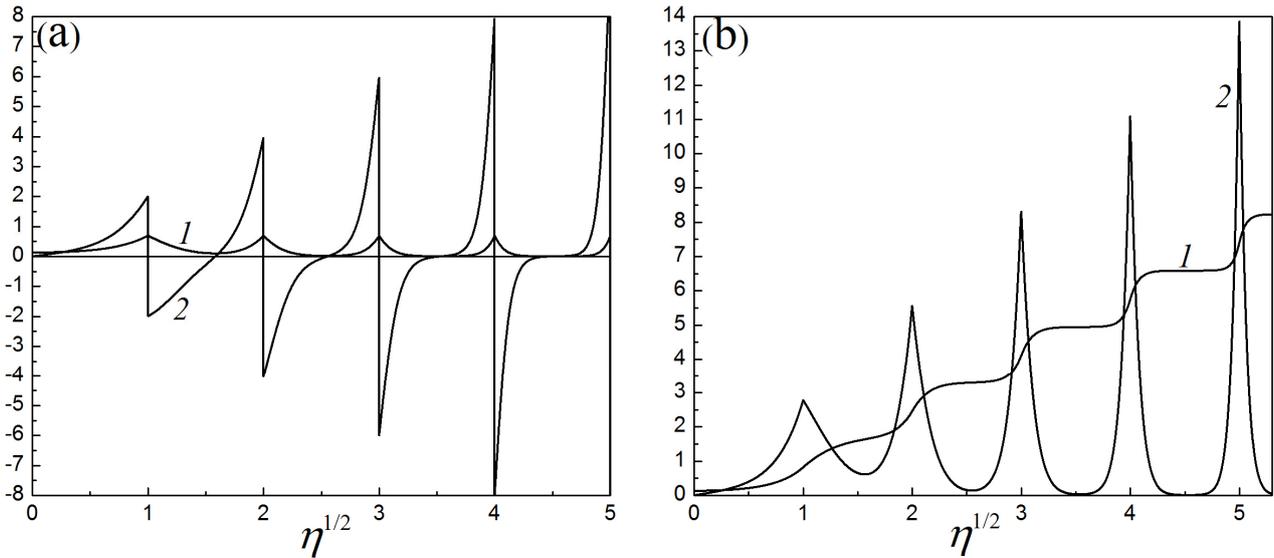


Fig. 4. Graphs of the functions  $\Psi_1''(\tau, \eta)$ ,  $\Psi_2''(\tau, \eta)$  and their derivatives at  $\tau = 0.5$ .

(a) The functions  $\Psi_1''(\tau, \eta)$  (1) and  $d\Psi_1''(\tau, \eta)/d\sqrt{\eta}$  (2); (b) The functions  $\Psi_2''(\tau, \eta)$  (1) and  $d\Psi_2''(\tau, \eta)/d\sqrt{\eta}$  (2).

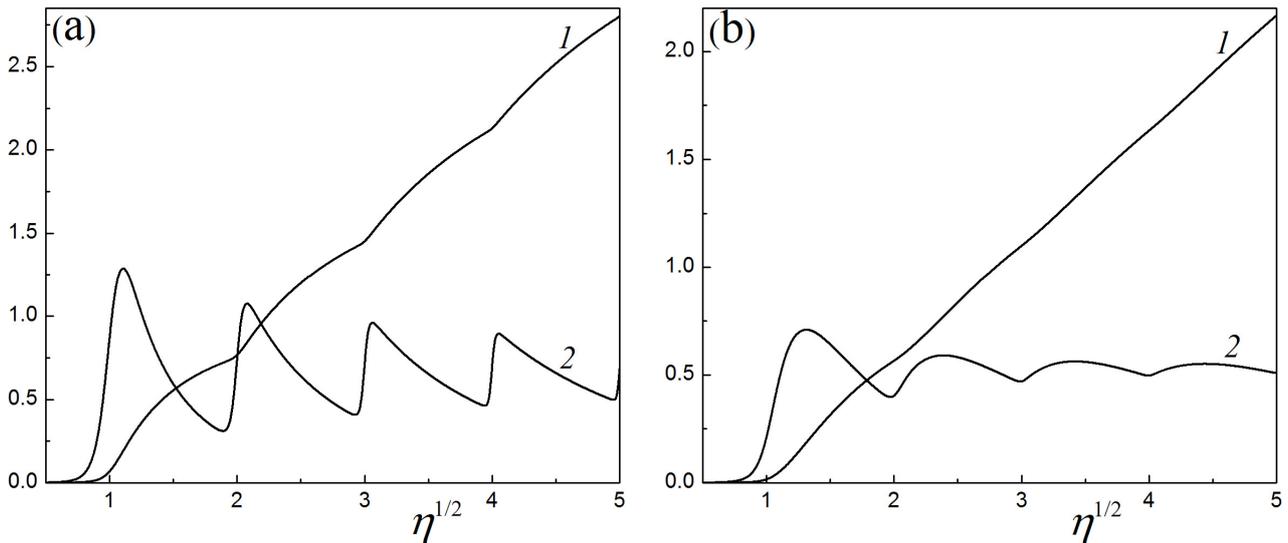


Fig. 5. Graphs of the functions  $(\tau/\eta)\Psi_1(\tau, \eta)$ ,  $2(\tau/\eta)^2\Psi_2(\tau, \eta)$  and their derivatives at  $\tau = 0.1$ .

