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COSMIC ACCELERATION AND STABILITY OF KALUZA-KLEIN UNIVERSE IN $f(Q,T)$ GRAVITY

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The $f(Q,T)$ gravity has been used in this research to study the Kaluza-Klein universe in the presence of matter distribution. In this theory of gravity, the action contains an arbitrary function $f(Q,T)$ where Q and T respectively denote the non-metricity and the trace of energy momentum tensor. The linear and additive form of $f(Q,T)$ gravity, $f(Q,T) = \alpha Q + \beta T$ where α and β are arbitrary constants, is taken into account in this work. To achieve a physically viable solution of the field equations, we have considered power law and exponential expansion law. We investigate the evolving character of the universe with physical and geometrical properties within this theoretical framework. To enhance clarity, statefinder diagnostic and EoS parameter are examined, to characterize different phases of the universe. The energy conditions of the models are also analyzed and found to be consistent with recent cosmological observations. Besides, one of our models inherently allows a phantom region for dark energy. Further the models examined in this work are confirmed through stability analysis.
Keywords: *$f(Q,T)$ gravity; Kaluza-Klein space-time; Equation of state; Energy conditions*

1. INTRODUCTION

The study of universe's accelerating expansion confirmed by numerous observations over the last two decades, is a major cosmological enigma among researchers. This concept of accelerated expansion of the universe is strongly backed by astronomical observations of type Ia Supernovae [1-3] and measurements of cosmic microwave background (CMB) from the Wilkinson Microwave Anisotropy Probe (WAMP) [4-5]. Over the past few years, many researchers have explored dark energy models and are probing the dynamic relationship between dark energy (DE) and cosmic evolution. Modifying the geometric part of the Einstein-Hilbert action is a promising approach for understanding DE. Researchers have proposed modified gravity theories to explain the universe's acceleration, offering alternative frameworks to understand this phenomenon. Several modified theories have emerged such as $f(G)$ gravity [6], $f(R)$ gravity [7], $f(R,T)$ gravity [8], $f(T)$ gravity [9], $f(G,T)$ gravity [10], $f(R,T,R_{\mu\nu}T^{\mu\nu})$ gravity [11], $f(Q)$ gravity [12] and the recently proposed $f(Q,T)$ gravity [13-14] to explore the dark energy and other cosmological challenges.

Among the several modified theories of gravitation, $f(Q,T)$ theory of gravity proposed by Xu et al. [13] has received a lot of interest in recent years. It has been suggested that cosmic acceleration can be achieved by replacing the Einstein - Hilbert action of general relativity with a function $f(Q,T)$ where Q and T denotes the non-metricity and the trace of energy momentum tensor respectively. The $f(Q,T)$ gravity is an extension of symmetric teleparallel gravity, which is built on the non-minimal coupling between the non-metricity Q and the trace T of matter energy momentum tensor. This coupling yields results in the non-conservation of the energy-momentum tensor possibly offering mechanisms for matter generation or decay in the early or late universe and potentially explaining cosmic acceleration. Inspired by this, we employ the $f(Q,T)$ gravity theory to explore plausible explanations for various cosmological phenomena occurring in the universe. Subsequently, $f(Q,T)$ gravity has been widely explored across cosmological and astrophysical contexts, including studies of investigations of cosmic acceleration [15-18], energy conditions [19], cosmological inflation [20-21], wormhole solution [22-24], viability of $f(Q,T)$ gravity [25-26], Baryogenesis [27], bounce cosmology [28-30], Rip cosmology [31].

In the present era, research on higher-dimensional space-time is a highly active area of investigation. Today's conventional four-dimensional space-time could have been preceded by a multi-dimensional phase. Higher dimensional cosmology may provide solutions to basic inquiries concerning the universe, encompassing its origin, structure and evolution. It also seeks to merge the two essential natural forces, gravity and electro-magnetism. The idea of an additional dimension was initially presented by Theodor Kaluza [32] and subsequently developed by Oskar Klein [33]. Kaluza added a fifth dimension to ordinary four-dimensional space-time and suggested that the geometry of higher dimensional space-time could account for both gravity and electromagnetism in single framework. Kaluza-Klein theory provides solutions to the flatness, horizon and cosmological constant issues found in the conventional cosmological model. This is achieved by introducing extra spatial dimensions, in which fundamental constants such as the gravitational constant and gauge couplings are treated as dynamical fields. These fields evolve during the

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compactification process in the early universe. Nowadays, higher dimensional cosmology has attracted the interest of numerous researchers. The five-dimensional Kaluza-Klein theory now captures the attention of many scholars [34-35]. Recently, Ray and Baruah [36] have delved into five-dimensional Kaluza-Klein cosmological model featuring two fluid sources. Pawar et al. [37], Katore et al. [38] have investigated the behavior of strange quark matter for Kaluza-Klein cosmological model in Lyra Geometry and $f(R,T)$ gravity respectively. Wankhade and Nimkar [39] studied the dynamic behavior of Kaluza-Klein universe in scalar tensor theory of gravitation. Authors of [40-41] have explored different aspects of Kaluza-Klein cosmological model in modified theories of gravity. Recently Ugale et al. [42] have investigated cosmological model with matter distribution in Brans-Dicke Theory of Gravitation. Wath and Nimkar [43], Nimkar and Hadole [44] have studied stability of cosmological model with matter distribution in Ruban's background and self-creation theory respectively.

In this paper, we propose to investigate five-dimensional Kaluza-Klein metric in presence of matter distribution within the framework of $f(Q,T)$ gravity by considering two different kinds of volumetric expansion namely power law and exponential expansion law. The paper is structured as follows: Section 1 contains a concise introduction and the motivation behind the current work. The framework of $f(Q,T)$ gravity is given in Section 2. In Section 3, we have derived the metric and field equations. In Section 4, we derived the solution of field equations. The physical and kinematical characteristics of power and exponential expansion law models are explored in Sections 5 and 6. Statefinder diagnostic, energy conditions and stability analysis are examined in Section 7. Conclusion is presented in Section 8.

2. THE FRAMEWORK OF $f(Q,T)$ GRAVITY

The field equations of modified $f(Q,T)$ gravity are derived from Hilbert Einstein variational action principle [13]

$$S = \int \left(\frac{1}{16\pi} f(Q,T) + L_m\right)\sqrt{-g}\,d^4x \tag{1}$$

where $f(Q,T)$ is a function of the non-metricity scalar Q and the trace of the matter-energy-momentum tensor T , also L_m is Lagrangian of the matter and $g = \det(g_{\mu\nu})$. The non-metricity scalar Q is defined as

$$Q = -g^{\mu\nu}\left(L^{\alpha}{}_{\beta\mu}L^{\beta}{}_{\nu\alpha} - L^{\alpha}{}_{\beta\alpha}L^{\beta}{}_{\mu\nu}\right) \tag{2}$$

where $L^{\alpha}{}_{\beta\gamma}$ represents the disformation tensor which is given by,

$$L^{\alpha}{}_{\beta\gamma} = -\frac{1}{2}g^{\alpha\delta}\left(\nabla_{\gamma}g_{\beta\delta} + \nabla_{\beta}g_{\delta\gamma} - \nabla_{\delta}g_{\beta\gamma}\right) \tag{3}$$

The non-metricity tensor is defined as

$$Q_{\gamma\mu\nu} = \nabla_{\gamma}g_{\mu\nu} \tag{4}$$

The trace of the non-metricity tensor is derived as follows

$$Q_{\alpha} = g^{\mu\nu}Q_{\alpha\mu\nu}\,, \tilde{Q}_{\alpha} = g^{\mu\nu}Q_{\mu\alpha\nu} \tag{5}$$

The superpotential or the non-metricity conjugate of the model is given by

$$P^{\alpha}{}_{\mu\nu} = -\frac{1}{2}L^{\alpha}{}_{\mu\nu} + \frac{1}{4}\left(Q^{\alpha} - \tilde{Q}^{\alpha}\right)g_{\mu\nu} - \frac{1}{4}\delta^{\alpha}{}_{(\mu}Q_{\nu)} \tag{6}$$

The energy-momentum tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}} \quad \text{and} \quad \theta_{\mu\nu} = g^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} \tag{7}$$

The field equation of $f(Q,T)$ gravity is obtained by varying the action (1) with respect to the metric tensor $g_{\mu\nu}$ as [13]

$$-\frac{2}{\sqrt{-g}}\nabla_{\alpha}\left(f_Q\sqrt{-g}P^{\alpha}{}_{\mu\nu}\right) - \frac{1}{2}f g_{\mu\nu} + f_T\left(T_{\mu\nu} + \theta_{\mu\nu}\right) - f_Q\left(P_{\mu\alpha\beta}Q_{\nu}{}^{\alpha\beta} - 2Q^{\alpha\beta}{}_{\mu}P_{\alpha\beta\nu}\right) = 8\pi T_{\mu\nu} \tag{8}$$

where $f_Q = \frac{\partial f}{\partial Q}$, $f_T = \frac{\partial f}{\partial T}$ and the non-metricity tensors $Q_{\nu}{}^{\alpha\beta} = -\nabla_{\nu}g^{\alpha\beta}$, $Q^{\alpha\beta}{}_{\mu} = g^{\alpha\gamma}g^{\beta\rho}\nabla_{\gamma}g_{\rho\mu}$

3. THE METRIC AND FIELD EQUATIONS

We have considered the five-dimensional Kaluza-Klein space-time as

$$ds^2 = dt^2 - A^2[dx^2 + dy^2 + dz^2] - B^2 d\psi^2 \tag{9}$$

where the metric potentials A , B are functions of cosmic time t only and the fifth coordinate ψ is taken to be space-like. The energy momentum tensor of matter (Landau and Lifshitz [45]) is given by

$$T_{ij} = (\varepsilon + p)u_i u_j - p g_{ij} \tag{10}$$

where p and ε are pressure and energy density respectively and $u_i = (1, 0, 0, 0, 0)$ is the velocity vector in comoving coordinates which satisfies the condition $u^i u_i = 1$.
From Eq. (10), we get

$$T_1^1 = T_2^2 = T_3^3 = T_4^4 = -p \ , \ T_0^0 = \varepsilon \, , \ T = \ \varepsilon - 4p \tag{11}$$

θ_j^i is derived as

$$\theta_j^i = \delta_j^i p - 2T_j^i = \mathrm{diag}\,(p - 2\varepsilon, 3p, 3p, 3p, 3p) \tag{12}$$

For given cosmological model, the non-metricity scalar Q is found to be

$$Q = -6\left(\frac{\dot{A}^2}{A^2} + \frac{\dot{A}\dot{B}}{AB}\right) \tag{13}$$

The field Eq. (8) for the line element (9) with energy momentum tensor (11), also using Eqs. (12) and (13) can be written as

$$\frac{1}{2}f + 6f_Q\left(\frac{\dot{A}^2}{A^2} + \frac{\dot{A}\dot{B}}{AB}\right) = 8\pi\tilde{G}p - 8\pi\left(1 + \tilde{G}\right)\varepsilon \tag{14}$$

$$\frac{1}{2}f + f_Q\left(2\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + 4\frac{\dot{A}^2}{A^2} + 5\frac{\dot{A}\dot{B}}{AB}\right) + \dot{f}_Q\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) = 8\pi\left(1 + 2\tilde{G}\right)p \tag{15}$$

$$\frac{1}{2}f + 3f_Q\left(\frac{\ddot{A}}{A} + 2\frac{\dot{A}^2}{A^2} + \frac{\dot{A}\dot{B}}{AB}\right) + 3\dot{f}_Q\frac{\dot{A}}{A} = 8\pi(1 + 2\tilde{G})p \tag{16}$$

where $f_T = 8\pi\tilde{G}$ and an overhead dot refers to derivative with respect to time t.
For the given cosmological model, the physical parameters are defined as follows. The spatial volume V is given by

$$V = a^4 = A^3 B \tag{17}$$

where a is the average scale factor.
The generalized mean Hubble's parameter H is expressed as

$$H = \frac{1}{4}\left(H_x + H_y + H_z + H_\psi\right) = \frac{1}{4}\left(3H_x + H_\psi\right) \tag{18}$$

where $H_x = \frac{\dot{A}}{A} = H_y = H_z$ and $H_\psi = \frac{\dot{B}}{B}$ are the directional Hubble parameters in the direction of x, y, z and ψ respectively.
The expansion scalar θ, mean anisotropy parameter A_m, the shear scalar σ^2 are expressed as

$$\theta = 4H \tag{19}$$

$$A_m = \frac{1}{4}\sum_{i=1}^{4}\left(\frac{H_i - H}{H}\right)^2 \tag{20}$$

$$\sigma^2 = \frac{3}{2}H^2 A_m \tag{21}$$

4. SOLUTIONS OF THE FIELD EQUATIONS

There are various forms of the function $f(Q,T)$ which is commonly used in the literature [13]. Here we shall focus on the linear and additive form of $f(Q,T)$ function [14] as $f(Q,T) = \alpha Q + \beta T$, where α and β are arbitrary constants. Here $f_Q = \alpha,\ f_T = \beta = 8\pi\tilde{G}$, $\tilde{G} = \frac{\beta}{8\pi}$.

Thus, the field Eqs. (14) - (16) can be reduced to

$$3\alpha\left(\frac{\dot{A}^2}{A^2} + \frac{\dot{A}\dot{B}}{AB}\right) = 3\beta p - \left(8\pi + \frac{3\beta}{2}\right)\varepsilon \tag{22}$$

$$\alpha\left(2\frac{d}{dt}\left(\frac{\dot{A}}{A}\right) + \frac{d}{dt}\left(\frac{\dot{B}}{B}\right) + 3\frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}\dot{B}}{AB}\right) = (8\pi + 4\beta)p - \frac{\beta}{2}\varepsilon \tag{23}$$

$$3\alpha\left(\frac{d}{dt}\left(\frac{\dot{A}}{A}\right) + 2\frac{\dot{A}^2}{A^2}\right) = (8\pi + 4\beta)p - \frac{\beta}{2}\varepsilon \tag{24}$$

Subtracting Eq. (24) from (23), we get

$$\frac{d}{dt}\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right) + \left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right)\frac{\dot{V}}{V} = 0 \tag{25}$$

which on integration yields

$$\frac{B}{A} = c_2 exp\left[c_1 \int \frac{dt}{V}\right] \tag{26}$$

where c_1 and c_2 are constants of integration.

Using Eq. (17), the scale factors A and B can be written explicitly as

$$A = c_2^{-\frac{1}{4}} V^{\frac{1}{4}} exp\left[-\frac{c_1}{4}\int \frac{dt}{V}\right] \tag{27}$$

$$B = c_2^{\frac{3}{4}} V^{\frac{1}{4}} exp\left[\frac{3c_1}{4}\int \frac{dt}{V}\right] \tag{28}$$

In terms of the directional Hubble parameters, we can express the fields Eqs. (22) – (24) as

$$3\alpha\left(H_x^2 + H_x H_\psi\right) = 3\beta p - \left(8\pi + \frac{3\beta}{2}\right)\varepsilon \tag{29}$$

$$\alpha\left(2\dot{H}_x + \dot{H}_\psi + 3H_x^2 + H_\psi^2 + 2H_x H_\psi\right) = (8\pi + 4\beta)p - \frac{\beta}{2}\varepsilon \tag{30}$$

$$3\alpha\left(2H_x^2 + \dot{H}_x\right) = (8\pi + 4\beta)p - \frac{\beta}{2}\varepsilon \tag{31}$$

Equations (29), (30) provide us the general formulations for the physical parameters of the $f(Q,T)$ model. Pressure and energy density for the model can be explicitly expressed in terms of the directional Hubble parameters as

$$p = \frac{\delta}{(8\pi+4\beta)}\left\{2(3\beta^2 + 32\pi^2 + 22\pi\beta)\dot{H}_x + (10\beta^2 + 128\pi^2 + 84\pi\beta)H_x^2 - \frac{\beta}{2}(8\pi + 4\beta)H_x H_\psi\right\} \tag{32}$$

$$\varepsilon = \delta\left\{3\beta\dot{H}_x - 2(4\pi - \beta)H_x^2 - (8\pi + 4\beta)H_x H_\psi\right\} \tag{33}$$

where

$$\delta = \frac{3\alpha}{\frac{9}{2}\beta^2 + 64\pi^2 + 44\pi\beta}$$

Equation of state is considered to be an important quantity for the description of dynamics of the universe. The equation of state parameter as a whole determines a relation between energy density and pressure. Using the expressions of p and ε , the equation of state parameter can be calculated as

$$\omega = \frac{2\left(3\beta^2+32\pi^2+22\pi\beta\right)\dot{H}_x+\left(10\beta^2+128\pi^2+84\pi\beta\right)H_x^2-\frac{\beta}{2}(8\pi+4\beta)H_x H_\psi}{(8\pi+4\beta)\left\{3\beta\dot{H}_x-2(4\pi-\beta)H_x^2-(8\pi+4\beta)H_x H_\psi\right\}} \tag{34}$$

Further, we have three independent equations (14) to (16) in four unknowns A, B, ε, p. To obtain viable cosmological model with determinate solutions, we require some additional condition. After thoroughly reviewing References [38] and [46], two different volumetric expansions are considered such as power law expansion and exponential law expansion.

5. POWER LAW EXPANSION MODEL

Here, we consider a power law volumetric expansion

$$V = t^{4m} \tag{35}$$

where m is a positive constant. The positivity of the exponent m aligns with empirical observations suggesting an expanding universe. Spatial volume of the universe starts with big-bang initially and with the increase of time, it expands. This demonstrates that universe begins its evolution with zero volume and expands with time t.

The power law expansion of the cosmos predicts a deceleration parameter $q = -1 + \frac{d}{dt}\left(\frac{1}{H}\right) = -1 + \frac{1}{m}$. Maintaining an outlook on the recent observational evidence favouring an accelerating cosmos, we choose to go in support of a negative deceleration parameter. For the current framework, this implies that the value of m should be greater than 1.

The metric potentials A and B can be obtained by using Eq. (35) in Eqs. (27) and (28) as

$$A = {c_2}^{-\frac{1}{4}}\, t^m \exp\left[-\frac{c_1 t^{1-4m}}{4(1-4m)}\right] \tag{36}$$

$$B = {c_2}^{\frac{3}{4}}\, t^m \exp\left[\frac{3c_1 t^{1-4m}}{4(1-4m)}\right] \tag{37}$$

It is seen that initially the scale factors A and B vanishes possessing initial singularity and at $t \to \infty$, both A and B diverge to infinity which aligns perfectly with the Big-Bang model of the universe.

Accordingly, the directional Hubble parameters in respective directions are

$$H_x = H_y = H_z = H - \frac{c_1}{4} t^{-4m} \tag{38}$$

$$H_\psi = H + \frac{3c_1}{4} t^{-4m} \tag{39}$$

The various cosmological parameters such as Hubble parameter H, expansion scalar θ, the shear scalar σ^2 and the mean anisotropic parameter A_m are obtained as

$$H = \frac{m}{t} \tag{40}$$

$$\theta = \frac{4m}{t} \tag{41}$$

$$\sigma^2 = \frac{9}{32} c_1^2 t^{-8m} \tag{42}$$

$$A_m = \frac{3c_1^2 t^{2(1-4m)}}{16m^2} \tag{43}$$

From the expression (40), (41) and (42), it is observed that Hubble parameter H, expansion scalar θ and shear scalar σ^2 are decreasing function of time. They diverge to infinity as $t \to 0$ and becomes zero as $t \to \infty$. The shear scalar diminishes but doesn't entirely disappear, implying that the universe's anisotropic nature gradually weakens, but never attains a perfectly isotropic state. The mean anisotropy parameter A_m is extremely high at $t = 0$ and approaches zero as $t \to \infty$. This rapid decline indicates that the universe is initially anisotropic but evolves towards isotropy at later times.

Pressure p, energy density ε for the model (9) can be obtained from Eqs. (32)-(33) and Eqs. (38)-(39) as

$$p = \frac{\delta}{(8\pi+4\beta)}\left\{\left[\frac{1}{2}(4m-3)\beta^2 + 16(2m-1)\pi^2 + (20m-11)\pi\beta\right]\frac{4m}{t^2} + (\beta^2 + 8\pi^2 + 6\pi\beta)c_1^2 t^{-8m}\right\} \tag{44}$$

$$\varepsilon = \frac{\delta}{8}\left\{(8\pi + 7\beta)c_1^2 t^{-8m} - \left[\beta\left(\frac{3}{2m} + 1\right) + 8\pi\right]\frac{16m^2}{t^2}\right\} \tag{45}$$

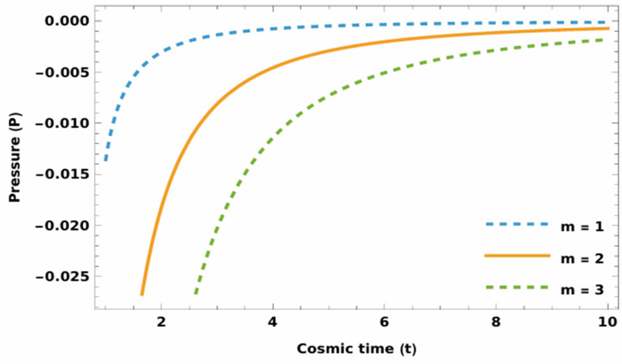

Figure 1. Plot of Pressure vs cosmic time $\boldsymbol{t}$ with $\boldsymbol{\alpha} = \boldsymbol{\beta} = -\mathbf{0.1}, \boldsymbol{c_1} = \mathbf{1}$

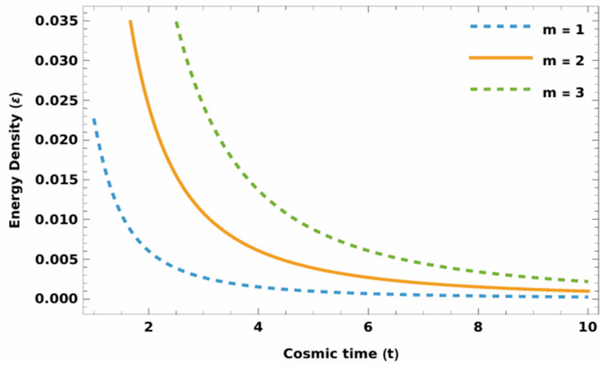

Figure 2. