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ELIMINATION OF SINGULARIRIES IN CAUSAL GREEN FUNCTIONS FOR GENERALIZED KLEIN-GORDON AND DIRAC EQUATIONS ON LIGHT CONE

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Klein-Gordon and Dirac equations are generalized to eliminate divergences in the integrals for Green functions of these equations. The generalized equations are presented as products of the operators for the Klein-Gordon equation with different masses and similarly for the operators of the Dirac equation. The homogeneous solutions of derived equations are sums of fields, corresponding to particles with the same values of the spin, the electric charge, the parities, but with different masses. Such particles are grouped into the kinds (families, dynasties) with members which are the particle generations. The Green functions of derived equations can be presented as sums of the products of Green functions for the Klein-Gordon equation (the Dirac equation) and the definite coefficients. The sums of these coefficients equal zero. The sums of the products of these coefficients and the particle masses to some powers equal zero too, i.e. for these coefficients some relations exist. In consequence of these relations the singularities in Green functions can be eliminated. It is shown that causal Green functions of derived equations can be finite in all the space-time. This is possible if minimal quantities of the generations N_b and N_f for the bosons and the fermions equal 3 and 6, respectively. An absence of singularities in the Green functions on light cone is related to an attenuation of particle interactions on short distances. It is shown explicitly for the generalization of the Yukawa potential.
KEY WORDS: convergence of integrals, differential equations, elimination of singularities, attenuation of interactions on short distances

УСУНЕННЯ СИНГУЛЯРНОСТЕЙ В ПРИЧИННИХ ФУНКЦІЯХ ГРІНА УЗАГАЛЬНЕНИХ РІВНЯНЬ КЛЕЙНА-ГОРДОНА І ДІРАКА НА СВІТЛОВОМУ КОНУСІ

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З метою усунення розбіжностей в інтегралах для функцій Гріна узагальнено рівняння Клейна-Гордона та Дірака. Узагальнені рівняння представляють собою добутки операторів рівняння Клейна-Гордона з різними масами і аналогічно для рівняння Дірака. Однорідні розв'язки одержаних рівнянь представляють собою суми полів, відповідних частинкам з однаковими значеннями спіну, електричного заряду, парностей, але з різними масами. Такі частинки групуються в роди (сім'ї, династії) а їхні члени є покоління. Функції Гріна одержаних рівнянь представляють собою суми добутків функцій Гріна рівняння Клейна-Гордона (рівняння Дірака) і визначених коефіцієнтів. Суми цих коефіцієнтів дорівнюють нулю. Суми добутків цих коефіцієнтів на маси частинок у деяких степенях теж дорівнюють нулю, тобто для цих коефіцієнтів існують деякі співвідношення. Внаслідок цих співвідношень стає можливим усунення сингулярностей у функціях Гріна. Показано, що причинні функції Гріна одержаних рівнянь можуть бути скінченими у всьому просторі-часі. Це можливе, якщо мінімальні кількості поколінь для бозонів N_b та ферміонів N_f дорівнюють 3 та 6, відповідно. Відсутність сингулярностей у функціях Гріна на світловому конусі пов'язана із послабленням взаємодій частинок на малих відстанях. Це показано явно для узагальнення потенціалу Юкави.
КЛЮЧОВІ СЛОВА: збіжність інтегралів, диференціальні рівняння, усунення сингулярностей, послаблення взаємодій на малих відстанях

УСТРАНЕНИЕ СИНГУЛЯРНОСТЕЙ В ПРИЧИННЫХ ФУНКЦИЯХ ГРИНА ОБОБЩЕННЫХ УРАВНЕНИЙ КЛЕЙНА-ГОРДОНА И ДИРАКА НА СВЕТОВОМ КОНУСЕ

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С целью устранения расходимостей в интегралах для функций Грина обобщены уравнения Клейна-Гордона и Дирака. Обобщенные уравнения представляют собой произведения операторов уравнения Клейна-Гордона с разными массами и аналогично для уравнения Дирака. Однородные решения полученных уравнений представляют собой суммы полей, соответствующих частицам с одними и теми же значениями спина, электрического заряда, четностей, но с разными массами. Такие частицы группируются в рода (семьи, династии) а их члены являются поколениями. Функции Грина полученных уравнений представляют собой произведения функций Грина уравнения Клейна-Гордона (уравнения Дирака) и определенных коэффициентов. Суммы этих коэффициентов равны нулю. Суммы произведений этих коэффициентов на массы частиц в некоторых степенях также равны нулю, то есть для этих коэффициентов существуют некоторые соотношения. Вследствие этих соотношений становится возможным устранение сингулярностей в функциях Грина. Показано, что причинные функции Грина полученных уравнений могут быть конечными во всем пространстве-времени.

Это возможно, если минимальные количества поколений для бозонов N_b и фермионов N_f равны 3 и 6, соответственно. Отсутствие сингулярностей в функциях Грина на световом конусе связано с ослаблением взаимодействий частиц на малых расстояниях. Это показано явно для обобщения потенциала Юкавы.
КЛЮЧЕВЫЕ СЛОВА: сходимость интегралов, дифференциальные уравнения, устранение сингулярностей, ослабление взаимодействий на малых расстояниях

Some physical values are expressed in terms of series or improper integrals. Magnitudes of these series and integrals must not depend on methods used for calculations. It means that series and integrals must be convergent. The convergence of series and improper integrals in theoretical approaches is one from necessary condition of a validity of these approaches. An existence of divergent series or integrals corresponds to wrong approaches. It may mean that such approach must be changed by new approach without divergent series or integrals.