(a) The functions  $(\tau/\eta)\Psi_1(\tau, \eta)$  (1) and  $d[(\tau/\eta)\Psi_1(\tau, \eta)]/d\sqrt{\eta}$  (2);

(b) The functions  $2(\tau/\eta)^2\Psi_2(\tau, \eta)$  (1) and  $d[2(\tau/\eta)^2\Psi_2(\tau, \eta)]/d\sqrt{\eta}$  (2).

Although, as it was noted, the function  $\Psi_1''(\tau, \eta)$  has an oscillation form, the total function  $\Psi_1(\tau, \eta)$  proves to be monotonically increasing (Fig. 5a, curve 1). Also monotonically increasing is the function  $\Psi_2(\tau, \eta)$  (Fig. 5b, curve 1). The derivatives of both of these functions have an oscillating character at not large values of the chemical potential (Fig. 5, curves 2).

Thus, the functions  $\Psi_1(\tau, \eta)$ ,  $\Psi_2(\tau, \eta)$  themselves through which the thermodynamic quantities are expressed are not oscillating, in particular there are absent oscillations of the thermodynamic potential (11) on the chemical potential  $\eta$ . However, as we will see, dependencies of some quantities on the chemical potential that include the derivatives of these functions, such as for example compressibilities, can have a nonmonotonic character.

### THERMODYNAMIC QUANTITIES AT LOW TEMPERATURES

The most interesting region where quantum effects can manifest themselves on the macroscopic level is the region of low temperatures. Let us consider the behavior of the observable characteristics at low temperatures, such that  $\tau \ll 1$ . In this limit, with account of the main exponential corrections

$$\Psi_1(\tau, \eta) = \frac{\eta}{\tau} \Psi'_1(\eta) + \exp\left[\tau^{-1}([x_0]^2 - \eta)\right] + \exp\left[\tau^{-1}(\eta - ([x_0] + 1)^2)\right], \quad (47)$$

$$\Psi_2(\tau, \eta) = \frac{\eta^2}{2\tau^2} \Psi'_2(\eta) + \frac{\pi^2}{6}[x_0] - \exp\left[\tau^{-1}([x_0]^2 - \eta)\right] + \exp\left[\tau^{-1}(\eta - ([x_0] + 1)^2)\right]. \quad (48)$$

For the reduced entropy, we obtain in this approximation:

$$\tilde{S} = \frac{\pi^2}{3}[x_0]\tau - (2\tau - a_1)e^{\frac{a_1}{\tau}} + (2\tau - a_2)e^{\frac{a_2}{\tau}}. \quad (49)$$

Here and below for brevity the designations are used  $a_1 \equiv [x_0]^2 - \eta$ ,  $a_2 \equiv \eta - ([x_0] + 1)^2$ . The dependencies of the entropy on temperature and density are shown in Figs. 6, 7. The dependence on temperature proves to be linear as in the volume case, but its slope changes by jumps as the energetic levels are being filled up (Fig. 6). The correction to the linear law, in contrast to the volume case where it is proportional to  $T^3$ , in this case is exponentially small.

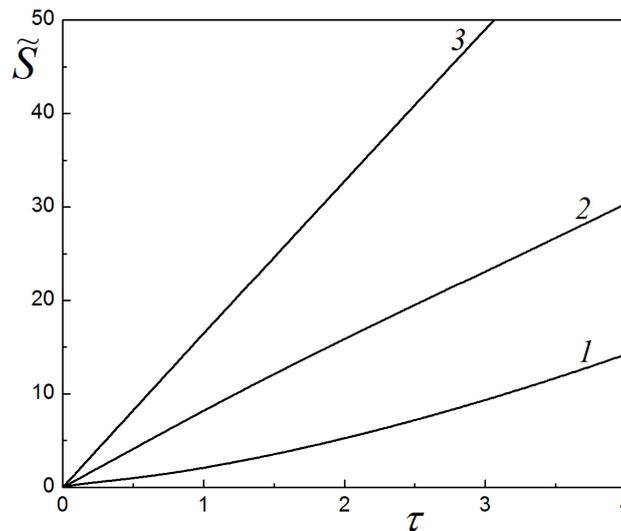


Fig. 6. The temperature dependencies of the reduced entropy  $\tilde{S}(\tau; x_0)$  at a fixed value of the chemical potential  $x_0 \equiv \sqrt{\eta}$ :  
 (1)  $x_0 = 1.1$ ; (2)  $x_0 = 3.0$ ; (3)  $x_0 = 5.5$ .

With varying the chemical potential or the density at low temperatures the entropy undergoes jumps, which are becoming more indistinct as temperature increases and entirely disappear at rather high temperatures (Fig. 7).

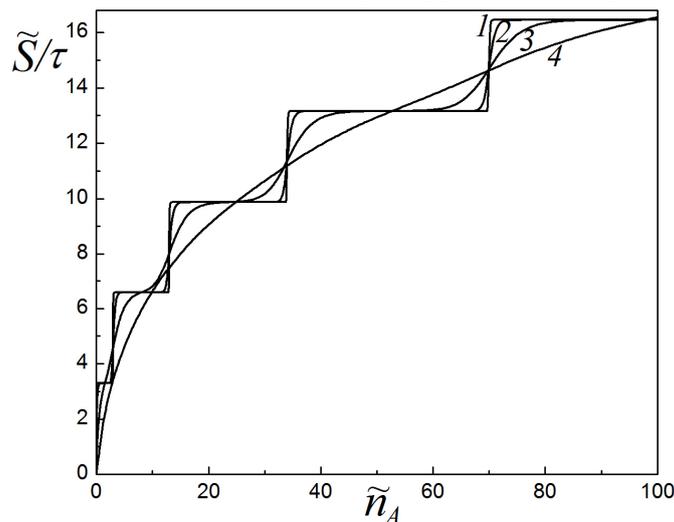


Fig. 7. The dependencies of the quantity  $\tilde{S}(\tilde{n}_A; \tau)/\tau$  on the reduced density at fixed temperature:  
 (1)  $\tau = 0.01$ ; (2)  $\tau = 0.1$ ; (3)  $\tau = 0.5$ ; (4)  $\tau = 2.0$ .

