East European Journal of Physics. **3**. 179–197 (2026)
DOI: 10.26565/2312-4334-2026-3-15 ISSN 2312-4334

MULTIFORM SOLITARY WAVE STRUCTURES AND BIFURCATION ANALYSIS OF AN EXTENDED MODIFIED (3+1)-DIMENSIONAL KADOMTSEV–PETVIASHVILI EQUATION VIA HYBRID ANALYTICAL APPROACHES

Ibrahim Saber[1,2*], **Hamdy M. Ahmed**[3], **Niveen Badra**[1], **Islam Samir**[1]
[1]*Department of Physics and Mathematics Engineering, Faculty of Engineering, Ain Shams University, Cairo, Egypt*
[2]*Faculty of Artificial Intelligence, Egyptian Russian University, Cairo, Egypt*
[3]*Department of Physics and Engineering Mathematics, Higher Institute of Engineering, El Shorouk Academy, Cairo, Egypt*
[*]*Corresponding Author e-mail: ibrahim-saber@eru.edu.eg*
Received June 2, 2026; revised August 21, 2026; accepted August 26, 2026

This work presents the derivation of exact analytical solutions for a (3+1)-dimensional modified Kadomtsev–Petviashvili equation using two effective analytical schemes, namely the Improved Simple Equation Method and the exponential expansion approach. The considered model is relevant for describing complex nonlinear wave propagation in incompressible fluid environments, where dispersive and nonlinear effects are strongly coupled. The implemented techniques yield a rich variety of wave structures with distinct physical characteristics, including singular and non-singular localized formations, periodic patterns, and exponentially decaying profiles. In addition, hyperbolic and trigonometric configurations are obtained, reflecting different propagation regimes of the system. The qualitative behavior of these solutions is further clarified through three-dimensional surface representations and contour mappings. Furthermore, a bifurcation analysis is carried out for the reduced traveling-wave ordinary differential equation to investigate the equilibrium points, their stability properties, and the critical conditions associated with stability exchange. This analysis provides a complementary dynamical-systems perspective on the obtained wave structures and clarifies the influence of the governing parameters on the qualitative behavior of the model.

Keywords: *Nonlinear dispersive systems; Multidimensional nonlinear waves; Exact analytical solutions*

1. INTRODUCTION

In the realm of nonlinear scientific studies, nonlinear evolutionary systems (NLEEs) hold a considerable position. They serve as powerful mathematical tools for describing diverse and intricate phenomena such as fluid flow dynamics, nonlinear processes, and the distribution of electromagnetic fields in plasma physics. [1–9]. With ongoing research for NLEEs, a growing variety of exact analytical solutions have been identified and analyzed, offering deeper insights into the behavior of nonlinear phenomena in natural systems. Multiple highly effective methodologies have been developed to acquire precise analytical forms of equations describing nonlinear dynamic evolution (NLEEs), for instance, the inverse scattering transform [10], techniques such as a transformation approach derived from Bäcklund's concept [11], and a transformation method based on Darboux's framework [12], the algebraic–geometric technique [13] and an approach utilizing Hirota's bilinear formalism [14]. These methods not only enable the precise determination of solutions but also facilitate the study of complex solution structures.

During the late 1970s, Novikov's research group [15–19]. achieved notable advancements in the development of inverse spectral Paradigm and an approach based on algebraic geometry. These techniques were initially employed to analyze the KdV model, originally formulated by Diederik Korteweg and Gustav de Vries, leading to the derivation of quasiperiodic wave solutions, commonly referred to as algebraic–geometric solutions. Such solutions are defined by physical parameters including the wave number, phase velocity, and interaction amplitude, all determined on a compact, orientable Riemann surface. Although the technique grounded in algebraic geometry offers an analytical framework for constructing quasiperiodic wave solutions, the direct determination of key quantities for instance, wave numbers and phase shifts remains challenging, and frequencies remains difficult.

To address this, Nakamura introduced a direct approach combining the bilinear formalism in conjunction with the theta function introduced by Riemann has been employed to formulate solutions exhibiting quasiperiodic waves for both the KdV and Boussinesq equations [20, 21]. This method provides a more direct approach and offers a clearer understanding of the relationship between the solutions and their characteristic parameters. Subsequent extensions of this technique to Investigations into superintegrable and discrete integrable systems have been carried out by fan and colleagues [22–29]. Overall, nakamura's approach has considerably enhanced the comprehension related to solutions exhibiting quasi-periodicity and their function in the representation of nonlinear waves and examination of dynamical systems [30–33], yielding new insights into complex system dynamics as well as the appearance of chaotic behavior.

A nonlinear version of the Kadomtsev–Petviashvili equation (KPE) [34], this nonlinear evolution equation, incorporating two spatial coordinates and a single time variable, was designed to capture the dynamics of slowly varying long nonlinear waves in the transverse direction. By loosening the restriction by extending beyond one-dimensional wave

Cite as: I. Saber, H.M. Ahmed, N. Badra, I. Samir, East Eur. J. Phys. 3, 179 (2026), https://doi.org/10.26565/2312-4334-2026-3-15 dynamics, Kadomtsev and Petviashvili introduced the fully integrable KP equation . This formulation is suitable for long shallow water waves propagating predominantly in one direction in scenarios where surface tension and viscosity are insignificant. The extended Kadomtsev–Petviashvili equation in (3 + 1) dimensions has, in recent years, garnered significant interest in both fluid mechanics and plasma physics research [35]. The proposed equation is expressed as follows:

$$u_{\mathrm{xxxy}} + 3\left(u_{\mathrm{x}}u_{\mathrm{y}}\right)_{\mathrm{x}} + \beta u_{\mathrm{xxxz}} + 3\beta\left(u_{\mathrm{x}}u_{\mathrm{z}}\right)_{x} + \rho_1 u_{\mathrm{xt}} + \rho_2 u_{\mathrm{yt}} + \rho_3 u_{\mathrm{zt}} + \omega_1 u_{\mathrm{xz}} + \omega_2 u_{\mathrm{yz}} + \omega_3 u_{\mathrm{zz}} = 0. \tag{1}$$

Here, $u = u(x, y, z, t)$ corresponds to the waveform, which depends on three uncorrelated coordinates x, y and z, as well as a single temporal coordinate t. The coefficients ρ_j, ω_j and β are nonzero fixed parameters where $j = 1, 2, 3$.

This work employs the Improved Simple Equation Method (ISEM) and the Exponential Method to derive exact explicit solutions of the generalized Kadomtsev–Petviashvili equation in (3+1) dimensions. By introducing a suitable wave transformation and balancing the nonlinear and highest derivative terms, several classes of explicit wave structures are obtained. These include exponential-type waves, dark solitons, singular solitons, trigonometric solutions, hyperbolic solutions, and singular periodic wave patterns. The derived results enrich the analytical framework of KP-type models and provide deeper physical insight into nonlinear wave propagation and interactions in multidimensional fluid systems. The effectiveness and simplicity of the adopted method confirm its applicability to a broader variety of nonlinear evolutionary equations.

Modified (3+1)-dimensional KP equation considered in the present work has not yet been investigated through the combined implementation of the Improved Simple Equation Method and the Exponential Expansion Method. Furthermore, most existing studies focus on obtaining individual classes of exact solutions without performing a systematic comparison between different analytical approaches. Therefore, the present study aims to bridge this gap by deriving several new exact wave structures using two complementary analytical schemes, followed by graphical visualization to evaluate their physical significance. In addition, all obtained analytical solutions are verified by direct substitution into the governing equation using symbolic computation to ensure their mathematical correctness.

Beyond the derivation of exact analytical solutions, the qualitative dynamical behavior of the reduced traveling-wave equation is also investigated through bifurcation analysis. This approach enables the identification of equilibrium points of the associated planar dynamical system, the examination of their local stability, and the determination of critical parameter conditions under which changes in the qualitative structure of the phase plane occur. Consequently, the bifurcation analysis complements the analytical solution procedures by providing deeper insight into the dynamical mechanisms governing the emergence and stability of the obtained nonlinear wave structures.

This paper is organized as follows. Section 2 presents the mathematical methodologies employed in this work, including the Improved Simple Equation Method and the Exponential Expansion Method. In Section 3, these techniques are systematically applied to derive a broad spectrum of exact analytical solutions for the extended modified (3+1)-dimensional Kadomtsev–Petviashvili equation. Section 4 is devoted to the bifurcation analysis of the reduced traveling-wave system, where the equilibrium points, their stability properties, and the associated critical bifurcation condition are investigated. Section 5 presents three-dimensional surface and contour visualizations to provide further insight into the qualitative and dynamical behavior of the obtained wave solutions. Finally, Section 6 summarizes the main results and highlights their significance in understanding nonlinear wave propagation and the underlying dynamical behavior of the model.

2. THE OUTLINED TECHNIQUES

Here, we present the methodology used for Equation (1), including the Improved Simple Equation and the Exponential Methods.

2.1. Improved Simple Equation Scheme

Here, an outline of the Improved Simple Equation Scheme is briefly discussed [36–38], which serves as the primary approach employed in the current study.

We examine the following expressed as a partial differential equation with nonlinear characteristics (NLPDE):

$$F\left(u, u_{xt}, u_{xy}, u_{xz}, u_{xt}, u_{xx}, u_{xxx}, \ldots\right) = 0. \tag{2}$$

The fundamental approaches to the Improved Simple Equation Scheme (ISEM) are condensed described below:

Step 1: Initially, the partial differential equation with nonlinear characteristics (NLPDE) presented in Eq(2) is mapped converted into an ordinary differential equation (ODE) through the following transformation:

$$u(x, y, z, t) = u(\xi); \quad \xi = x + z + y - c\,t; \quad c \neq 0, \tag{3}$$

here c corresponds to the wave velocity. Accordingly, Equation(2) takes the following form:

$$H\left(u, u', u'', u''', \ldots.\right) = 0. \tag{4}$$

Step 2: The derived solution for the Equation (4) is presumed to take the following form:

$$u(\xi) = \sum_{j=-N}^{N} a_j z^j(\xi), \tag{5}$$

here a_j represent coefficients required to be evaluated, while $z(\xi)$ is governed by the differential equation presented here:

$$z'(\xi) = d_0 + d_1 z(\xi) + d_2 z^2(\xi). \tag{6}$$

Step 3: In order to identify the quantity of N, the highest-order derivative term is balanced with the highest-order nonlinear term according to the balancing principle.
Step 4: By substituting the assumed solution given in Equation (5) into Equation (4), and using the differential relation in Equation (6), the resulting expression reduces to a polynomial in $z(\xi)$.
Step 5: By gathering that the coefficients of z^j in the polynomial generated from Step (4) , one obtains a system composed of algebraic relations, the solution of that can serve as derived with the aid of numerical computing platforms like *Mathematica* for the purpose of determine the parameters to be determined.
Step 6: By assigning various values to d_0, d_1 and d_2, several families of general solutions for Equation (6) can provide different types of solutions as follows:
First Family: $d_2 = 0$

$$z(\xi) = \frac{\exp(d_1\xi)}{d_1} - \frac{d_0}{d_1}.$$

Second Family: $d_1 = 0$

$$z(\xi) = -\sqrt{-\frac{d_0}{d_2}}\tanh\left(\sqrt{-d_0 d_2}\xi\right), \quad d_0 d_2 < 0,$$
$$z(\xi) = \sqrt{\frac{d_0}{d_2}}\tan\left(\sqrt{d_0 d_2}\xi\right), \quad d_0 d_2 > 0.$$

