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ACCELERATING FIVE-DIMENSIONAL BIANCHI TYPE-I COSMOLOGY IN SAEZ–BALLESTER THEORY WITH HYPERBOLIC EXPANSION

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We present a five-dimensional anisotropic Bianchi type-I cosmological model in Saez–Ballester scalar–tensor theory employing the hyperbolic expansion law $a(t) = \sinh^{\alpha}(\beta t)$. Exact solutions of the field equations are derived and their cosmological behavior is examined. The model evolves from an initial singular state to a late-time accelerated phase, with anisotropy diminishing and isotropization achieved asymptotically. The deceleration parameter approaches ($q = -1$), while ($\omega_{de} \approx -1.308$) indicates phantom dark-energy behavior. The null, weak, and dominant energy conditions remain satisfied, whereas the strong energy condition is violated, supporting the observed cosmic acceleration. Furthermore, the statefinder and Om diagnostics reveal that the model asymptotically approaches the ΛCDM scenario.

Keywords: *Saez–Ballester theory; Bianchi type-I universe; Higher-dimensional cosmology; Hyperbolic expansion law; Dark energy; Cosmic acceleration*

1. INTRODUCTION

The discovery of the late-time accelerated expansion of the Universe through observations of type Ia supernovae revolutionized modern cosmology and established the existence of an exotic cosmic component known as dark energy [1, 2]. Subsequent observations of the cosmic microwave background radiation, baryon acoustic oscillations, and large-scale structure surveys have further confirmed this accelerating scenario and constrained the cosmological parameters with remarkable precision [3]. Within the framework of the standard cosmological model, the accelerated expansion is attributed to the cosmological constant Λ. Although the ΛCDM model successfully explains a wide range of observational data, it suffers from the well-known fine-tuning and coincidence problems, thereby motivating the exploration of alternative descriptions of dark energy and gravity [4, 5].

In recent years, modified theories of gravity have emerged as promising candidates for explaining the accelerated expansion of the Universe without introducing an ad hoc cosmological constant. Extensive investigations have been carried out in $f(R)$ gravity, scalar–tensor theories, teleparallel gravity, and other geometrically modified frameworks [6–10]. These theories provide richer gravitational dynamics and often admit cosmological solutions capable of describing both the early inflationary epoch and the present accelerated phase. Among the various alternatives to Einstein's theory, scalar–tensor theories occupy a central position because scalar fields naturally arise in higher-dimensional theories, supergravity, and string-inspired models. One particularly interesting scalar–tensor formulation is the Saez–Ballester theory proposed by Saez and Ballester [11]. This theory introduces a dimensionless scalar field coupled to gravity through a non-standard kinetic term and has been employed to study several cosmological scenarios. Owing to its mathematical simplicity and physical richness, the theory has attracted considerable attention in investigations of anisotropic universes, exact cosmological solutions, and dark-energy dynamics [12–15].

Anisotropic cosmological models play an important role in understanding the early stages of cosmic evolution. Although present observations indicate that the Universe is highly isotropic on large scales, it is plausible that anisotropies existed during the primordial epoch. The study of anisotropic space-times therefore provides valuable insight into the isotropization process and the dynamical behaviour of the Universe at early times [16–18]. In this context, Bianchi cosmological models have proved particularly useful because they represent homogeneous but anisotropic generalizations of the Friedmann–Lemaître–Robertson–Walker geometry. Among them, the Bianchi type-I model is the simplest anisotropic cosmological model and has been extensively employed in the study of dark energy, scalar fields, and modified gravity theories [19, 20]. Another important development in modern cosmology is the investigation of higher-dimensional space-times. The concept of extra dimensions originated from the pioneering works of Kaluza and Klein [21, 22] and subsequently became an important ingredient of unified field theories. Higher-dimensional cosmological models naturally arise in string theory, supergravity, and braneworld scenarios, providing a broader framework for investigating gravitational interactions and the large-scale dynamics of the Universe [23, 24]. In such models, the additional dimension can introduce

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extra geometric degrees of freedom and modify the effective four-dimensional gravitational dynamics obtained after dimensional reduction. Depending on the underlying higher-dimensional framework, these effects may manifest themselves through modifications of the effective energy density and pressure, the evolution of the scale factors, the gravitational coupling, or additional scalar-field-like degrees of freedom [25, 26]. Consequently, extra-dimensional models can provide alternative mechanisms for influencing the expansion history of the Universe and the evolution of primordial anisotropies. Recent investigations of higher-dimensional and braneworld cosmologies have demonstrated that their parameters can be constrained using cosmological observations, including Hubble-rate measurements, Type Ia supernovae, baryon acoustic oscillations, Big Bang nucleosynthesis, and the growth of large-scale structure [27, 28].

The physical verification of extra-dimensional effects, however, remains challenging because the additional dimension may be compactified at a scale beyond the direct reach of current experiments or may interact only weakly with ordinary matter. Nevertheless, possible signatures of extra dimensions can be investigated indirectly through precision tests of gravity and cosmological observations. In particular, laboratory experiments searching for deviations from Newton's inverse-square law at submillimetre scales provide constraints on possible compact extra dimensions and other short-range gravitational modifications [29]. Cosmological observations can provide complementary constraints, since extra-dimensional scenarios may modify the expansion history or introduce additional effective components that leave signatures in the cosmic microwave background, baryon acoustic oscillations, Type Ia supernovae, Big Bang nucleosynthesis, and the growth of cosmic structure [27, 30]. Such observations can therefore be used to place constraints on extra-dimensional scenarios, although the resulting signatures are generally model-dependent and may be degenerate with those arising from dark energy or other modified-gravity mechanisms. In this context, five-dimensional cosmological models provide a relatively tractable framework for studying the possible dynamical consequences of an additional dimension while retaining sufficient mathematical simplicity for obtaining and analyzing exact cosmological solutions. A five-dimensional Bianchi type-I space-time is particularly useful for examining the interplay between anisotropic expansion, higher-dimensional geometry, and scalar-field dynamics.

