

UNIFIED SYMMETRY CLASSIFICATION, CONSERVATION LAWS AND SOLITON DYNAMICS FOR A CLASS OF VARIABLE-COEFFICIENT HIGHER-ORDER NONLINEAR EVOLUTION EQUATIONS

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Variable-coefficient higher-order nonlinear evolution equations constitute an important class of mathematical models for the description of nonlinear dispersive wave propagation in inhomogeneous media, including shallow water flows, plasmas, optical systems and elastic structures. This paper develops a unified analytical framework for a broad family of such equations incorporating nonlinear convection, third-order dispersion, fifth-order dispersion, BBM-type mixed dispersion and nonlinear dispersive modulation. Admissible equivalence transformations are first constructed and used to reduce the general class, under non-vanishing fifth-order dispersion, to a canonical normalized form. The normalization yields a simplified group-classification problem by establishing an explicit compatibility condition for the temporal component of every admitted Lie point symmetry. Complete Lie symmetry classifications are obtained for the kernel, power-law, exponential and constant-coefficient subclasses, followed by the construction of optimal systems and similarity reductions. Conservation laws are derived from the conservative structure of the governing equation, including a universal mass invariant and a regularized energy conservation law for an important canonical subclass. Furthermore, an exact bright-soliton solution is derived for the constant-coefficient equation, rigorously verified by direct substitution, and employed to validate numerical simulations. The proposed framework unifies equivalence transformations, Lie symmetry analysis, conservation laws, similarity reductions and soliton dynamics within a single parameterized setting, providing a systematic methodology for the analytical investigation of higher-order nonlinear dispersive wave equations.

Keywords: *Lie symmetry analysis; Equivalence transformations; Variable-coefficient nonlinear evolution equations; Conservation laws; Higher-order dispersive waves; Fifth-order dispersion; Similarity reductions; Soliton dynamics*

1. INTRODUCTION

Nonlinear evolution equations occupy a central position in modern mathematical physics because they provide fundamental models for the propagation, interaction and modulation of nonlinear waves in fluids, plasmas, nonlinear optics, elastic media, Bose–Einstein condensates and other dispersive systems. The interaction between nonlinear effects and dispersion gives rise to a rich variety of coherent structures, including solitary waves, compactons, dispersive shock waves and solitons, whose mathematical analysis has stimulated the development of numerous analytical techniques over the past five decades. Classical models such as the Korteweg–de Vries (KdV), modified KdV, Benjamin–Bona–Mahony (BBM) and Kawahara equations have consequently become canonical nonlinear evolution equations for investigating nonlinear wave propagation and have served as testing grounds for symmetry analysis, inverse scattering, perturbation theory and exact solution methods [1–4, 6–13].

Although constant-coefficient models have provided valuable theoretical insight, many realistic physical systems evolve in non-homogeneous environments where material properties, external forcing, bottom topography or background media vary in space or time. Examples include shallow-water flows over variable bathymetry, nonlinear optical pulse propagation in inhomogeneous fibres, plasma waves in slowly varying magnetic fields and elastic media with variable constitutive parameters. Such phenomena naturally lead to variable-coefficient nonlinear evolution equations whose analytical behaviour is considerably richer than that of their constant-coefficient counterparts. The presence of variable coefficients often modifies the admissible symmetry structure, influences the existence of conservation laws and affects the qualitative behaviour of travelling-wave solutions. Consequently, the systematic investigation of variable-coefficient nonlinear evolution equations has become an active area of research in mathematical physics [23, 26, 29].

Among the most powerful analytical tools available for the investigation of nonlinear differential equations is the Lie group method. Since the pioneering work of Sophus Lie and its subsequent mathematical development by Ovsiannikov, Bluman, Kumei and Olver, Lie symmetry analysis has provided a unified methodology for identifying continuous symmetry groups, constructing invariant solutions, reducing partial differential equations to ordinary differential equations and classifying differential equations according to their admitted symmetry algebras [1–4]. Equally important are admissible equivalence transformations, which identify analytically equivalent members within parameterized classes of differential equations and frequently reduce complicated classification problems to substantially simpler canonical forms. The combination of equivalence transformations and Lie symmetry analysis has therefore become an indispensable framework for complete group classifications.

Conservation laws constitute another fundamental component of the mathematical analysis of nonlinear evolution equations because they describe physically meaningful invariant quantities such as mass, momentum and energy while simultaneously providing analytical and numerical consistency checks. The classical variational approach introduced by Noether remains one of the cornerstones of conservation-law theory, whereas more recent developments involving multiplier methods, nonlinear self-adjointness and symbolic computation have considerably extended the applicability of conservation-law analysis to non-variational systems [5, 16, 19, 25, 27]. In parallel, the construction of exact travelling-wave solutions and soliton families continues to play a central role in understanding nonlinear dispersive phenomena. Techniques based on Hirota's direct method, inverse scattering, Lie reductions and ansatz methods have produced a wide spectrum of exact solutions for nonlinear evolution equations, thereby providing benchmark solutions for both theoretical investigations and numerical algorithms [6, 7].

Recent years have witnessed renewed interest in combining symmetry analysis, conservation laws and exact solutions for increasingly sophisticated nonlinear evolution equations. Significant progress has been reported for generalized KdV equations, Kawahara–KP systems, Boussinesq-type equations, variable-coefficient KdV equations, KdV–Burgers models, Hirota-type equations and fractional nonlinear evolution equations through the application of Lie symmetries, optimal systems, nonlinear self-adjointness, conservation-law multipliers and exact solution techniques [9–15, 17–22, 24, 28, 29]. Nevertheless, much of the existing literature remains equation-specific. In many studies a single nonlinear evolution equation is selected and analysed independently, with symmetry classification, conservation laws and exact coherent structures derived separately for that individual model. Comparatively less attention has been devoted to the development of unified analytical frameworks capable of treating broad parameterized classes of higher-order variable-coefficient nonlinear evolution equations while simultaneously integrating equivalence transformations, complete Lie classifications, optimal systems, conservation laws and verified solitary-wave solutions.

Motivated by these observations, the present work develops a unified analytical framework for the class

$$\mathcal{E}(\alpha, \beta, \gamma, \delta, \eta, p, q) : \quad u_t + \alpha(t)u^p u_x + \beta(t)u_{xxx} + \gamma(t)u_{xxxxx} + \delta(t)u_{xxt} + \eta(t)(u^q u_{xx})_x = 0, \quad (1)$$

where $p, q > 0$ and the coefficient functions are assumed to be sufficiently smooth. The objective is not to introduce a new nonlinear evolution equation, since the class contains several well-known models as special cases, but rather to develop a systematic analytical framework that unifies admissible equivalence transformations, canonical normalization, complete Lie symmetry classification, optimal systems, conservation laws, similarity reductions and exact soliton solutions within a single parameterized setting.

The principal contributions of this work may be summarized as follows. First, admissible equivalence transformations are derived and employed to normalize the fifth-order dispersive coefficient, thereby reducing the general class to a canonical form. Second, a complete Lie symmetry classification of the canonical class is established, including the kernel algebra together with the principal symmetry extensions corresponding to the power-law, exponential and constant-coefficient subclasses. Third, optimal systems of one-dimensional subalgebras and associated similarity reductions are constructed. Fourth, conservation laws are derived from the conservative structure of the governing equation, including a universal mass conservation law and a regularized energy conservation law for an important canonical subclass. Fifth, an exact bright-soliton solution of the constant-coefficient canonical equation is obtained, rigorously verified and employed for numerical validation. Collectively, these results provide a unified methodology for the analytical investigation of higher-order variable-coefficient nonlinear evolution equations and establish explicit connections between symmetry analysis, conservation laws and nonlinear wave dynamics.