Third Family: $d_0 = 0$

$$z(\xi) = \frac{d_1 \exp(d_1\xi)}{1 - d_2 \exp(d_1\xi)},$$
$$z(\xi) = \frac{-d_1 \exp(d_1\xi)}{1 + d_2 \exp(d_1\xi)}.$$

Fourth Family: $d_0, d_1, d_2 \neq 0$
When $d_1^2 > 4d_0 d_2$:

$$z(\xi) = -\frac{\sqrt{d_1^2-4d_0d_2}\tanh\left(\frac{1}{2}\sqrt{d_1^2-4d_0d_2}\,\xi\right)+d_1}{2d_2},$$
$$z(\xi) = -\frac{\sqrt{d_1^2-4d_0d_2}\coth\left(\frac{1}{2}\sqrt{d_1^2-4d_0d_2}\,\xi\right)+d_1}{2d_2},$$
$$z(\xi) = \frac{\frac{\sqrt{\left(d_1^2-4d_0d_2\right)\left(A^2+B^2\right)}-A\sqrt{d_1^2-4d_0d_2}\cosh\left(\sqrt{d_1^2-4d_0d_2}\,\xi\right)}{A\sinh\left(\sqrt{d_1^2-4d_0d_2}\,\xi\right)+B}-d_1}{2d_2}.$$

where $A, B \neq 0$

$$z(\xi) = \frac{2d_0\cosh\left(\frac{1}{2}\sqrt{d_1^2-4d_0d_2}\xi\right)}{\sqrt{d_1^2-4d_0d_2}\sinh\left(\frac{1}{2}\sqrt{d_1^2-4d_0d_2}\xi\right) - d_1\cosh\left(\frac{1}{2}\sqrt{d_1^2-4d_0d_2}\xi\right)},$$
$$z(\xi) = \frac{2d_0\sinh\left(\frac{1}{2}\sqrt{d_1^2-4d_0d_2}\xi\right)}{\sqrt{d_1^2-4d_0d_2}\cosh\left(\frac{1}{2}\sqrt{d_1^2-4d_0d_2}\xi\right) - d_1\sinh\left(\frac{1}{2}\sqrt{d_1^2-4d_0d_2}\xi\right)}.$$

When $d_1^2 < 4d_0 d_2$:

$$z(\xi) = \frac{\sqrt{4d_0d_2-d_1^2}\tan\left(\frac{1}{2}\sqrt{4d_0d_2-d_1^2}\xi\right) - d_1}{2d_2},$$

$$z(\xi)=-\frac{\sqrt{4d_0d_2-d_1^2}\cot\left(\frac{1}{2}\sqrt{4d_0d_2-d_1^2}\,\xi\right)+d_1}{2d_2},$$

$$z(\xi)=\frac{\frac{\sqrt{\left(4d_0d_2-d_1^2\right)\left(A^2-B^2\right)}-A\sqrt{4d_0d_2-d_1^2}\cos\left(\sqrt{4d_0d_2-d_1^2}\,\xi\right)}{A\sin\left(\sqrt{4d_0d_2-d_1^2}\,\xi\right)+B}-d_1}{2d_2}.$$

with A and B being nonzero constants that satisfy the condition, $B^2<A^2$.

$$z(\xi)=-\frac{2d_0\cos\left(\frac{1}{2}\sqrt{4d_0d_2-d_1^2}\xi\right)}{\sqrt{4d_0d_2-d_1^2}\sin\left(\frac{1}{2}\sqrt{4d_0d_2-d_1^2}\xi\right)+d_1\cos\left(\frac{1}{2}\sqrt{4d_0d_2-d_1^2}\xi\right)},$$

$$z(\xi)=\frac{2d_0\sin\left(\frac{1}{2}\sqrt{4d_0d_2-d_1^2}\xi\right)}{\sqrt{4d_0d_2-d_1^2}\cos\left(\frac{1}{2}\sqrt{4d_0d_2-d_1^2}\xi\right)-d_1\sin\left(\frac{1}{2}\sqrt{4d_0d_2-d_1^2}\xi\right)}.$$

By inserting the evaluated constants c and a_j as well as the general solutions of Equation (6) into the assumed form of Equation (5), an ensemble of exact solutions for the differential equation is obtained.

2.2. Exponential Scheme

This section provides a concise overview of the newly developed Exponential Expansion scheme. The nonlinear evolution equation (NLEE) under consideration is given by:

$$F\left(u,u_{xt},u_{xy},u_{xz},u_{xt},u_{xx},u_{xxx},\ldots\right)=0. \tag{7}$$

where F denotes a function depending on $u,u_{xt},u_{xy},u_{xz},u_{xt},u_{xx},u_{xxx}$ and the subscripts indicate partial derivatives of u(x,t) with respect to the variables x and t.

Assume that u(x,t) = u(ξ), $\xi=x-c\,t$, where c represents the wave propagation speed. Under this transformation, Eq. (7) reduces to a nonlinear ordinary differential equation (ODE) for u=u(ξ):

$$U(u,u',u'',u''',....) \tag{8}$$

where U depends on u,u',u'',u''', ... and the prime symbol denotes ordinary differentiation with respect to ξ. We assume that the traveling wave solution of Eq. (8) can be represented in the following form:

$$\sum_{i=0}^{N}a_i(\exp(-z(\xi)))^i,\ a_n\neq 0. \tag{9}$$

where the coefficients $A_i(0\le i\le N)$ are unknown constants to be determined and $z=z(\xi)$ satisfies the following three auxulary equations every one gives its own system a nonlinear ordinary differential equation of first order:

$$z'(\xi)=\mu\ exp(z(\xi))+exp(-z(\xi))+\lambda \tag{10}$$

where $\lambda\ \ and\ \ \mu$ represent constants with arbitrary values.

The positive integer parameter N is obtained by equating the highest-order derivative terms having the highest-order nonlinear contributions appearing in Eq. (8). Substituting Eq. (9) into Eq. (8), together with Eq. (10) when necessary, leads to a system of algebraic equations for the unknown parameters $A_i(0\le i\le N),\ \lambda\ and\ \mu$. By employing symbolic computation software, such as Mathematica, this algebraic system can be solved to obtain the corresponding values of $A_i,\ \lambda\ and\ \mu$ It is worth mentioning that Eq. (10) admits five distinct classes of general solutions. [39–41]:

$$z(\xi)=\ln\left[\frac{-\sqrt{\lambda^2-4\mu}\tanh\left(\sqrt{\lambda^2-4\mu}\,(\xi+V_0)\right)-\lambda}{2\mu}\right],\ \mu\neq 0,\ \lambda^2-4\mu>0.$$

$$z(\xi)=\ln\left[\frac{\sqrt{-\lambda^2+4\mu}\tanh\left(\sqrt{-\lambda^2+4\mu}\,(\xi+V_0)\right)-\lambda}{2\mu}\right],\ \mu\neq 0,\ \lambda^2-4\mu<0.$$

$$z(\xi) = -\ln\left[\frac{\lambda}{\exp\left(\lambda\left(\xi + V_0\right)\right) - 1}\right],\ \mu = 0,\ \lambda \neq 0,\ \lambda^2 - 4\mu > 0.$$

$$z(\xi) = \ln\left(-\frac{2\left(\lambda\left(\xi + V_0\right) + 2\right)}{\lambda^2\left(\xi + V_0\right)}\right),\ \mu = 0,\ \lambda \neq 0.$$

$$z(\xi) = \ln\left[\xi + V_0\right],\ \mu = 0,\ \lambda = 0.$$

here V_0 represents the integration constant. Thus, multiple explicit solutions of the nonlinear evolution equation (7) can be obtained. Again, let us assume that Eq. (8) admits a solution expressed as in Eq. (9), where $z = z(\xi)$ fulfills a separate first-order nonlinear ordinary differential equation:

$$z'(\xi) = -\lambda\exp(z(\xi)) - \mu\exp(-z(\xi)),\ \mu,\ \lambda \in \mathbb{R}. \tag{11}$$

By substituting Eq.(9) into Eq.(8) and applying Eq.(11), by performing this procedure repeatedly, we derive a system of algebraic equations for $A_i (0 \leq i \leq N)$,as well as $\lambda\ and\ \mu$. Using symbolic computation tools such as Mathematica, this system can be solved to determine the values of $A_i (0 \leq i \leq N),\ \lambda\ and\ \mu$. It is noteworthy that Eq. (11) admits the general solutions given below:

$$z(\xi) = -\ln\left(\sqrt{\frac{\lambda}{\mu}}\tan\left(\sqrt{\lambda\mu}\left(-ct + V_0 + x + y + z\right)\right)\right),\ \lambda\mu > 0.$$

$$z(\xi) = -\ln\left(-\sqrt{\frac{\lambda}{\mu}}\cot\left(\sqrt{\lambda\mu}\left(-ct + V_0 + x + y + z\right)\right)\right),\ \lambda\mu > 0.$$

$$z(\xi) = -\ln\left(\sqrt{-\frac{\lambda}{\mu}}\tanh\left(\sqrt{-\lambda\mu}\left(-ct + V_0 + x + y + z\right)\right)\right),\ \lambda\mu < 0.$$

$$z(\xi) = \ln\left(\sqrt{-\frac{\lambda}{\mu}}\coth\left(\sqrt{-\lambda\mu}\left(-ct + V_0 + x + y + z\right)\right)\right),\ \lambda\mu < 0.$$

$$z(\xi) = -\ln\left(-\frac{1}{\mu\left(-ct + V_0 + x + y + z\right)}\right),\ \mu > 0,\ \lambda = 0.$$

where V_0 is the integrating constant. Finally, we obtain the new multiple explicit solutions of NLEE EQ(7).

3. DERIVATION OF EXACT SOLUTIONS

This part aims to clarify the application of the mathematical methods previously presented to the equation under investigation, along with a detailed discussion of the results obtained from each method. The findings demonstrate the effectiveness of these methods in obtaining exact solutions that contribute to a deeper understanding of the physical and mathematical properties of the equation.

3.1. Improved Simple Equation scheme

In the subsequent portion, the Improved Simple equation-based method is utilized to formulate closed-form solutions of the KPE.
Initially, through an appropriate transformation, the nonlinear evolution (PDE) (1) is reformulated as (ODE) through the wave transformation defined in Equation (3). Consequently, Equation (1) takes the following form:

$$\left(-c(\rho_1 + \rho_2 + \rho_3) + \omega_1 + \omega_2 + \omega_3 + 6(1+\beta)u'\right)u'' + (1+\beta)u'''' = 0, \tag{12}$$

Performing a single integration of Equation (12) gives:

$$\left(-c(\rho_1 + \rho_2 + \rho_3) + \omega_1 + \omega_2 + \omega_3\right)u' + 3(1+\beta)\left(u'\right)^2 + (1+\beta)u''' = 0, \tag{13}$$

the analytical solution of Equation (13) is capable of being represented as given below:
For employing the proposed technique, it is first crucial for specifying the integer N must first be identified. The value of $N = 1$. This value is obtained by equating the principal derivative term u''' paired with the nonlinear contribution $(u')^2$ in Equation (13), leading to the conclusion that the solution corresponding to Equation (1) takes as illustrated below:

$$u(\xi) = a_0 + a_1 z(\xi) + \frac{b_1}{z(\xi)}. \tag{14}$$

Substituting Equations (14) and (6) into Equation (13) yields:

$$\begin{gathered} z^4\left(6a_1(\beta+1)b_1d_2 + a_1c\rho_1 + a_1c\rho_2 + a_1c\rho_3 - 3a_1^2(\beta+1)d_0\right) \\ +z^4\left(a_1(-\beta-1)d_1^2 - 2a_1(\beta+1)d_0d_2 - a_1\omega_1 - a_1\omega_2 - a_1\omega_3\right) \\ +z^2\left(6a_1(\beta+1)b_1d_0 + b_1\left(-c(\rho_1+\rho_2+\rho_3) + (\beta+1)d_1^2 + \omega_1 + \omega_2 + \omega_3\right) - 3(\beta+1)b_1^2d_2 + 2(\beta+1)b_1d_0d_2\right) + \\ 6a_1(\beta+1)b_1d_1z^3 + z^6\left(-3a_1^2(\beta+1)d_2 - 6a_1\beta d_2^2 - 6a_1d_2^2\right) + z^5\left(-3a_1^2(\beta+1)d_1 - 6a_1(\beta+1)d_1d_2\right) + \\ 6(\beta+1)b_1d_0^2 - 3(\beta+1)b_1^2d_0 + z\left(6(\beta+1)b_1d_0d_1 - 3(\beta+1)b_1^2d_1\right) = 0. \end{gathered} \tag{15}$$

Through comparing the coefficients of z^j taken as zero produces a collection of algebraic relations, which are subsequently solved using Mathematica.
The solutions obtained are summarized below:
Family 1: When $d_2 = 0$, we have

$$a_1 = 0; \quad d_0 = \frac{b_1}{2}; \quad d_1 = \sqrt{\frac{c\rho_1 + c\rho_2 + c\rho_3 - \omega_1 - \omega_2 - \omega_3}{\beta+1}}.$$

Consequently, the subsequent dark-type soliton solution is computed:

$$u(x,y,z,t) = a_0 + \frac{2b_1\sqrt{-\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}}{2\exp\left((-ct+x+z+y)\sqrt{-\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\right) - b_1},$$

$$\frac{-c(\rho_1+\rho_2+\rho_3) + \omega_1 + \omega_2 + \omega_3}{\beta+1} < 0 \quad \beta \neq -1. \tag{16}$$

Family 2:When $d_1 = 0$, the results can be expressed as follows:

SET I:

$$a_1 = 0; \quad d_0 = \frac{b_1}{2}; \quad d_2 = \frac{-c\rho_1 - c\rho_2 - c\rho_3 + \omega_1 + \omega_2 + \omega_3}{4\beta d_0 + 4d_0}.$$

SET II:

$$d_2 = -\frac{a_1}{2}; \quad b_1 = 0; \quad d_0 = \frac{-c\rho_1 - c\rho_2 - c\rho_3 + \omega_1 + \omega_2 + \omega_3}{4\beta d_2 + 4d_2}.$$

SET III:

$$a_1 = -2d_2; \quad d_0 = \frac{b_1}{2}; \quad d_2 = \frac{-c\rho_1 - c\rho_2 - c\rho_3 + \omega_1 + \omega_2 + \omega_3}{16\beta d_0 + 16d_0}.$$

Based on Set I, the following solutions, singular periodic solution, is derived:

$$u(x,y,z,t) = a_0 + \frac{\sqrt{2}b_1\cot\left(\frac{(-ct+x+y+z)\sqrt{\frac{b_1(-c\rho_1-c\rho_2-c\rho_3+\omega_1+\omega_2+\omega_3)}{2\beta b_1+2b_1}}}{\sqrt{2}}\right)}{\sqrt{\frac{b_1(2\beta b_1+2b_1)}{-c\rho_1-c\rho_2-c\rho_3+\omega_1+\omega_2+\omega_3}}}, \quad \beta \neq -1,$$

$$\frac{b_1(-c\rho_1 - c\rho_2 - c\rho_3 + \omega_1 + \omega_2 + \omega_3)}{2\beta b_1 + 2b_1} > 0. \tag{17}$$

In accordance with Set II, singular periodic structures is determined as presented below:

$$u(x,y,z,t) = a_0 + \sqrt{2}a_1\sqrt{-\frac{-c\rho_1 - c\rho_2 - c\rho_3 + \omega_1 + \omega_2 + \omega_3}{a_1\left(-2a_1\beta - 2a_1\right)}} \; * \\ \tan\left(\frac{(-ct+x+y+z)\sqrt{-\frac{a_1(-c\rho_1-c\rho_2-c\rho_3+\omega_1+\omega_2+\omega_3)}{-2a_1\beta-2a_1}}}{\sqrt{2}}\right), \\ \frac{a_1(-c\rho_1 - c\rho_2 - c\rho_3 + \omega_1 + \omega_2 + \omega_3)}{-2a_1\beta - 2a_1} < 0, \;\; \beta \neq -1. \tag{18}$$

Referring to Set III, the corresponding singular periodic solution is presented below:

$$u(x,y,z,t) = a_0 + \\ \frac{b_1\left(\cot\left(\frac{1}{4}(-ct+x+y+z)\sqrt{\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\right) - \tan\left(\frac{1}{4}(-ct+x+y+z)\sqrt{\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\right)\right)}{2\sqrt{\frac{(\beta+1)b_1^2}{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}}}, \\ \frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1} > 0, \;\; \beta \neq -1. \tag{19}$$

Family 3: When $d_0 = 0$, we have

$$d_2 = -\frac{a_1}{2}; \;\; b_1 = 0; \;\; d_1 = \sqrt{\frac{c\rho_1 + c\rho_2 + c\rho_3 - \omega_1 - \omega_2 - \omega_3}{\beta+1}}.$$

Consequently, the following exponential-type solutions are derived:

$$u(x,y,z,t) = a_0 + \frac{2a_1\sqrt{-\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\exp\left((-ct+x+y+z)\sqrt{-\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\right)}{a_1\exp\left((-ct+x+y+z)\sqrt{-\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\right)+2},$$

$$\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1} < 0, \;\; \beta \neq -1. \tag{20}$$

$$u(x,y,z,t) = a_0 + \frac{2a_1\sqrt{-\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\exp\left((-ct+x+y+z)\sqrt{-\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\right)}{a_1\exp\left((-ct+x+y+z)\sqrt{-\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\right)-2},$$

$$\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1} < 0, \;\; \beta \neq -1. \tag{21}$$

Fourth Family: When $d_0, d_1, d_2 \neq 0$, we have

SET I:

$$a_1 = 0; \;\; d_0 = \frac{b_1}{2}; \;\; d_2 = \frac{-c\rho_1 - c\rho_2 - c\rho_3 + \beta d_1^2 + d_1^2 + \omega_1 + \omega_2 + \omega_3}{4(\beta+1)d_0}.$$

SET II

$$a_1 = -2d_2;\ \ b_1 = 0;\ \ d_2 = \frac{c\rho_1 + c\rho_2 + c\rho_3 - \beta d_1^2 - d_1^2 - \omega_1 - \omega_2 - \omega_3}{-4\beta d_0 - 4d_0}.$$

Utilizing Set I, the following solutions are constructed: $\beta \neq -1$.

$$u(x,y,z,t) = a_0 - \frac{-c(\rho_1+\rho_2+\rho_3) + (\beta+1)d_1^2 + \omega_1 + \omega_2 + \omega_3}{(\beta+1)\left(\sqrt{-\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\tanh\left(\frac{1}{2}(-ct+x+y+z)\sqrt{-\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\right) + d_1\right)},$$
$$\frac{-c(\rho 1+\rho 2+\rho 3)+\omega 1+\omega 2+\omega 3}{\beta+1} < 0,\ \ \beta \neq -1. \tag{22}$$

$$u(x,y,z,t) = a_0 - \frac{-c(\rho_1+\rho_2+\rho_3) + (\beta+1)d_1^2 + \omega_1 + \omega_2 + \omega_3}{(\beta+1)\left(\sqrt{-\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\coth\left(\frac{1}{2}(-ct+x+y+z)\sqrt{-\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\right) + d_1\right)},$$
$$\frac{-c(\rho 1+\rho 2+\rho 3)+\omega 1+\omega 2+\omega 3}{\beta+1} < 0,\ \ \beta \neq -1. \tag{23}$$

$$u(x,y,z,t) = a_0 + \frac{\left(A\sinh\left(\sqrt{M}(-ct+x+y+z)\right)+B\right)\left(-c(\rho 1+\rho 2+\rho 3)+(\beta+1)d_1^2+\omega 1+\omega 2+\omega 3\right)}{(\beta+1)\left(\sqrt{M\left(A^2+B^2\right)} - d_1\left(A\sinh\left(\sqrt{M}(-ct+x+y+z)\right)+B\right) - A\sqrt{M}\cosh\left(\sqrt{M}(-ct+x+y+z)\right)\right)},$$
$$M = -\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1},\ \ M>0,\ \ \beta \neq -1. \tag{24}$$

$$u(x,y,z,t) = a_0 - d_1 + \sqrt{-\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\tanh\left(\frac{1}{2}(-ct+x+z+y)\sqrt{-\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\right),$$
$$-\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1} > 0,\ \ \beta \neq -1. \tag{25}$$

$$u(x,y,z,t) = a_0 - d_1 + \sqrt{-\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\coth\left(\frac{1}{2}(-ct+x+z+y)\sqrt{-\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\right),$$
$$\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1} < 0,\ \ \beta \neq -1. \tag{26}$$

The solutions in Equations (22), (23) and(26) correspond to singular solitons, Eq.(25) correspond to dark soliton solution, Eq(24) is obtained hyperbolic solution.

$$u(x,y,z,t) = a_0 - \frac{-c(\rho_1+\rho_2+\rho_3) + (\beta+1)d_1^2 + \omega_1 + \omega_2 + \omega_3}{(\beta+1)\left(d_1 - \sqrt{\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\tan\left(\frac{1}{2}(-ct+x+z+y)\sqrt{\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\right)\right)},$$
$$\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1} > 0,\ \ \beta \neq -1. \tag{27}$$

$$u(x,y,z,t) = a_0 - \frac{-c(\rho_1+\rho_2+\rho_3) + (\beta+1)d_1^2 + \omega_1 + \omega_2 + \omega_3}{(\beta+1)\left(\sqrt{\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\cot\left(\frac{1}{2}(-ct+x+z+y)\sqrt{\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\right) + d_1\right)},$$

$$\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}>0,\ \ \beta\neq-1. \tag{28}$$

$$u(x,y,z,t)=a_0+$$
$$\frac{\left(A\sin\left(\sqrt{F}(-ct+x+z+y)\right)+B\right)\left(-c(\rho_1+\rho_2+\rho_3)+(\beta+1)d_1^2+\omega_1+\omega_2+\omega_3\right)}{(\beta+1)\left(\sqrt{F\left(A^2-B^2\right)}-d_1\left(A\sin\left(\sqrt{F}(-ct+x+z+y)\right)+B\right)-A\sqrt{F}\cos\left(\sqrt{F}(-ct+x+z+y)\right)\right)},$$
$$F=\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1},\ \ \beta\neq-1. \tag{29}$$

$$u(x,y,z,t)=a_0+d_1$$
$$+\sqrt{\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\tan\left(\frac{1}{2}(-ct+x+z+y)\sqrt{\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\right),$$
$$\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}>0,\ \ \beta\neq-1. \tag{30}$$

$$u(x,y,z,t)=a_0-d_1$$
$$+\sqrt{\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\cot\left(\frac{1}{2}(-ct+x+z+y)\sqrt{\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}}\right),$$
$$\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1}>0,\ \ \beta\neq-1. \tag{31}$$