To obtain exact solutions of the nonlinear gravitational field equations, it is often necessary to impose physically motivated constraints on the cosmic dynamics. One useful approach is the adoption of specific expansion laws capable of describing realistic cosmological evolution. Hyperbolic expansion laws have received significant attention because they naturally describe a transition from an early decelerating phase to a late-time accelerating epoch. Such behaviour is consistent with observational evidence and has been successfully employed in a variety of cosmological investigations [31]. Furthermore, these expansion laws often lead to exact analytical solutions, enabling a detailed examination of the evolution of physical and geometrical parameters.

Motivated by the above considerations, we investigate a five-dimensional anisotropic Bianchi type-I cosmological model in the framework of Saez–Ballester scalar–tensor theory by adopting the hyperbolic expansion law $a(t) = \sinh^{\alpha}(\beta t)$, where α and β are positive constants. Exact solutions of the field equations are obtained and the resulting cosmological behaviour is analyzed through the evolution of the Hubble parameter, expansion scalar, shear scalar, anisotropy parameter, energy density, pressure, and equation-of-state parameter. The physical viability of the model is further examined through the energy conditions, while statefinder and Om diagnostics are employed to characterize the dark-energy dynamics and assess the asymptotic behaviour of the model relative to the standard ΛCDM cosmology.

The paper is organized as follows. In Section 2, the field equations of the Saez–Ballester theory for a five-dimensional Bianchi type-I space-time are presented. Exact solutions are obtained in Section 3. The physical properties and cosmological implications of the model are discussed in subsequent sections. Finally, the main conclusions are summarized in the last section.

2. SAEZ–BALLESTER THEORY AND FIELD EQUATIONS

We consider the five-dimensional anisotropic Bianchi type-I space-time described by the line element

$$ds^2 = -dt^2 + A^2(dx^2 + dy^2) + B^2 dz^2 + C^2 d\psi^2, \tag{1}$$

where the metric potentials $A(t)$, $B(t)$, and $C(t)$ depend only on the cosmic time t. The coordinates (x, y, z) represent the ordinary spatial dimensions, while ψ denotes the additional spatial dimension. The model is spatially homogeneous but anisotropic and reduces to the spatially flat Friedmann–Robertson–Walker (FRW) universe in the isotropic limit $A = B = C$.

The gravitational dynamics are governed by the Saez–Ballester scalar–tensor theory, whose action is given by

$$S = \int \sqrt{-g}\left[R - \omega\phi^m g^{ij}\phi_{,i}\phi_{,j} + L_m\right] d^5x, \tag{2}$$

where R is the Ricci scalar, ϕ is a dimensionless scalar field, ω and m are coupling constants, and L_m denotes the matter Lagrangian density. Throughout this work, geometrized units are adopted such that $8\pi G = c = 1$.

Variation of the action (2) with respect to the metric tensor g_{ij} yields the gravitational field equations

$$R_{ij} - \frac{1}{2}g_{ij}R - \omega\phi^m\left(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}\right) = -T_{ij}, \tag{3}$$

where T_{ij} is the energy–momentum tensor of the cosmic fluid.
Similarly, variation of the action with respect to the scalar field ϕ gives the scalar field equation

$$2\phi^m\phi^{,i}_{;i} + m\phi^{m-1}\phi_{,k}\phi^{,k} = 0 \tag{4}$$

The matter content of the Universe is assumed to be a perfect fluid whose energy–momentum tensor is

$$T_{ij} = (\rho + p)u_i u_j + p g_{ij}, \tag{5}$$

where ρ and p denote the energy density and isotropic pressure, respectively, and the five-velocity vector satisfies $u^i u_i = -1$.
The covariant conservation law $T^{ij}_{;j} = 0$ leads to the continuity equation

$$\dot{\rho} + (\rho + p)\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = 0 \tag{6}$$

To characterize the expansion dynamics of the anisotropic universe, we define the average scale factor as

$$a = (A^2BC)^{1/4}, \tag{7}$$

so that the spatial volume becomes

$$V = a^4 = A^2BC. \tag{8}$$

The mean Hubble parameter is

$$H = \frac{\dot{a}}{a} = \frac{1}{4}\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) \tag{9}$$

The directional Hubble parameters along the principal axes are defined by

$$H_1 = \frac{\dot{A}}{A}, \qquad H_2 = \frac{\dot{B}}{B}, \qquad H_3 = \frac{\dot{C}}{C} \tag{10}$$

The deceleration parameter is given by

$$q = -\frac{a\ddot{a}}{\dot{a}^2} \tag{11}$$

which characterizes the acceleration or deceleration of the cosmic expansion.
Substituting the metric (1) into the field equations (3) and using the perfect-fluid energy–momentum tensor (5), we obtain the independent gravitational field equations

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} = -p + \frac{\omega}{2}\phi^m\dot{\phi}^2, \tag{12}$$

$$2\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} + \frac{\ddot{C}}{C} + 2\frac{\dot{A}\dot{C}}{AC} = -p + \frac{\omega}{2}\phi^m\dot{\phi}^2, \tag{13}$$

$$2\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}^2}{A^2} + 2\frac{\dot{A}\dot{B}}{AB} = -p + \frac{\omega}{2}\phi^m\dot{\phi}^2, \tag{14}$$

$$\frac{\dot{A}^2}{A^2} + 2\frac{\dot{A}\dot{B}}{AB} + 2\frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} = \rho - \frac{\omega}{2}\phi^m\dot{\phi}^2, \tag{15}$$

while the scalar field equation reduces to

$$\ddot{\phi} + \dot{\phi}\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) + \frac{m}{2}\frac{\dot{\phi}^2}{\phi} = 0 \tag{16}$$

3. EXACT COSMOLOGICAL SOLUTIONS

The field equations (12)–(16) constitute a highly nonlinear system involving the metric potentials A, B, and C, the scalar field ϕ, the energy density ρ, and the pressure p. Since the number of unknown functions exceeds the number of independent equations, an additional physically motivated condition is required to obtain exact solutions.