The value of the entropy jump per unit of area depends only on temperature and is determined by the formula

$$\frac{\Delta S}{A} = \frac{\pi m}{3\hbar^2} T. \quad (50)$$

Pay attention that this quantity does not explicitly depend on the distance  $L$ , though certainly this parameter enters into the condition of applicability of the formula (50)  $T \ll \pi^2 \hbar^2 / 2mL^2$ . The entropy jumps are accompanied by the absorption of heat  $\Delta Q = T\Delta S$ . For electrons the heat absorbed at the jump of the entropy per unit of area  $\Delta Q/A \approx 1.6 \cdot 10^{-5} T^2 \text{ erg}/(\text{cm}^2 \text{K}^2)$  (temperature in Kelvins). For  ${}^3\text{He}$  atoms this quantity is by three orders greater.

The density at low temperatures has only the exponentially small, depending on temperature, correction

$$\tilde{n} = \tilde{n}_A = \eta \Psi'_1(\eta) + \tau \left( e^{\frac{a_1}{\tau}} + e^{\frac{a_2}{\tau}} \right). \quad (51)$$

Neglecting the exponential corrections, the chemical potential in the expression for the entropy (49) and in other thermodynamic quantities can be taken with a good accuracy at zero temperature. In this approximation the chemical potential is connected with the reduced densities by the relation

$$\tilde{n} = \tilde{n}_A = [x_0] \left\{ \eta - \frac{1}{6} ([x_0] + 1)(2[x_0] + 1) \right\}. \quad (52)$$

The density monotonically increases with increasing the chemical potential, undergoing breaks (discontinuities in the derivative) at the specific points  $x_0 = [x_0]$ . In the limit of rather high density when  $x_0 \gg 1$ , we can set  $[x_0] \approx x_0$  and from (52) there follows the usual formula that relates the density of the bulk Fermi gas with the chemical potential at zero temperature:  $n = (2m\mu)^{3/2} / 3\pi^2 \hbar^3$ . The thermodynamical potential in this limit also acquires the usual form  $\Omega = -AL(4\sqrt{2}/15\pi^2)(m^{3/2}\mu^{5/2}/\hbar^3)$ . The condition  $x_0 \gg 1$  is equivalent to the condition  $N^{1/3} \gg (A/L^2)^{1/3}$ . In the volume case  $A \approx L^2$  it is equivalent to the condition  $N^{1/3} \gg 1$ , which is always true in a system of large number of particles. The exact formulas should be used under fulfilment of the condition  $N \sim A/L^2$ .

If thermodynamic quantities are taken in the reduced form (24), and the dimensionless chemical potential  $\eta$  and the dimensionless temperature  $\tau$  are used as independent variables, then as it was shown the geometrical dimensions fall out of the thermodynamic relations, in particular the distance between plates  $L$  falls out (or the thickness of a film, from experimentalist's point of view). Meanwhile, exactly the dependencies of the observable quantities on the thickness of a film are of interest in experiment. To obtain such dependencies, the relations derived above should be presented in the dimensional form. At that, the form of dependence of the thermodynamic quantities on the thickness of a film will essentially depend on what quantity is being fixed when studying such dependencies: the total density  $n$  or the surface density  $n_A$ . Let us show it on the example of dependence of the chemical potential on the thickness of a film.

The formula (52) can be written in the form

$$\frac{2n}{\pi} L^3 = \frac{2n_A}{\pi} L^2 = [x_0] \left\{ x_0^2 - \frac{1}{6} ([x_0] + 1)(2[x_0] + 1) \right\}. \quad (53)$$

As we see, the dependence of the distance  $L$  on the parameter  $x_0$  will be different depending on what is fixed – the volume or the surface density. The chemical potential is expressed through  $L$  and the parameter  $x_0$  by the formula

$$\mu = \frac{\pi^2 \hbar^2}{2mL^2} x_0^2. \quad (54)$$

The formulas (53) and (54) define parametrically (parameter  $x_0$ ) the dependence of the chemical potential on the thickness of a film at fixed  $n$  or  $n_A$ . It is easy to check that at a fixed surface density  $(d\mu/dL)_{n_A} < 0$  and, therefore, the chemical potential monotonically decreases with increasing  $L$ . At a fixed volume density the derivative  $(d\mu/dL)_n$  turns into zero at the minimum points determined by the equation  $x_0^2 - (1/2)([x_0] + 1)(2[x_0] + 1) = 0$ . The distance between planes at the minimum points of the chemical potential is determined by the formula

$$\frac{2n}{\pi} L_{\min}^3 = \frac{1}{3} [x_0] ([x_0] + 1)(2[x_0] + 1). \quad (55)$$

At the points  $x_0 = [x_0] \geq 2$  the sign of the derivative  $(d\mu/dL)_n$  changes by a jump. These points correspond to the local maximums of the chemical potential, so that

$$\frac{2n}{\pi} L_{\max}^3 = \frac{1}{6} [x_0] ([x_0] - 1)(4[x_0] + 1). \quad (56)$$

Thus, the dependence of the chemical potential on  $L$  at a fixed volume density has an oscillating character. The dependencies  $\mu = \mu(L)$  at fixed surface (curve 1) and volume (curve 2) densities are shown in Fig. 8.