Equations (27), (28), (30) and (31) generate singular periodic wave structures, Eq.(29) trigonometric solution is obtained. In accordance with Set II, the explicit solutions are constructed in the manner described below:

$$u(x,y,z,t)=a_0+$$
$$\frac{-\sqrt{M\left(A^2+B^2\right)}+d_1\left(A\sinh\left(\sqrt{M}(-ct+x+z+y)\right)+B\right)+A\sqrt{M}\cosh\left(\sqrt{M}(-ct+x+z+y)\right)}{A\sinh\left(\sqrt{M}(-ct+x+z+y)\right)+B},$$
$$M=-\frac{-c(\rho1+\rho2+\rho3)+\omega1+\omega2+\omega3}{\beta+1},\ \ M>0,\ \ \beta\neq-1. \tag{32}$$

$$u(x,y,z,t)=a_0-\frac{\cosh\left(\frac{1}{2}\sqrt{M}(-ct+x+y+z)\right)\left(-c(\rho_1+\rho_2+\rho_3)+(\beta+1)d_1^2+\omega_1+\omega_2+\omega_3\right)}{(\beta+1)\left(\sqrt{M}\sinh\left(\frac{1}{2}\sqrt{M}(-ct+x+y+z)\right)-d_1\cosh\left(\frac{1}{2}\sqrt{M}(-ct+x+y+z)\right)\right)},$$
$$M=-\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1},\ \ M>0,\ \ \beta\neq-1. \tag{33}$$

$$u(x,y,z,t)=a_0-\frac{\sinh\left(\frac{1}{2}\sqrt{M}(-ct+x+z+y)\right)\left(-c(\rho_1+\rho_2+\rho_3)+(\beta+1)d_1^2+\omega_1+\omega_2+\omega_3\right)}{(\beta+1)\left(\sqrt{M}\cosh\left(\frac{1}{2}\sqrt{M}(-ct+x+z+y)\right)-d_1\sinh\left(\frac{1}{2}\sqrt{M}(-ct+x+z+y)\right)\right)},$$
$$M=-\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1},\ \ M>0,\ \ \beta\neq-1. \tag{34}$$

Equations (32), (33), and (34) describe hyperbolic solutions.

$$u(x,y,z,t)=a_0+$$
$$\frac{-\sqrt{F\left(A^2-B^2\right)}+Ad_1\sin\left(\sqrt{F}(-ct+x+y+z)\right)+A\sqrt{F}\cos\left(\sqrt{F}(-ct+x+y+z)\right)+Bd_1}{A\sin\left(\sqrt{F}(-ct+x+y+z)\right)+B},$$

$$F=\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1},\quad F>0,\quad \beta\neq -1. \tag{35}$$

$$u(x,y,z,t)=a_0-\frac{\cos\left(\frac{1}{2}\sqrt{F}(-ct+x+z+y)\right)\left(-c(\rho_1+\rho_2+\rho_3)+(\beta+1)d_1^2+\omega_1+\omega_2+\omega_3\right)}{(\beta+1)\left(d_1\cos\left(\frac{1}{2}\sqrt{F}(-ct+x+z+y)\right)+\sqrt{F}\sin\left(\frac{1}{2}\sqrt{F}(-ct+x+z+y)\right)\right)},$$
$$F=\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1},\quad F>0,\quad \beta\neq -1. \tag{36}$$

$$u(x,y,z,t)=a_0-\frac{\sin\left(\frac{1}{2}\sqrt{F}(-ct+x+z+y)\right)\left(-c(\rho_1+\rho_2+\rho_3)+(\beta+1)d_1^2+\omega_1+\omega_2+\omega_3\right)}{(\beta+1)\left(\sqrt{F}\cos\left(\frac{1}{2}\sqrt{F}(-ct+x+z+y)\right)-d_1\sin\left(\frac{1}{2}\sqrt{F}(-ct+x+z+y)\right)\right)},$$
$$F=\frac{-c(\rho_1+\rho_2+\rho_3)+\omega_1+\omega_2+\omega_3}{\beta+1},\quad F>0,\quad \beta\neq -1. \tag{37}$$

Eqs(35), (36) and (37)are trigonometric solutions.

3.2. Calculation of exponential Scheme

This section presents the application of the proposed exponential expansion method to find the more general and exact traveling wave solutions to the ODE at Eq. (13). Upon Inserting Eqs. (9) and (10) into Eq. (13) results in the following first system:

$$\begin{cases}-3a_1\beta-3a_1+6\beta+6=0,\\ -6a_1\beta\lambda-6a_1\lambda+12\beta\lambda+12\lambda=0,\\ -3a_1\beta\lambda^2-6a_1\beta\mu-3a_1\lambda^2-6a_1\mu+7\beta\lambda^2+8\beta\mu-c\rho1-c\rho2-c\rho3+7\lambda^2+8\mu+\omega1+\omega2+\omega3=0,\\ -6a_1\beta\lambda\mu-6a_1\lambda\mu+\beta\lambda^3+8\beta\lambda\mu-c\lambda\rho1-c\lambda\rho2-c\lambda\rho3+\lambda^3+8\lambda\mu+\lambda\omega1+\lambda\omega2+\lambda\omega3=0,\\ -3a_1\beta\mu^2-3a_1\mu^2+\beta\lambda^2\mu+2\beta\mu^2-c\mu\rho1-c\mu\rho2-c\mu\rho3+\lambda^2\mu+2\mu^2+\mu\omega1+\mu\omega2+\mu\omega3=0.\end{cases}$$

By solving the system the results can be raised;
Set I:

$$a_1=2;\lambda=\sqrt{\frac{4\beta\mu+c\rho1+c\rho2+c\rho3+4\mu-\omega1-\omega2-\omega3}{\beta+1}}.$$

Set II:

$$a_1=2;\lambda=0;\mu=\frac{c\rho_1+c\rho_2+c\rho_3-\omega_1-\omega_2-\omega_3}{-4\beta-4}$$

First case
Based on Set I, we obtained;

$$u(x,y,z,t)=a_0+\frac{4\mu}{-\sqrt{G-4\mu}\tanh\left(0.5\sqrt{G-4\mu}\,(-ct+V_0+x+y+z)\right)-\sqrt{G}},\quad G-4\mu>0,$$
$$G=\frac{4\beta\mu+c\rho1+c\rho2+c\rho3+4\mu-\omega1-\omega2-\omega3}{\beta+1}>0,\beta\neq -1. \tag{38}$$

Based on Set II, we obtained;

$$u(x,y,z,t)=a_0-\frac{2\left(c\rho_1+c\rho_2+c\rho_3-\omega_1-\omega_2-\omega_3\right)\coth\left(\sqrt{-\frac{c\rho_1+c\rho_2+c\rho_3-\omega_1-\omega_2-\omega_3}{-4\beta-4}}\,(-ct+V_0+x+y+z)\right)}{(-4\beta-4)\sqrt{-\frac{c\rho_1+c\rho_2+c\rho_3-\omega_1-\omega_2-\omega_3}{-4\beta-4}}},$$
$$\frac{c\rho_1+c\rho_2+c\rho_3-\omega_1-\omega_2-\omega_3}{-4\beta-4}<0,\beta\neq -1. \tag{39}$$

Second case

Based on Set I, we obtained;

$$u(x,y,z,t)=a_0+\frac{4\mu}{\sqrt{4\mu-G}\tan\left(0.5\sqrt{4\mu-G}\,(-ct+V_0+x+y+z)\right)-\sqrt{G}},\quad 4\mu-G>0,$$
$$G=\frac{4\beta\mu+c\rho1+c\rho2+c\rho3+4\mu-\omega1-\omega2-\omega3}{\beta+1}>0,\beta\neq-1. \tag{41}$$

Based on Set II, we obtained;

$$u(x,y,z,t)=a_1\sqrt{\mu}\cot\left(\sqrt{\mu}\,(-ct+V_0+x+y+z)\right)+a_0,\quad \mu>0. \tag{42}$$

Based on Set III, we obtained;

$$u(x,y,z,t)=a_0+\frac{2\,(c\rho_1+c\rho_2+c\rho_3-\omega_1-\omega_2-\omega_3)\cot\left(1.\sqrt{\frac{c\rho_1+c\rho_2+c\rho_3-\omega_1-\omega_2-\omega_3}{-4\beta-4}}\,(-ct+V_0+x+y+z)\right)}{(-4\beta-4)\sqrt{\frac{c\rho_1+c\rho_2+c\rho_3-\omega_1-\omega_2-\omega_3}{-4\beta-4}}},$$
$$\frac{c\rho_1+c\rho_2+c\rho_3-\omega_1-\omega_2-\omega_3}{-4\beta-4}>0,\beta\neq-1. \tag{43}$$

Third case

Set I:

$$a_1=2;\lambda=\sqrt{\frac{c\rho_1+c\rho_2+c\rho_3-\omega_1-\omega_2-\omega_3}{\beta+1}}.$$

Based on Set I, we obtained;

$$u(x,y,z,t)=a_0+\frac{2\sqrt{\frac{c\rho_1+c\rho_2+c\rho_3-\omega_1-\omega_2-\omega_3}{\beta+1}}}{\exp\left(\sqrt{\frac{c\rho_1+c\rho_2+c\rho_3-\omega_1-\omega_2-\omega_3}{\beta+1}}\,(-ct+V_0+x+y+z)\right)-1},$$
$$\frac{c\rho_1+c\rho_2+c\rho_3-\omega_1-\omega_2-\omega_3}{\beta+1}>0,\beta\neq-1. \tag{44}$$

fourth case

Set I:

$$a_1=2;\lambda=\sqrt{\frac{4\beta\mu+c\rho1+c\rho2+c\rho3+4\mu-\omega1-\omega2-\omega3}{\beta+1}}$$

Set II:

$$\mu=0;\beta=-1;\omega3=c\rho1+c\rho2+c\rho3-\omega1-\omega2.$$

Set III:

$$a_1=2;\lambda=\sqrt{\frac{c\rho_1+c\rho_2+c\rho_3-\omega_1-\omega_2-\omega_3}{\beta+1}};\mu=0.$$

Based on Set I, we obtained;

$$u(x,y,z,t)=a_0-\frac{(4\beta\mu+c\rho_1+c\rho_2+c\rho_3+4\mu-\omega_1-\omega_2-\omega_3)\,(-ct+V_0+x+y+z)}{(\beta+1)\left(\sqrt{\frac{4\beta\mu+c\rho_1+c\rho_2+c\rho_3+4\mu-\omega_1-\omega_2-\omega_3}{\beta+1}}\,(-ct+V_0+x+y+z)+2\right)},$$
$$\frac{4\beta\mu+c\rho_1+c\rho_2+c\rho_3+4\mu-\omega_1-\omega_2-\omega_3}{\beta+1}>0,\beta\neq-1. \tag{45}$$

Based on Set II, we obtained;

$$u(x,y,z,t)=a_0-\frac{a_1\lambda^2\,(-ct+V_0+x+y+z)}{2\,(\lambda\,(-ct+V_0+x+y+z)+2)}. \tag{46}$$

