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FINITE-SIZE SCALING ANALYSIS FOR THE QCD DECONFINING PHASE TRANSITION TO A COLOR-SINGLET QUARK-GLUON PLASMA WITH THE THREE u, d AND s QUARK FLAVORS

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In this work, we calculate the partition function of the Quark-Gluon plasma (QGP) projected on the SU(3) color-singlet representation, using the projection method, within a density of states for quarks and gluons given by the Multiple Reflexion Expansion (MRE) approximation. We examine the impact of incorporating the area and curvature terms, in addition to the volume term in the density of states, on the behavior of several physical quantities characterizing the deconfining phase transition from a hadronic gas phase of massive pions, to a Quark-Gluon plasma (QGP) phase made up of gluons, massless up and down quarks, and massive strange quarks along with their antiquarks, within the Bag model and a phenomenological phase coexistence model.By means of a finite-size scaling analysis, we investigate the behavior of some scaling exponents relevant to the occurring deconfining phase transition in a finite volume, especially that of the scaling exponent relative to the shift of the effective transition temperature, by considering contributions of the curvature term only, the area term only in the density of states additionally to the volume term, then the contribution of all three terms, by analyzing the value of the shift critical exponent λ compared to that obtained previously using the volume term only in the quarks and gluons density of states.

Keywords: *Deconfining phase transition; Quark-Gluon Plasma; Color-singletness; Projection method; Finite-size scaling analysis*

1. INTRODUCTION

Under ordinary conditions of temperature and/or density, quarks and gluons, which make up protons, neutrons and all hadrons [1, 2], are bound together by the strong interaction, which is described by Quantum Chromodynamics (QCD) (For a recent review see for example [3, 4]). This means that quarks and gluons are confined within hadrons. However, at high temperatures and/or densities, the strong interactions become negligible, and hadronic matter undergoes a transition to a deconfined phase, where quarks and gluons are no longer confined and move freely in a hot and dense state [5, 6]. This new state of nuclear matter is called the Quark-Gluon Plasma (QGP) [7, 8]. The color confinement suggests that this QGP phase may exist in a color-singlet state [9–11]. It is thought that the QGP exists in the cores of neutron stars, also known as dense stars [12, 13], where the density is so high. It also dominated the universe in the first 10^{-5} seconds after the Big Bang when the temperature was extremely high [14, 15]. However, as the temperature rapidly decreased, these particles began to combine and form hadrons. To study the QGP phase, a "little bang" is recreated in laboratories by conducting ultra-relativistic heavy-ion collisions at high energy in accelerators [16, 17]. This was first proposed in 1983 by J. D. Bjorken, who gave an initial description of the evolution of the QGP [18].

This work is a continuation of the previous studies of one of us [19–21], where we examined the thermally driven deconfinement phase transition (DPT) in a finite size, at zero chemical potential ($\mu = 0$), and of [22] where we examined the density driven deconfinement phase transition (DPT) at fixed temperature, from a hadronic gas (HG) phase to the QGP phase projected on the color-singlet SU(3) representation using the projection method [23, 24]. Therein, only the contribution of the volume term was considered in the quarks and gluons density of states, and the behavior of several physical quantities has been studied near the transition. Then, by means of a finite-size scaling analysis [25–27], we investigated the universal quantities called critical exponents, considered as indicators of the order of the occurring phase transition [28–30]. Our goal is to improve the value of the shift scaling exponent, which was found in our previous work [19] considerably deviating from the expected value 1, known to characterize a first-order phase transition, and which was obtained for the three other scaling exponents characterizing the width of the transition region as well as the heights of the peaks of both thermal susceptibility and specific heat density. This will be achieved by considering more realistic conditions in the studied system, treating the pions in the hadronic phase as massive particles and including the massive strange quarks in addition to the massless up and down quarks and the gluons in the QGP phase. Also, we consider the area and curvature contributions, in addition to the volume contribution, in the quarks and gluons density of states determined by the Multiple Reflection Expansion (MRE) approximation [31, 32], commonly called the asymptotic expansion of the density of states, used in the projection method for the calculation of the color-singlet QGP partition function.

We describe the thermally driven QCD deconfinement phase transition at zero chemical potential using the MIT bag model [7, 33], which states that at low density and/or temperature the real vacuum exerts an external pressure, called the

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bag constant B, on the QCD perturbative vacuum of the hadron bag, keeping the quarks and gluons confined inside the bag, for further details see [34, 35]. We also assume the coexistence of the HG and QGP phases in a finite volume, using a phenomenological model proposed in [36]. Initially, we investigate the behavior of several thermodynamic quantities, such as the order parameter and its first and second derivatives with respect to temperature, for different volumes at zero chemical potential ($\mu = 0$). We aim to study how the area and curvature terms affect the temperature at which the phase transition occurs, and also the scaling critical exponent that describes the variations of the shift of this transition temperature with the system volume. We compare the obtained results with those of our previous works [19, 20], where only the volume term was included in the density of states, and with those obtained in [21] with the contribution of the volume and curvature terms in the quarks and gluons density of states.

2. PARTITION FUNCTIONS OF THE HADRONIC GAS AND THE COLOR-SINGLET QGP PHASES

Let us recall that as in our previous works [19–21], we consider a mixed system of HG and QGP phases in a finite total volume: $V = V_{\mathrm{HG}} + V_{\mathrm{QGP}}$. In this system, the hadronic phase occupies a volume V_{HG} with a fraction h of the total system volume, such that $V_{\mathrm{HG}} = hV$. The parameter h represents the quantitative characterization of the macroscopic state of the system, and the QGP phase occupies a volume V_{QGP}, such that $V_{\mathrm{QGP}} = (1-h)V$, where $h = 1$ corresponds to the system being entirely in the hadronic phase, and $h = 0$ corresponds to the system being entirely in the QGP phase. In such a phenomenological model used in [36], the probability density to find the system in the state h is given by:

$$p(h, T, \mu, V) = \frac{Z(h, T, \mu, V)}{\int_0^1 Z(h, T, \mu, V)\, dh} \tag{1}$$

and the mean value of any thermodynamic quantity $A(T, \mu, V)$ of the system can then be written in the following form:

$$\langle A(T, \mu, V)\rangle = \int_0^1 A(h, T, \mu, V)\, p(h, T, \mu, V)\, dh \tag{2}$$

where $A(h, T, \mu, V)$ is the thermodynamic quantity in the state h, given in the case of an intensive quantity by:

$$A(h, T, \mu, V) = hA_{\mathrm{HG}}(T, \mu, hV) + (1-h)A_{\mathrm{QGP}}(T, \mu, (1-h)V) \tag{3}$$

with A_{HG} and A_{QGP} the thermodynamic quantities relative to the individual HG and QGP phases respectively; $Z(h, T, \mu, V)$ is the total partition function of the mixed HG-QGP system:

$$Z(h, T, \mu, V) = Z_{\mathrm{QGP}}(h, T, \mu, V)\, Z_{\mathrm{HG}}(h, T, \mu, V)\, Z_{\mathrm{vac}}(h, T, \mu, V) \tag{4}$$