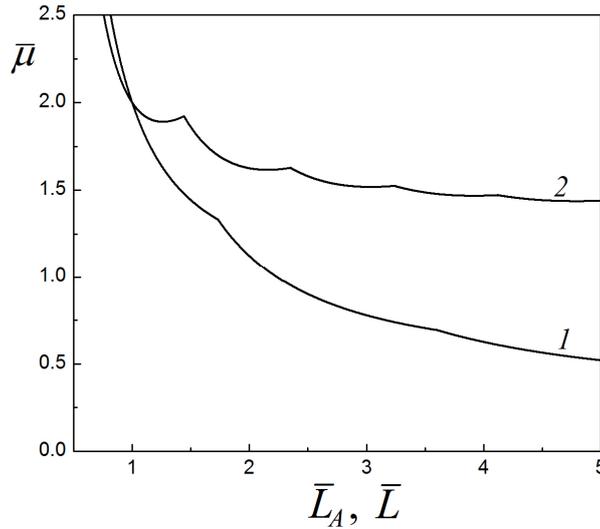


Fig. 8. The dependencies of the chemical potential on the distance between planes:  
 (1)  $\bar{\mu} = \bar{\mu}(\bar{L}_A)$  at a fixed surface density  $n_A$ , here  $\bar{\mu} \equiv (m/\pi\hbar^2 n_A)\mu$ ,  $\bar{L}_A \equiv (2n_A/\pi)^{1/2} L$ ;  
 (2)  $\bar{\mu} = \bar{\mu}(\bar{L})$  at a fixed volume density  $n$ , here  $\bar{\mu} \equiv (2/\pi^4)^{1/3} (m/\hbar^2 n^{2/3})\mu$ ,  $\bar{L} \equiv (2n/\pi)^{1/3} L$ .

The pressures at low temperatures are determined by the formulas:

$$\tilde{p}_{\parallel} = \frac{\eta^2}{4} \Psi'_2(\eta) + \frac{\pi^2}{12} [x_0] \tau^2 + \frac{\tau^2}{2} \left( -e^{\frac{a_1}{\tau}} + e^{\frac{a_2}{\tau}} \right), \tag{57}$$

$$\tilde{p}_{\perp} = \eta^2 [\Psi'_1(\eta) - \Psi'_2(\eta)] + \tau \left( [x_0]^2 e^{\frac{a_1}{\tau}} + ([x_0] + 1)^2 e^{\frac{a_2}{\tau}} \right). \tag{58}$$

As seen, the parallel pressure, in addition to the exponential temperature correction, also contains the power correction proportional to the square of temperature, and the perpendicular pressure contains only the exponential temperature correction. The dependencies of the reduced energy and pressures on the reduced density at zero temperature are shown in Fig. 9. Both the energy and the pressures monotonically increase with increasing the chemical potential or the density. At the points where the filling of discrete levels begins the dependence  $\tilde{p}_{\perp} = \tilde{p}_{\perp}(\tilde{n}_A)$  undergoes breaks. In the limit  $x_0 \gg 1$  at zero temperature the both pressures  $p_{\parallel}, p_{\perp}$  prove to be equal to the pressure of the bulk degenerate Fermi gas.

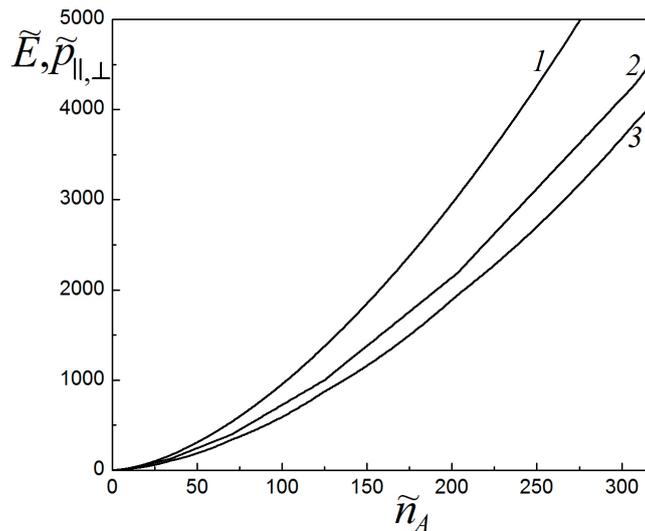


Fig. 9. The dependencies of the reduced energy and pressures on the reduced density at zero temperature:  
 (1)  $\tilde{E}(\tilde{n}_A)$ ; (2)  $\tilde{p}_{\perp}(\tilde{n}_A)$ ; (3)  $\tilde{p}_{\parallel}(\tilde{n}_A)$ .

There are of interest the dependencies of the perpendicular pressure on the distance between planes, presented in Fig. 10. The pressure at fixed  $n_A$  decreases with increasing  $L$  (curve 1). At one point, corresponding to  $x_0 = 2$ , this curve undergoes

a break: the derivative  $dp_{\perp}/dL$  on the left at this point is negative and on the right it equals to zero. At a fixed  $n$  the dependence  $p_{\perp} = p_{\perp}(L)$  has an oscillating form (curve 2). The extremum points of this function can be found in the same way as for the dependence  $\mu(L)$  at fixed  $n$  and are given by the expressions:

$$\begin{aligned} \frac{2n}{\pi} L_{\min}^3 &= \frac{1}{6} [x_0] ([x_0] - 1) (4[x_0] + 1), \\ \frac{2n}{\pi} L_{\max}^3 &= \frac{1}{12} [x_0] ([x_0] - 1) (8[x_0] + 11). \end{aligned} \quad (59)$$

On the basis of the performed analysis, seemingly, a general conclusion can be made that the oscillating dependencies of thermodynamic quantities on the width  $L$  take place for the case of the fixed total density and they are absent when the surface density is fixed.

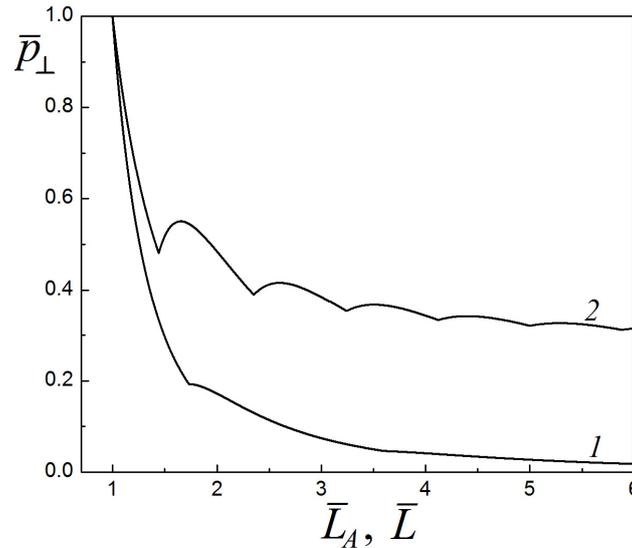


Fig. 10. The dependencies of the perpendicular pressure on the distance between planes:

- (1)  $\bar{p}_{\perp} = \bar{p}_{\perp}(\bar{L}_A)$  at a fixed surface density  $n_A$ , here  $\bar{p}_{\perp} \equiv (2m/\pi^3 \hbar^2)(\pi/2n_A)^{5/2} p_{\perp}$ ,  $\bar{L}_A \equiv (2n_A/\pi)^{1/2} L$ ;  
 (2)  $\bar{p}_{\perp} = \bar{p}_{\perp}(\bar{L})$  at a fixed volume density  $n$ , here  $\bar{p}_{\perp} \equiv (2m/\pi^3 \hbar^2)(\pi/2n)^{5/3} p_{\perp}$ ,  $\bar{L} \equiv (2n/\pi)^{1/3} L$ .

He we give also the formulas, following from the general relations (34) – (38), for the compressibilities at zero temperature:

$$\tilde{\gamma}_{\parallel} = \frac{[x_0]}{\{\eta \Psi'_1(\eta)\}^2} = \frac{1}{[x_0] \left\{ x_0^2 - \frac{1}{6} ([x_0] + 1) (2[x_0] + 1) \right\}^2}, \quad (60)$$

$$\tilde{\gamma}_{\perp} = \frac{[x_0]}{2\eta^2 \left\{ (3\Psi'_1(\eta) - 5\Psi'_2(\eta)) [x_0] + 2(\Psi'_1(\eta))^2 \right\}} = \frac{1}{[x_0] ([x_0] + 1) (2[x_0] + 1) \left\{ x_0^2 - \frac{7}{9} [x_0]^2 - \frac{2}{3} [x_0] + \frac{4}{9} \right\}}. \quad (61)$$

The dependencies of compressibilities on the volume density at zero temperature are shown in Fig. 11. As seen, at some values of density, at which the filling of levels begins, the compressibilities undergo jumps. On approaching to the point that corresponds to  $x_0 = 2$  from the side of large densities, the perpendicular compressibility tends to infinity. The nature of this divergence is similar to the nature of the break at this point in the dependence of the perpendicular pressure  $p_{\perp} = p_{\perp}(L)$  (Fig. 10, curve 1). In the limit  $x_0 \gg 1$  we have  $\gamma_{\parallel} = \gamma_{\perp} = 3^{1/3}/\pi^{4/3} m/\hbar^2 n^{5/3}$ . The quantity  $u^2 = 1/mn\gamma_{\parallel} = (\pi^{4/3}/3^{1/3}) \hbar^2 n^{2/3}/m^2$  determines in this case the square of the speed of sound in the bulk Fermi gas.

In conclusion of this section, we proceed to the analysis of the low temperature behavior of heat capacities. For the first time, the calculation of the electron heat capacity for particles of small size with account of the discreteness of the energy levels was made by Fröhlich [13]. He showed that, in contrast to the bulk Fermi gas which has the linear temperature dependence of the heat capacity, in the case of small in all coordinates particles the heat capacity decreases exponentially with decreasing temperature. In the case considered in the present paper, in addition to the discrete levels there is possible a free motion of particles along planes, that leads to the maintenance of the linear law in the temperature dependence in given conditions.