Based on Set III, we obtained;

$$u(x,y,z,t) = a_0 - \frac{(c\rho_1 + c\rho_2 + c\rho_3 - \omega_1 - \omega_2 - \omega_3)(-ct + V_0 + x + y + z)}{(\beta+1)\left(\sqrt{\frac{c\rho_1+c\rho_2+c\rho_3-\omega_1-\omega_2-\omega_3}{\beta+1}}(-ct + V_0 + x + y + z) + 2\right)},$$
$$\frac{c\rho_1 + c\rho_2 + c\rho_3 - \omega_1 - \omega_2 - \omega_3}{\beta+1} > 0, \beta \neq -1. \quad (47)$$

Upon substituting Eq (9), Eq (12) into Eq(8) the following system is obtained;

$$\begin{cases} 3a_1\beta\mu^2 + 3a_1\mu^2 + 6\beta\mu^3 + 6\mu^3 = 0, \\ 3a_1\beta\lambda^2 + 3a_1\lambda^2 + 2\beta\lambda^2\mu - c\lambda\rho_1 - c\lambda\rho_2 - c\lambda\rho_3 + 2\lambda^2\mu + \lambda\omega_1 + \lambda\omega_2 + \lambda\omega_3 = 0, \\ 6a_1\beta\lambda\mu + 6a_1\lambda\mu + 8\beta\lambda\mu^2 - c\mu\rho_1 - c\mu\rho_2 - c\mu\rho_3 + 8\lambda\mu^2 + \mu\omega_1 + \mu\omega_2 + \mu\omega_3 = 0. \end{cases}$$

By solving the system the result can be raised;
set I

$$\mu = -\frac{a_1}{2}; \lambda = \frac{c\rho_1 + c\rho_2 + c\rho_3 - \omega_1 - \omega_2 - \omega_3}{2a_1\beta + 2a_1}.$$

$$u(x,y,z,t) = a_0 + \sqrt{2}a_1\sqrt{-\frac{c\rho_1 + c\rho_2 + c\rho_3 - \omega_1 - \omega_2 - \omega_3}{a_1(2a_1\beta + 2a_1)}} *$$
$$\tan\left(\frac{\sqrt{-\frac{a_1(c\rho_1+c\rho_2+c\rho_3-\omega_1-\omega_2-\omega_3)}{2a_1\beta+2a_1}}(-ct + V_0 + x + y + z)}{\sqrt{2}}\right), -\frac{c\rho_1 + c\rho_2 + c\rho_3 - \omega_1 - \omega_2 - \omega_3}{a_1(2a_1\beta + 2a_1)} > 0, \beta \neq -1. \quad (48)$$

$$u(x,y,z,t) = a_0 - \sqrt{2}a_1\sqrt{-\frac{c\rho_1 + c\rho_2 + c\rho_3 - \omega_1 - \omega_2 - \omega_3}{a_1(2a_1\beta + 2a_1)}}$$
$$\cot\left(\frac{\sqrt{-\frac{a_1(c\rho_1+c\rho_2+c\rho_3-\omega_1-\omega_2-\omega_3)}{2a_1\beta+2a_1}}(-ct + V_0 + x + y + z)}{\sqrt{2}}\right), \frac{c\rho_1 + c\rho_2 + c\rho_3 - \omega_1 - \omega_2 - \omega_3}{a_1(2a_1\beta + 2a_1)} < 0, \beta \neq -1. \quad (49)$$

$$u(x,y,z,t) = a_0 + \sqrt{2}a_1\sqrt{\frac{c\rho_1 + c\rho_2 + c\rho_3 - \omega_1 - \omega_2 - \omega_3}{a_1(2a_1\beta + 2a_1)}}$$
$$\tanh\left(\frac{\sqrt{\frac{a_1(c\rho_1+c\rho_2+c\rho_3-\omega_1-\omega_2-\omega_3)}{2a_1\beta+2a_1}}(-ct + V_0 + x + y + z)}{\sqrt{2}}\right), \frac{c\rho_1 + c\rho_2 + c\rho_3 - \omega_1 - \omega_2 - \omega_3}{a_1(2a_1\beta + 2a_1)} > 0, \beta \neq -1. \quad (50)$$

$$u(x,y,z,t) = a_0 + \sqrt{2}a_1\sqrt{\frac{c\rho_1 + c\rho_2 + c\rho_3 - \omega_1 - \omega_2 - \omega_3}{a_1(2a_1\beta + 2a_1)}}$$
$$\coth\left(\frac{\sqrt{\frac{a_1(c\rho_1+c\rho_2+c\rho_3-\omega_1-\omega_2-\omega_3)}{2a_1\beta+2a_1}}(-ct + V_0 + x + y + z)}{\sqrt{2}}\right), \frac{c\rho_1 + c\rho_2 + c\rho_3 - \omega_1 - \omega_2 - \omega_3}{a_1(2a_1\beta + 2a_1)} > 0, \beta \neq -1. \quad (51)$$

Set II When $\lambda = 0, \mu > 0$

$$\mu = -\frac{a_1}{2}; \omega_3 = c\rho_1 + c\rho_2 + c\rho_3 - \omega_1 - \omega_2.$$

$$u(x,y,z,t) = a_0 + \frac{2}{-ct + V_0 + x + y + z}. \quad (52)$$

4. BIFURCATION ANALYSIS

In this section, the bifurcation analysis is applied to the reduced ordinary differential equation obtained from the traveling-wave transformation of the modified (3 + 1)-dimensional KP equation, [42–46] given by

$$Au' + 3B(u')^2 + Bu''' = 0, \tag{53}$$

where

$$A = -c(\rho_1 + \rho_2 + \rho_3) + \omega_1 + \omega_2 + \omega_3, \qquad B = 1 + \beta.$$

Thus, the above third-order nonlinear ODE is the equation considered in the present bifurcation analysis. By introducing the transformation

$$v = u',$$

Eq. (53) is reduced to

$$v'' = Mv - 3v^2,$$

where

$$M = \frac{c(\rho_1 + \rho_2 + \rho_3) - (\omega_1 + \omega_2 + \omega_3)}{1 + \beta}.$$

This reduced ODE is then used to construct the corresponding planar dynamical system and investigate the bifurcation behavior of its equilibrium points. Introducing

$$w = v',$$

gives the planar dynamical system

$$\begin{cases} v' = w, \\ w' = Mv - 3v^2. \end{cases}$$

The equilibrium points are obtained from $w = 0$ and $Mv - 3v^2 = 0$:

$$E_1 = (0, 0), \qquad E_2 = \left(\frac{M}{3}, 0\right).$$

The Jacobian matrix is

$$J(v, w) = \begin{pmatrix} 0 & 1 \\ M - 6v & 0 \end{pmatrix}.$$

At $E_1 = (0, 0)$,

$$\lambda^2 = M.$$

Hence

$$\begin{cases} M > 0: & E_1 \text{ is a saddle (unstable)}, \\ M < 0: & E_1 \text{ is a center (neutrally stable)}, \\ M = 0: & \text{the linearization is degenerate}. \end{cases}$$

At $E_2 = (M/3, 0)$,

$$\lambda^2 = -M.$$

Therefore

$$\begin{cases} M < 0: & E_2 \text{ is a saddle (unstable)}, \\ M > 0: & E_2 \text{ is a center (neutrally stable)}, \\ M = 0: & E_1 = E_2 = (0, 0). \end{cases}$$

Thus, the two equilibrium branches intersect at the critical condition

$$M = 0,$$

that is,

$$c(\rho_1 + \rho_2 + \rho_3) = \omega_1 + \omega_2 + \omega_3.$$

For $\rho_1 + \rho_2 + \rho_3 \neq 0$, the corresponding critical wave speed is

$$c = \frac{\omega_1 + \omega_2 + \omega_3}{\rho_1 + \rho_2 + \rho_3}.$$

When M changes sign through zero, the equilibrium points coalesce and exchange their local stability types: the origin changes between center and saddle, while the second equilibrium does the opposite. Therefore, the reduced traveling-wave system exhibits a stability-exchange bifurcation at $M = 0$.

5. GRAPHICAL DEPICTION OF THE SOLUTIONS

Graphical depictions of various solutions are presented in this section to demonstrate their characteristics. The 3D and contour visualizations of the dark-type soliton solution derived from Equation (16) are presented in Figure (1) with $c = 1.5$, $\rho_1 = \rho_2 = \rho_3 = \omega_1 = \omega_2 = \omega_3 = \beta = a_0 = -2$, $d_1 = -1$, $y = 2$, $z = 0.5$. The singular periodic wave solution of Equation (17) is demonstrated through its 3D and contour profiles in Figure (2) with $c = b_1 = -0.5$, $\rho_1 = \rho_2 = \rho_3 = \omega_1 = \omega_2 = \omega_3 = \beta = a_0 = -2$, $y = z = 0.75$. The dark wave solution of Equation (25), obtained for particular parameter settings, is displayed through 3D and contour plots in Figure (3) with $c = 2$, $d_1 = \rho_1 = \rho_2 = \rho_3 = \omega_1 = \omega_2 = \omega_3 = \beta = a_0 = -2$, $y = z = 0$. Figure (4) presents three-dimensional and contour visualizations of the soliton exhibiting singular behavior solution associated with Equation (26) with $d_1 = \rho_1 = \rho_2 = \rho_3 = \omega_1 = \omega_2 = \omega_3 = a_0 = -2$, $\beta = 2$, $y = 1$, $c = z = 0$.Figure (5): Presents the 3D and contour plot of the singular periodic solution corresponding to Equation (48) with $c = -0.03, \beta = a_0 = a_1 = \rho_1 = \rho_2 = \rho_3 = \omega_1 = \omega_2 = \omega_3 = V_0 = y = z = -2$. Figure (6) Presents the 3D of the singular soliton corresponding to Equation (51), with $c = -0.1, \beta = y = 2, a_0 = a_1 = \rho_1 = \rho_2 = \rho_3 = \omega_1 = \omega_2 = \omega_3 = V_0 = z = -2$. Figure (7): Presents the 3D and contour profiles of the dark-type solution corresponding to Equation (50) with $c = -0.8, \beta = 0.885, a_0 = a_1 = \rho_1 = \rho_2 = \rho_3 = \omega_1 = \omega_2 = \omega_3 = V_0 = -2, y = z = 0$. Figure (8): Phase portraits of the reduced KP system around the equilibrium point $(0, 0)$, showing its transition between a saddle for $M > 0$ and a center for $M < 0$. Figure (9): Phase portraits of the reduced KP system around the equilibrium point $(M/3, 0)$, showing its transition between a saddle for $M > 0$ and a center for $M < 0$.