where $Z_{\mathrm{QGP}}(h, T, \mu, V)$ and $Z_{\mathrm{HG}}(h, T, \mu, V)$ are the partition functions of the QGP and HG phases respectively, and $Z_{\mathrm{vac}}(h, T, \mu, V)$ reads for the real vacuum pressure B as mentioned above:

$$Z_{\mathrm{vac}} = \exp\left(-\frac{BV_{\mathrm{QGP}}}{T}\right) \tag{5}$$

For the calculation of $Z_{\mathrm{HG}}(h, T, \mu, V)$, we consider a hadronic gas consisting of massive pions ($m_\pi = 138\,\mathrm{MeV}$), and we obtain:

$$Z_{\mathrm{HG}} = \exp\left(\frac{d_\pi V_{\mathrm{HG}}}{2\pi^2 T} \int_0^\infty \frac{k^4\, dk}{\sqrt{k^2 + m_\pi^2}\left(e^{\beta\sqrt{k^2+m_\pi^2}} - 1\right)}\right) \tag{6}$$

where k is the momentum, d_π the pion degeneracy factor, and $\beta = 1/T$.

To calculate $Z_{\mathrm{QGP}}(h, T, \mu, V)$ projected on the color-singlet SU(3) representation, we implement the color-singletness constraint into the quantum statistical description of the QGP phase, using the group theoretical projection method formulated by Redlich and Turko [22, 23], and for this we consider a QGP phase consisting of gluons, massless up and down quarks and massive strange quarks with their antiquarks, and we assume non-interaction between these particles. Such a partition function of a color-singlet QGP, in a volume V, at temperature T and chemical potential μ, can be written as in [37, 38]:

$$Z_{\mathrm{QGP}}(T, V, \mu) = \frac{4}{9\pi^2} \int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi} d\phi\, d\psi\, M(\phi, \psi)\, \tilde{Z}_{\mathrm{QGP}}(T, V, \mu, \phi, \psi) \tag{7}$$

where $M(\phi, \psi)$ is the weight function or the Haar measure, given by:

$$M(\phi, \psi) = \left(\sin\left(\frac{1}{2}\left(\psi + \frac{\phi}{2}\right)\right) \sin\left(\frac{\phi}{2}\right) \sin\left(\frac{1}{2}\left(\psi - \frac{\phi}{2}\right)\right)\right)^2 \tag{8}$$

and $\tilde{Z}_{\text{QGP}}$ the generating function, given for a system of quarks, antiquarks and gluons, at a temperature T and chemical potential μ in a volume V by:

$$\tilde{Z}_{\text{QGP}}(T,V,\mu,\phi,\psi) = \tilde{Z}_{\text{quark}}(T,V,\mu,\phi,\psi)\,\tilde{Z}_{\text{gluon}}(T,V,\mu,\phi,\psi) \tag{9}$$

where $\tilde{Z}_{\text{quark}}$ and $\tilde{Z}_{\text{gluon}}$ are the generating functions corresponding to the quarks-antiquarks and gluons respectively, expressed as:

$$\ln \tilde{Z}_{\text{quark}} = \sum_{q=r,b,g}\sum_{\alpha}\left\{\ln\left[1+e^{-\beta(\sqrt{k^2+m_q^2}-\mu)+i\theta_q}\right]+\ln\left[1+e^{-\beta(\sqrt{k^2+m_q^2}+\mu)-i\theta_q}\right]\right\} \tag{10}$$

$$\ln \tilde{Z}_{\text{gluon}} = \sum_{G=1}^{4}\sum_{\alpha}\left\{\ln\left[1+e^{-\beta k+i\theta_G}\right]^{-1}+\ln\left[1+e^{-\beta k-i\theta_G}\right]^{-1}\right\} \tag{11}$$

m_q being the quark mass. From [37], the explicit expressions of the angles θ_q $(q=r,b,g)$ and θ_G $(G=1,\dots,4)$ in SU(3) are respectively:

$$\theta_r = \frac{\phi}{2}+\frac{\psi}{3},\quad \theta_g = -\frac{\phi}{2}+\frac{\psi}{3},\quad \theta_b = -\frac{2\psi}{3} \tag{12}$$

$$\theta_1 = \theta_r-\theta_g,\quad \theta_2=\theta_g-\theta_b,\quad \theta_3=\theta_b-\theta_r,\quad \theta_4=0 \tag{13}$$

In the continuum approximation for single-particle energy levels, the summation $\sum_\alpha$ can be replaced by an integral using a density of states, expressed for quarks-antiquarks and gluons respectively as:

$$\ln \tilde{Z}_{\text{quark}} = \sum_{q=r,b,g} d_Q\int_0^\infty\left\{\ln\left[1+e^{-\beta(\sqrt{k^2+m_q^2}-\mu)+i\theta_q}\right]+\ln\left[1+e^{-\beta(\sqrt{k^2+m_q^2}+\mu)-i\theta_q}\right]\right\}\rho_Q(k,V)\,dk \tag{14}$$

$$\ln \tilde{Z}_{\text{gluon}} = \sum_{G=1}^{4} d_G\int_0^\infty\left\{\ln\left[1+e^{-\beta k+i\theta_G}\right]^{-1}+\ln\left[1+e^{-\beta k-i\theta_G}\right]^{-1}\right\}\rho_G(k,V)\,dk \tag{15}$$

d_Q and d_G are the spin-isospin degeneracy factors for quarks-antiquarks and gluons respectively; $\rho_Q(k,V)$, $\rho_G(k,V)$ are the density of states for quarks and gluons respectively, with a spherical volume $V_{\text{QGP}} = \frac{4}{3}\pi R^3$ described by the bag model [32, 38], R being the radius of the bag [14]. In the following, we shall use the density of states ρ_i $(i=Q,G)$ given by the Multiple Reflection Expansion (MRE) approximation proposed by Balian and Bloch [27], and used for example in [39], expressed by the sum of a volume term, an area term and a curvature term on the form:

$$\rho_i(k,V) = \frac{V_{\text{QGP}}k^2}{2\pi^2}+f_{A,i}4\pi R^2 k+f_{C,i}8\pi R+\cdots,\quad i=Q,G \tag{16}$$

where $f_{A,i}$ and $f_{C,i}$ are the area and curvature coefficients respectively [41–44], given by:

$$f_{A,Q} = -\frac{1}{8\pi}\left(1-\frac{1}{2\pi}\arctan\frac{k}{m}\right)\to f_{A,Q}=0 \text{ when } m=0 \tag{17}$$

$$f_{C,Q} = \frac{1}{12\pi^2}\left[1-\frac{3k}{2m}\left(\frac{\pi}{2}-\arctan\frac{k}{m}\right)\right]\to f_{C,Q} = -\frac{1}{24\pi^2}\text{ when } m=0 \tag{18}$$

$$f_{A,G}=0,\qquad f_{C,G}=-\frac{1}{6\pi^2} \tag{19}$$

After proper calculations and simplifications, we find:

$$Z_{\text{QGP}}(T,V,\mu=0) = \frac{4}{9\pi^2}\int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi}d\phi\,d\psi\,M(\phi,\psi)\,e^{g_1+g_2+g_3} \tag{20}$$

where g_1, g_2, g_3 result respectively from the contributions of the volume, area and curvature terms in the quarks and gluons density of states, and they are given by:

$$\begin{aligned} g_1 = {} & d_Q V_{\text{QGP}}\sum_{q=r,b,g}\left\{\frac{\pi^2}{6}T^3\left[\frac{7}{30}-\left(\frac{\theta_q}{\pi}\right)^2+\frac{1}{2}\left(\frac{\theta_q}{\pi}\right)^4\right]\right.\\ & \left.+\frac{(m_S T)^{3/2}}{\sqrt{2}\,\pi^{3/2}}\left(1-\frac{\theta_q^2}{2!}+\frac{\theta_q^4}{4!}\right)e^{-m_S/T}\right\}\\ & + d_G V_{\text{QGP}}T^3\sum_{G=1}^{4}\frac{\pi^2}{12}\left[-\frac{7}{30}+\left(\frac{\theta_G-\pi}{\pi}\right)^2-\frac{1}{2}\left(\frac{\theta_G-\pi}{\pi}\right)^4\right]\end{aligned} \tag{21}$$

$$g_2 = d_Q\left(\frac{3V_{\mathrm{QGP}}}{4\pi}\right)^{2/3}\left(\sqrt{\frac{2m_S}{\pi}}T^{3/2} - m_S T\right)\sum_{q=r,b,g}\left(1-\frac{\theta_q^2}{2!}+\frac{\theta_q^4}{4!}\right)e^{-m_S/T} \tag{22}$$

$$\begin{aligned} g_3 = d_Q\left(\frac{3V_{\mathrm{QGP}}}{4\pi}\right)^{1/3}&\sum_{q=r,b,g}\left\{\frac{\pi}{3}T\left(-\frac{1}{3}+\left(\frac{\theta_q}{\pi}\right)^2\right)\right. \\ &\left.+\left(\frac{2}{3}\sqrt{\frac{2T}{\pi m_S}}-T+\sqrt{\frac{2}{\pi m_S}}T^{3/2}\right)\left(1-\frac{\theta_q^2}{2!}+\frac{\theta_q^4}{4!}\right)e^{-m_S/T}\right\} \\ &+d_G\left(\frac{3V_{\mathrm{QGP}}}{4\pi}\right)^{1/3}\sum_{G=1}^{4}\frac{2\pi}{3}T\left[\frac{1}{3}-\left(\frac{\theta_G-\pi}{\pi}\right)^2\right] \end{aligned} \tag{23}$$

Let us note that we have used some approximations in the calculation of the PF of the massive strange quarks [39].

3. BEHAVIOR OF THE ORDER PARAMETER AND ITS FIRST AND SECOND DERIVATIVES WITHIN DIFFERENT CONTRIBUTIONS IN THE DENSITY OF STATES

In a first step, we will study the variations of the mean value of the order parameter with temperature T for different system sizes V, as in the previous work of one of us [19], but now by adding the area and curvature terms in the quarks and gluons density of states, then studying the effect of these terms on the variations of the order parameter, which is nothing but the mean value of the hadronic volume fraction $\langle h(T,V)\rangle$, then on the transition temperature at finite volume $T_c(V)$ deduced from $\langle h(T,V)\rangle$. We find the following expression of the order parameter:

$$\langle h(T,V)\rangle = \frac{\int_0^1 h\,dh\int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi} d\phi\,d\psi\,M(\phi,\psi)\,e^{X_1}}{\int_0^1 dh\int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi} d\phi\,d\psi\,M(\phi,\psi)\,e^{X_1}} \tag{24}$$

where

$$X_1 = -\frac{(1-h)VB}{T} + g_1 + g_2 + g_3 + \frac{hVd_\pi}{2\pi^2 T}I_1 \tag{25}$$

and

$$I_1 = \int_0^\infty \frac{k^4\,dk}{\sqrt{k^2+m_\pi^2}\left(e^{\beta\sqrt{k^2+m_\pi^2}}-1\right)} \tag{26}$$

The order parameter is calculated using a suitable numerical integration method, and the integrals are evaluated at different values of the system volume V, over a temperature interval around the transition temperature. The variations of the order parameter as a function of temperature for different volumes are shown in Figure 1, with a bag constant value $B^{1/4} = 225\,\mathrm{MeV}$, at zero chemical potential ($\mu = 0$), considering different contributions to the quarks and gluons density of states: the volume term only, volume and area terms, volume and curvature terms, and volume, area and curvature terms.

From the various curves on Figure 1, it can clearly be observed that for large system volumes, there is a sharp jump of $\langle h(T,V)\rangle$ from the value 1 in the HG phase to zero in the QGP phase, at a temperature approaching the true transition temperature, noted $T_c(\infty)$, and found to have the value $T_c(\infty) = 158.522\,\mathrm{MeV}$, for $B^{1/4} = 225\,\mathrm{MeV}$. This jump, mathematically described by the Heaviside function, reflects the existence of a latent heat during the transition, signifying that the transition is of first order. For small system volumes, this discontinuity becomes smoothed and rounded off, around an effective transition temperature, noted $T_c(V)$, which can be defined as the temperature at which the order parameter reaches the value $\langle h\rangle = 0.5$ [36], indicating that the HG and QGP phases contribute equally to the system. This effective transition temperature in a finite volume is shifted to temperatures higher than $T_c(\infty)$. This shift is due to the color-singletness condition, which reduces the effective number of internal degrees of freedom as the volume decreases, causing the system to reach the necessary pressure for the phase transition at higher temperatures in smaller volumes [37].

It can also be noted from Figure 1 that the general shape of the order parameter $\langle h(T,V)\rangle$ is similar in all four cases, with a difference in the effective transition temperature at which the transition occurs, since it is well expected that the transition temperature at a fixed volume varies with the contributions to the quarks and gluons density of states. The obtained results show that it takes its highest value when all three terms are included (volume, area, and curvature) and its lowest value when only the volume term is included, with:

$$T_c^{(\mathrm{Vol})}(V) < T_c^{(\mathrm{Vol+Area})}(V) < T_c^{(\mathrm{Vol+Curv})}(V) < T_c^{(\mathrm{Vol+Area+Curv})}(V). \tag{27}$$

Table 1 illustrates the effective transition temperature $T_c(V)$ for different volumes, for all considered contributions to the density of states.