We adopt the hyperbolic expansion law

$$a(t) = \sinh^{\alpha}(\beta t), \qquad \alpha > 0, \quad \beta > 0, \tag{17}$$

where α and β are positive constants.

Using equation (8), the corresponding spatial volume is

$$V = A^2BC = a^4 = \sinh^{4\alpha}(\beta t) \tag{18}$$

The Hubble parameter corresponding to the hyperbolic law becomes

$$H = \frac{\dot{a}}{a} = \alpha\beta \coth(\beta t) \tag{19}$$

and the deceleration parameter is

$$q = -1 + \frac{1}{\alpha}\,\mathrm{sech}^2(\beta t) \tag{20}$$

Equation (20) shows that the model evolves from an initially decelerating phase to a late-time accelerating epoch. In particular, $\lim_{t\to\infty} q = -1$, which corresponds to a de Sitter-type accelerated expansion.

Subtracting equation (12) from equations equation (13), (14) and integrating the resulting differential equations twice, we obtain

$$\frac{A^2}{B} = l_1 \exp\left(c_1 \int a^{-4} dt\right), \tag{21}$$

$$\frac{A^2}{C} = l_2 \exp\left(c_2 \int a^{-4} dt\right), \tag{22}$$

and

$$\frac{B}{C} = l_3 \exp\left(c_3 \int a^{-4} dt\right), \tag{23}$$

where l_1, l_2, l_3, c_1, c_2, and c_3 are integration constants.

Combining the above relations with the volume condition, the metric potentials can be written as

$$A = x_1 a^{\frac{2}{3}} \exp\left(d_1 \int a^{-4} dt\right) = x_1 \sinh^{\frac{2\alpha}{3}}(\beta t) \exp\left[d_1 \int \sinh^{-4\alpha}(\beta t) dt\right], \tag{24}$$

$$B = x_2 a^{\frac{4}{3}} \exp\left(d_2 \int a^{-4} dt\right) = x_2 \sinh^{\frac{4\alpha}{3}}(\beta t) \exp\left[d_2 \int \sinh^{-4\alpha}(\beta t) dt\right], \tag{25}$$

$$C = x_3 a^{\frac{4}{3}} \exp\left(d_3 \int a^{-4} dt\right) = x_3 \sinh^{\frac{4\alpha}{3}}(\beta t) \exp\left[d_3 \int \sinh^{-4\alpha}(\beta t) dt\right] \tag{26}$$

where

$$x_1 = (l_1 l_2)^{1/6}, \qquad x_2 = \left(\frac{l_3}{l_1}\right)^{1/3}, \qquad x_3 = (l_2 l_3)^{-1/3}, \tag{27}$$

and

$$d_1 = \frac{c_1 + c_2}{6}, \qquad d_2 = \frac{c_3 - c_1}{3}, \qquad d_3 = -\frac{c_2 + c_3}{3}. \tag{28}$$

These constants satisfy the consistency conditions

$$x_1^2 x_2 x_3 = 1, \qquad d_1 + d_2 + d_3 = 0 \tag{29}$$

From equation (16) we get,

$$\phi(t) = \left[\frac{h(m+2)}{2} \int a^{-4} dt\right]^{\frac{2}{m+2}} = \left[\frac{h(m+2)}{2} \int \sinh^{-4\alpha}(\beta t)\, dt\right]^{\frac{2}{m+2}} \tag{30}$$

4. PHYSICAL AND KINEMATICAL PROPERTIES OF THE MODEL

Using the exact solutions obtained in the previous section, we now investigate the kinematical evolution and physical characteristics of the cosmological model.

The mean Hubble parameter corresponding to the hyperbolic expansion law is

$$H = \alpha\beta \coth(\beta t). \tag{31}$$

The expansion scalar is given by

$$\theta = 4H = 4\alpha\beta \coth(\beta t) \tag{32}$$

The directional Hubble parameters along the principal directions are obtained from the metric potentials as

$$H_x = H_y = \frac{\dot{A}}{A} = \frac{2\alpha\beta}{3}\coth(\beta t) + d_1 \sinh^{-4\alpha}(\beta t), \tag{33}$$

$$H_z = \frac{\dot{B}}{B} = \frac{4\alpha\beta}{3}\coth(\beta t) + d_2 \sinh^{-4\alpha}(\beta t), \tag{34}$$

$$H_\psi = \frac{\dot{C}}{C} = \frac{4\alpha\beta}{3}\coth(\beta t) + d_3 \sinh^{-4\alpha}(\beta t), \tag{35}$$

The shear scalar and mean anisotropy parameter is given by

$$\sigma^2 = \frac{2}{9}\alpha^2\beta^2\coth^2(\beta t) + \frac{1}{2}\left(2d_1^2 + d_2^2 + d_3^2\right)\sinh^{-8\alpha}(\beta t) \tag{36}$$

$$\Delta = \frac{2d_1^2 + d_2^2 + d_3^2}{4\alpha^2\beta^2\coth^2(\beta t)}\ \sinh^{-8\alpha}(\beta t). \tag{37}$$

Since $\lim_{t\to\infty} \sinh^{-8\alpha}(\beta t) = 0$, it follows that $\lim_{t\to\infty} \Delta = 0$, and consequently $\lim_{t\to\infty} \frac{\sigma}{\theta} = 0$. Therefore, the anisotropy gradually decreases with cosmic time and the universe approaches isotropy at late times.

The deceleration parameter corresponding to the hyperbolic expansion law is

$$q = -1 + \frac{1}{\alpha}\operatorname{sech}^2(\beta t) \tag{38}$$

At the initial epoch, $q(0) = -1 + \frac{1}{\alpha}$, while at late times $\lim_{t\to\infty} q = -1$. Hence, the model describes a universe evolving from an early decelerating phase to a late-time accelerated expansion phase approaching the de Sitter regime.

Furthermore, the average scale factor satisfies $a = \sinh^{\alpha}(\beta t)$, which implies $a \to 0$ as $t \to 0$, and $a \to \infty$ as $t \to \infty$. Therefore, the obtained cosmological model represents an expanding universe beginning with a Big-Bang-type singularity and evolving towards an accelerating and isotropic state at late times.

From equation (15) we get,

$$\rho(t) = \frac{52}{9}\alpha^2\beta^2\coth^2(\beta t) + \left(d_1^2 + 2d_1d_2 + 2d_1d_3 + d_2d_3 + \frac{\omega h^2}{2}\right)\sinh^{-8\alpha}(\beta t) \tag{39}$$

Using equation (12) we obtain

$$\begin{aligned} p(t) = &\left(\frac{10}{3}\alpha\beta^2 - \frac{68}{9}\alpha^2\beta^2\right)\coth^2(\beta t) - \frac{10}{3}\alpha\beta^2 - \frac{4}{3}\alpha\beta d_1 \coth(\beta t)\sinh^{-4\alpha}(\beta t) \\ &+ \left[\frac{\omega h^2}{2} - \left(d_1^2 + d_1d_3 + d_3^2\right)\right]\sinh^{-8\alpha}(\beta t) \end{aligned} \tag{40}$$

The equation of state parameter is defined as

$$\omega_{de}(t) = \frac{p(t)}{\rho(t)}. \tag{41}$$

Therefore,

$$\omega_{de}(t) = \frac{\left(\frac{10}{3}\alpha\beta^2 - \frac{68}{9}\alpha^2\beta^2\right)\coth^2(\beta t) - \frac{10}{3}\alpha\beta^2 - \frac{4}{3}\alpha\beta d_1 \coth(\beta t)\sinh^{-4\alpha}(\beta t)}{\frac{52}{9}\alpha^2\beta^2\coth^2(\beta t) + \left(d_1^2 + 2d_1d_2 + 2d_1d_3 + d_2d_3 + \frac{\omega h^2}{2}\right)\sinh^{-8\alpha}(\beta t)} + \frac{\left[\frac{\omega h^2}{2} - \left(d_1^2 + d_1d_3 + d_3^2\right)\right]\sinh^{-8\alpha}(\beta t)}{\frac{52}{9}\alpha^2\beta^2\coth^2(\beta t) + \left(d_1^2 + 2d_1d_2 + 2d_1d_3 + d_2d_3 + \frac{\omega h^2}{2}\right)\sinh^{-8\alpha}(\beta t)} \tag{42}$$

At late cosmic times ($t \to \infty$), the hyperbolic functions satisfy $\coth(\beta t) \to 1$, while the terms $\sinh^{-4\alpha}(\beta t)$ and $\sinh^{-8\alpha}(\beta t)$ vanish exponentially for $\alpha > 0$. Consequently,

$$\lim_{t\to\infty} \omega_{de}(t) = -\frac{17}{13} \approx -1.308. \tag{43}$$

Since $\omega_{de} < -1$, the effective dark-energy fluid enters the phantom regime at late times. This result indicates that the universe undergoes a phase of accelerated expansion stronger than that produced by a cosmological constant ($\omega = -1$). Furthermore, the exponential decay of the anisotropic and scalar-field terms implies that their influence becomes negligible in the asymptotic future, leading to a dark-energy dominated cosmological evolution. Therefore, within the framework of the Saez–Ballester scalar–tensor theory, the present model predicts that the Universe evolves toward a stable phantom-like accelerated phase, with the late-time dynamics governed primarily by the effective dark-energy sector.

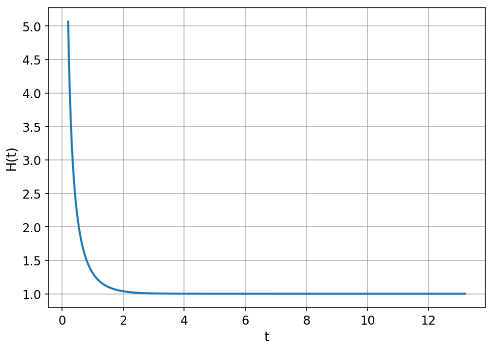

Figure 1. H vs t

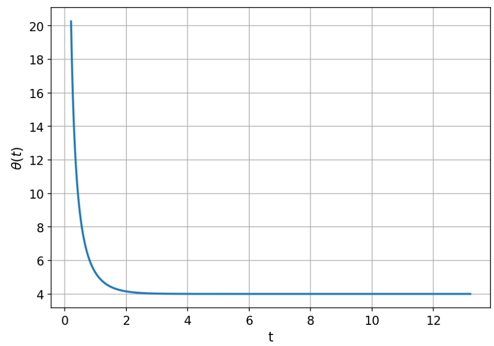

Figure 2. θ vs t

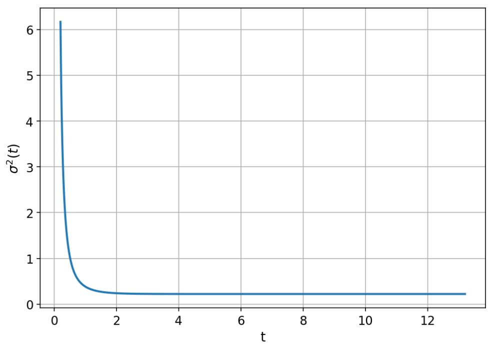

Figure 3. σ^2 vs t

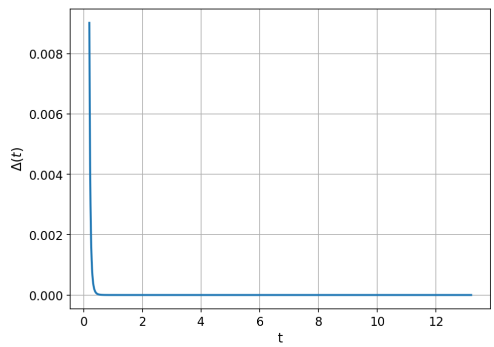

Figure 4. Δ vs t

For the purpose of numerical analysis and graphical illustration, we choose the parameter values $\alpha = 1.0$, $\beta = 1.0$, $d_1 = 0.001$, $d_2 = -0.0005$, $d_3 = -0.0005$, $\omega = 5.0$, $h = 1.0$, subject to the condition $d_1 + d_2 + d_3 = 0$. This set of parameters yields a physically acceptable cosmological model and is used throughout the paper to examine its evolutionary characteristics.