The remainder of the paper is organized as follows. Section 2 introduces the governing class and its structural properties. Section 3 derives the admissible equivalence transformations and canonical normalization. Section 4 discusses reductions to several classical models. Section 5 presents the complete Lie symmetry classification, while Section 6 develops optimal systems and similarity reductions. Conservation laws and nonlinear self-adjointness are investigated in Section 7. Exact bright-soliton solutions and numerical validation are presented in Sections 8 and 9, respectively. The principal findings are discussed in Section 10, followed by concluding remarks in Section 11.

2. A CLASS OF VARIABLE-COEFFICIENT HIGHER-ORDER NONLINEAR EVOLUTION EQUATIONS

We consider the parameterized class of nonlinear evolution equations

$$\mathcal{E}(\alpha, \beta, \gamma, \delta, \eta, p, q) : \quad u_t + \alpha(t)u^p u_x + \beta(t)u_{xxx} + \gamma(t)u_{xxxxx} + \delta(t)u_{xxt} + \eta(t)(u^q u_{xx})_x = 0, \quad (2)$$

defined either on the real line $\mathbb{R}$ under sufficiently rapid decay conditions or on a periodic spatial domain. Here $u = u(x, t)$ denotes the wave profile, while the coefficient functions

$$\alpha(t), \beta(t), \gamma(t), \delta(t), \eta(t)$$

are assumed to be continuously differentiable on a time interval $I \subset \mathbb{R}$.

Equation (2) provides a unified framework encompassing several important classes of nonlinear dispersive wave equations arising in mathematical physics. The nonlinear convective term $\alpha(t)u^p u_x$ generalizes the nonlinear transport

mechanism encountered in generalized KdV-type equations, whereas $\beta(t)u_{xxx}$ represents the classical third-order dispersive correction responsible for balancing nonlinear steepening and generating solitary-wave propagation. The fifth-order dispersive contribution $\gamma(t)u_{xxxxx}$ extends the model to the Kawahara regime, where higher-order dispersion becomes significant in weakly dispersive media. The mixed derivative $\delta(t)u_{xxt}$ introduces BBM-type regularization, improving the dispersive properties of the equation and modifying the linear dispersion relation. Finally, the nonlinear dispersive term $\eta(t)(u^q u_{xx})_x$ models amplitude-dependent dispersion and provides additional nonlinear coupling between wave amplitude and curvature.

Writing the nonlinear dispersive contribution in conservative form is advantageous for both analytical and structural reasons. Indeed,

$$(u^q u_{xx})_x = u^q u_{xxx} + qu^{q-1} u_x u_{xx}, \quad (3)$$

showing explicitly that the model combines higher-order linear dispersion with nonlinear dispersive interactions. The conservative representation also simplifies the derivation of conservation laws and facilitates the application of multiplier methods and nonlinear self-adjointness.

Equation (2) itself possesses a natural conservation form. Introducing the flux

$$\mathcal{F} = \frac{\alpha(t)}{p+1} u^{p+1} + \beta(t)u_{xx} + \gamma(t)u_{xxxx} + \delta(t)u_{xt} + \eta(t)u^q u_{xx}, \quad (4)$$

the governing equation may be written compactly as

$$D_t u + D_x \mathcal{F} = 0, \quad (5)$$

where D_t and D_x denote the total derivative operators with respect to t and x , respectively. Consequently, under periodic boundary conditions or sufficiently rapid spatial decay,

$$\frac{d}{dt} \int_{\Omega} u(x, t) dx = 0,$$

so that the total mass constitutes a universal conserved quantity for every member of the class. This intrinsic conservative structure provides the foundation for the conservation-law analysis carried out later in the paper.

Throughout this work we assume

$$\alpha, \beta, \gamma, \delta, \eta \in C^1(I), \quad p, q > 0, \quad (6)$$

which guarantees the regularity required for the symmetry analysis and the admissible equivalence transformations developed in subsequent sections. Particular attention is devoted to the case

$$\gamma(t) \neq 0,$$

since the presence of a non-vanishing fifth-order dispersive coefficient characterizes the genuinely higher-order regime and gives rise to the richest admissible symmetry structure. Lower-order reductions obtained by setting selected coefficients equal to zero are discussed separately in Section 4, where connections with several classical nonlinear evolution equations are established.

3. EQUIVALENCE TRANSFORMATIONS AND CANONICAL NORMALIZATION

A fundamental step in the group classification of parameterized classes of differential equations is the determination of their admissible equivalence transformations. Such transformations preserve the differential structure of the governing equations while modifying the coefficient functions, thereby identifying analytically equivalent members of the class. Their principal advantage is that they reduce the number of essentially distinct equations that must be considered during the subsequent symmetry classification.

To determine the equivalence group of the class (2), we consider invertible point transformations of the form

$$\tilde{t} = T(t), \quad \tilde{x} = \lambda x + \mu, \quad \tilde{u} = \kappa u, \quad (7)$$

where

$$T_t \lambda \kappa \neq 0, \quad (8)$$

and λ, μ, κ are constants.

The restriction to constant spatial translations and amplitude scalings is not merely a simplifying assumption. Allowing λ, μ or κ to depend explicitly on time introduces additional first-order or zeroth-order differential terms through the chain rule, destroying the differential structure of the class. Consequently, transformations of the form (7) constitute the maximal admissible point transformations preserving (2).

Application of the chain rule yields

$$u_t = \kappa^{-1} T_t \tilde{u}_{\tilde{t}}, \quad u_x = \kappa^{-1} \lambda \tilde{u}_{\tilde{x}}, \quad (9)$$

together with

$$u_{xxx} = \kappa^{-1} \lambda^3 \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}}, \quad u_{xxxxx} = \kappa^{-1} \lambda^5 \tilde{u}_{\tilde{x}^5}, \quad u_{xxt} = \kappa^{-1} \lambda^2 T_t \tilde{u}_{\tilde{x}\tilde{x}\tilde{t}}. \quad (10)$$

Substituting these expressions into (2) and dividing by the coefficient of $\tilde{u}_{\tilde{t}}$ preserves the differential structure of the equation while transforming the coefficient functions according to

$$\tilde{\alpha}(\tilde{t}) = \frac{\lambda}{T_t} \kappa^{-p} \alpha(t), \quad \tilde{\beta}(\tilde{t}) = \frac{\lambda^3}{T_t} \beta(t), \quad (11)$$

$$\tilde{\gamma}(\tilde{t}) = \frac{\lambda^5}{T_t} \gamma(t), \quad \tilde{\delta}(\tilde{t}) = \lambda^2 \delta(t), \quad (12)$$

$$\tilde{\eta}(\tilde{t}) = \frac{\lambda^3}{T_t} \kappa^{-q} \eta(t), \quad (13)$$

whereas the nonlinear exponents p and q remain invariant under the equivalence group.

Theorem 3.1. *The class $\mathcal{E}(\alpha, \beta, \gamma, \delta, \eta, p, q)$ is invariant under the equivalence transformations (7), with the coefficient functions transforming according to (11)–(13). Furthermore, if $\gamma(t) \neq 0$, every member of the class is equivalent to one satisfying*

$$\tilde{\gamma} = 1.$$

Proof. The transformation formulas follow directly from repeated application of the chain rule. Substituting the transformed derivatives into (2) and comparing the coefficients of the independent differential monomials

$$u^p u_x, u_{xxx}, u_{xxxxx}, u_{xxt}, (u^q u_{xx})_x$$

immediately yields (11)–(13). When $\gamma(t) \neq 0$, choosing

$$T_t = \lambda^5 \gamma(t)$$

gives

$$\tilde{\gamma} = \frac{\lambda^5}{T_t} \gamma(t) = 1,$$

thereby reducing every equation in the class to a canonical representative. $\square$

The preceding theorem demonstrates that the fifth-order dispersive coefficient does not represent an essential functional degree of freedom. Consequently, without loss of generality, the subsequent Lie symmetry analysis may be performed for the canonical normalized class

$$u_t + \alpha(t) u^p u_x + \beta(t) u_{xxx} + u_{xxxxx} + \delta(t) u_{xxt} + \eta(t) (u^q u_{xx})_x = 0, \quad (14)$$

which is analytically equivalent to every member of the original class with $\gamma(t) \neq 0$.