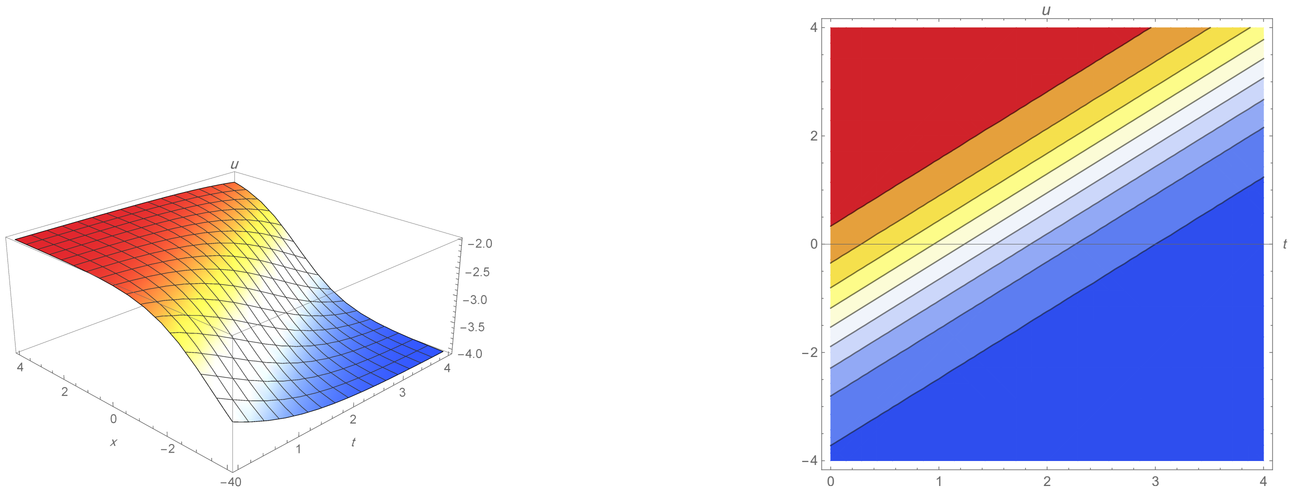

Figure 1. Provides the 3D and contour visualizations illustrating the dark-type soliton solution related to Equation (16)
.

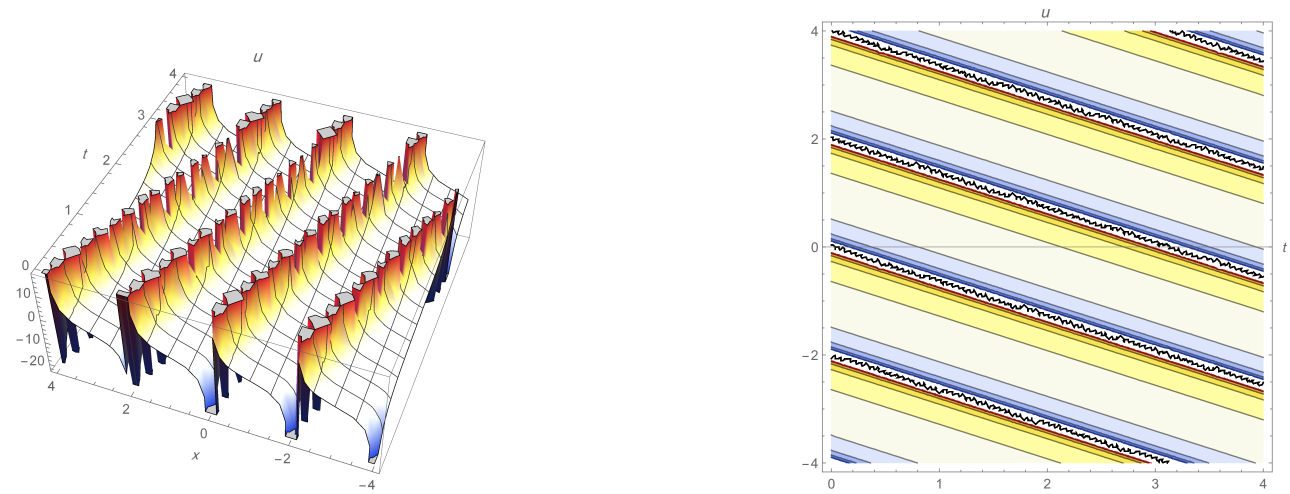

Figure 2. Provides the 3D and contour visualizations illustrating the periodic structures exhibiting singular behavior wave solution related to Equation (17)

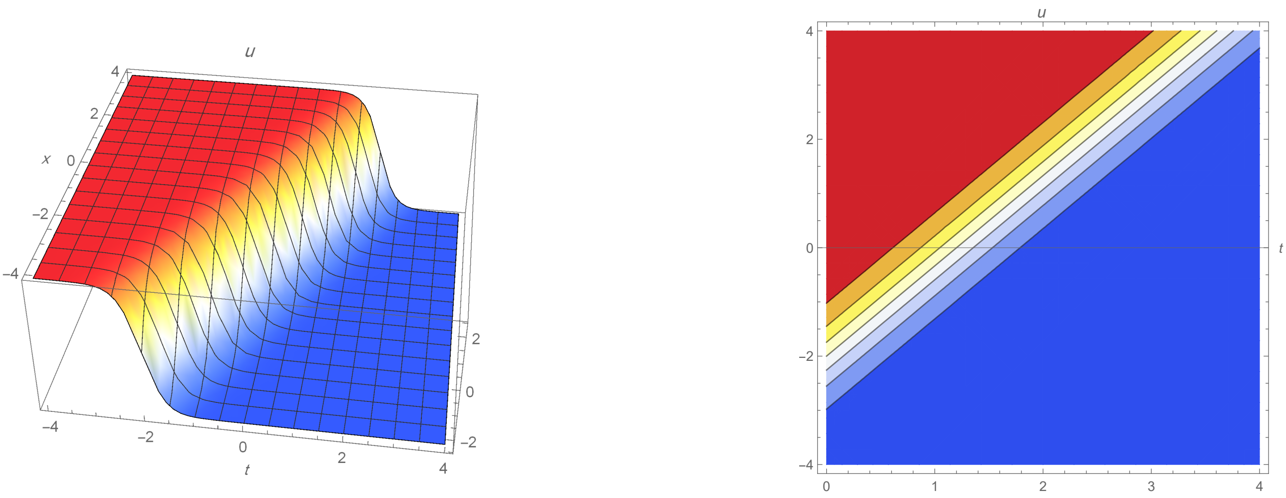

Figure 3. Presents the 3D and contour profiles of the dark-type solution corresponding to Equation (25)

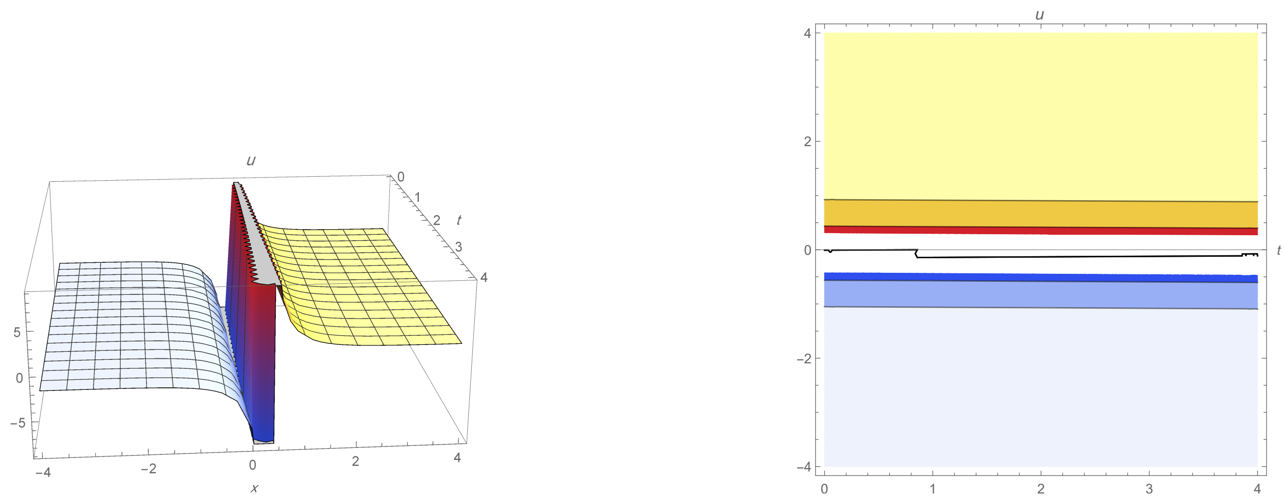

Figure 4. Depicts the 3D and contour views of the soliton exhibiting singular behavior solution associated with Equation (26)

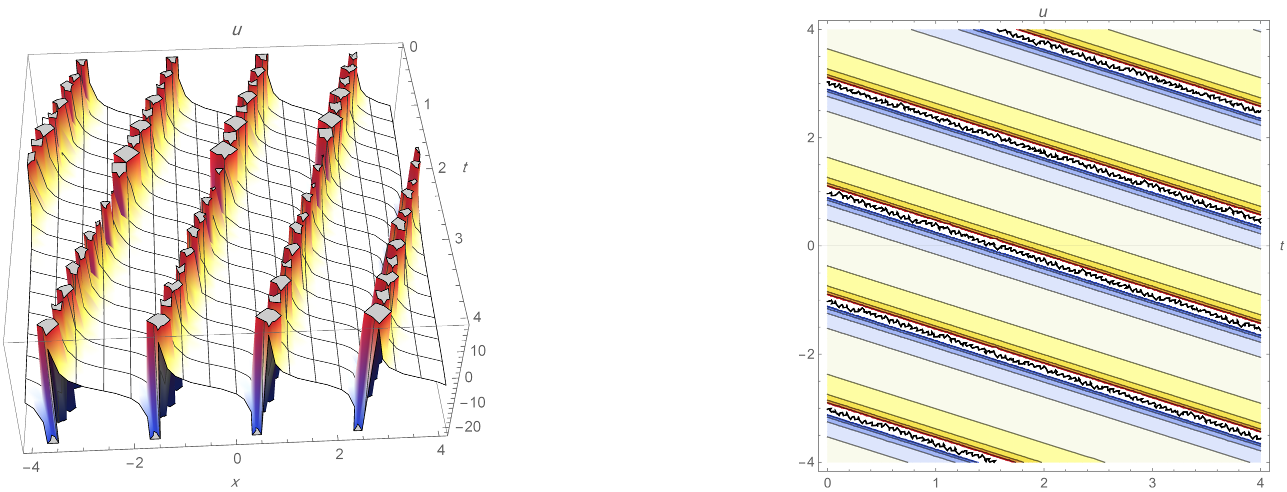

Figure 5. Presents the 3D and contour plot of the singular periodic solution corresponding to Equation (48)

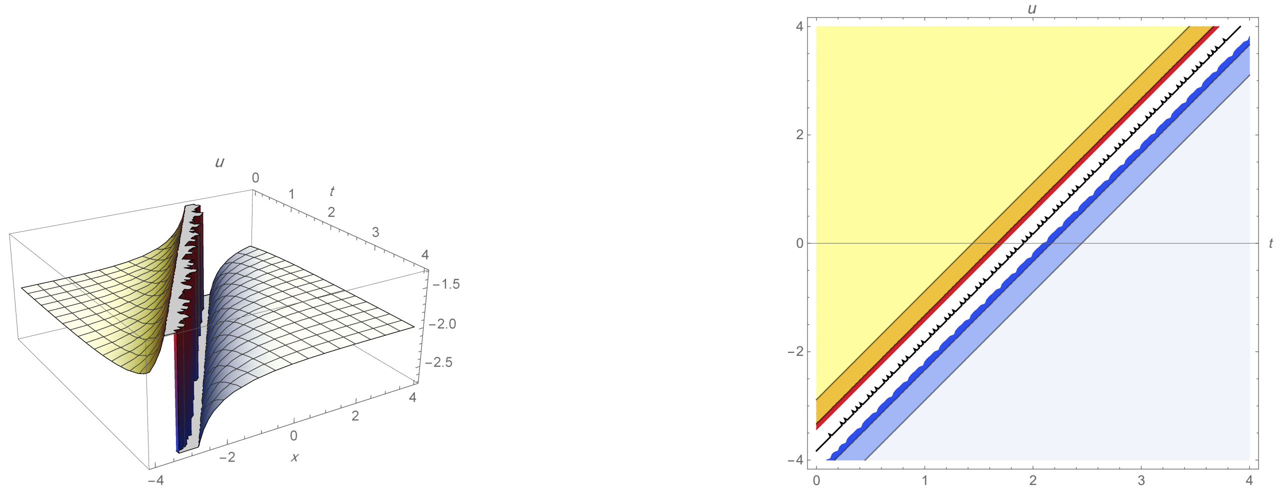

Figure 6. Presents the 3D and contour plot of the singular soliton corresponding to Equation (51)

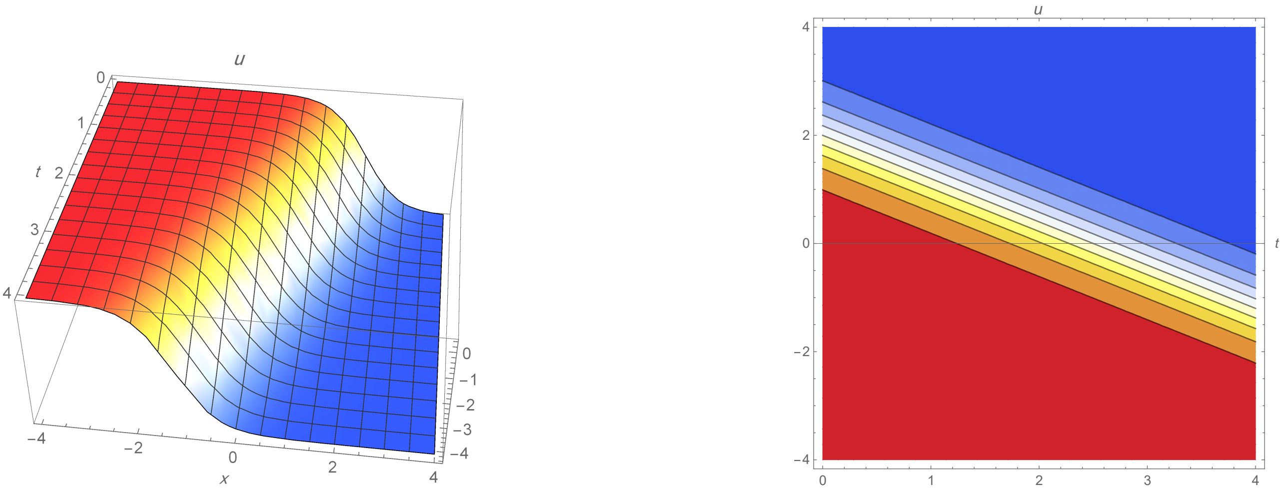

Figure 7. Presents the 3D and contour profiles of the dark-type solution corresponding to Equation (50)

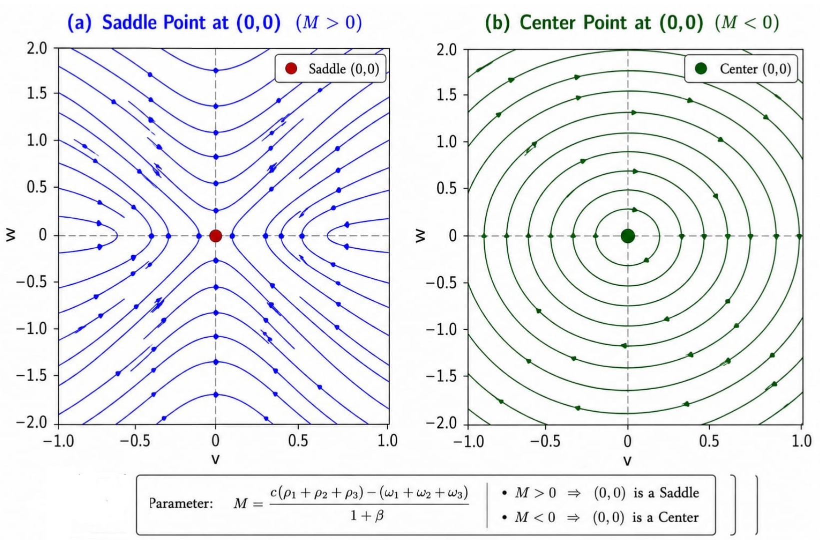

Figure 8. Phase portraits of the reduced KP system around the equilibrium point $(0, 0)$, showing its transition between a saddle for $M > 0$ and a center for $M < 0$

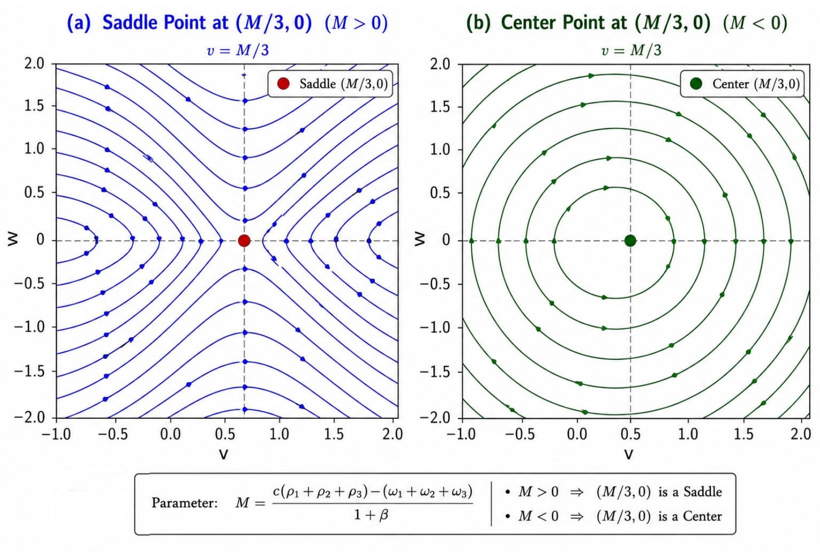

Figure 9. Phase portraits of the reduced KP system around the equilibrium point $(M/3, 0)$, showing its transition between a saddle for $M > 0$ and a center for $M < 0$

CONCLUSIONS

In the present work, the (3+1)-dimensional generalized Kadomtsev–Petviashvili equation, which characterizes wave spread in incompressible fluids, is analytically investigated using the Improved Simple Equation Method (ISEM) and Exponential Method These methods are employed to construct several classes of exact wave solutions, including exponential-type solutions, dark-type soliton solutions, exhibiting singular solution, singular periodic patterns, as well as hyperbolic forms. Furthermore, three-dimensional representations along with contour plots are provided to visualize and analyze the dynamic and physical characteristics of the obtained solutions. Moreover, the bifurcation analysis of the reduced traveling-wave system provides a dynamical interpretation of the obtained nonlinear wave structures. The equilibrium points and their stability properties are examined, revealing a critical condition at which the equilibrium branches coalesce and exchange their stability characteristics. This bifurcation behavior clarifies how variations in the governing parameters influence the qualitative dynamics of the modified (3+1)-dimensional KP equation. Therefore, the combination of exact analytical solution methods and bifurcation analysis offers a more comprehensive understanding of both the explicit wave structures and their underlying dynamical behavior.

ORCID

Ibrahim Saber, https://orcid.org/0009-0000-4262-3271; **Hamdy M. Ahmed**, https://orcid.org/0000-0001-8772-3663; **Niveen Badra**, https://orcid.org/0000-0002-8831-0577; **Islam Samir**, https://orcid.org/0000-0002-8172-3782

REFERENCES

[1] Feng L, and Zhu Z. "Darboux transformation and soliton solutions for a nonlocal two-component complex mKdV equation," J. Nonlinear Math. Phys. **30**, 112–128 (2023). https://doi.org/10.1007/s44198-023-00112-w

[2] Alraddadi I, and Alsharif F. "Innovative soliton solutions for a (2+1)-dimensional generalized KdV equation using two effective approaches," AIMS Math. **9**, 1664–1685 (2024). https://doi.org/10.3934/math.2024082

[3] Alam N., Ullah M.S., Nofal T.A., Ahmed H.M., Ahmed K.K., and Al-Nahhas M.A. "Novel dynamics of the fractional KFG equation through the unified and unified solver schemes with stability and multistability analysis," Nonlinear Eng. **13**, 20240034 (2024). https://doi.org/10.1515/nleng-2024-0034

[4] Liao S. "A new non-perturbative approach in quantum mechanics for time-independent Schrödinger equations," Sci. China Phys. Mech. Astron. **63**, 234612 (2020). https://doi.org/10.1007/s11433-019-9430-4

[5] Ahmed K.K., Ahmed H.M., Badra N.M., Mirzazadeh M., Rabie W.B., and Eslami M. "Diverse exact solutions to Davey–Stewartson model using modified extended mapping method," Nonlinear Analysis: Modelling and Control. **29**(5), 983-1002 (2024). https://doi.org/10.15388/namc.2024.29.36103

[6] Biswas A, Vega-Guzman J, Kara A, Qin Z, Ekici M, Yıldırım Y, Alshehri H, and Belic M. "Conservation laws for solitons in magneto-optic waveguides with dual-power law nonlinearity," Phys Lett. A, **416**, 127667 (2021). https://doi.org/10.1016/j.physleta.2021.127667

[7] Samir I., Abd-Elmonem A., and Ahmed H.M. "General solitons for eighth-order dispersive nonlinear Schrödinger equation with ninth-power law nonlinearity using improved modified extended tanh method," Optical and Quantum Electronics, **55**(5), 470 (2023). https://doi.org/10.1007/s11082-023-04753-5

[8] Horikis T, Frantzeskakis D, and Smyth N. "Extended shallow water wave equations," Wave Motion, **112**, 102934 (2022). https://doi.org/10.1016/j.wavemoti.2022.102934

[9] Khalifa A.S., Ahmed H.M., Badra N.M., and Rabie W.B. "Exploring solitons in optical twin-core couplers with Kerr law of nonlinear refractive index using the modified extended direct algebraic method," Optical and Quantum Electronics, **56**(6), 1060 (2024). https://doi.org/10.1007/s11082-024-06882-x

[10] Shakeel K, et al. "Construction of soliton solutions of time-fractional CDGSK equation with Painlevé analysis in plasma physics." Symmetry, **16**, 824 (2024). https://doi.org/10.3390/sym16070824

[11] Alharbi M., Elmandouh A., and Elbrolosy M. "Soliton, bifurcation, and quasi-periodic behaviour in fractional coupled Schrödinger equations," Mathematics, **13**, 3174 (2025). https://doi.org/10.3390/math13193174

[12] Yao F., and Younas U. "Dynamics of soliton solutions to nonlinear dynamical equations via neural-symbolic methods," Mathematics, **13**, 3546 (2025). https://doi.org/10.3390/math13213546

[13] Chen, X., Xia, T., and Zhu, L., "A (2+1)-Dimensional Integrable Breaking Soliton Equation and Its Algebro-Geometric Solutions," Mathematics, **12**(13), 2034 (2024). https://doi.org/10.3390/math12132034

[14] Zhang Y, and Chen S. "Advanced Hirota bilinear approach for multi-component integrable systems," Appl Math Comput. **455**, 128123 (2024). https://doi.org/10.1016/j.amc.2023.128123