Figures 1–8 collectively demonstrate the evolutionary characteristics of the proposed five-dimensional anisotropic cosmological model from the early universe to the late-time accelerated epoch. The Hubble parameter H and the expansion scalar θ begin with very large values near the initial singularity and decrease with cosmic time, indicating a

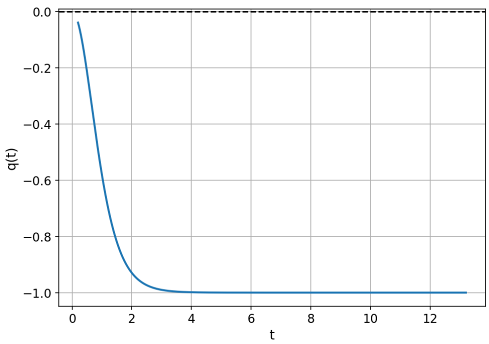

Figure 5. q vs t

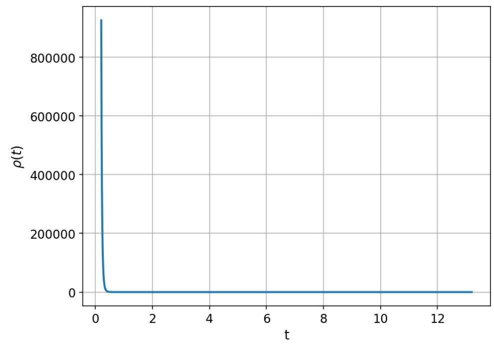

Figure 6. ρ vs t

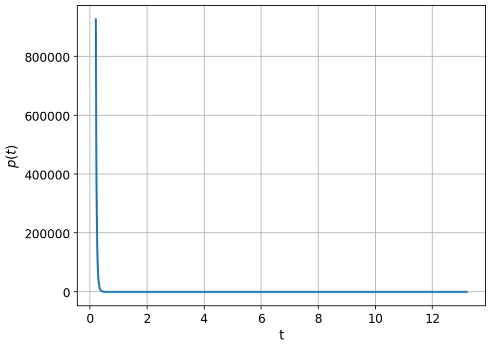

Figure 7. p vs t

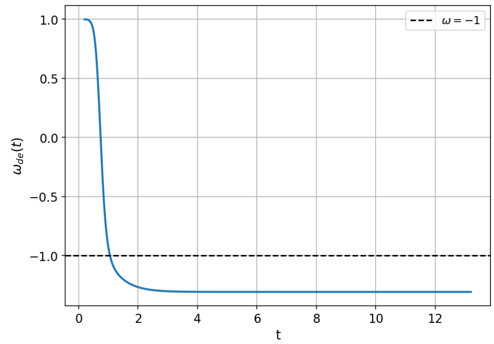

Figure 8. ω_{de} vs t

rapidly expanding early universe that gradually evolves toward a stable de Sitter-like expansion phase. Simultaneously, the shear scalar σ^2 and the mean anisotropy parameter Δ decrease monotonically and tend to zero at late times, demonstrating that the initial anisotropy of the universe gradually disappears and the model approaches isotropy in agreement with present cosmological observations. The deceleration parameter q evolves toward the value -1, confirming the existence of a late-time accelerated expansion phase. The energy density ρ remains positive throughout the cosmic evolution while decreasing from a very large initial value due to the continuous expansion of the universe. The pressure p attains negative values during the evolution, generating the repulsive gravitational effect required to drive cosmic acceleration. Furthermore, the equation-of-state parameter ω_{de} approaches a value below -1, specifically $\omega_{de} \approx -1.308$, indicating that the effective dark-energy component enters the phantom regime at late times. Overall, the figures reveal that the universe evolves from an initially dense, rapidly expanding, and anisotropic state toward an isotropic, dark-energy-dominated, phantom-like accelerated phase, thereby providing a physically consistent description of the observed late-time cosmic acceleration.

5. ENERGY CONDITIONS

The energy conditions provide important criteria for examining the physical viability of cosmological models and the nature of the cosmic fluid driving the expansion of the Universe.

5.1. Null Energy Condition

The NEC requires

$$\rho + p \geq 0. \tag{44}$$

Using the above expressions, one obtains

$$\begin{aligned}\rho + p = &\left(\frac{10}{3}\alpha\beta^2 - \frac{16}{9}\alpha^2\beta^2\right)\coth^2(\beta t) - \frac{10}{3}\alpha\beta^2 \\ &- \frac{4}{3}\alpha\beta d_1 \coth(\beta t)\sinh^{-4\alpha}(\beta t) \\ &+ \left(d_1 d_2 + d_1 d_3 + d_2 d_3 - d_3^2 + \omega h^2\right)\sinh^{-8\alpha}(\beta t)\end{aligned} \tag{45}$$

At late times ($t \to \infty$),

$$\lim_{t\to\infty}(\rho + p) = \frac{2}{9}\alpha\beta^2(15 - 8\alpha). \tag{46}$$

Hence, the NEC remains valid asymptotically for $\alpha \leq 15/8$.