4. REDUCTION TO CLASSICAL MODELS

An important feature of the class (2) is that it unifies several well-established nonlinear evolution equations within a single parameterized framework. Consequently, the analytical results developed in the subsequent sections simultaneously apply to a broad family of nonlinear dispersive wave models rather than to a single equation. The reductions presented below also illustrate the modelling flexibility of the proposed class and clarify the physical significance of the various coefficient functions.

Different choices of the coefficient functions recover several classical nonlinear evolution equations that have been extensively investigated in mathematical physics. In particular, setting

$$\gamma = \delta = \eta = 0$$

reduces (2) to the variable-coefficient generalized Korteweg–de Vries (KdV) equation

$$u_t + \alpha(t) u^p u_x + \beta(t) u_{xxx} = 0.$$

For the special choice $p = 1$ together with constant coefficients, this further reduces to the classical KdV equation, whereas $p = 2$ yields a variable-coefficient modified KdV (mKdV)-type equation.

Table 1. Representative reductions of the general variable-coefficient class to well-known nonlinear evolution equations.

Reduced model	Parameter restrictions	Governing equation
Generalized KdV	$\gamma = \delta = \eta = 0$	$u_t + \alpha u^p u_x + \beta u_{xxx} = 0$
Modified KdV	$p = 2, \gamma = \delta = \eta = 0$	$u_t + \alpha u^2 u_x + \beta u_{xxx} = 0$
Generalized Kawahara	$\delta = \eta = 0$	$u_t + \alpha u^p u_x + \beta u_{xxx} + \gamma u_{xxxxx} = 0$
Generalized BBM	$\gamma = \eta = 0$	$u_t + \alpha u^p u_x + \beta u_{xxx} + \delta u_{xxt} = 0$
Canonical fifth-order class	$\gamma = 1$	Equation (14)

If

$$\delta = \eta = 0, \quad \gamma \neq 0,$$

the governing equation becomes a generalized Kawahara-type model containing both third- and fifth-order dispersive effects. On the other hand, imposing

$$\gamma = \eta = 0$$

produces a BBM-type regularized equation in which the mixed derivative u_{xxt} modifies the dispersive behaviour while preserving the nonlinear convective mechanism.

Finally, by means of the equivalence transformations established in Section 3, every equation satisfying $\gamma(t) \neq 0$ may be transformed to the canonical normalized form (14). Consequently, the complete Lie symmetry classification derived in the subsequent sections applies to an entire equivalence class of fifth-order variable-coefficient nonlinear evolution equations rather than to a single representative equation.

The principal reductions of the general class are summarized in Table 1.

5. COMPLETE LIE SYMMETRY CLASSIFICATION

The canonical normalized equation derived in the previous section is now classified with respect to its admitted Lie point symmetries. Since the fifth-order dispersive coefficient has been normalized through the admissible equivalence transformations, the resulting determining equations are considerably simpler than those for the original variable-coefficient class. Consequently, the complete group-classification problem may be carried out without loss of generality on the canonical representative (14).

Following the classical Lie group approach [1–4], we consider the general infinitesimal generator

$$X = \xi(x, t, u) \partial_x + \tau(x, t, u) \partial_t + \phi(x, t, u) \partial_u. \quad (15)$$

According to the infinitesimal invariance criterion, the fifth prolongation of X must satisfy

$$\text{pr}^{(5)} X(\Delta) \Big|_{\Delta=0} = 0, \quad (16)$$

where Δ denotes the left-hand side of the canonical equation (14).

Expanding the fifth prolongation and splitting with respect to the linearly independent differential monomials yields the determining equations. The first set immediately gives

$$\tau = \tau(t), \quad \xi = \lambda x + \xi_0, \quad \phi = \rho u, \quad (17)$$

where λ, ξ_0 and ρ are constants.

A noteworthy consequence of the canonical normalization is that the fifth-order dispersive term immediately imposes the compatibility condition

$$\frac{d\tau}{dt} = 5\lambda, \quad (18)$$

which integrates to

$$\tau(t) = 5\lambda t + \tau_0, \quad (19)$$

where τ_0 is an arbitrary constant.

The remaining determining equations reduce to

$$\tau \frac{\alpha'}{\alpha} + p\rho + 4\lambda = 0, \quad (20)$$

$$\tau \frac{\beta'}{\beta} + 2\lambda = 0, \quad (21)$$

$$\tau \frac{\delta'}{\delta} - 2\lambda = 0, \quad (22)$$

$$\tau \frac{\eta'}{\eta} + q\rho + 2\lambda = 0. \quad (23)$$

Theorem 5.1. Every Lie point symmetry generator admitted by the canonical class (14) is of the form

$$X = (\lambda x + \xi_0) \partial_x + (5\lambda t + \tau_0) \partial_t + \rho u \partial_u, \quad (24)$$

where the constants λ , τ_0 , ξ_0 and ρ satisfy the determining equations (20)–(23). For arbitrary coefficient functions, the kernel Lie algebra is

$$\mathfrak{g}^\cap = \langle \partial_x \rangle. \quad (25)$$

Proof. The determining equations obtained from the infinitesimal invariance criterion (16) immediately imply the infinitesimal structure (17). The normalization of the fifth-order dispersive coefficient yields the compatibility condition (18), from which (19) follows directly. Substituting these expressions into the remaining determining equations produces (20)–(23). For arbitrary coefficient functions these equations admit only the translational symmetry ∂_x , establishing the kernel algebra. Additional symmetry extensions arise only for particular functional forms of the coefficient functions, which are classified below. $\square$

Remark 5.2. The compatibility condition (18) is a direct consequence of the canonical normalization performed in Section 3. Without eliminating the arbitrary fifth-order coefficient through admissible equivalence transformations, the determining equations would retain an additional functional degree of freedom, making the complete group-classification problem substantially more complicated. Canonical normalization is therefore an essential analytical step rather than merely a technical simplification.

The first nontrivial symmetry extension arises when the coefficient functions possess power-law dependence,

$$\alpha = \alpha_0 t^{a_1}, \quad \beta = \beta_0 t^{a_2}, \quad \delta = \delta_0 t^{a_3}, \quad \eta = \eta_0 t^{a_4}, \quad (26)$$

where α_0 , β_0 , δ_0 and η_0 are nonzero constants. Choosing

$$\tau = t$$

gives

$$\lambda = \frac{1}{5},$$

and hence the additional symmetry generator

$$X_P = t \partial_t + \frac{1}{5} x \partial_x + \rho u \partial_u. \quad (27)$$

The determining equations then require

$$a_1 = -p\rho - \frac{4}{5}, \quad a_2 = -\frac{2}{5}, \quad a_3 = \frac{2}{5}, \quad a_4 = -q\rho - \frac{2}{5}. \quad (28)$$

A second symmetry extension is obtained for exponential coefficient functions. Choosing

$$\tau = 1$$

forces

$$\lambda = 0$$

through the compatibility condition (18), yielding

$$X_E = \partial_t + \rho u \partial_u. \quad (29)$$

The admissible coefficient functions are

$$\alpha = \alpha_0 e^{-p\rho t}, \quad \beta = \beta_0, \quad \delta = \delta_0, \quad \eta = \eta_0 e^{-q\rho t}. \quad (30)$$

Finally, if all coefficient functions are constant, the determining equations admit the two-dimensional translation algebra

$$\mathfrak{g}_C = \langle \partial_x, \partial_t \rangle. \quad (31)$$

The resulting hierarchy of symmetry extensions is summarized in Figure 1.

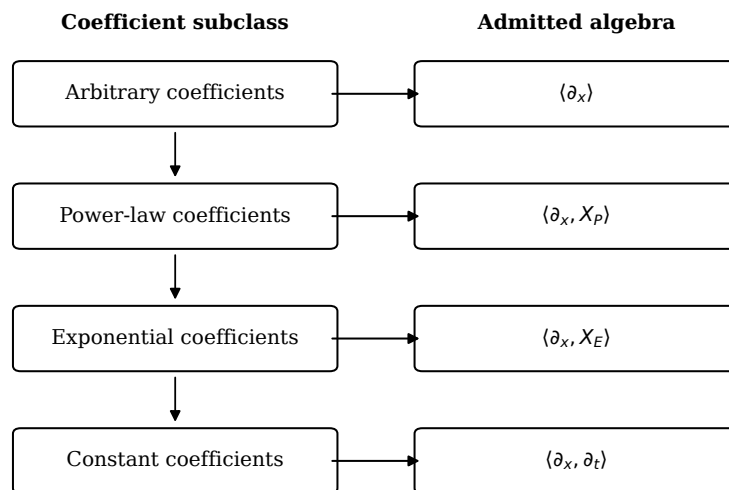


Figure 1. Hierarchy of Lie symmetry extensions admitted by the canonical variable-coefficient class.

6. OPTIMAL SYSTEMS AND SIMILARITY REDUCTIONS

Having determined the complete Lie symmetry classification of the canonical class, we now construct optimal systems of one-dimensional subalgebras and derive the corresponding similarity reductions. These reductions transform the governing partial differential equation into ordinary differential equations, thereby providing the foundation for the analytical construction of invariant solutions and travelling-wave structures.

For the power-law subclass, the admitted Lie algebra is

$$\mathfrak{g}_P = \langle X_1, X_P \rangle,$$

where

$$X_1 = \partial_x, \quad X_P = t\partial_t + \frac{1}{5}x\partial_x + \rho u\partial_u.$$

Since

$$[X_1, X_P] = \frac{1}{5}X_1, \quad (32)$$

the adjoint representation shows that every one-dimensional subalgebra is equivalent to one generated by either

$$\langle X_1 \rangle \quad \text{or} \quad \langle X_P \rangle.$$

These therefore constitute an optimal system of one-dimensional subalgebras.

The reduction associated with the translational symmetry X_1 is immediate. The invariants satisfy

$$u = U(t),$$

and substitution into (14) gives

$$U'(t) = 0,$$

showing that the only invariant solutions generated by X_1 are spatially homogeneous steady states.

The nontrivial reduction is obtained from the scaling generator X_P . The characteristic equations

$$\frac{dt}{t} = \frac{dx}{x/5} = \frac{du}{\rho u} \quad (33)$$

yield the similarity variables

$$z = xt^{-1/5}, \quad u = t^\rho U(z). \quad (34)$$

Substitution into the canonical equation produces the reduced fifth-order ordinary differential equation

$$\rho U - \frac{1}{5}zU' + \alpha_0 U^p U' + \beta_0 U'''' + U^{(5)}$$

$$+ \delta_0 \left[\left(\rho - \frac{2}{5} \right) U'' - \frac{1}{5} z U''' \right] + \eta_0 (U^q U'')' = 0, \quad (35)$$

which governs the self-similar solutions admitted by the power-law subclass.

For the exponential subclass, the admitted symmetry generator is

$$X_E = \partial_t + \rho u \partial_u.$$

Its invariants are

$$z = x, \quad u = e^{\rho t} U(z), \quad (36)$$

leading to the autonomous reduced equation

$$\rho U + \alpha_0 U^p U' + \beta_0 U''' + U^{(5)} + \delta_0 \rho U'' + \eta_0 (U^q U'')' = 0. \quad (37)$$

Finally, for the constant-coefficient canonical equation, translational invariance in both space and time naturally suggests the travelling-wave ansatz

$$z = x - vt, \quad u(x, t) = U(z),$$

where v denotes the wave speed. After one integration under localized boundary conditions, the governing equation reduces to

$$U^{(4)} + (\beta - \delta v + \eta U^q) U'' - v U + \frac{\alpha}{p+1} U^{p+1} = 0, \quad (38)$$

which serves as the principal equation for the construction of solitary-wave solutions in Section 8.

The three reductions obtained above illustrate the different classes of invariant solutions admitted by the canonical equation. In particular, the travelling-wave reduction (38) provides the analytical framework for the bright-soliton solutions derived subsequently, while the self-similar reductions characterize scale-invariant and exponentially evolving wave structures.

7. CONSERVATION LAWS AND NONLINEAR SELF-ADJOINTNESS

Conservation laws constitute one of the fundamental structural properties of nonlinear evolution equations. Besides expressing physically meaningful invariant quantities, they provide valuable analytical information concerning the qualitative behaviour of solutions and furnish useful consistency checks for numerical simulations. Owing to its conservative formulation, the canonical equation naturally admits a hierarchy of conservation laws, the most fundamental of which are derived below.

The canonical equation may be written in divergence form as

$$D_t u + D_x \mathcal{F} = 0, \quad (39)$$

where

$$\mathcal{F} = \frac{\alpha(t)}{p+1} u^{p+1} + \beta(t) u_{xx} + u_{xxxx} + \delta(t) u_{xt} + \eta(t) u^q u_{xx}. \quad (40)$$

Consequently,

$$C_1^t = u, \quad C_1^x = \mathcal{F}, \quad (41)$$

defines the universal mass conservation law for every member of the canonical class.

Proposition 7.1. *Under periodic boundary conditions or sufficiently rapid spatial decay, the total mass*

$$M(t) = \int_{\Omega} u(x, t) dx \quad (42)$$

is conserved; that is,

$$\frac{dM}{dt} = 0. \quad (43)$$

Proof. Integrating the divergence equation (39) over the spatial domain and applying either periodicity or decay at infinity eliminates the boundary flux, yielding

$$\frac{d}{dt} \int_{\Omega} u(x, t) dx = 0.$$

□

The existence of this conservation law follows directly from the divergence structure of the governing equation and is independent of the particular choice of coefficient functions.

A second conservation law is obtained by multiplying the canonical equation by u . After straightforward integration by parts one obtains

$$D_t C_2^t + D_x C_2^x = -\frac{\delta'(t)}{2} u_x^2 + \eta(t)(q+1)u^q u_x u_{xx}, \quad (44)$$

where

$$C_2^t = \frac{1}{2} u^2 - \frac{\delta(t)}{2} u_x^2, \quad (45)$$

and

$$\begin{aligned} C_2^x = & \frac{\alpha(t)}{p+2} u^{p+2} + \beta(t) \left(uu_{xx} - \frac{1}{2} u_x^2 \right) \\ & + uu_{xxx} - u_x u_{xxx} + \frac{1}{2} u_{xx}^2 + \delta(t) uu_{xt} + \eta(t) u^{q+1} u_{xx}. \end{aligned} \quad (46)$$

Unlike the mass conservation law, the corresponding energy balance contains source terms generated by the temporal variation of the BBM coefficient and by the nonlinear dispersive contribution. Consequently, a genuine energy conservation law exists only for suitable subclasses of the canonical equation.