As first example we can remember the problem of the absolutely black body radiation with the divergent integral in classical physics. This problem has been solved by Planck. He proposed the hypothesis on quanta. From this hypothesis the quantum physics was began.

The second interesting example is related to the Adler – Bell – Jackiw axial anomaly in the electroweak theory. The study of the axial Adler – Bell – Jackiw anomaly shows that a contribution of one $1/2$ - spin particle (a quark or a lepton) leads to the linear divergence [1]. But taking into account of some sets of the leptons and the quark such as e, ν_e, u, d (first generation of the particles) or μ, ν_μ, c, s (second generation) or τ, ν_τ, t, b (third generation) allow to eliminate this divergence. Thus, the convergence of the axial anomaly leads the relation between the quarks and the leptons.

In relation with this the next questions arise:

1. Which divergent improper integrals are in the field theory?
2. Why do the particle generations exist?
3. How many of the particle generations must exist?

In relation with these questions the divergent integrals for the Green functions of the Klein-Gordon and Dirac equations are studied in Refs. [2, 3]. To avoid these divergences the generalizations of the Klein-Gordon and the Dirac equations are proposed in Refs. [2, 3].

PARADOX OF GREEN FUNCTIONS

It is well known that in the static case the exchange by the massless particle gives the Coulomb (Newton) potential

$$V(r,0) = \frac{1}{4\pi r}, \tag{1}$$

where r is the distance between the point charge and the point of observation.
The exchange by the particle of the m mass gives the Yukawa potential

$$V(r,m) = \frac{1}{4\pi}\frac{e^{-mr}}{r}. \tag{2}$$

These potentials are the Green functions

$$V(r,m) = G(\vec{x},m) = \frac{1}{(2\pi)^3}\int \frac{e^{i\vec{q}\vec{x}}}{\vec{q}^{\,2} + m^2} d^3 q, \tag{3}$$

where $r = |\vec{x}|$. Note that we can put $m = 0$ in Eqs. (2), (3) for the Coulomb potential. In the relativistic case the exchange by the boson of the m mass can be expressed with means of the Green function for the Klein-Gordon-Fock equation

$$D(x,m) = \frac{1}{(2\pi)^4}\int \frac{e^{-iqx} d^4 q}{-q^2 + m^2}. \tag{4}$$

For the $1/2$ - spin particle the Green function of the Dirac equation has a form

$$S(x,m)=\frac{1}{(2\pi)^4}\int\frac{(\hat{q}+m)e^{-iqx}d^4q}{-q^2+m^2}. \tag{5}$$

Usually the expressions (1) and (2) are derived from (3) by the calculations of the integrals in the spherical frame. Note that the integral in (3) is the infinite threefold integral. As it is known the improper (in particular infinite) integral converges in that case only if the calculations of it give the same finite result by any possible methods. The convergences of improper one-fold and multiple integrals have some distinctions. In particular, the conditional convergence does not exist for multiple improper integrals. In Refs [4, 5] it is proved that the double improper integral converges only if it converges absolutely (i.e., the improper double integral with the module of the integrand converges). This is valid for any multiple improper integrals also [5]. Thus for the multiple improper integrals the convergence and the absolute convergence are equivalent [5]. Therefore, multiple improper integral converges then and only then when this integral converges absolutely. Thus the integral in (3) converges only in case of the convergence of the integral

$$\frac{1}{(2\pi)^3}\int\frac{d^3q}{\vec{q}^{\,2}+m^2}. \tag{6}$$

But this integral diverges. Therefore the integral in (3) diverges. To see this we shall integrate (3) in the cylindrical frame. We take $\vec{x}=(0,0,r)$. Then $\vec{q}\vec{x}=q_3r$ and $d^3q=\frac{1}{2}d|\vec{q}_\perp|^2\,d\varphi dq_3$. We shall integrate in the next order: with respect to the φ angle, $|\vec{q}_\perp|$, q_3, respectively. Thus we derive

$$\begin{aligned}G(\vec{x},m)&=\frac{1}{8\pi^2}\int_{-\infty}^{+\infty}e^{iq_3r}dq_3\int_0^{\infty}\frac{d\vec{q}_\perp^{\,2}}{\vec{q}_\perp^{\,2}+q_3^2+m^2}=\\&=\frac{1}{8\pi^2}\int_{-\infty}^{+\infty}e^{iq_3r}dq_3\left[\lim_{\vec{q}_\perp^{\,2}\to\infty}\ln\left(\vec{q}_\perp^{\,2}+q_3^{\,2}+m^2\right)-\ln\left(q_3^2+m^2\right)\right]=\\&=\frac{1}{2\pi}\delta(r)\lim_{|\vec{q}_\perp|\to\infty}\ln|\vec{q}_\perp|-\frac{1}{2\pi^2r}\lim_{q_3\to\infty}\ln q_3\sin q_3r+\frac{1}{4\pi}\cdot\frac{e^{-mr}}{r}.\end{aligned} \tag{7}$$

We see that this integral diverges as the first term is indefinite and the limit in the second term does not exist, but these divergent terms do not depend on the particle mass m.

Thus we derive the paradox (paradox of the Green functions.) Similarly the divergences of integrals (4) and (5) can be shown [3].

From the mathematical point of view the use of the Green functions (1)-(5) is incorrect, but these Green functions (calculated by some fashion) give fairly good description of different experimental data.

We may assume that the solution of the Green function paradox is possible by two ways: 1) We can conclude that existing theory is wrong and we must find new theoretical approach based on new mathematical methods; 2) we can try to modify existing theory.

Generalizations of Klein–Gordon and Dirac equations

In Refs. [2, 3] the second way was investigated by means of proper modification of the Green functions and corresponding generalization of the Klein-Gordon and Dirac equations. It was proposed:

The generalizations of the Klein-Gordon and Dirac equations must have some simple form;

The existing expressions (such as (1), (2)) can be derived from new generalized Green functions as some approximation.

To obtain convergent threefold integral instead of (3) the next integral was considered:

$$\bar{V}(\vec{r})=\frac{1}{(2\pi)^3}\int\frac{e^{i\vec{q}\vec{r}}d^3q}{(\vec{q}^{\,2}+m_1^2)(\vec{q}^{\,2}+m_2^2)\cdots(\vec{q}^{\,2}+m_N^2)}. \tag{8}$$

Instead of (4) the integral (the Green function)

$$\bar{G}(x)=\frac{1}{(2\pi)^4}\int\frac{e^{-iqx}d^4q}{\left(-q^2+m_1^2\right)\left(-q^2+m_2^2\right)....\left(-q^2+m_N^2\right)}=\frac{1}{(2\pi)^4}\int\frac{e^{-iqx}d^4q}{P_N\left(q^2\right)}, \quad (9)$$

where $P_N\left(q^2\right)$ is the polynomial of the N degree with respect to q^2,was proposed. The Green function (9) corresponds to generalized non-homogeneous Klein-Gordon equation of the $2N$ degree

$$\left(\Box+m_1^2\right)\left(\Box+m_2^2\right)........\left(\Box+m_N^2\right)\Phi(x)=\eta(x), \quad (10)$$

where $\Phi(x)$ is the field and $\eta(x)$ is the current (the field source). We consider the case of the polynomial with real non-negative different roots $m_1<m_2<m_3<....<m_N$. Note that for the advanced, retarded and causal Green functions we must write the corresponding imaginary infinitesimal term to each m_k^2.

The general classical solution $\Phi_{cl}(x)$ of the linear equation (10) is the sum of the general solution of the corresponding homogeneous equation $\Phi(x)_{free}$ and partial solution $\Phi(x)_{nh}$ of non-homogeneous equation:

$$\Phi(x)_{free}=\int d^4q\sum_{k=1}^{N}\delta\left(q^2-m_k^2\right)\left[c_k e^{-iqx}+\tilde{c}_k e^{iqx}\right] \quad (11)$$

$$\Phi(x)_{nh}=\int\bar{G}(x-y)\eta(y)d^4y, \quad (12)$$

where c_k and $\tilde{c}_k$ are the arbitrary constants. Thus $\Phi(x)_{free}$ is the sum of the terms corresponding to particles with the same charge, parity, spin but with different masses. Each term in (11) corresponding to number k is the solution of the homogeneous Klein – Gordon equation as $\left(\Box+m_k^2\right)\left(c_k e^{-iqx}+\tilde{c}_k e^{-iqx}\right)\ \delta\left(q^2-m_k^2\right)=0$. In Ref. [3] it is shown that the case of equal masses in Eq. (10) must be excluded. It was shown that the functions $\Phi(x)_{free}$ are non-normalizable if at least two masses are equal. Thus the masses in the generalized Klein Gordon equation must be different. For the rational fraction in (9) the expansion can be written [2, 3]

$$\frac{1}{P_N\left(q^2\right)}=\frac{1}{\left(-q^2+m_1^2\right)\left(-q^2+m_2^2\right)...\left(-q^2+m_N^2\right)}=\sum_{k=1}^{N}\frac{A_k}{-q^2+m_k^2},$$
$$A_k=-\frac{1}{P_N'\left(m_k^2\right)}=\lim_{q^2\to m_k^2}\frac{-q^2+m_k^2}{P_N\left(q^2\right)},\quad A_k=(-1)^{k+1}\left|A_k\right|. \quad (13)$$

The A_k coefficients obey the relations:

$$\sum_{k=1}^{N}A_k m_k^{2l}=0,\qquad l=0,1,2,...,N-2 \quad (14)$$

$$\sum_{k=1}^{N}A_k m_k^{2N-2}=(-1)^{N+1}, \quad (15)$$

Using the equality (13) we can express the Green function (9) of Eq. (10) in terms of the Green functions (4)

$$\bar{G}(x)=\sum_{k=1}^{N}A_k D\left(x,m_k\right). \quad (16)$$

As the dimension of the time-space is equal to four the integral (9) can be convergent at $N\geq 3$. Consequently for each spinless particle two (or greater) particles with the same charges, isospin, C - and P parity, but different masses, must exist in addition. We may say that such particles are members of some set (a family or a kind or a dynasty). In Eqs. (11), (13) k is the number of the particle generation. We may assume that the quantity of members in kinds for the elementary particle is less than the quantity of member in kinds for the composite particle. Each particle belongs to some kind and some generation.