5.2. Weak Energy Condition

The WEC requires

$$\rho \geq 0, \qquad \rho + p \geq 0 \tag{47}$$

Since the leading term in the energy density is positive,

$$\rho(t) = \frac{52}{9}\alpha^2\beta^2 \coth^2(\beta t) + O\Big(\sinh^{-8\alpha}(\beta t)\Big), \tag{48}$$

the condition $\rho \geq 0$ is satisfied for suitable choices of the model parameters. Therefore, the validity of the WEC is determined by the sign of $\rho + p$ and coincides with the NEC at late times.

5.3. Strong Energy Condition

The SEC requires

$$\rho + 3p \geq 0. \tag{49}$$

For the present model,

$$\begin{aligned}\rho + 3p = &\left(10\alpha\beta^2 - \frac{152}{9}\alpha^2\beta^2\right)\coth^2(\beta t) - 10\alpha\beta^2 \\ &- 4\alpha\beta d_1 \coth(\beta t)\sinh^{-4\alpha}(\beta t) \\ &+ \left(-d_1^2 + 2d_1d_2 - d_1d_3 + d_2d_3 - 3d_3^2 + 2\omega h^2\right)\sinh^{-8\alpha}(\beta t).\end{aligned} \tag{50}$$

As $t \to \infty$,

$$\lim_{t\to\infty}(\rho + 3p) = -\frac{2}{9}\alpha\beta^2(31 + 76\alpha) \tag{51}$$

Since $\alpha > 0$, it follows that

$$\rho + 3p < 0 \tag{52}$$

Therefore, the SEC is violated during the late-time evolution of the Universe. Such a violation is a characteristic feature of accelerating cosmological models and supports the observed cosmic acceleration.

5.4. Dominant Energy Condition

The DEC requires

$$\rho \geq 0, \qquad \rho - p \geq 0 \tag{53}$$

The quantity $\rho - p$ becomes

$$\begin{aligned}\rho - p = &\left(\frac{40}{3}\alpha^2\beta^2 - \frac{10}{3}\alpha\beta^2\right)\coth^2(\beta t) + \frac{10}{3}\alpha\beta^2 \\ &+ \frac{4}{3}\alpha\beta d_1 \coth(\beta t)\sinh^{-4\alpha}(\beta t) \\ &+ \left(2d_1^2 + 2d_1d_2 + 3d_1d_3 + d_2d_3 + d_3^2\right)\sinh^{-8\alpha}(\beta t)\end{aligned} \tag{54}$$

In the asymptotic limit,

$$\lim_{t\to\infty}(\rho - p) = \frac{40}{3}\alpha^2\beta^2 > 0 \tag{55}$$

Thus, the DEC remains satisfied at late times.

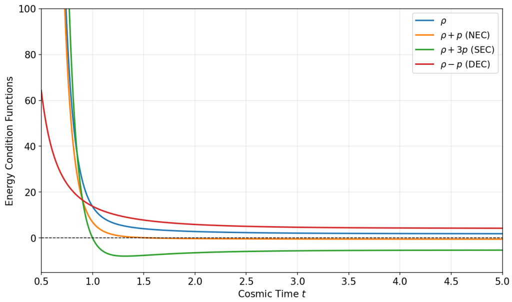

Figure 9. Variation of the energy density ρ, the null energy condition (NEC) $(\rho + p)$, the strong energy condition (SEC) $(\rho + 3p)$, and the dominant energy condition (DEC) $(\rho - p)$ with cosmic time t.

The above analysis shows that the energy density remains positive throughout the cosmic evolution for admissible values of the model parameters. The NEC and WEC can be satisfied depending on the value of α, whereas the SEC is violated in the late-time epoch. The violation of the SEC indicates the presence of a repulsive gravitational effect responsible for the accelerated expansion of the Universe. Furthermore, the DEC remains valid asymptotically, ensuring that the energy flux propagates within the causal structure of spacetime. These results support the physical viability of the present cosmological model and are consistent with the phantom-like behavior of the effective dark-energy component obtained from the equation-of-state parameter.

Figure 9 illustrates the evolution of the energy density ρ, the Null Energy Condition (NEC) $(\rho + p)$, the Strong Energy Condition (SEC) $(\rho + 3p)$, and the Dominant Energy Condition (DEC) $(\rho - p)$ as functions of cosmic time. The figure shows that the energy density remains positive throughout the cosmic evolution, indicating the physical viability of the model and the presence of a positive effective energy content in the universe. The quantity $(\rho + p)$ remains positive during the evolution, implying that the NEC is satisfied and consequently the Weak Energy Condition (WEC) is also fulfilled. In contrast, $(\rho + 3p)$ becomes negative and remains below zero at late times, demonstrating the violation of the SEC. Such a violation is a characteristic signature of accelerated expansion, since it corresponds to the emergence of a repulsive gravitational effect capable of overcoming the attractive nature of ordinary matter. Furthermore, the quantity $(\rho - p)$ remains positive throughout the evolution, confirming the validity of the DEC and ensuring that the energy flux propagates causally within spacetime. Consequently, Figure 9 shows that the present model satisfies the NEC, WEC, and DEC, while exhibiting a violation of the SEC at late times, a feature commonly associated with an accelerating universe. This behavior is consistent with the phantom-like dark-energy nature of the model and provides strong evidence that the observed cosmic acceleration is driven by an effective repulsive dark-energy component.

6. STATEFINDER DIAGNOSTIC

The statefinder parameters $\{r, s\}$ are defined as

$$r = \frac{\dddot{a}}{aH^3}, \qquad s = \frac{r-1}{3\left(q - \frac{1}{2}\right)}. \tag{56}$$

Using

$$r = 1 + 3\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3}, \tag{57}$$

we obtain

$$r = 1 + \left(\frac{2}{\alpha^2} - \frac{3}{\alpha}\right)\operatorname{sech}^2(\beta t). \tag{58}$$

$$s = \frac{\left(\frac{2}{\alpha} - 3\right)\operatorname{sech}^2(\beta t)}{3\left[\operatorname{sech}^2(\beta t) - \frac{3\alpha}{2}\right]}. \tag{59}$$

At late times,

$$\lim_{t\to\infty} \operatorname{sech}^2(\beta t) = 0, \tag{60}$$

and consequently,

$$\lim_{t\to\infty} r = 1, \qquad \lim_{t\to\infty} s = 0. \tag{61}$$

Therefore,

$$(r, s) \to (1, 0), \tag{62}$$

which corresponds to the ΛCDM fixed point. Hence, the present model asymptotically approaches the standard cosmological constant dominated universe at late times.