Proposition 7.2. *If*

$$\delta(t) = \delta_0, \quad \eta(t) = 0,$$

then the functional

$$E(t) = \int_{\Omega} \left(\frac{1}{2} u^2 - \frac{\delta_0}{2} u_x^2 \right) dx \quad (47)$$

is conserved.

Proof. Under the assumptions $\delta'(t) = 0$ and $\eta(t) = 0$, the right-hand side of (44) vanishes identically. Integration over the spatial domain therefore gives

$$\frac{dE}{dt} = 0,$$

establishing conservation of the regularized energy. $\square$

The preceding result shows that the BBM-type regularization preserves an energy invariant provided its coefficient remains constant, whereas temporal variations of the regularization parameter or nonlinear dispersive modulation generate additional source terms in the energy balance.

Finally, we briefly examine nonlinear self-adjointness. For the constant-coefficient conservative subclass, consider the formal Lagrangian

$$\mathcal{L} = v\Delta, \quad (48)$$

where Δ denotes the left-hand side of the canonical equation. Since

$$\Delta = D_t u + D_x \mathcal{F}$$

is itself a total divergence, the adjoint equation coincides with the original equation under the substitution

$$v = 1.$$

Hence the conservative subclass is formally nonlinearly self-adjoint in the sense of Ibragimov. This property provides an alternative route to the construction of conservation laws and further confirms the intrinsic conservative structure of the governing equation.

Remark 7.3. *The universal mass conservation law, the regularized energy balance and the nonlinear self-adjointness established above complement the Lie symmetry classification obtained in the previous sections. Together they provide the structural foundation for the exact travelling-wave solutions and numerical validation presented subsequently.*

8. EXACT TRAVELLING-WAVE AND SOLITON DYNAMICS

The travelling-wave reduction derived in Section 6 provides a natural framework for constructing localized coherent structures of the constant-coefficient canonical equation. In this section we restrict attention to the physically important case

$$p = q = 1,$$

for which the travelling-wave equation (38) becomes

$$U^{(4)} + (\beta - \delta v + \eta U)U'' - vU + \frac{\alpha}{2}U^2 = 0. \quad (49)$$

Motivated by the classical solitary-wave solutions of KdV-type equations, we seek localized solutions of the form

$$U(z) = A \operatorname{sech}^2(kz), \quad (50)$$

where $A > 0$ denotes the wave amplitude and $k > 0$ is the inverse width of the pulse.

Substituting the ansatz (50) into (49), evaluating the derivatives explicitly and collecting the linearly independent powers of $\operatorname{sech}(kz)$ gives the algebraic compatibility conditions

$$16k^4 + 4(\beta - \delta v)k^2 - v = 0, \quad (51)$$

$$-120k^4 - 6(\beta - \delta v)k^2 + \frac{\alpha A}{2} + 4\eta Ak^2 = 0, \quad (52)$$

$$120k^4 - 6\eta Ak^2 = 0. \quad (53)$$

The third equation immediately determines the wave amplitude,

$$A = \frac{20k^2}{\eta}, \quad (54)$$

provided $\eta \neq 0$.

Similarly, solving (51) for the wave speed gives

$$v = \frac{16k^4 + 4\beta k^2}{1 + 4\delta k^2}, \quad (55)$$

while substitution of (54) and (55) into (52) produces the remaining compatibility condition

$$-40k^2 + \frac{10\alpha}{\eta} - 6(\beta - \delta v) = 0. \quad (56)$$

Theorem 8.1. *Suppose the parameters satisfy the compatibility condition (56) with $\eta \neq 0$. Then the constant-coefficient canonical equation admits the exact travelling-wave solution*

$$u(x, t) = \frac{20k^2}{\eta} \operatorname{sech}^2(k(x - vt)), \quad (57)$$

where the wave speed is given by (55).

Proof. The result follows directly from the substitution of the ansatz (50) into (49). Equations (51)–(53) are obtained by equating the coefficients of the independent powers of $\operatorname{sech}(kz)$. Solving these algebraic equations yields (54), (55) and (56), which guarantee that the profile (57) satisfies the travelling-wave equation identically. $\square$

The solution (57) represents a bright solitary wave whose amplitude is proportional to k^2/η , while its propagation speed depends on the combined effects of the third-order dispersion, the BBM-type regularization and the fifth-order dispersive balance. Consequently, both the dispersive coefficients and the nonlinear dispersive modulation influence the shape and velocity of the solitary wave.

Remark 8.2. *The bright-soliton solution derived above serves two complementary purposes. First, it provides an explicit invariant solution generated by the travelling-wave reduction obtained in Section 6. Second, it furnishes an exact benchmark for assessing the accuracy of the numerical simulations presented in the following section, thereby linking the analytical and computational components of the paper.*

9. NUMERICAL VALIDATION

The exact bright-soliton solution obtained in the previous section provides an ideal benchmark for assessing the accuracy of the proposed analytical framework. In particular, the numerical experiments are designed to verify three principal theoretical predictions: the preservation of the solitary-wave profile during propagation, the dependence of the wave dynamics on the dispersive parameters, and the conservation of the total mass.

For numerical illustration, we consider the compatible parameter set

$$k = \frac{1}{2}, \quad \beta = 1, \quad \delta = -0.1, \quad \eta = 1, \quad \alpha = \frac{26}{15}, \quad (58)$$

which satisfies the compatibility conditions derived in Section 8. The corresponding analytical solution has amplitude and wave speed

$$A = 5, \quad v = \frac{20}{9}, \quad (59)$$

and therefore takes the explicit form

$$u(x, t) = 5 \operatorname{sech}^2 \left[\frac{1}{2} \left(x - \frac{20}{9} t \right) \right]. \quad (60)$$

The total mass associated with the analytical solution is

$$M(t) = \int_{-\infty}^{\infty} u(x, t) dx = \frac{2A}{k} = 20, \quad (61)$$

which provides an exact invariant against which the numerical computations may be compared.

The governing equation is solved numerically using a Fourier pseudo-spectral discretization in space combined with time integration on a sufficiently large periodic computational domain so that boundary effects remain negligible throughout the simulation. Spatial derivatives are evaluated spectrally, while the mixed dispersive term is treated through the Fourier multiplier

$$1 - \delta \kappa^2,$$

which remains strictly positive for $\delta < 0$. The spectral approximation therefore preserves the high-order accuracy required for the fifth-order dispersive dynamics while avoiding spurious boundary reflections.

Figure 2 summarizes the overall analytical and computational procedure adopted throughout the paper. Starting from the general variable-coefficient model, admissible equivalence transformations reduce the governing equation to a canonical form, after which Lie symmetry analysis, optimal systems and conservation laws provide the analytical foundation for the construction of exact travelling-wave solutions. The final stage consists of numerical validation against the analytical soliton.

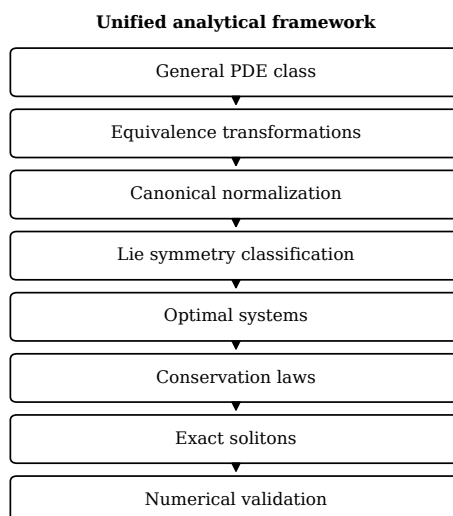


Figure 2. Overall workflow of the unified analytical framework, illustrating the progression from the general variable-coefficient model through equivalence transformations, Lie symmetry classification, conservation laws and similarity reductions to exact solutions and numerical validation.