For the $1/2$ - spin particles the next generalization of the non-homogeneous Dirac equation is proposed in Refs. [2, 3]

$$\left(-i\hat{\partial}+m_1\right)\left(-i\hat{\partial}+m_2\right).....\left(-i\hat{\partial}+m_N\right)\Psi(x)=\chi(x). \tag{17}$$

The classical solution of the homogeneous equation (17) is given by analogy with (11)

$$\Psi^{\alpha}(x)_{free}=\sum_{s}\sum_{k=1}^{N}\int d^4p\delta\left(q^2-m_k^2\right)\left[C_k u^{\alpha}{}_{k,s}(q)e^{-iqx}+\tilde{C}_k v^{\alpha}{}_{k,s}(q)e^{iqx}\right], \tag{18}$$

where α is the bispinor index, s corresponds to spin projection, $u^{\alpha}{}_{k,s}(q)$ and $v^{\alpha}{}_{k,s}(q)$ are the bispinors, C_k and $\tilde{C}_k$ are arbitrary constants. The Green functions (which are 4×4 -matrixes) for this equation may be written as

$$\overline{S}(x)=\frac{1}{(2\pi)^4}\int\frac{\left(\hat{q}+m_1\right)\left(\hat{q}+m_2\right)....\left(\left(\hat{q}+m_N\right)\right)}{\left(-q^2+m_1^2\right)\left(-q^2+m_2^2\right)...\left(-q^2+m_N^2\right)}d^4q. \tag{19}$$

For the integrand in (19) the equation may be written

$$\begin{aligned}
R_N(\hat{q})&=\frac{\left(\hat{q}+m_1\right)}{\left(-q^2+m_1^2\right)}\cdot\frac{\left(\hat{q}+m_2\right)}{\left(-q^2+m_2^2\right)}\cdots\frac{\left(\hat{q}+m_N\right)}{\left(-q^2+m_N^2\right)}=\\
&=\frac{1}{\left(-\hat{q}+m_1\right)\left(-\hat{q}+m_2\right)\cdots\left(-\hat{q}+m_N\right)}=\frac{1}{Q_N(\hat{q})}=\\
&=\sum_{k=1}^{N}B_k\frac{\hat{q}+m_k}{-q^2+m_k^2},\quad B_k=-\frac{1}{Q_N^{'}(m_k)}=\lim_{\hat{q}\to m_k}\frac{-\hat{q}+m_k}{Q_N(\hat{q})},\quad B_1>1.
\end{aligned} \tag{20}$$

The relations similar to (14), (15) are valid for the B_k coefficients:

$$\sum_{k=1}^{N}B_k m_k^l=0,\ l=0,1,2,\dots,N-2, \tag{21}$$

$$\sum_{k=1}^{N}B_k m_k^{N-1}=(-1)^{N+1}. \tag{22}$$

The Green functions (19) are given by

$$\overline{S}(x)=\sum_{k=1}^{N}B_k S\left(x,m_k\right)=\sum_{k=1}^{N}B_k\left(i\hat{\partial}+m_k\right)D\left(x,m_k\right). \tag{23}$$

The integral (19) can be convergent at $N\ge 5$ only. Thus for each the spin - $1/2$ particle four (or greater) particles with the same charges, P - parity, but with different masses, must exist in addition.

Elimination of singularities in Green functions on light cone

1. Consider the generalization of the Yukawa potential

$$\overline{V}\left(\vec{x}\right)=\sum_{k=1}^{N}A_k V\left(\vec{x},m_k\right)=\frac{1}{4\pi}\sum_{k=1}^{N}A_k\frac{e^{-m_k r}}{r}. \tag{24}$$

Note that the result (24) can be derived by two methods: a) with an use of the theorem on residues; b) with an use of the expansion of fraction (13), the formula (7), and the relation (14) at $l=0$ (because first and second terms in the final expression (7) do not depend on the m mass).

Each term of the sum in (24) has singularity at $r=\left|\vec{x}\right|=0$ (i.e. on the light cone, $x^2=0-r^2=0$). Using the expansion $e^{-m_k r} = 1 - m_k r + \frac{m_k^2 r^2}{2} - \frac{m_k^3 r^3}{6} + \dots$ at small r and relations (14) for $l = 0$ and 1, we derive

$$\overline{V}\left(\vec{x}\right) = -\frac{1}{4\pi}\sum_{k=1}^{N} A_k \left(m_k + \frac{m_k^3 r^2}{6} \right). \tag{25}$$

The continuous $\overline{V}\left(\vec{x}\right)$ is given by

$$\overline{V}\left(\vec{x}\right)_{cont} = \begin{cases} -\frac{1}{4\pi}\sum_{k=1}^{N} A_k m_k, & r = 0 \\ \frac{1}{4\pi}\sum_{k=1}^{N} A_k \frac{e^{-m_k r}}{r}, & r \neq 0 \end{cases}. \tag{26}$$

This $\overline{V}\left(\vec{x}\right)_{cont}$ has no any singularities, as contrast with the Coulomb and Yukawa potentials. From (25) we see that on short distances the potential must have the form of harmonic oscillator. The oscillatory potentials are widely used in the nuclear physics and in the quark models [6, 7]. But in these cases the parameters of the oscillatory potentials are determined from the experimental data. The interaction force at small r is given by

$$\vec{F}\left(\vec{x}\right) = -grad\overline{V}\left(\vec{x}\right) = \frac{\vec{x}}{12\pi}\sum_{k=1}^{N} A_k m_k^3 , \tag{27}$$

and it has no any singularities. It is interesting to note, that $\vec{F}\left(0\right) = 0$. Therefore, we may assume that all the interactions must be relaxed at short distances. It is similar to asymptotic freedom.