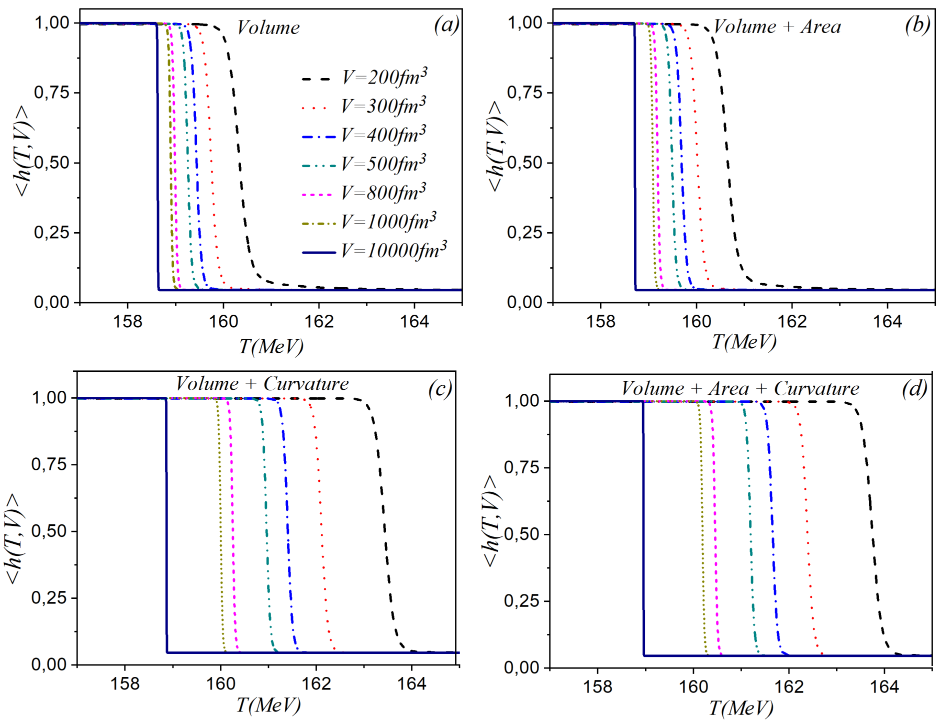

Figure 1. Variations of the order parameter $\langle h(T,V)\rangle$ as a function of temperature T, for different volumes, at $\mu = 0$ and $B^{1/4} = 225\,\text{MeV}$, with the four contributions in the density of states.

Table 1. Values of the effective transition temperature $T_c(V)$ for various volumes, with different terms contributing to the density of states.

System Volume (fm^3)	$T_c^{(\text{Vol})}(V)$ (MeV)	$T_c^{(\text{Vol+Area})}(V)$ (MeV)	$T_c^{(\text{Vol+Curv})}(V)$ (MeV)	$T_c^{(\text{Vol+Area+Curv})}(V)$ (MeV)
200	160.338	160.654	163.434	163.747
300	159.746	160.021	162.111	162.387
500	159.255	159.489	160.961	161.196
800	158.984	159.186	160.253	160.455
1000	158.897	159.085	159.999	160.188
2000	158.737	158.888	159.442	159.593
3000	158.690	158.821	159.228	159.360
4000	158.666	158.785	159.111	159.230
5000	158.652	158.763	159.035	159.146
6000	158.642	158.747	158.982	159.087
8000	158.631	158.725	158.911	159.006
10000	158.624	158.712	158.865	158.954

From the above analysis, it follows that the first thermal derivative of the order parameter, known as the thermal susceptibility $\chi(T,V)$, reaches its minimum value at the effective transition temperature $T_c(V)$, corresponding to the value $\langle h\rangle = 0.5$ of the order parameter, and subsequently to a vanishing second derivative $\langle h(T,V)\rangle''$ of the order parameter with respect to temperature.

Consequently, one can also investigate the influence of various contributions by analyzing the first and second derivatives of the order parameter with respect to temperature T, defined as follows:

$$\chi(T,V) = \frac{\partial\langle h(T,V)\rangle}{\partial T} \tag{28}$$

$$\langle h(T,V)\rangle'' = \frac{\partial^2\langle h(T,V)\rangle}{\partial T^2} = \frac{\partial \chi(T,V)}{\partial T} \tag{29}$$

Our results for the variations of the thermal susceptibility are illustrated in Figure 2. It is noted that the behavior of the susceptibility $\chi(T,V)$ as a function of temperature T at a fixed system volume V, is similar for the four cases of the quarks and gluons density of states: with the contributions of the volume term only, volume and area terms, volume and curvature terms, and all three terms. The key difference is the shift of the curves to higher temperatures when the area term is included, and the temperature further increases when the curvature term is added, reaching the highest values when all terms are included. For example, at $V = 200\,\mathrm{fm}^3$, the transition temperature is $T_c^{(\mathrm{Vol})} = 160.865\,\mathrm{MeV}$ with the volume term only, and it increases to $T_c^{(\mathrm{Vol+Area})} = 161.215\,\mathrm{MeV}$, while with the volume and curvature terms it attains $T_c^{(\mathrm{Vol+Curv})} = 164.652\,\mathrm{MeV}$, and it reaches its highest value $T_c^{(\mathrm{Vol+Area+Curv})} = 164.997\,\mathrm{MeV}$ with the volume, area and curvature terms contribution.

From Figure 2, we observe that in all four cases, the singularity of the Dirac delta function in $\chi(T,V)$, which occurs at the thermodynamic limit, spreads out into a finite peak acquiring a finite width $\delta T(V)$ in small volumes. This spreading corresponds to the rounding of the order parameter and is due to the finite probability of presence for the QGP phase below the transition temperature $T_c(V)$ and of the HG phase above $T_c(V)$, induced by significant thermodynamic fluctuations. This result is expected since the finite discontinuity of the order parameter, described by the Heaviside function, becomes a delta function δ at the level of the first derivative, i.e., in the susceptibility [19].

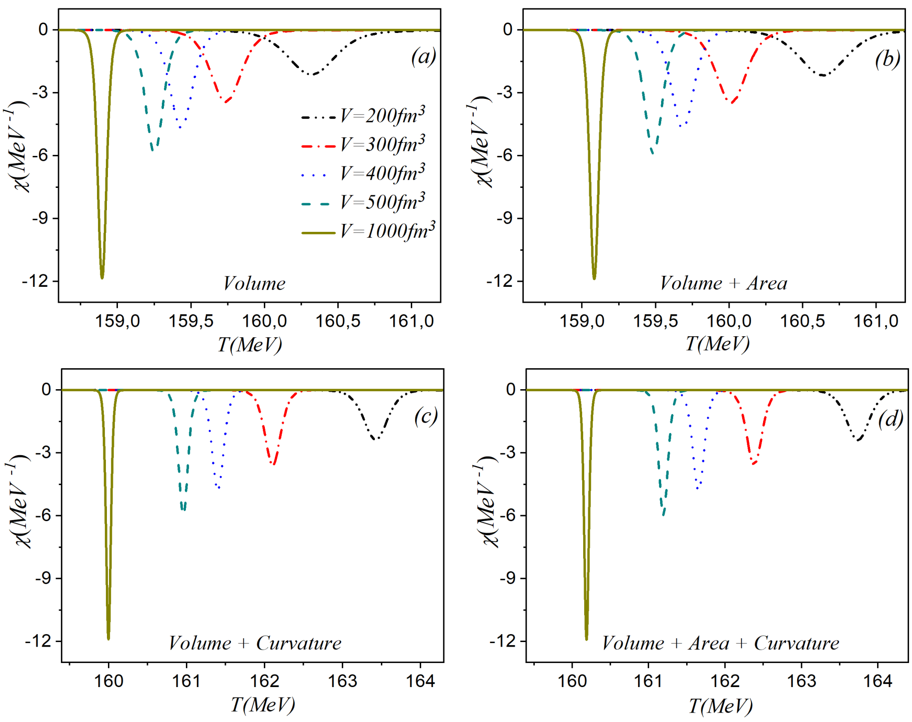

Figure 2. Variations of the susceptibility $\chi(T,V)$ versus temperature, for different system volumes, at $\mu = 0$ and $B^{1/4} = 225\,\mathrm{MeV}$, with the four contributions in the density of states.

To better understand the width $\delta T(V)$ of the region over which the transition is rounded off, we plot the second derivative of the order parameter $\langle h(T,V)\rangle''$ as a function of temperature T for different system sizes in Figure 3, across all contributions. The second derivative is zero at the transition temperature $T_c(V)$. The width of the transition region $\delta T(V)$ is defined as the difference between the temperatures $T_1(V)$ and $T_2(V)$ at which the second derivative reaches its minimum and maximum respectively [19], i.e., $\delta T(V) = T_2(V) - T_1(V)$.

For further clarity, Figure 4 illustrates the order parameter $\langle h(T,V)\rangle$ (black solid line), its first thermal derivative $\chi(T,V)$ (red dashed line), and its second thermal derivative $\langle h(T,V)\rangle''$ (blue dotted line) versus temperature for $V = 200\,\mathrm{fm}^3$, considering the volume term only in the quarks and gluons density of states. According to Figures 2, 3 and 4, the shift of the transition temperature $\tau_T(V)$ defined as $\tau_T(V) = T_c(V) - T_c(\infty)$ decreases with increasing volume V, as noted in previous works [19–21]. The same behavior holds for the width of the transition region $\delta T(V)$, while the height of the susceptibility peak $|\chi_T|^{\max}(V)$ increases with increasing volume V. This is because the fluctuations driving the transition are weaker in larger systems.

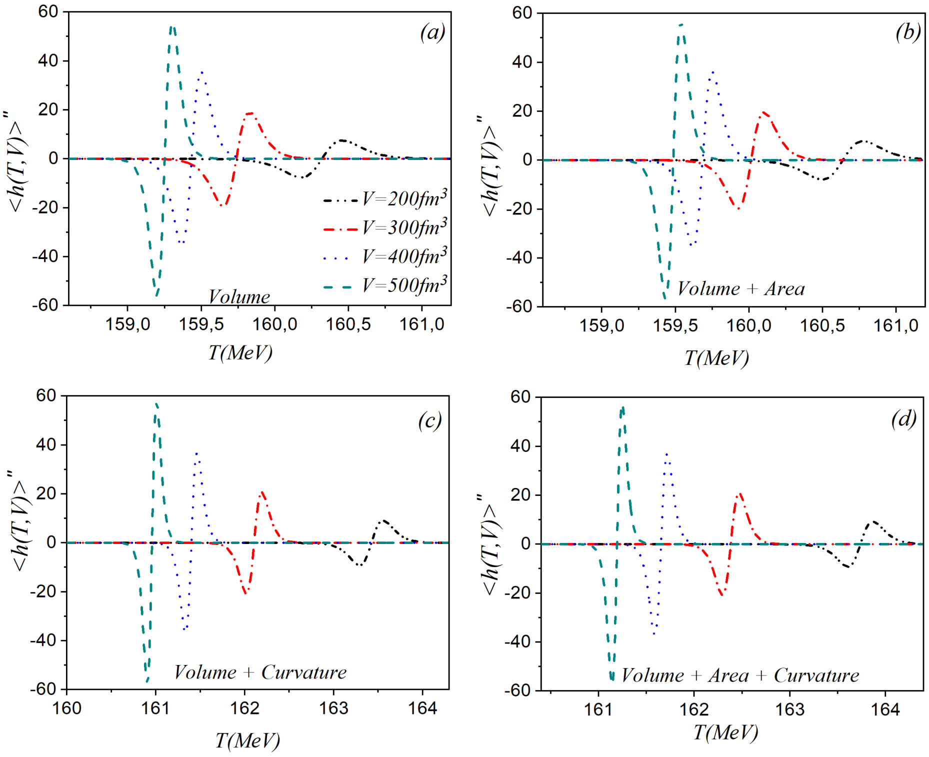

Figure 3. Variations of the second derivative of the order parameter $\langle h(T,V)\rangle''$ versus temperature, for different system sizes, at $\mu = 0$ and $B^{1/4} = 225\,\mathrm{MeV}$, with the four contributions in the density of states.

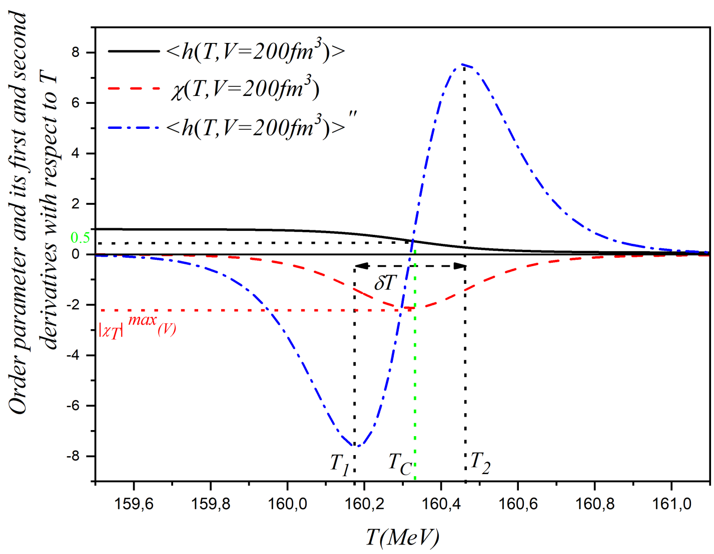

Figure 4. Plot of the order parameter $\langle h(T,V)\rangle$ (black solid line), its first thermal derivative $\chi(T,V)$ (red dashed line), and its second thermal derivative $\langle h(T,V)\rangle''$ (blue dotted line) versus temperature for a volume $V = 200\,\mathrm{fm}^3$, with the contribution of the volume term only in the density of states.

4. EFFECT OF THE CURVATURE AND THE AREA TERMS ON THE CRITICAL EXPONENTS FOR THE THERMAL DECONFINEMENT PHASE TRANSITION

It is well known in the theory of phase transitions that characteristic quantities such as the shift of the effective transition temperature $T_c(V)$ relative to the true transition temperature $T_c(\infty)$, the width of the transition region $\delta T(V)$ and the susceptibility maxima $|\chi_T|^{\max}(V)$ follow scaling behaviors, expressed as power laws of the volume V, characterized by critical scaling exponents. The power laws for a first-order phase transition are given by:

$$\begin{cases} \tau_T(V) = T_c(V) - T_c(\infty) \sim V^{-\lambda}, \\ \delta T(V) \sim V^{-\theta}, \\ |\chi_T|^{\max}(V) \sim V^{\gamma}. \end{cases} \tag{30}$$

where λ, θ and γ are the scaling exponents, which are expected to be equal to unity for a first-order phase transition, as shown in finite-size scaling (FSS) theory [45–47].

In this part of the work, we study the behavior of these three quantities with varying volume V, by incorporating the contribution of curvature and area terms in the density of states, additionally to the volume term, which has been previously considered alone in [19]. A finite-size scaling analysis, using a numerical parametrization of the data with the power-law forms defined in Eq. (30), yields very interesting results, which we illustrate in the following and compare with previous findings from [19].

4.1. The shift scaling exponent

In the aim of exploring the impact of the terms contributing to the density of states on the shift scaling exponent λ, associated with the shift of the transition temperature, we first determine the effective transition temperature $T_c(V)$ in a finite volume, and calculate its deviation from $T_c(\infty)$, to obtain the shift: $\tau_T(V) = T_c(V) - T_c(\infty)$. Table 2 shows the values of the shift of the transition temperature at various volumes, for the four cases of contributions in the quarks and gluons density of states.

Table 2. Values of the shift of the transition temperature for different terms contributing to the density of states.

System Volume (fm^3)	$\tau_T^{(\mathrm{Vol})}(V)$ (MeV)	$\tau_T^{(\mathrm{Vol+Area})}(V)$ (MeV)	$\tau_T^{(\mathrm{Vol+Curv})}(V)$ (MeV)	$\tau_T^{(\mathrm{Vol+Area+Curv})}(V)$ (MeV)
200	1.816	2.132	4.912	5.225
300	1.223	1.500	3.589	3.865
500	0.732	0.966	2.439	2.673
1000	0.375	0.562	1.477	1.666
3000	0.167	0.307	0.706	0.837
5000	0.129	0.240	0.513	0.624
6000	0.120	0.224	0.460	0.564
8000	0.108	0.203	0.389	0.484
9000	0.104	0.196	0.364	0.455
10000	0.102	0.189	0.343	0.431

As mentioned earlier, the shift of the effective transition temperature $\tau_T(V)$, relative to the true transition temperature $T_c(\infty)$, is expected to behave as a power of the system volume, according to Eq. (30), where the shift scaling exponent λ is expected to be equal to one. The obtained data for the shift $\tau_T(V)$ are plotted in Figure 5, and the values of the shift scaling exponent λ for the four contributions are summarized in Table 3.

Table 3. Values of the shift scaling exponent for the four cases of contributing terms to the density of states.

Shift scaling exponent	λ_1	λ_2	λ_3	λ_4
Contributions	Vol	Vol+Area	Vol+Curv	Vol+Area+Curv
Values	0.960 ± 0.018	0.788 ± 0.022	0.734 ± 0.007	0.692 ± 0.008

At first glance, we notice that for the volume term contribution only in the density of states, we obtain a shift scaling exponent: $\lambda_1(\mathrm{Vol}) = 0.960 \pm 0.018$, which is approximately equal to 1. When comparing this value to that of our previous work [19], in which: $\lambda = 0.876 \pm 0.041$ with the volume contribution only, our findings turn out to be an improvement. It can also be noted that with the addition of the area and curvature terms, the value of the shift exponent decreases consistently to $\lambda_4(\mathrm{Vol + Area + Curv}) = 0.692 \pm 0.008$. Initially, we assumed that adding the area and curvature terms would refine the result for the shift scaling exponent λ, making it more accurate or insightful. However, after completing the calculations, we found oppositely that incorporating these terms actually decreased the λ value.

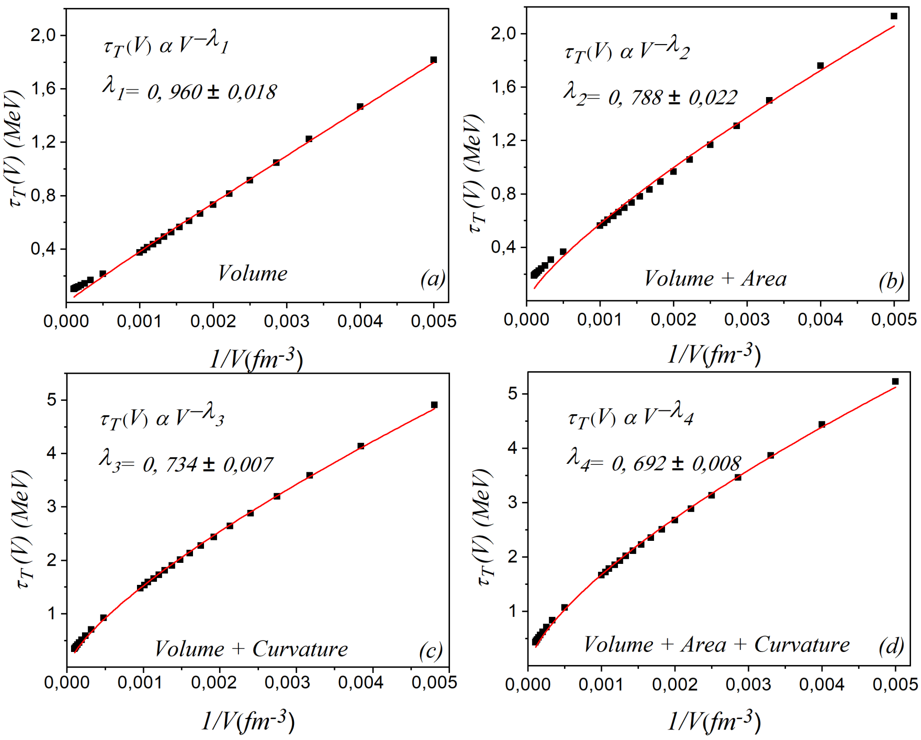

Figure 5. Variations of the shift of the transition temperature $\tau_T(V)$ with inversed volume, for the four contributions in the density of states.

This outcome could be attributed to several factors, such as the increasing complexity of the calculation or inherent limitations of the model. This discrepancy is partly due to the mass of the particles involved. Let us recall that our previous work [19] considered only massless up and down quarks in the QGP phase, and massless pions in the hadronic gas phase, while in the present work, we introduced a third massive quark (the strange quark) in the QGP phase and accounted for the mass of the pions in the HG phase.

We also observed that the shift scaling exponent has different values when the fit of the data of the shift of the transition temperature is carried out in the interval of small volumes or that of large ones. This is illustrated in Figure 6, which shows the shift of the transition temperature as a function of inverse volume, both for (a) small and (b) large volumes, when considering the volume term only in the density of states. The obtained result of the shift exponent $\lambda_1(\mathrm{Vol}) = 0.997 \pm 0.005$, with a fit in the small-volumes interval, suggests that our model matches better in small volumes, where the color-singletness condition applies, and where there are fewer numerical difficulties. When the fit is carried out in the interval of large volumes, the obtained value of the shift exponent $\lambda'_1(\mathrm{Vol}) = 0.620 \pm 0.030$, is rather far from the value 1 expected for a first-order phase transition. To fully understand the influence of the area and curvature terms on the overall behavior of the critical exponents, we investigate the width of the transition region $\delta T(V)$ and the height of the susceptibility peak $|\chi_T|^{\max}(V)$ within the contribution of these terms in the density of states and we illustrate our findings in the following.