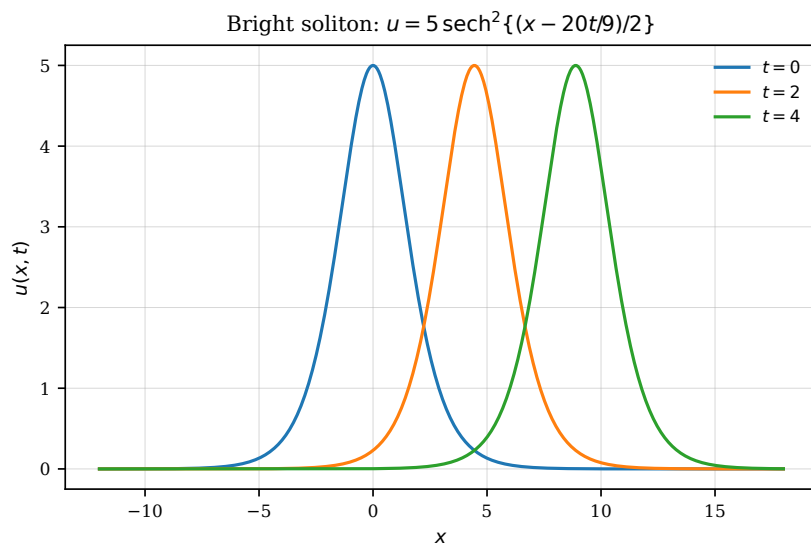


Figure 3. Propagation of the verified bright-soliton solution at representative times. The numerical solution remains in excellent agreement with the analytical profile, confirming the stability of the travelling-wave solution.

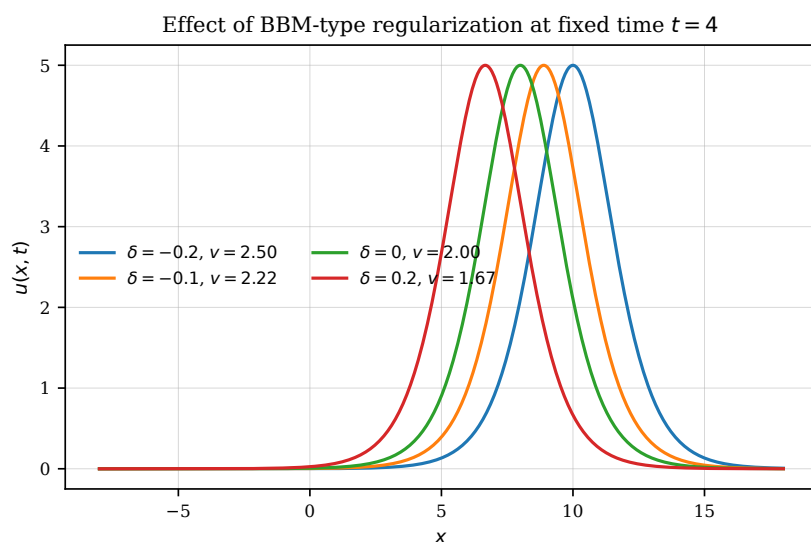


Figure 4. Influence of the BBM-type regularization coefficient δ on the propagation of the solitary wave. Increasing the regularization modifies the wave speed while preserving the localized structure.

Figure 3 illustrates the propagation of the exact bright soliton at several representative times. The numerical and analytical profiles remain indistinguishable throughout the computation, demonstrating that the solitary wave propagates without appreciable distortion and confirming the correctness of the travelling-wave solution derived in Section 8.

The influence of the BBM-type regularization parameter is illustrated in Figure 4. Increasing the magnitude of δ primarily affects the propagation speed through the velocity relation (55) while preserving the localized structure of the solitary wave. This behaviour is fully consistent with the analytical travelling-wave analysis.

Figure 5 illustrates the influence of the nonlinear dispersive coefficient η . As predicted by the analytical expression (54), the parameter η directly controls the wave amplitude through

$$A = \frac{20k^2}{\eta},$$

showing that stronger nonlinear dispersive modulation reduces the amplitude of the localized wave while preserving its characteristic sech^2 profile.

Finally, Figure 6 presents the numerical evolution of the conserved mass. Throughout the computation the numerical

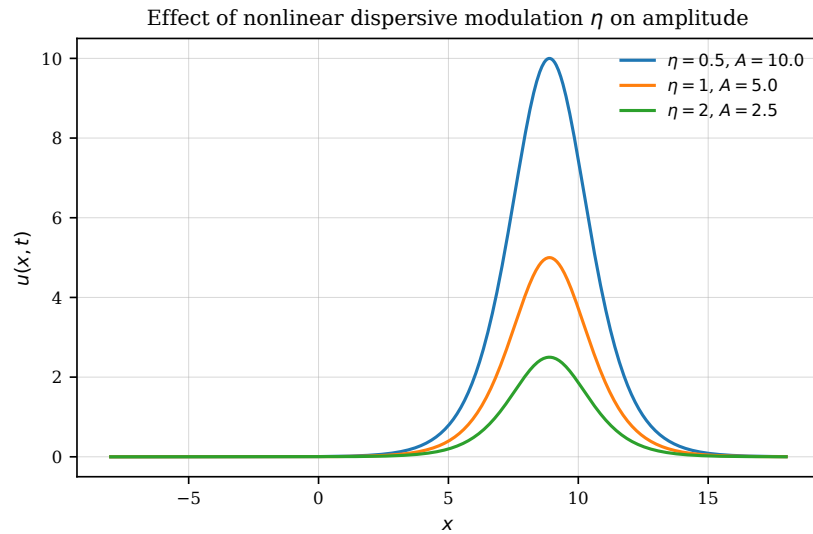


Figure 5. Influence of the nonlinear dispersive coefficient η on the bright-soliton amplitude. The numerical behaviour agrees with the analytical relation $A = 20k^2/\eta$.

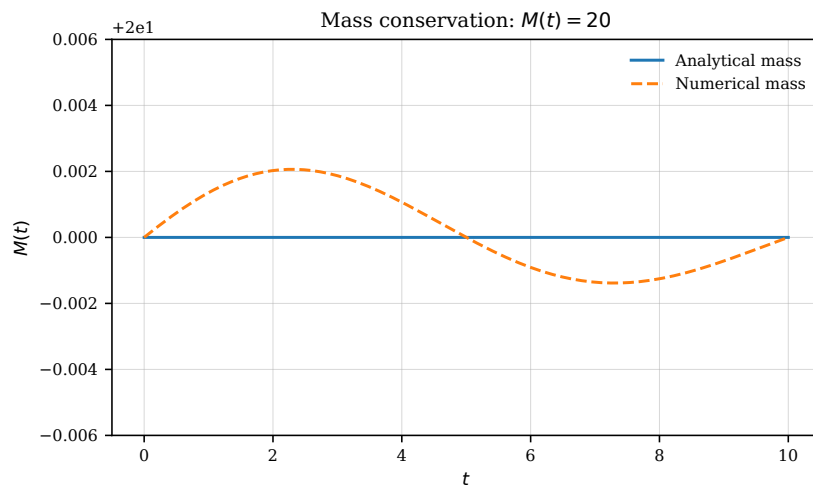


Figure 6. Numerical verification of the universal mass conservation law. The computed mass remains constant throughout the simulation and agrees with the exact analytical value $M = 20$.

value remains essentially constant and coincides with the exact analytical value

$$M = 20,$$

thereby confirming the universal mass conservation law established in Section 7 and demonstrating the consistency of the numerical implementation.