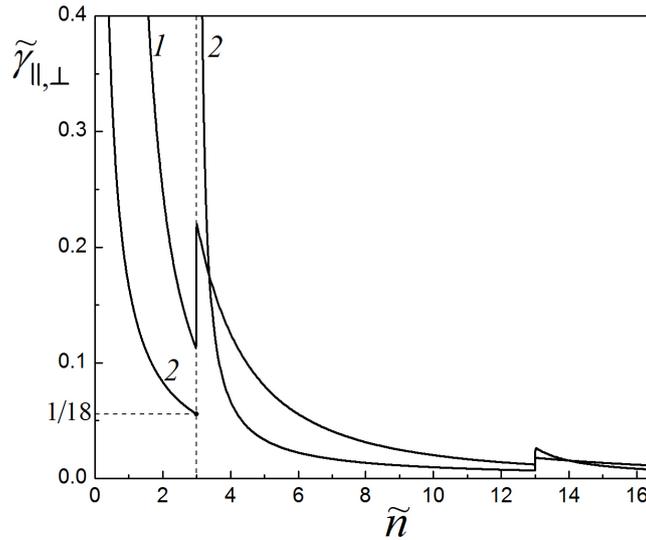


Fig. 11. The dependencies of compressibilities on the reduced density at zero temperature: (1)  $\tilde{\gamma}_{\parallel} = \tilde{\gamma}_{\parallel}(\tilde{n})$ ; (2)  $\tilde{\gamma}_{\perp} = \tilde{\gamma}_{\perp}(\tilde{n})$ .

In the main approximation all introduced above heat capacities (29) – (32), as it had to be expected, prove to be identical and proportional to temperature

$$\tilde{C} \approx \frac{\pi^2}{3} [x_0] \tau. \tag{62}$$

Under fulfilment of the condition  $x_0 \gg 1$  the formula (62), naturally, turns into the expression for the low temperature heat capacity of the bulk Fermi gas  $C = (\pi/3)^{2/3} (mT/\hbar^2) n^{1/3} LA$ . The heat capacity as a function of the chemical potential and density undergoes jumps at the points, in which the filling of new discrete levels begins (Fig. 12). The account for the corrections to the formula (62) smoothes out the steps. The value of the jump of the heat capacity is the same as that of the entropy (50):  $\Delta C/A = \pi mT/3\hbar^2$ .

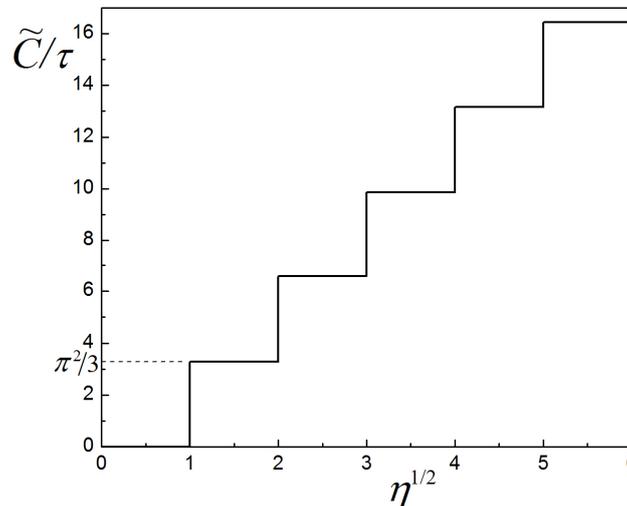


Fig. 12. The dependence of the ratio of the reduced heat capacity to the dimensionless temperature

$$\tilde{C}/\tau = \frac{\pi^2}{3} [x_0] \text{ at } \tau \ll 1 \text{ on the chemical potential } \eta^{1/2}.$$

For the heat capacities  $\tilde{C}_{AL}, \tilde{C}_{Ap_{\parallel}}, \tilde{C}_{Lp_{\perp}}, \tilde{C}_{Ap_{\perp}}$  the corrections to the linear law (62) have the exponential character, and for two heat capacities these corrections are proportional to  $\tau^3$ :

$$\tilde{C}_{p_{\parallel}p_{\perp}} \approx \frac{\pi^2}{3} [x_0] \tau \left\{ 1 + \frac{2\pi^2 [x_0]}{15 \eta^2} \frac{(5[x_0]\Psi'_2 - 3[x_0]\Psi'_1 - 2\Psi_1'^2)}{\Psi_1'([x_0]\Psi'_2 + \Psi_1'\Psi'_2 - 2\Psi_1'^2)} \tau^2 \right\}, \tag{63}$$

$$\tilde{C}_{Lp_{\parallel}} \approx \frac{\pi^2}{3} [x_0] \tau \left\{ 1 + \frac{\pi^2}{3} [x_0] \frac{\tau^2}{(\eta\Psi'_1)^2} \right\}. \tag{64}$$

The differences between the heat capacities  $\tilde{C}_{p_{\parallel}p_{\perp}}$ ,  $\tilde{C}_{Lp_{\parallel}}$  and other heat capacities are proportional to  $\tau^3$ .

**THERMODYNAMIC QUANTITIES AT HIGH TEMPERATURES**

Let us consider the area of high temperatures, where the de Broglie wavelength is much less than the average distance between particles:  $\Lambda/l = \Lambda n^{1/3} \ll 1$ . This condition is fulfilled if the parameter  $t = \eta/\tau$  is negative and large by absolute value  $|t| \gg 1$ . At high temperatures and macroscopic distances between planes, such that  $\Lambda/L = 2/\sqrt{\pi\tau} \ll 1$ , also the condition  $\sqrt{\tau} \gg 1$  holds. In this approximation, in the sum of the formula (40) it is sufficient to account for the main term with  $l = 1$ . Taking into account the relation  $\theta_3(0, q) = [2K(m)/\pi]^{1/2}$ , we obtain

$$\Psi_s(\tau, \eta) = \frac{1}{2} \left[ \sqrt{\frac{2K(m)}{\pi}} - 1 \right] e^{\frac{\eta}{\tau}}. \tag{65}$$

Here  $K(m)$  is the full elliptic integral of the first kind, and  $q = q(m) = \exp[-\pi K(1-m)/K(m)]$  is the Jacobi parameter [17]. Considering the definition of the theta-function in (40), we find that the parameter  $m$  and the dimensionless temperature are connected by the relation

$$\frac{1}{\tau} = \pi \frac{K(1-m)}{K(m)}. \tag{66}$$

For considered large values of  $\tau$ , the parameter  $m$  is close to unity. Using expansions in the small parameter  $m_1 = 1-m$  and taking into account that  $K(m) \approx \frac{1}{2} \ln \frac{16}{m_1}$ , from (66) we have:  $m_1 = 16e^{-\pi^2\tau}$ . For  $\sqrt{\tau} \gg 1$ , taking account of the main terms, we obtain

$$\Psi_s(\tau, \eta) = \frac{1}{2} [\sqrt{\pi\tau} - 1] e^{\frac{\eta}{\tau}}. \tag{67}$$

Taking account of only the first term in square brackets in (67) leads to the relations for the classical ideal gas, and accounting for the second term in brackets gives the correction on the finite width  $L$  proportional to the ratio  $\Lambda/L$ . The thermodynamic potential in the classical limit with account of such correction acquires the form

$$\Omega = -\frac{2TAL}{\Lambda^3} \left( 1 - \frac{\Lambda}{2L} \right) e^{\frac{\mu}{T}}. \tag{68}$$

From the formula for the number of particles

$$N = -\left( \frac{\partial \Omega}{\partial \mu} \right)_{T,A,L} = \frac{2AL}{\Lambda^3} \left( 1 - \frac{\Lambda}{2L} \right) e^{\frac{\mu}{T}} \tag{69}$$

there follows the dependence of the chemical potential on the density and temperature:

$$\frac{\mu}{T} = \ln \left( \frac{n\Lambda^3}{2} \right) + \frac{\Lambda}{2L}. \tag{70}$$

For the entropy we have the expression that generalizes the Sackur-Tetrode formula [6] to account for the quantum size effect:

$$S = N \ln \frac{2e^{5/2}}{n\Lambda^3} - N \frac{\Lambda}{4L}. \tag{71}$$

In the parallel pressure the linear with respect to  $\Lambda/L$  correction is absent, and it is present in the perpendicular pressure:

$$p_{\parallel} = nT, \quad p_{\perp} = nT \left( 1 + \frac{\Lambda}{2L} \right). \tag{72}$$

Here is also the formula for the energy with account of such correction:

$$E = \frac{3}{2} NT \left( 1 + \frac{\Lambda}{6L} \right). \tag{73}$$

Let us write down in the high temperature limit the expressions for all heat capacities, which were defined above, with account of the quantum size correction:

$$\begin{aligned}
 C_{AL} &= N \left( \frac{3}{2} + \frac{\Lambda}{8L} \right), & C_{p_{\parallel}p_{\perp}} &= \frac{5}{2} N, \\
 C_{Lp_{\parallel}} &= N \left( \frac{5}{2} + \frac{\Lambda}{8L} \right), & C_{Ap_{\parallel}} &= N \left( \frac{5}{2} + \frac{3\Lambda}{8L} \right), \\
 C_{Lp_{\perp}} &= N \left( \frac{5}{2} - \frac{\Lambda}{8L} \right), & C_{Ap_{\perp}} &= N \left( \frac{5}{2} - \frac{3\Lambda}{8L} \right).
 \end{aligned} \tag{74}$$

As seen, in the volume limit  $C_{AL}$  turns into the heat capacity at a constant volume of the classical ideal gas  $C_V = (3/2)N$ , and the other five heat capacities turn into the heat capacity of the classical gas at a constant pressure  $C_p = (5/2)N$ . It should be also noted that the heat capacity  $C_{p_{\parallel}p_{\perp}}$  does not contain the linear in the parameter  $\Lambda/L$  correction.

### CONCLUSION

In the paper there have been derived the exact formulas for calculation of the thermodynamic functions of the ideal Fermi gas in the quantum well formed by two parallel walls. It is shown that all thermodynamic quantities, written in the dimensionless reduced form not containing the geometric dimensions, can be expressed through some standard functions of the dimensionless temperature and the dimensionless chemical potential and their derivatives. These functions themselves do not oscillate with varying the chemical potential or density, but the derivatives of these functions have oscillations (Fig. 5). Through the introduced standard functions there are calculated the thermodynamic potential, energy, density, entropy, equations of state, heat capacities and compressibilities of the Fermi gas at arbitrary temperatures in the considered conditions of the confined geometry. It is shown that owing to the anisotropy the Fermi gas in this case has two equations of state since the pressures perpendicular and parallel to planes are different, and also is characterized by a set of several heat capacities. At low temperatures the entropy and all heat capacities depend on temperature in the same linear way and undergo jumps at the beginning of the filling of new discrete energy levels. It is shown that the character of dependence of thermodynamic quantities on the distance between planes essentially depends on whether this dependence is considered at a fixed surface or at a fixed volume density. At a fixed surface density the thermodynamic quantities vary monotonically with the distance between planes, and at a fixed volume density they undergo oscillations. In the area of high temperatures the quantum corrections to thermodynamic quantities are obtained, which are proportional to the ratio of the thermal de Broglie wavelength to the distance between planes.