4.2. The smearing scaling exponent

Figure 7 shows the width of the transition region $\delta T(V)$ as a function of inverse volume, for the four cases of contributing terms in the quarks and gluons density of states.

A numerical parametrization with the power-law form $\delta T(V) \sim V^{-\theta}$, gives the values of the smearing critical exponent shown in Table 4.

Table 4. Values of the smearing scaling exponent for the four cases of contributing terms to the density of states.

Smearing scaling exponent	θ_1	θ_2	θ_3	θ_4
Contributions	Vol	Vol+Area	Vol+Curv	Vol+Area+Curv
Values	1.038 ± 0.006	1.033 ± 0.005	0.993 ± 0.004	0.992 ± 0.002

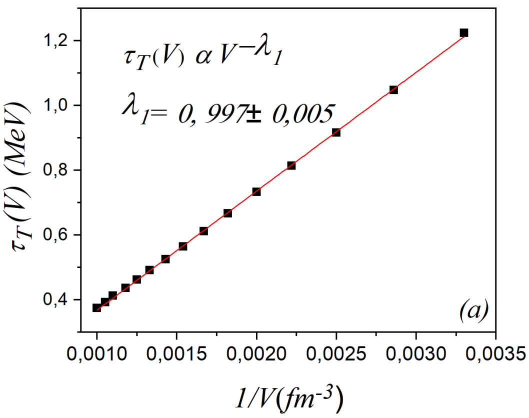

Figure 6. Variations of the shift of the transition temperature $\tau_T(V)$ with inversed volume, for (a) small and (b) large volumes, with the contribution of the volume term only in the density of states.

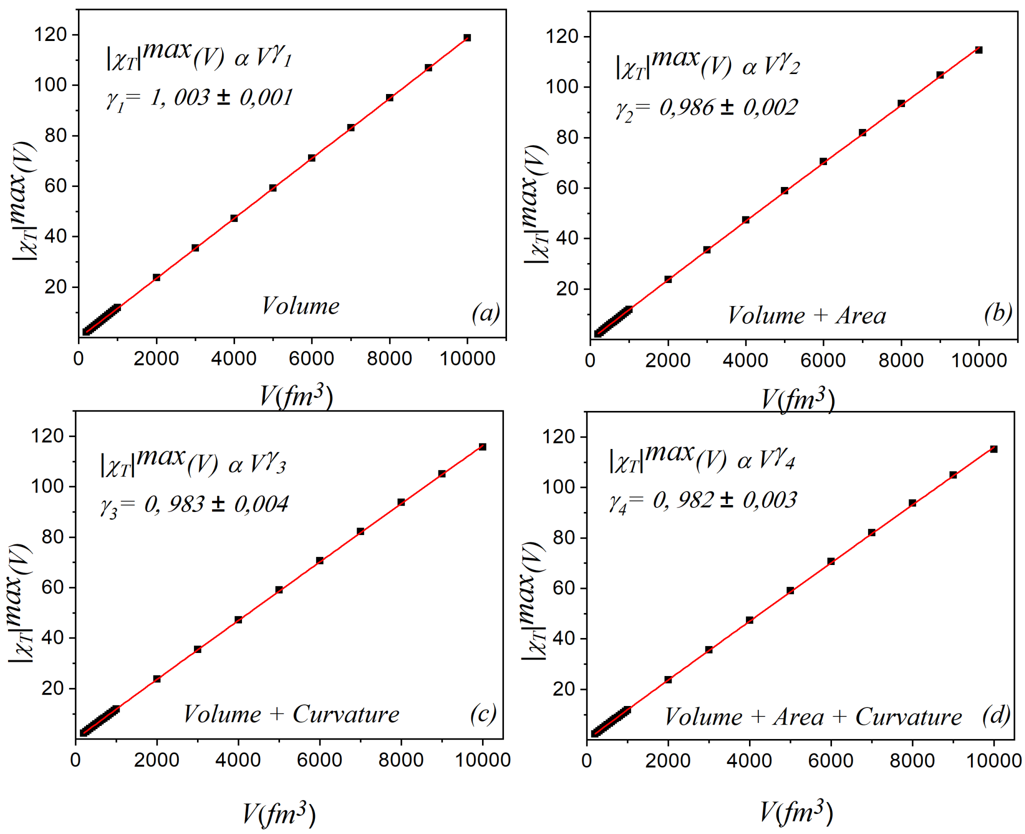

Figure 7. Variations of the width $\delta T(V)$ of the temperature region over which the transition is smeared out, with inversed volume, for the four contributions in the density of states.

4.3. The susceptibility scaling exponent

Figure 8 shows the variations of the height of the susceptibility peak $|\chi_T|^{\max}(V)$, with varying volume V, for the four cases of contributing terms in the quarks and gluons density of states.

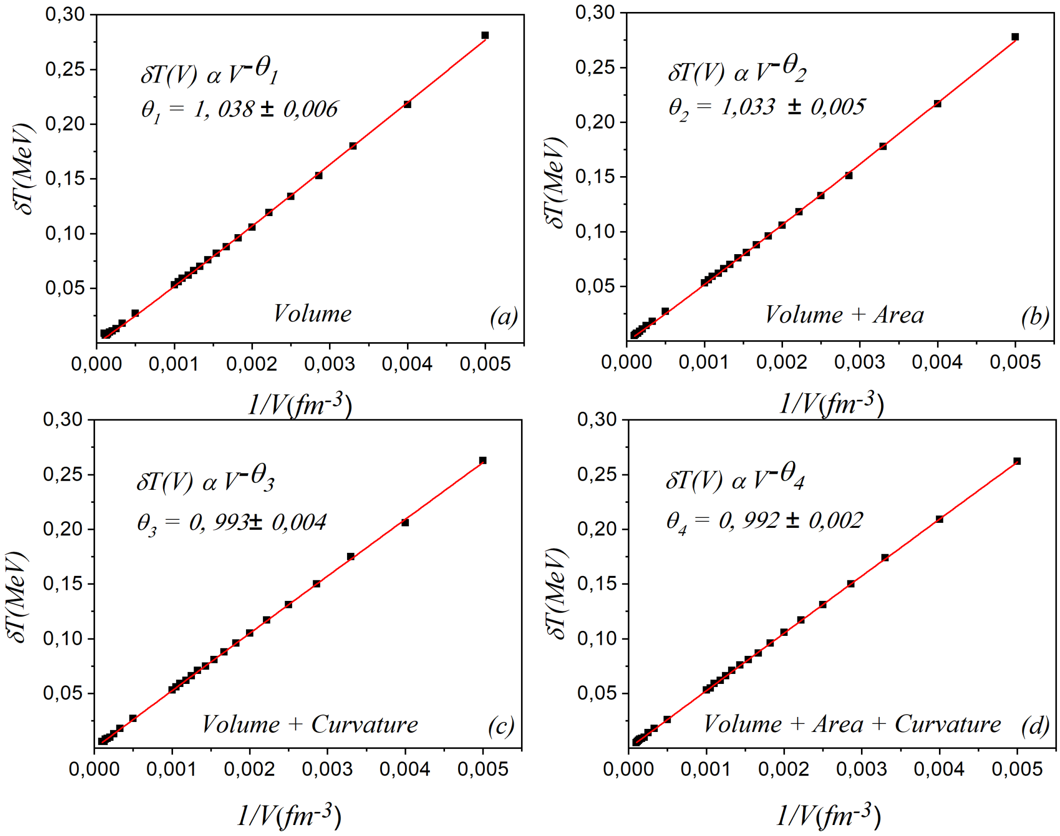

Figure 8. Variations of the susceptibility maxima $|\chi_T|^{\max}(V)$ versus volume, for the four contributions in the density of states.