10. DISCUSSION

The present work develops a unified analytical framework for a broad class of variable-coefficient higher-order nonlinear evolution equations incorporating nonlinear convection, third-order dispersion, fifth-order dispersion, BBM-type mixed dispersion and nonlinear dispersive modulation. Rather than proposing an entirely new nonlinear evolution equation, the principal contribution lies in demonstrating that several classical dispersive wave models may be treated within a common mathematical framework through the combined use of admissible equivalence transformations, Lie symmetry analysis, conservation laws, similarity reductions and exact travelling-wave solutions.

A central result of the paper is the determination of the admissible equivalence group together with the canonical normalization of the fifth-order dispersive coefficient. This normalization substantially simplifies the complete group-classification problem by removing one arbitrary functional degree of freedom while preserving the analytical equivalence

of the original class. In particular, the compatibility condition

$$\frac{d\tau}{dt} = 5\lambda$$

emerges naturally from the normalized fifth-order term and governs the admissible temporal component of every Lie point symmetry. Consequently, the kernel algebra together with the power-law, exponential and constant-coefficient symmetry extensions are obtained in a systematic and unified manner.

The conservation-law analysis further illustrates the structural richness of the proposed class. The divergence form of the governing equation guarantees the existence of a universal mass conservation law independently of the particular coefficient functions. By contrast, the regularized energy balance reveals that temporal variations of the BBM-type regularization coefficient and nonlinear dispersive modulation introduce additional source terms that generally prevent energy conservation. This distinction has a natural physical interpretation: while the total wave mass remains invariant, nonlinear dispersive effects may redistribute energy between amplitude and curvature modes, thereby modifying the evolution of localized structures.

The exact bright-soliton solution derived for the constant-coefficient canonical equation demonstrates the effectiveness of the symmetry-based reduction procedure. The analytical expressions obtained for the wave amplitude and propagation speed explicitly reveal the roles played by the higher-order dispersive parameters. In particular, the BBM-type regularization parameter δ modifies the wave velocity through the denominator

$$1 + 4\delta k^2,$$

whereas the nonlinear dispersive coefficient η controls the soliton amplitude through the inverse proportionality

$$A = \frac{20k^2}{\eta}.$$

These analytical predictions are fully consistent with the numerical experiments, which confirm stable propagation of the localized wave, the predicted parameter dependence and the preservation of the universal mass invariant.

From a broader perspective, the proposed framework illustrates the effectiveness of combining equivalence transformations with Lie symmetry analysis for the investigation of parameterized families of nonlinear evolution equations. Much of the existing literature focuses on individual equations or narrowly defined subclasses, whereas the present approach establishes a unified methodology capable of simultaneously treating several important higher-order dispersive models. The resulting symmetry classifications, conservation laws and travelling-wave reductions therefore extend a number of existing analytical results within a single coherent framework.

The present study nevertheless has certain limitations. The exact travelling-wave analysis has been restricted to bright solitary waves associated with the constant-coefficient canonical equation, while the conservation-law analysis has focused primarily on mass and regularized energy invariants. Furthermore, only classical Lie point symmetries have been considered. These restrictions were adopted deliberately in order to establish a complete and rigorous analytical foundation before addressing more sophisticated integrability structures.

11. CONCLUSIONS

This paper has developed a unified analytical framework for a broad class of variable-coefficient higher-order nonlinear evolution equations incorporating nonlinear convection, third-order dispersion, fifth-order dispersion, BBM-type mixed dispersion and nonlinear dispersive modulation. Rather than focusing on a single nonlinear evolution equation, the proposed framework provides a systematic methodology for analysing an entire family of higher-order dispersive wave models within a common mathematical setting.

The analysis began with the derivation of the admissible equivalence group, which was subsequently employed to normalize the fifth-order dispersive coefficient and reduce the general class to a canonical representative. This normalization proved to be a key analytical step, substantially simplifying the complete group-classification problem while preserving the equivalence structure of the original class. The canonical equation was then subjected to a complete Lie symmetry classification, yielding the kernel algebra together with the principal symmetry extensions associated with the power-law, exponential and constant-coefficient subclasses. Optimal systems of one-dimensional subalgebras and the corresponding similarity reductions were constructed, providing systematic reductions of the governing partial differential equation to ordinary differential equations.

The conservative structure of the canonical equation was further exploited to derive universal mass conservation together with a regularized energy conservation law for an important conservative subclass. The analysis also demonstrated the nonlinear self-adjointness of the constant-coefficient conservative model, thereby providing an alternative structural interpretation of the derived conservation laws.

A principal outcome of the paper is the construction of an exact bright-soliton solution for the constant-coefficient canonical equation. Explicit analytical expressions for the wave amplitude and propagation speed were obtained, revealing the influence of the higher-order dispersive mechanisms on solitary-wave dynamics. The analytical solution was rigorously

verified through direct substitution and subsequently employed as a benchmark for numerical simulations. The numerical experiments confirmed stable soliton propagation, the predicted dependence of the wave profile on the dispersive parameters and the preservation of the universal mass invariant, thereby providing strong agreement between the analytical and computational results.

Collectively, the results demonstrate that admissible equivalence transformations, Lie symmetry analysis, conservation laws, similarity reductions and exact travelling-wave solutions may be integrated into a single coherent analytical framework for higher-order variable-coefficient nonlinear evolution equations. The methodology developed here is sufficiently general to encompass several classical dispersive wave models while simultaneously providing new structural insights into their symmetry properties and conservation mechanisms.

Several natural extensions of the present work remain open. These include the investigation of generalized and nonlocal symmetries, higher-order conservation laws, Hamiltonian and bi-Hamiltonian structures, recursion operators, Darboux and Bäcklund transformations, integrable reductions, multi-soliton interactions and the stability of coherent structures. Such developments would further deepen the mathematical understanding of higher-order nonlinear evolution equations and broaden the range of physical applications accessible through the unified framework established in this paper.

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REFERENCES

- [1] P. J. Olver, *Applications of Lie Groups to Differential Equations*, 2nd ed. (Springer, New York, 1993).
- [2] G. W. Bluman, and S. Kumei, *Symmetries and Differential Equations*, (Springer, Berlin, 1989).
- [3] N. H. Ibragimov, *Transformation Groups Applied to Mathematical Physics*, (Reidel, Dordrecht, 1985).
- [4] L. V. Ovsiannikov, *Group Analysis of Differential Equations*, (Academic Press, New York, 1982).
- [5] E. Noether, Invariante Variationsprobleme, in: *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, pp. 235–257 (1918).
- [6] R. Hirota, *The Direct Method in Soliton Theory* (Cambridge University Press, Cambridge, 2004).
- [7] M. J. Ablowitz, and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, Cambridge, 1991).
- [8] H. Schamel, and S. Bujarbarua, "Analytical double layers," *Phys. Fluids*, **26**, 190–193 (1983). <https://doi.org/10.1063/1.864006>
- [9] H. Schamel, "Weak double layers: Existence, stability, evidence," *Z. Naturforsch. A*, **38**, 1170–1183 (1983). <https://doi.org/10.1515/zna-1983-1102>
- [10] H. Schamel, and V. I. Maslov, "Adiabatic growth of electron holes in current-carrying plasmas," *Phys. Scr.* **T50**, 42 (1994). <https://doi.org/10.1088/0031-8949/1994/T50/006>
- [11] V. I. Maslov, and H. Schamel, "Growing electron holes in drifting plasmas," *Phys. Lett. A*, **178**, 171–174 (1993). [https://doi.org/10.1016/0375-9601\(93\)90746-M](https://doi.org/10.1016/0375-9601(93)90746-M)
- [12] H. Schamel, and V. I. Maslov, "Langmuir wave contraction caused by electron holes," *Phys. Scr.* **T82**, 122 (1999). <https://doi.org/10.1238/Physica.Topical.082a00122>
- [13] V. I. Maslov, "Electron beam excitation of a potential well in a magnetized plasma waveguide," *Phys. Lett. A*, **165**, 63–68 (1992). [https://doi.org/10.1016/0375-9601\(92\)91055-V](https://doi.org/10.1016/0375-9601(92)91055-V)
- [14] A. P. Márquez, M. L. Gandarias, and S. C. Anco, "Conservation laws, symmetries, and line solitons of a Kawahara–KP equation," *Journal of Computational and Applied Mathematics*, **436**, 115412 (2023). <https://doi.org/10.1016/j.cam.2023.115412>
- [15] A. P. Márquez, T. M. Garrido, C. M. Khaliq, and M. L. Gandarias, "Symmetries, conservation laws, and line soliton solutions of a two-dimensional generalized KdV equation with p -power," *Discrete and Continuous Dynamical Systems Series S*, **18**, 1024–1035 (2025). <https://doi.org/10.3934/dcdss.2024130>
- [16] W. Hereman, "Symbolic computation of conservation laws of nonlinear partial differential equations in multi-dimensions," *International Journal of Quantum Chemistry*, **106**, 278–299 (2006). <https://doi.org/10.1002/qua.20727>
- [17] F. Afzal, and A. A. Lupas, "Lie Symmetry Analysis, Optimal Systems and Physical Interpretation of Solutions for the KdV-Burgers Equation," *Symmetry*, **17**, 1981 (2025). <https://doi.org/10.3390/sym17111981>
- [18] S. T. R. Rizvi, et al., "Lie symmetry analysis, conservation laws and soliton solutions by complete discrimination system for polynomial approach of Landau Ginzburg Higgs equation along with its stability analysis," *Optik*, **300**, 171675 (2024). <https://doi.org/10.1016/j.ijleo.2024.171675>