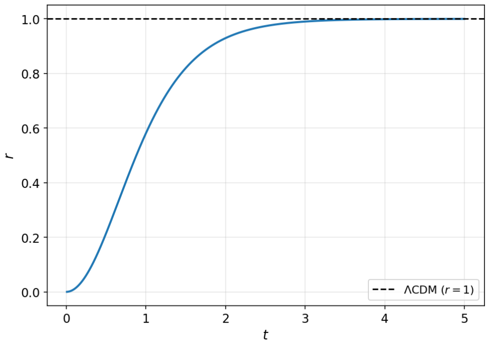

Figure 10. r vs t

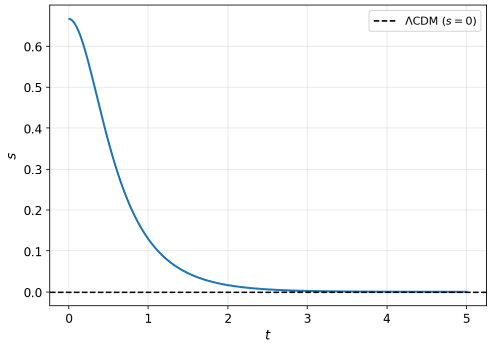

Figure 11. s vs t

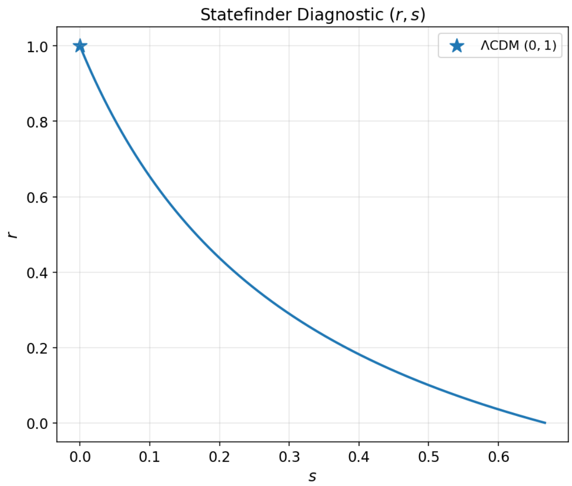

Figure 12. Statefinder trajectory in the (s, r) plane

The evolution of the statefinder parameters is illustrated in Figs. 10–12. Figure 10 shows that the parameter r increases with cosmic time and asymptotically approaches the ΛCDM value $r = 1$. Similarly, Fig. 11 demonstrates that the parameter s gradually tends to zero as the universe evolves. The convergence of r and s to their respective ΛCDM values indicates that the present model evolves toward a cosmological-constant-dominated phase at late times.

Furthermore, the trajectory in the (s, r) plane, shown in Fig. 12, approaches the fixed point $(0, 1)$ corresponding to the standard ΛCDM model. This behavior confirms that although the model may exhibit deviations from the ΛCDM scenario during the early stages of evolution, it gradually converges to the standard cosmological picture in the asymptotic future. Therefore, the statefinder diagnostic reveals that the proposed model successfully describes the late-time accelerated expansion of the universe and ultimately approaches a de Sitter-like phase.

7. OM DIAGNOSTIC ANALYSIS

The Om diagnostic is a useful geometrical tool for distinguishing different dark energy models and is defined as

$$Om(z) = \frac{E^2(z) - 1}{(1+z)^3 - 1}, \tag{63}$$

where

$$E(z) = \frac{H(z)}{H_0}. \tag{64}$$

Using the scale factor

$$a = \sinh^{\alpha}(\beta t) = \frac{1}{1+z}, \tag{65}$$

we obtain

$$\sinh(\beta t) = (1+z)^{-1/\alpha}. \tag{66}$$

Since

$$H = \alpha\beta \coth(\beta t), \tag{67}$$

and

$$\coth^2(\beta t) = 1 + \sinh^{-2}(\beta t), \tag{68}$$

it follows that

$$H(z) = \alpha\beta\sqrt{1 + (1+z)^{2/\alpha}}. \tag{69}$$

At the present epoch ($z = 0$),

$$H_0 = H(0) = \alpha\beta\sqrt{2}. \tag{70}$$

Therefore,

$$E(z) = \frac{H(z)}{H_0} = \frac{\sqrt{1 + (1+z)^{2/\alpha}}}{\sqrt{2}}, \tag{71}$$

which gives

$$E^2(z) = \frac{1 + (1+z)^{2/\alpha}}{2}. \tag{72}$$

Substituting into the definition of the Om diagnostic, we obtain

$$Om(z) = \frac{\frac{1}{2}\left[(1+z)^{2/\alpha} - 1\right]}{(1+z)^3 - 1}, \tag{73}$$

or equivalently,

$$Om(z) = \frac{(1+z)^{2/\alpha} - 1}{2\left[(1+z)^3 - 1\right]}. \tag{74}$$

The behaviour of $Om(z)$ provides information about the underlying dark energy dynamics. In the ΛCDM model, $Om(z)$ remains constant and is equal to the present matter density parameter Ω_{m0}. Any deviation from constancy indicates departure from the standard cosmological constant scenario.

For the present model,

$$\lim_{z\to -1} Om(z) = \frac{-1}{-2} = \frac{1}{2}, \tag{75}$$

showing that the Om parameter approaches a constant value at late times. This behaviour indicates that the model asymptotically evolves towards a de Sitter-like accelerated phase. Nevertheless, since $Om(z)$ remains redshift dependent during the cosmic evolution, the model generally differs from the standard ΛCDM cosmology except in the asymptotic future limit.

The evolution of the $Om(z)$ parameter is shown in Fig. 13. The parameter decreases monotonically with increasing redshift, indicating that it is not constant throughout the cosmic evolution. Since a constant Om is a characteristic feature of the ΛCDM model, the observed redshift dependence signifies a departure from the standard cosmological constant scenario and reflects the dynamical nature of dark energy in the present model. As the universe evolves, $Om(z)$ tends toward the asymptotic value 0.5, suggesting that the model approaches a stable de Sitter-like accelerated phase at late times.

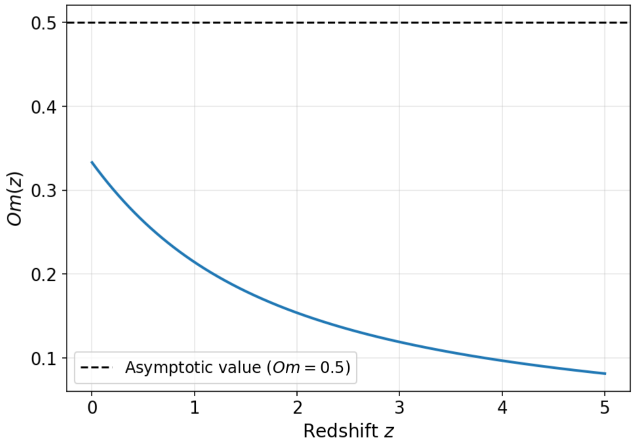

Figure 13. Evolution of the Om diagnostic parameter $Om(z)$ as a function of redshift z for $\alpha = 1.0$.

8. CONCLUSIONS

In the present work, we have constructed and analyzed an exact five-dimensional Bianchi type-I cosmological model in the framework of Saez–Ballester scalar–tensor theory by adopting the hyperbolic expansion law $a(t) = \sinh^{\alpha}(\beta t)$. This choice enables the complete integration of the field equations and provides a realistic cosmological scenario capable of describing the observed accelerated expansion of the universe. The obtained solution corresponds to an expanding universe that begins with a Big-Bang-type singularity, where the spatial volume vanishes while the expansion rate becomes extremely large. The Hubble parameter and expansion scalar decrease monotonically with cosmic time and approach finite values at late times, indicating a smooth transition toward a stable accelerated epoch. The deceleration parameter evolves toward ($q = -1$), showing that the model naturally approaches a de Sitter-like expansion phase. The anisotropic nature of the early universe is reflected through the nonvanishing shear scalar and anisotropy parameter. However, both quantities decay with cosmic time and asymptotically vanish, demonstrating that the universe evolves toward isotropy. This behaviour is consistent with current astronomical observations, which indicate a highly isotropic universe on large scales.

The physical properties of the cosmic fluid reveal that the energy density remains positive throughout the evolution, while the pressure becomes negative and generates the repulsive gravitational effect required for acceleration. The effective equation-of-state parameter evolves into the phantom region and asymptotically approaches ($\omega_{de} \approx -1.308$), indicating that the late-time dynamics are dominated by a phantom-like dark-energy component. Consequently, the model predicts an accelerated expansion stronger than that produced by a cosmological constant. The analysis of the energy conditions further supports the physical viability of the model. The null energy condition, weak energy condition, and dominant energy condition remain satisfied for admissible parameter values, whereas the strong energy condition is violated during the late-time evolution. Such a violation is a characteristic feature of accelerating cosmologies and confirms the existence of an effective repulsive gravitational mechanism.

To further characterize the model, statefinder and Om diagnostics were employed. The statefinder parameters converge to the fixed point $\{r, s\} = (1, 0)$, showing that the model asymptotically approaches the standard (Λ)CDM cosmology. The Om diagnostic remains redshift dependent during the cosmic evolution, indicating deviations from the standard cosmological-constant scenario, although it tends to a constant value in the asymptotic future. These diagnostics demonstrate that the model can successfully reproduce the observed late-time acceleration while retaining distinctive dynamical features. Therefore, the proposed five-dimensional Saez–Ballester cosmological model provides a mathematically consistent and physically viable description of the universe, successfully explaining the transition from an initially anisotropic state to an isotropic, phantom-dark-energy-dominated accelerated phase.

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ПРИСКОРЕННЯ П'ЯТИВИМІРНОЇ КОСМОЛОГІЇ Б'ЯНКІ ТИПУ I В ТЕОРІЇ САЕЗА-БАЛЛЕСТЕРА З ГІПЕРБОЛІЧНИМ РОЗШИРЕННЯМ

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Ми представляємо п'ятивимірну анізотропну космологічну модель Б'янкі типу I в скалярно-тензорній теорії Саез-Баллестера, що використовує закон гіперболічного розкладання $a(t) = \sinh^{\alpha}(\beta t)$. Отримано точні розв'язки рівнянь поля та досліджено їх космологічну поведінку. Модель еволюціонує від початкового сингулярного стану до пізньої прискореної фази, зі зменшенням анізотропії та асимптотично досягнутою ізотропізацією. Параметр уповільнення наближається до ($q = -1$), тоді як ($\omega_{de} \approx -1.308$) вказує на фантомну поведінку темної енергії. Умови нульової, слабкої та домінуючої енергії залишаються виконаними, тоді як умова сильної енергії порушується, що підтверджує спостережуване космічне прискорення. Крім того, метод визначення стану та діагностика Om показують, що модель асимптотично наближається до сценарію ΛCDM.

Ключові слова: *теорія Саеза-Баллестера; Всесвіт Біанкі I типу; космологія вищих вимірів; гіперболічний закон розширення; темна енергія; космічне прискорення*