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APPENDIX

CALCULATION OF FUNCTIONS  $\Psi_s(\tau, \eta)$

First, let us consider the case when  $\eta < 1$ . Using the formula at  $t < 0$  that is correct in this case

$$\Phi_s(t) = \sum_{l=1}^{\infty} (-1)^{l+1} \frac{e^{lt}}{l^s}, \tag{A1}$$

we obtain

$$\Psi_s(\tau, \eta) = \frac{1}{2} \sum_{l=1}^{\infty} \frac{(-1)^{l+1} e^{\frac{l\eta}{\tau}}}{l^s} \theta_3\left(0, \frac{l}{\pi^2 \tau}\right) - \frac{1}{2} \Phi_s\left(\frac{\eta}{\tau}\right), \tag{A2}$$

where  $\theta_3(v, x) \equiv 1 + 2 \sum_{k=1}^{\infty} e^{-k^2 \pi^2 x} \cos 2\pi v x$  is the theta-function, so that

$$\theta_3\left(0, \frac{l}{\pi^2 \tau}\right) \equiv 1 + 2 \sum_{n=1}^{\infty} e^{-\frac{ln^2}{\tau}}. \tag{A3}$$

Note the useful relations:

$$\Phi_1(t) = \ln(1 + e^t), \quad \frac{d\Phi_1(t)}{dt} = \frac{e^t}{1 + e^t}, \quad \Phi_2(t) = \frac{t^2}{2} \left(1 + \frac{\pi^2}{3t^2}\right) - \Phi_2(-t). \tag{A4}$$

Somewhat more complex is the case  $\eta > 1$ , realized at low temperatures. We consider that for  $t > 0$  and  $s \geq 1$ :

$$\Phi_s(t) = \frac{t^s}{\Gamma(s+1)} [1 + \chi_s(t)] + (-1)^{s-1} \Phi_s(-t), \tag{A5}$$

where

$$\chi_s(t) \equiv s(s-1)! \sum_{l=1}^{s-1} \frac{[(-1)^l - 1](2^{-l} - 1)\zeta(l+1)}{(s-1-l)! t^{l+1}}, \tag{A6}$$

$\Gamma(s)$  is the gamma function,  $\zeta(l)$  is the Riemann zeta function.

The functions (12) can be written in the form

$$\Psi_s(\tau, \eta) \equiv \sum_{n=1}^{\infty} \Phi_s[\tau^{-1}(\eta - n^2)] = \sum_{n=1}^{[x_0]} \Phi_s[\tau^{-1}(\eta - n^2)] + \sum_{n=[x_0]+1}^{\infty} \Phi_s[\tau^{-1}(\eta - n^2)], \tag{A7}$$

where the designation is used  $x_0 \equiv \sqrt{\eta} = \sqrt{\mu/\varepsilon_L}$ . In the first term the argument  $\tau^{-1}(\eta - n^2) > 0$  and the formula (A5) can be used, and in the second term  $\tau^{-1}(\eta - n^2) < 0$  and the expansion (A1) is valid, so that

$$\Psi_s(\tau, \eta) = \frac{1}{\Gamma(s+1)} \sum_{n=1}^{[x_0]} \tau^{-s} (\eta - n^2)^s \{1 + \chi_s[\tau^{-1}(\eta - n^2)]\} + \Psi_s^{\text{exp}}(\tau, \eta), \tag{A8}$$

where the function

$$\Psi_s^{\text{exp}}(\tau, \eta) \equiv (-1)^{s+1} \sum_{n=1}^{[x_0]} \Phi_s[\tau^{-1}(n^2 - \eta)] + \sum_{n=[x_0]+1}^{\infty} \Phi_s[\tau^{-1}(\eta - n^2)] \tag{A9}$$

contains the exponential terms, which are small for  $\tau \ll 1$  but, however, they can be considerable near the specific points at  $\eta = n^2$ . In (A9) the main contribution is given by the terms with  $n = [x_0]$  and  $n = [x_0] + 1$ , and other terms give the exponentially small contribution of higher order. Therefore, after having extracted the main contribution, the formula (A9) can be written in the form

$$\Psi_s^{\text{exp}}(\tau, \eta) = (-1)^{s+1} \Phi_s[\tau^{-1}([x_0]^2 - \eta)] + \Phi_s[\tau^{-1}(\eta - ([x_0] + 1)^2)] + \bar{\Psi}_s^{\text{exp}}(\tau, \eta), \tag{A10}$$

where

$$\bar{\Psi}_s^{\text{exp}}(\tau, \eta) \equiv (-1)^{s+1} \sum_{n=1}^{[x_0]-1} \Phi_s[\tau^{-1}(n^2 - \eta)] + \sum_{n=[x_0]+2}^{\infty} \Phi_s[\tau^{-1}(\eta - n^2)]. \tag{A11}$$

In the considered problem we need the functions at  $s = 1$  and  $s = 2$ , which for  $\eta > 1$  can be represented in the form (41), (42).