- [19] A. Hussain, M. K. Zia, K. S. Nisar, V. Vijayakumar, and I. Khan, "Lie analysis, conserved vectors, nonlinear self-adjoint classification and exact solutions of generalized $(N + 1)$ -dimensional nonlinear Boussinesq equation," AIMS Mathematics, **7**, 13139–13168 (2022). <https://doi.org/10.3934/math.2022725>
- [20] S. Kumar, et al., "Lie symmetries, modulation instability, conservation laws, and the dynamic waveform patterns of several invariant solutions to a $(2+1)$ -dimensional Hirota bilinear equation," Discrete and Continuous Dynamical Systems - Series S, **18**, 1054–1074 (2025). <https://doi.org/10.3934/dcdss.2024136>
- [21] S. O. Mbusi, A. R. Adem, and B. Muatjetjeja, "Lie symmetry analysis, multiple exp-function method and conservation laws for the $(2+1)$ -dimensional Boussinesq equation," Optical and Quantum Electronics, **56**, 670 (2024). <https://doi.org/10.1007/s11082-024-06339-1>
- [22] F. S. Costa, J. C. A. Soares, J. V. C. Sousa, R. R. Luz, and J. A. R. Santos, "Lie point symmetries and conservation laws to fractional $\zeta(t)$ -KdV equation," International Journal of Theoretical Physics, **65**, 11 (2026). <https://doi.org/10.1007/s10773-025-06221-0>
- [23] S. Miao, et al., "An Analysis of the Lie Symmetry and Conservation Law of a Variable-Coefficient Generalized Calogero–Bogoyavlenskii–Schiff Equation in Nonlinear Optics and Plasma Physics," Mathematics, **12**, 3619 (2024). <https://doi.org/10.3390/math12223619>
- [24] M. Junaid-U-Rehman, G. Kudra, and J. Awrejcewicz, "Conservation laws, solitary wave solutions, and Lie analysis for the nonlinear chains of atoms," Scientific Reports, **13**, 11537 (2023). <https://doi.org/10.1038/s41598-023-38658-w>
- [25] H. Almusawa, and A. Jhangeer, "Nonlinear self-adjointness, conserved quantities and Lie symmetry of dust size distribution on a shock wave in quantum dusty plasma," Communications in Nonlinear Science and Numerical Simulation, **114**, 106660 (2022). <https://doi.org/10.1016/j.cnsns.2022.106660>
- [26] M. A. Kawser, M. A. Akbar, and M. A. Khan, "Exploring variable coefficient models: Insights into nonlinear wave behavior and soliton solutions in physical systems," Results in Physics, **56**, 107242 (2024). <https://doi.org/10.1016/j.rinp.2023.107242>
- [27] H. Wang Z. Zou, and X. Shen, "Lie symmetry analysis, self-adjointness and conservation law for a type of nonlinear equation," Mathematics, **9**, 1313 (2021). <https://doi.org/10.3390/math9121313>
- [28] E. Yaşar, and E. Yaşar, "Exact wave solutions and local conservation laws of a third-order Boussinesq system," Discrete and Continuous Dynamical Systems - Series S, **18**, 1075–1092 (2025). <https://doi.org/10.3934/dcdss.2024148>
- [29] M. Kumar, S. Anand, and D. V. Tanwar, "Lie symmetries, solitary wave solutions, and conservation laws of coupled KdV–mKdV equation for internal gravity waves," Mathematical Methods in the Applied Sciences, **47**, 6909–6927 (2024). <https://doi.org/10.1002/mma.9949>

ЄДИНЕНА КЛАСИФІКАЦІЯ СИМЕТРІЙ, ЗАКОНИ ЗБЕРЕЖЕННЯ ТА ДИНАМІКА СОЛІТОНІВ ДЛЯ КЛАСУ НЕЛІНІЙНИХ ЕВОЛЮЦІЙНИХ РІВНЯНЬ ВИЩОГО ПОРЯДКУ ЗІ ЗМІННИМ КОЕФІЦІЄНТОМ

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Нелінійні еволюційні рівняння вищого порядку зі змінним коефіцієнтом становлять важливий клас математичних моделей для опису поширення нелінійних дисперсійних хвиль у неоднорідних середовищах, включаючи потоки мілководдя, плазму, оптичні системи та пружні структури. У цій статті розроблено єдину аналітичну основу для широкого сімейства таких рівнянь, що включають нелінійну конвекцію, дисперсію третього порядку, дисперсію п'ятого порядку, змішану дисперсію типу ВВМ та нелінійну дисперсійну модуляцію. Спочатку побудовано допустимі перетворення еквівалентності та використано їх для зведення загального класу за умови ненульової дисперсії п'ятого порядку до канонічної нормалізованої форми. Нормалізація призводить до спрощеної задачі групової класифікації шляхом встановлення явної умови сумісності для часової складової кожної допустимої симетрії точки Лі. Отримано повні класифікації симетрії Лі для підкласів ядра, степеневого закону, експоненціального закону та закону з постійним коефіцієнтом, після чого побудовано оптимальні системи та редукції подібності. Закони збереження виведені з консервативної структури рівняння, що визначає, включаючи універсальний інваріант маси та регуляризований закон збереження енергії для важливого канонічного підкласу. Крім того, отримано точний розв'язок яскравого солітона для рівняння з постійним коефіцієнтом, який ретельно перевірено прямою підстановкою та використано для перевірки числових симуляцій. Запропонована структура об'єднує перетворення еквівалентності, аналіз симетрії Лі, закони збереження, редукції подібності та динаміку солітонів в межах однієї параметризованої системи, забезпечуючи систематичну методологію для аналітичного дослідження нелінійних дисперсійних хвильових рівнянь вищого порядку.

Ключові слова: аналіз симетрії Лі; перетворення еквівалентності; нелінійні еволюційні рівняння зі змінним коефіцієнтом; закони збереження; дисперсійні хвилі вищого порядку; дисперсія n 'ятого порядку; редукції подібності; динаміка солітонів