Note that if to use for $V\left(r, m_k\right)$ the result (7) derived in the cylindrical frame instead (2) then the contributions of the diverging terms vanish, as consequence of the relation (14) at $l = 0$. Thus we derive (22), (23) for $V\left(r, m_k\right)$ (2) and (7). This confirms the convergence of the $\overline{G}\left(\vec{x}\right)$-function.

Compare the r-dependences of the potentials (1) and (24) for $N_b = 3$. Consider the functions $\varphi_{NC}(r) = 4\pi V(r,0) = 1/r$ and $\varphi(r) = 4\pi\overline{G}(\vec{x})/A_1$. For the $f(r) = r\varphi(r)$ it is easy to see that $f(r) \to 1$ at $r \to \infty$. We have got $m_1 = 0$, $0 < m_2 < m_3$. The A_k coefficients in (13) are given by

$$A_1 = \frac{1}{m_2^2 m_3^2}, A_2 = -\frac{1}{m_2^2 (m_3^2 - m_2^2)}, A_3 = \frac{1}{m_3^2 (m_3^2 - m_2^2)}. \tag{28}$$

According to (28) for $\varphi(r)$ at small r it may be written

$$\varphi(r) = \frac{m_2^2 m_3^2}{m_2 + m_3}\left(\frac{1}{m_2 m_3} - \frac{r^2}{6} \right). \tag{29}$$

We see that

$$\varphi(0) = \varphi_0 = \frac{m_2 m_3}{m_2 + m_3} > 0. \tag{30}$$

The $f'(r)$ derivative is positive for $r \in [0, \infty)$, as

$$f'(r) = \frac{m_2 m_3}{m_3^2 - m_2^2}(m_3 e^{-m_2 r} - m_2 e^{-m_3 r}).$$

Therefore, the $f(r)$-function increases from 0 to 1 and the $\varphi(r)$ function is positive for $r \in [0, \infty)$. Thus, $\varphi(r)$ function is a ratio of two increasing functions. The $\varphi(r)$ function can have got monotonic decrease. But a possibility of an existence of extrema for the $\varphi(r)$ function at $r \in (0, \infty)$ cannot be excluded. Indeed, the extremum points are the solutions of the equation

$$f'(r)r - f(r) = \frac{m_2 m_3}{m_3^2 - m_2^2}(m_3 e^{-m_2 r} - m_2 e^{-m_3 r})r \text{ - } \left[1 + \frac{1}{m_3^2 - m_2^2}\left(-m_3^2 e^{-m_2 r} + m_2^2 e^{-m_3 r}\right)\right] = 0. \quad (31)$$

As this equation cannot be solved analytically, an existence of extrema for the $\varphi(r)$ function cannot be proved at arbitrary masses m_2, m_3. We consider also the functions $F(r) = -\varphi'(r)$ which are proportional to the forces. In Figure the $\varphi(r)$ potentials and the $F(r)$ forces for monotonic decrease $\varphi(r)$ (A) and $\varphi(r)$ with extrema (B) are presented. Inflection points of graph for the $\varphi(r)$ potentials can be points of extrema for the $F(r)$ forces. In the case of the monotonic decreasing $\varphi(r)$ (without of extrema) the $F(r)$ forces has got one extremum and does not change the sign at $r \in (0, \infty)$. But in the case of the $\varphi(r)$-potential with extrema the corresponding $F(r)$ force has got a few extrema and changes the sign.