A numerical parametrization with the power-law form $|\chi_T|^{\max}(V) \sim V^{\gamma}$, gives the values of the susceptibility critical exponent shown in Table 5.

Table 5. Values of the susceptibility scaling exponent for the four cases of contributing terms to the density of states.

Susceptibility scaling exponent	γ_1	γ_2	γ_3	γ_4
Contributions	Vol	Vol+Area	Vol+Curv	Vol+Area+Curv
Values	1.003 ± 0.001	0.986 ± 0.002	0.983 ± 0.004	0.982 ± 0.003

We notice from Table 4 and Table 5 that the values of the smearing scaling exponent θ and the susceptibility scaling exponent γ are equal to 1 when considering only the volume term contribution: $\theta_1(\text{Vol}) = 1.038 \pm 0.006$ and $\gamma_1(\text{Vol}) = 1.003 \pm 0.001$. However, with the addition of area and curvature terms, both values decrease slightly to $\theta_4(\text{Vol} + \text{Area} + \text{Curv}) = 0.992 \pm 0.002$, and $\gamma_4(\text{Vol} + \text{Area} + \text{Curv}) = 0.982 \pm 0.003$. This confirms that adding the area and curvature terms reduces the values of all critical exponents, including θ and γ, as well as λ.

5. CONCLUSIONS

In the present work, we studied the thermal DPT in a finite volume, at zero chemical potential $\mu = 0$ and with $B^{1/4} = 225$ MeV for the bag constant, based on the equations of state of a hadronic gas phase consisting of massive pions and of a color-singlet QGP phase containing gluons, massless up and down quarks and massive strange quarks along with their antiquarks. The primary aim was to investigate the influence of the area and curvature contributions to the density of states used in the calculation of the QGP equation of state on physical quantities characterizing the DPT occurring in a finite volume, and specifically the scaling exponent relative to the shift of the transition temperature, and then compare these findings with previous results from [19], obtained therein with the volume term contribution only in the density of states and reporting $\lambda = 0.876 \pm 0.041$.

Initially, we hypothesized that adding the area and curvature terms would improve the accuracy or significance of the shift scaling exponent λ deviating from the value 1, expected for a first-order phase transition. However, after completing our analysis, the obtained results revealed that including these terms actually led to a decrease in the value of the shift critical exponent λ. This could be attributed to various factors, such as the increasing complexity of the calculation or limitations in the model. To verify this, we determined the width of the transition region $\delta T(V)$ and the maximum of the susceptibility peak $|\chi_T|^{\max}(V)$ from the calculations incorporating the curvature and area terms into the quarks and gluons density of states. We found that the smearing scaling exponent θ and the susceptibility scaling exponent γ both equal unity, with the contribution of the volume term only to the density of states, but their values decreased when the area and curvature terms were added to the quarks and gluons density of states.

Our current result for the shift scaling exponent, $\lambda_1 = 0.960 \pm 0.018$, is an improvement over the previous result, $\lambda = 0.876 \pm 0.041$ [19], for the volume term contribution only in the density of states. This improvement in our results can be partially explained by the mass of the particles involved, since previous studies assumed that the QGP phase contained only massless up and down quarks, while the hadronic gas phase contained massless pions. In contrast, this work introduced a third quark to the QGP phase, namely the massive strange quark, and accounted for the pion mass in the HG phase. Finally, we observed that the value of the shift critical exponent changes when the parametrization $\tau_T(V) \sim V^{-\lambda}$ is performed on an interval of small volumes or large volumes. For small volumes, we recovered the expected value and found $\lambda_1(\mathrm{Vol}) = 0.997 \pm 0.005$, whereas for large volumes, the obtained value is $\lambda_1'(\mathrm{Vol}) = 0.620 \pm 0.030$. This suggests that our model matches better for small volumes, where there are fewer numerical difficulties and where the "color-singletness" condition applies, since it is well known that the finite-size corrections between a color-unprojected QGP and a color-projected one disappear as the size and/or temperature of the system increase, because the color projection no longer plays any role in this limit [9]. This obtained result for the shift scaling exponent agrees very well with those of finite-size scaling (FSS) theory, which expects the characteristic scaling exponents to be equal to unity for a first-order phase transition [45–47].

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АНАЛІЗ СКІНЧЕННО-РОЗМІРНОГО МАШТАБУВАННЯ ДЛЯ ФАЗОВОГО ПЕРЕХОДУ КХД З ДЕКОНФАЙНМЕНТОМ ДО КОЛЬОРОВО-СИНГЛЕТНОЇ КВАРК-ГЛЮОННОЇ ПЛАЗМИ З ТРЬОМА КВАРКОВИМИ АРОМАТАМИ u, d, s

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У цій роботі ми обчислюємо функцію розподілу кварк-глюонної плазми (QGP), спроектованої на кольорове синглетне представлення SU(3), використовуючи метод проекції, в межах густини станів для кварків та глюонів, заданої наближенням множинного відбиття (MRE). Ми досліджуємо вплив включення членів площі та кривизни, на додаток до члена об'єму в густину станів, на поведінку кількох фізичних величин, що характеризують деконфінуючий фазовий перехід від адронної газової фази масивних піонів до фази кварк-глюонної плазми (КГП), що складається з глюонів, безмасових верхніх та нижніх кварків, масивних дивних кварків разом з їхніми антикварками, в рамках моделі Бега та феноменологічної моделі співіснування фаз. За допомогою аналізу масштабування скінченних розмірів ми досліджуємо поведінку деяких показників масштабування, що стосуються деконфінуючого фазового переходу, що відбувається в скінченному об'ємі, особливо показника масштабування відносно зсуву ефективної температури переходу, розглядаючи внески лише члена кривизни, лише члена площі в густину станів, крім члена об'єму, а потім внесок усіх трьох членів, аналізуючи значення критичного показника зсуву λ порівняно з тим, що було отримано раніше з використанням лише члена об'єму в густині станів кварків та глюонів.
Ключові слова: *деконфінуючий фазовий перехід; кварк-глюонна плазма; синглетність кольору; проекційний метод; аналіз масштабування скінченних розмірів*