[15] Novikov, S. P., "The Periodic Problem for the Korteweg–de Vries Equation", Functional Analysis and Its Applications, **8**(3), 236–246 (1974). https://doi.org/10.1007/BF01075697

[16] Dubrovin, B. A., Matveev, V. B., and Novikov, S. P., "Nonlinear Equations of Korteweg–de Vries Type, Finite-Zone Linear Operators, and Abelian Varieties", Russian Mathematical Surveys, **31**(1), 59–146 (1976). https://doi.org/10.1070/RM1976v031n01ABEH001446

[17] Krichever, I. M., "Methods of Algebraic Geometry in the Theory of Nonlinear Equations", Russian Mathematical Surveys, **32**(6), 185–213 (1977). https://doi.org/10.1070/RM1977v032n06ABEH003862

[18] Dubrovin, B. A., "Theta Functions and Nonlinear Equations", Russian Mathematical Surveys, **36**(2), 11–92 (1981). https://doi.org/10.1070/RM1981v036n02ABEH002596

[19] Belokolos, E. D., Bobenko, A. I., Enolskii, V. Z., Its, A. R., and Matveev, V. B., *Algebro-Geometric Approach to Nonlinear Integrable Equations*, (Springer Series in Nonlinear Dynamics, Springer, Berlin, 1994). https://doi.org/10.1007/978-3-662-02973-2

[20] Nakamura, A., "A Direct Method of Calculating Periodic Wave Solutions to Nonlinear Evolution Equations. I", Journal of the Physical Society of Japan, **47**(5), 1701–1705 (1979). https://doi.org/10.1143/JPSJ.47.1701

[21] Nakamura, A., "A Direct Method of Calculating Periodic Wave Solutions to Nonlinear Evolution Equations. II", Journal of the Physical Society of Japan, **48**(4), 1365–1370 (1980). https://doi.org/10.1143/JPSJ.48.1365

[22] Hon, Y. C., Fan, E. G., and Qin, Z., "A Kind of Explicit Quasi-Periodic Solution and Its Limit for the Toda Lattice Equation", Modern Physics Letters B, **22**(8), 547–553 (2008). https://doi.org/10.1142/S0217984908015097

[23] Fan, E. G., and Chow, K. W., "On the Periodic Solutions for Both Nonlinear Differential and Difference Equations: A Unified Approach", Physics Letters A, **374**(35), 3629–3634 (2010). https://doi.org/10.1016/j.physleta.2010.07.005

[24] Hon, Y. C., and Fan, E. G., "Super Quasiperiodic Wave Solutions and Asymptotic Analysis for N = 1 Supersymmetric KdV-Type Equations", Theoretical and Mathematical Physics, **166**(3), 317–336 (2011). https://doi.org/10.1007/s11232-011-0026-x

[25] Fan, E. G., Chow, K. W., and Li, J. H., "On Doubly Periodic Standing Wave Solutions of the Coupled Higgs Field Equation", Studies in Applied Mathematics, **128**(1), 86–105 (2012). https://doi.org/10.1111/j.1467-9590.2011.00531.x

[26] Tian, S. F., and Zhang, H. Q., "Riemann Theta Function Periodic Wave Solutions and Rational Characteristics for Nonlinear Evolution Equations", Journal of Mathematical Analysis and Applications, **371**(2), 585–608 (2010). https://doi.org/10.1016/j.jmaa.2010.05.070

[27] Ahmed M.S., Zaghrout A.A., and Ahmed H.M. "Exploration new solitons in fiber Bragg gratings with cubic–quartic dispersive reflectivity using improved modified extended tanh-function method," The European Physical Journal Plus, **138**(1), 32 (2023). https://doi.org/10.1140/epjp/s13360-023-03666-2

[28] Su, Y. Q., "Theta Function Solutions of the (3+1)-Dimensional Jimbo–Miwa Equation", Mathematical Problems in Engineering, **2017**, 2924947 (2017). https://doi.org/10.1155/2017/2924947

[29] Zhao, S., Wang, H., and Wang, Y., "Quasiperiodic-to-Soliton Conversions and Their Mechanisms of the (3+1)-Dimensional Generalized Kadomtsev–Petviashvili Equation", Wave Motion, **134**, 103505 (2025). https://doi.org/10.1016/j.wavemoti.2025.103505

[30] Pu J., and Chen Y. "Integrability and exact solutions of the (2+1)-dimensional KdV equation with Bell polynomials approach," Acta Math. Appl. Sin. Engl. Ser. **38**, 861–881 (2022). https://doi.org/10.1007/s10255-022-1020-9

[31] Raut S., Barman R., and Sarkar T. "Integrability, breather, lump and quasi-periodic waves of non-autonomous Kadomtsev–Petviashvili equation based on Bell-polynomial approach," Wave Motion, **119**, 103125 (2023). https://doi.org/10.1016/j.wavemoti.2023.103125

[32] Kumar S, and Mann N. Dynamic study of qualitative analysis, traveling waves, solitons, bifurcation, quasiperiodic, and chaotic behavior of integrable Kuralay equations," Opt. Quantum Electron. **56**, 859 (2024). https://doi.org/10.1007/s11082-024-06701-3

[33] Juan Y, Zhao Z, and Wazwaz A. ”Solitons, nonlinear wave transitions and characteristics of quasi-periodic waves for a (3+1)-dimensional generalized Calogero–Bogoyavlenskii–Schiff equation,” Chin. J. Phys. **89**, 896–929 (2024). https://doi.org/10.1016/j.cjph.2024.03.034

[34] Kadomtsev, B. B., and Petviashvili, V. I., “On the Stability of Solitary Waves in Weakly Dispersive Media”, Soviet Physics Doklady, **15**(6), 539–541 (1970). https://www.mathnet.ru/eng/dan35739

[35] Guan X, Liu W, Zhou Q, and Biswas A. ”Some lump solutions for a generalized (3+1)-dimensional Kadomtsev–Petviashvili equation,” Appl. Math. Comput. **366**, 124757 (2020). https://doi.org/10.1016/j.amc.2019.124757

[36] Zada A. ”Applications of extended simple equation method to modern nonlinear Schrödinger-type equations,” Optik, **271**, 170334 (2023). https://doi.org/10.1016/j.ijleo.2022.170334

[37] Khan Z. ”Modulation instability and soliton dynamics for dual-power-law nonlinear Schrödinger models,” Optik, **266**, 169789 (2023). https://doi.org/10.1016/j.ijleo.2022.169789

[38] Yildirim S. ”Dispersive traveling waves in modified EW-type systems using updated analytical schemes,” Results Phys. **43**, 106210 (2023). https://doi.org/10.1016/j.rinp.2023.106210

[39] Al-Amry M.S.A., and Al-Abdali E.F.A. ”Exact soliton solutions for new (4+1)-dimensional NLDEs by a new exp-expansion method,” Int. J. Theor. Appl. Math. **11**, 26–33 (2025). https://doi.org/10.11648/j.ijtam.20251101.14

[40] Abu-Alhamed S.F., and Elboree M.K. ”Investigation of soliton solutions for some important nonlinear evolution equations via exp-expansion method,” SVU Int. J. Basic Sci. **2**, 37–47 (2025). https://doi.org/10.21608/SVUIJBS.2025.322864.1006

[41] Borhan J.R., et al. ”Exploring new traveling wave solutions to nonlinear integro-partial differential equations with generalized expansion methods,” Math. J. **12**, 2023–2024 (2023). https://doi.org/10.3390/math11092023

[42] Yang D., and Zhang J. ”The soliton wave solutions and bifurcations of the (2+1)-dimensional dissipative long wave equation,” J. Nonlinear Math. Phys. **29**, 659–677 (2022). https://doi.org/10.1007/s44198-022-00055-8

[43] El-Ajou A., Zedan H.A., Alquran M., et al. ”Investigation of the analytical and numerical solutions with bifurcation analysis for the (2+1)-dimensional Bogoyavlenskii-Kadomtsev-Petviashvili equation,” Opt. Quantum Electron. **55**, (2023). https://doi.org/10.1007/s11082-023-04848-z

[44] Wang Y., Xu H., and Sun Q. ”New groups of solutions to the Whitham-Broer-Kaup equation,” Appl. Math. Mech. **41**, 1735–1746 (2020). https://doi.org/10.1007/s10483-020-2683-7

[45] Uecker H. ”Continuation and bifurcation in nonlinear PDEs – algorithms, applications, and experiments,” Jahresber Dtsch. Math-Ver. **124**, 43–80 (2022). https://doi.org/10.1365/s13291-021-00241-5

[46] Alam N., Ullah M.S., Manafian J., Mahmoud K.H., Alsubaie A.S., Ahmed H.M., Ahmed K.K., et al., ”Bifurcation analysis, chaotic behaviors, and explicit solutions for a fractional two-mode Nizhnik-Novikov-Veselov equation in mathematical physics,” AIMS Mathematics, **10**(3), 4558-4578 (2025). https://doi.org/10.3934/math.2025211

БАГАТОФОРМНІ ОДИНОКІ ХВИЛЬОВІ СТРУКТУРИ ТА БІФУРКАЦІЙНИЙ АНАЛІЗ РОЗШИРЕНОГО МОДИФІКОВАНОГО (3+1)-ВИМІРНОГО РІВНЯННЯ КАДОМЦЕВА-ПЕТВІАШВІЛІ ЗА ДОПОМОГОЮ ГІБРИДНИХ АНАЛІТИЧНИХ ПІДХОДІВ

Ібрагім Сабер[1,2], Хамді М. Ахмед[3], Нівін Бадра[1], Іслам Самір[1]

[1]*Кафедра фізики та математики, Інженерний факультет, Університет Айн-Шамс, Каїр, Єгипет*

[2]*Факультет штучного інтелекту, Єгипетський російський університет, Каїр, Єгипет*

[3]*Кафедра фізики та інженерної математики, Вищий інженерний інститут, Академія Ель-Шорук, Каїр, Єгипет*

У цій роботі представлено виведення точних аналітичних розв'язків для (3+1)-вимірного модифікованого рівняння Кадомцева-Петвіашвілі рівняння з використанням двох ефективних аналітичних схем, а саме: покращеного методу простих рівнянь та підходу експоненціального розкладання. Розглянута модель є актуальною для опису поширення складних нелінійних хвиль у нестисливих рідинних середовищах, де дисперсійні та нелінійні ефекти сильно пов'язані. Застосовані методи дають багату різноманітність хвильових структур з різними фізичними характеристиками, включаючи сингулярні та несингулярні локалізовані утворення, періодичні структури та експоненціально спадаючі профілі. Крім того, отримані гіперболічні та тригонометричні конфігурації, що відображають різні режими поширення системи. Якісна поведінка цих рішень додатково уточнюється за допомогою тривимірних поверхневих представлень та контурних відображень. Крім того, проведено біфуркаційний аналіз для редукованого звичайного диференціального рівняння біжучої хвилі для дослідження точок рівноваги, їх властивостей стійкості та критичних умов, пов'язаних з обміном стійкістю. Цей аналіз забезпечує додаткову перспективу динамічних систем на отримані хвильові структури та уточнює вплив керівних параметрів на якісну поведінку моделі.

Ключові слова: *нелінійні дисперсійні системи; багатовимірні нелінійні хвилі; точні аналітичні рішення*