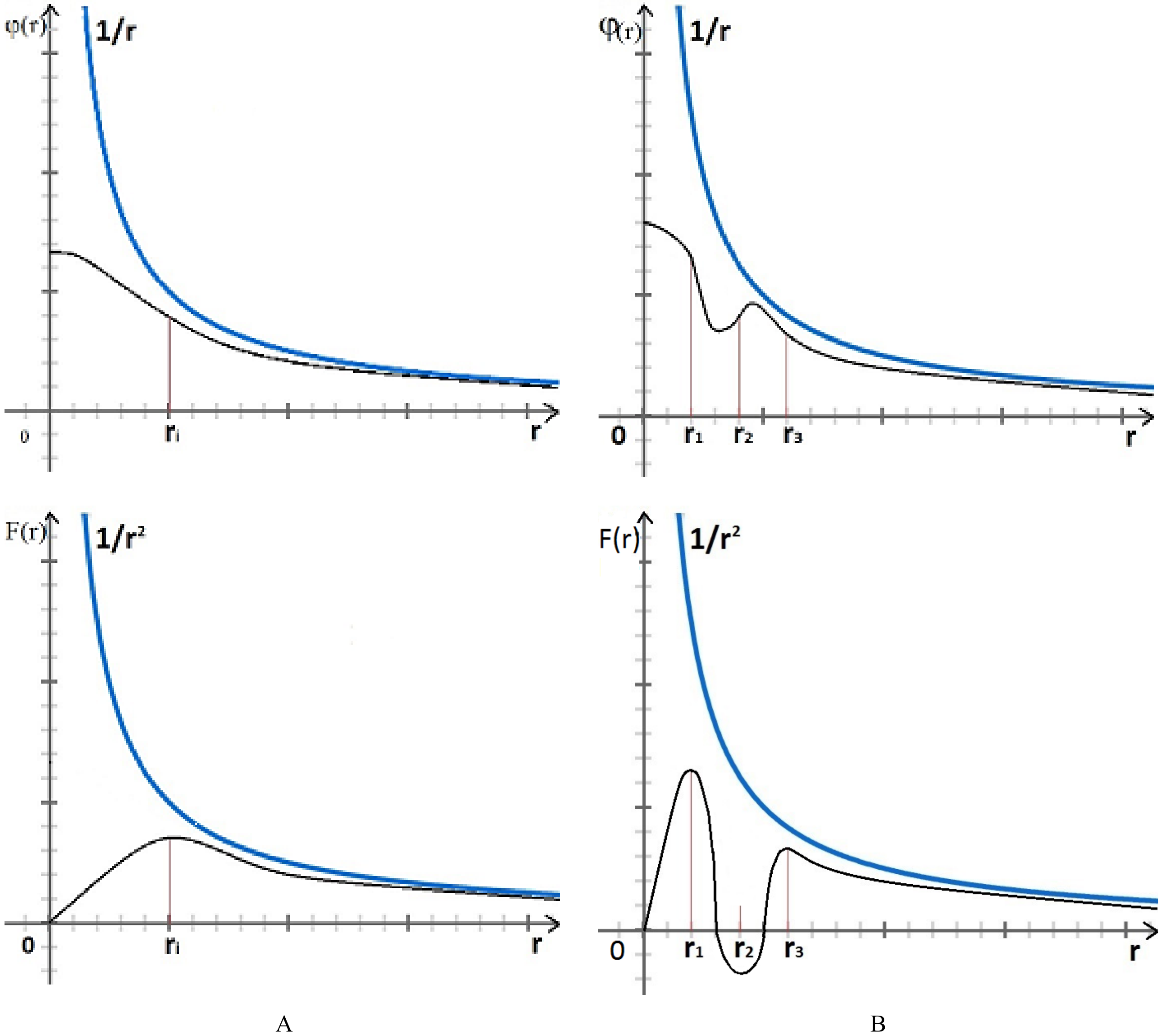

Figure. The $r-$ dependences of the $\varphi(r)$ potentials and corresponding the $F(r)$ forces for $\varphi(r)$: without extrema (A) and for $\varphi(r)$ with extrema (B). The $\varphi(r)$ potentials and the corresponding $F(r)$ forces are compared with the Newton-Coulomb potential $\varphi_{NC}(r) = 1/r$ and the $F_{NC}(r) = 1/r^2$ force. The points r_1, r_2, r_3 are the inflection points of the graph for $\varphi(r)$ potentials.

It may be assumed that the potential (24) at $m_1 = 0$ is a potential of a classic electric field induced by a q-charge. Then the energy of such electric field is finite in contrast with an infinite energy for the Coulomb potential. Indeed the energy of charge for the Coulomb potential is given by an integral

$$W=\frac{1}{8\pi}\int\vec{E}^2(\vec{x})d^3x=\frac{q^2}{32\pi^2\varepsilon_o^2}\int_0^\infty\frac{dr}{r^2}\ , \tag{32}$$

where $\vec{E}(r)=-\dfrac{q}{4\pi\varepsilon_0 r^3}\vec{r}$ is the strength of the electric field. As this integral diverges at $r=0$ the energy of the field is infinite. For the potential (24) the strength of the electric field is proportional to the $F(r)$ -force, which is continuous for $r\in[0,\infty)$. For the $F(r)$ energy of electric field equals

$$W=\frac{q^2A_1^2}{32\pi^2\varepsilon_0^2}\int_0^\infty r^2F(r)^2dr \tag{33}$$

The integral (33) converges at $r\to 0$ and at $r\to\infty$. Therefore the energy of electric field determined by the potential (24) is finite.

2. Since the generalized Klein-Gordon equation (10) has degree greater than four their Green functions and their first partial derivatives can be continuous function of the time - and spatial variables, i.e. these Green functions cannot have any singularities (more precisely these Green functions can have the removable discontinuity). Note that the Green functions of the Klein-Gordon equation have singularities on the light cone, such as $\delta\left(x^2\right), 1/x^2, \Theta\left(x^2\right), \ln\left|x^2\right|$ [8, 9]. Consider the elimination of the singularities in the causal Green function of the Eq. (10). Using the Eq. (16) and the expression for the causal Green function of the Klein-Gordon equation from the Ref. [8] we can write

$$\overline{G}_c(x)=\sum_{k=1}^{N_b}A_kD_c(x,m_k)=\sum_{k=1}^{N_b}A_k[\frac{\delta(x^2)}{4\pi}+\frac{m_k{}^2i}{4\pi^2}\frac{K_1(z_k)}{z_k}], \tag{34}$$

$$z_k=m_k\sqrt{-x^2+i\varepsilon},$$

where $K_1(z_k)$ is the Macdonald function [10, 11]. To derive a behaviour of the $\overline{G}_c(x)$ -function at low x^2 we exploit the expansion of the $K_1(z_k)$ -function [10]. It allows derive:

$$\begin{aligned}\overline{G}_c(x) = -\frac{i}{8\pi^2}\{A\ln_2+\frac{x^2}{8}[-\frac{1}{2}Am_4\ln(M^2\left|x^2\right|/4)-A\ln_4+Am_4(-i\pi\theta(x^2)-\gamma+\frac{5}{4})]+\\ +\frac{x^4}{192}[\frac{1}{2}Am_6\ln(M^2\left|x^2\right|/4)+A\ln_6+Am_6(i\pi\theta(x^2)+\gamma-\frac{5}{3})+\\ +\frac{x^6}{9216}[-\frac{1}{2}Am_8\ln(M^2\left|x^2\right|/4)-A\ln_8+Am_8(-i\pi\theta(x^2)-\gamma+\frac{47}{24})]\}+O(x^8)+O(x^8\ln M^2\left|x^2\right|),\end{aligned} \tag{35}$$

where $\gamma=0.5772157$ is the Euler-Mascheront constant, M is an arbitrary mass (e.g., $M=m_1$ or $M=m_N$). In the Eq. (35) products of the A_k -coefficients (13) and the m_k -masses are denoted as:

$$Am_l=\sum_{k=1}^{N_b}A_km_k{}^l, A\ln_l=\sum_{k=1}^{N_b}A_km_k{}^l\ln(\frac{m_k}{M}) \tag{36}$$

According to (14), (15) the numbers in (36) equal: $Am_0=Am_2=0, Am_4=1$ for $N_b=3$ and $Am_0=Am_2=Am_4=0, Am_6=-1$ for $N_b=4$. From the Eq. (35) it can be seen that the Green function of the Eq. (10) does not exist on the light cone (i.e. at $x^2=0$). It is induced by terms including the $x^2\ln M^2\left|x^2\right|$-, $x^4\ln M^2\left|x^2\right|$-, $x^6\ln M^2\left|x^2\right|$- functions. But the left and right limits of these functions exist and equal zero in the $x^2=0$-point. Thus, the $\overline{G}_c(x)$ -function has got the removable discontinuity in the $x^2=0$-point. Therefore, the continuous causal Green functions of the generalized Klein-Gordon equation (10) can be introduced at $N_b\geq 3$. Similarly continuous advanced $\overline{G}\left(x\right)_{adv}$ and the retarded $\overline{G}\left(x\right)_{ret}$ Green functions of the equation (10) can be introduced.

The expression (35) for the $\overline{G}_c(x)$ -function includes the $Am_4x^2\ln M^2\left|x^2\right|$ term. At $N_b=3$ the limits of the partial derivatives for this term with respect to x_μ tend to infinity when $x^2 \to 0$. Therefore, the partial derivatives of the Green functions of the generalized Klein-Gordon equation (10) cannot be continuous functions on the light cone at $N_b =$ 3. As at $N_b \geq 4$ the Am_4-coefficient vanishes. Therefore, at $N_b \geq 4$ the partial derivatives of the Green functions for the generalized Klein-Gordon equation (10) can be determined as continuous functions in all the space-time.

Note that the $-q^2 + m_k{}^2$-factors in Eqs (4), (5), (9), (13), (16), (19), (20), (23) must be changed by the $-q^2 + m_k{}^2 - i\varepsilon$-, $-q^2 + m_k{}^2 + 2i\varepsilon q_0$-, and $-q^2 + m_k{}^2 - 2i\varepsilon q_0$- factors for the causal, advanced, and retarded Green functions, respectively.

3. Consider the behavior of the causal Green functions for the generalized Dirac equation (17) near the $x^2 = 0$-point. In analogy with (36) we denote the sums including the B_k-coefficients as

$$Bm_l = \sum_{k=1}^{N_b} B_k m_k{}^l, B\ln_l = \sum_{k=1}^{N_b} B_k m_k{}^l \ln(\frac{m_k}{M}). \tag{37}$$

According to (21), (22) the numbers in (37) equal: $Bm_0 = Bm_1 = Bm_2 = Bm_3 = 0, Bm_4 = 1$ for $N_f = 5$ and $Bm_0 = Bm_1 = Bm_2 = Bm_3 = Bm_4 = 0, Bm_5 = -1$ for $N_f = 6$. The causal Green functions $\overline{S}_c(x)$ for the generalized Dirac equation (17) can be derived from Eq. (23) and Eq. (35) with substitutions $Bm_l, B\ln_l$ instead of $Am_l, A\ln_l$. Then from (35) we can see that the $\overline{S}_c(x)$ Green function includes partial derivatives of the $Bm_4 x^2 \ln M^2 \left|x^2\right|$ with respect to x_μ. At $N_f = 5$ these partial derivatives have got infinite limits in the $x^2 = 0$-point. At $N_f \geq 6$ this dangerous term disappear in the partial derivatives for the $\overline{S}(x)_c$ Green function. Therefore, the degree in the generalized Dirac equation (17) must be $N_f \geq 6$.

Near the $x^2 = 0$-point at $N_f \geq 6$ the causal Green functions $\overline{S}_c(x)$ is given by

$$\begin{aligned}
\overline{S}_c(x) = -\frac{i}{8\pi^2}\{ & -i\frac{\hat{x}}{4}B\ln_4 + B\ln_3 + \frac{x^2}{48}[(i\frac{\hat{x}}{2}Bm_6 - 6Bm_5)\ln(M^2\left|x^2\right|/4) + \\
& + i\hat{x}(B\ln_6 + Bm_6(i\pi\theta(x^2) + \gamma - \frac{17}{24}) - 6B\ln_6 + 6Bm_5(-i\pi\theta(x^2) - \gamma + \frac{5}{4}] + \\
& + B\ln_7 + Bm_7(i\pi\theta(x^2) + \gamma - \frac{5}{3})] + \frac{x^6}{9216}[-\frac{1}{2}Bm_9\ln(M^2\left|x^2\right|/4) - \\
& - B\ln_9 + Bm_9(-i\pi\theta(x^2) - \gamma + \frac{47}{24})]\} + O(x^8) + O(x^8\ln M^2\left|x^2\right|).
\end{aligned} \tag{38}$$

From (38) it is seen that all elements of the $\overline{S}_c(x)$- matrix can be determined as the continuous functions of coordinates in all the space-time. The elements of the $\overline{S}_c(x)$-matrix redefined similarly to $\overline{V}\left(\vec{x}\right)_{cont}$ (26) exist in arbitrary point of the space-time. It means also that requirement of the continuity of the elements of the $\overline{S}_c(x)$-matrix allow to determine degree of the generalized Dirac equation (17), $N_f \geq 6$. The continuity of the $\overline{S}_c(x)$ matrix elements can be considered as a proof of a convergence of the integrals (19) for $N_f \geq 6$.

Solution of Green function paradox

Consider the topic about the reproduction of the results derived early (such as (1) and (2)) in our approach. It is easy to see from (24) that at relatively large r in the sum the term including m_1 is important only, i.e. at relatively large r $\overline{G}\left(\vec{x}\right)$ approximately is equal to the Yukawa potential with the m_1 mass. Simultaneously large r

corresponds to small components of the q - momentum. Assume that $m_1 / m_k << 1$ for $k = 2,3,...,N$. Then we can rewrite approximately the equations (10) and (17) in forms

$$\begin{aligned} &\left(\Box + m_1^2\right) m_2^2 ... m_N^2 \Phi(x) = \eta(x) \\ &\left(-i\hat{\partial} + m_1\right) m_2 ... m_N \Psi(x) = \chi(x). \end{aligned} \tag{39}$$

These equations practically coincide with the non-homogeneous Klein-Gordon and Dirac equations for the particles with the m_1 mass.

We can reduce the equations (10) and (17) at large distances (i.e. in low – energy approximation) to the non-homogeneous Klein –Gordon and Dirac equations, respectively, by means of the redefinitions of the interaction currents. We have seen from (7) that the calculations of the Coulomb and Yukawa potentials (1), (2) by means of the integral (3) are incorrect. But we have derived these potentials as large-distance limit of the Green function for the generalization of the Klein – Gordon equation in the static case (10). In consequence of this and approximate validity of the Klein –Gordon equation at low energies (at large distances) we may assume that the use of the Coulomb and Yukawa potentials in the low – energy physics is admissible. In particular, the results derived in the solid state physics, the plasma physics, the statistical physics, the atomic physics, and low – energy nuclear physics are valid.

CONCLUSIONS

It has shown that the integrals for the Green functions of the Klein-Gordon and Dirac equations diverge as they have got different values at different fashions of the calculations. The partial differential equations of the degree greater than four must be considered to derive the Green functions with convergent integrals.

The generalizations of the Klein-Gordon and Dirac equations have been proposed. Orders of the proposed partial differential equations are related to the quantities of the particle generations N_b and N_f. The order of the generalized Klein-Gordon equation equals $2N_b$ and for the generalized Dirac equation it equals N_f. The causal Green functions of the Klein-Gordon and the Dirac equations are the Feynman propagators for the 0- and $\frac{1}{2}$-spin particles. Similarly the causal Green functions of generalized Klein-Gordon and the Dirac equations are the Feynman propagators for kinds of the 0- and $\frac{1}{2}$-spin particles. These functions are important for calculations of reactions amplitudes and they must be continuous functions. The continuity of the Green functions and their partial derivatives are determined by the orders of the differential equations. From explicit forms for the generalized Yukawa potential (24), and the generalized Green functions (35), (38) it is seen that they do not exist on the light cone ($x^2 = 0$). However, for $N_b \geq 3, N_f \geq 6$ the limits of these functions exist in the $x^2 = 0$-point. Therefore, the Green functions of the proposed equations can be defined as continuous ones (as for the generalized Yukawa potential in (26)). It means that the Green functions of proposed equations can be considered for $N_b \geq 3, N_f \geq 6$ as continuous functions and they have no any singularities in all the time – space. In particular, from Figure it is seen that the interaction potentials and corresponding forces have no singularities.

It has been shown that in our approach the interaction forces must be proportional to the distances between particles at the short distances. We have derived that the interaction potentials must have the oscillatory form at short distances. It may be assumed that the exchange by all particles from the boson kind leads to the attenuation of corresponding interaction at short distances. It is similar to asymptotic freedom.

Note that at $N_b > 3, N_f > 6$ causal Green functions of proposed equations and some partial derivatives can be determined as continuous functions. It is of interest to study behavior of the retarded and advanced Green functions of proposed equations.

For the photon and the gluon two (or greater) massive particles with the quantum numbers of the photon and the gluon must exist, respectively.

In spite of necessary existence of heavy photons the results developed in large distance physics (in such as the solid state physics, the plasma physics, the statistical physics, the atomic physics, the low energy nuclear physics) practically do not change.

From the consideration of this paper can be concluded that the dimension of the space-time, the orders of the differential equations for a description of particles interactions, minimal quantities of particle generations, and the convergences of the integrals for the Green functions are related.

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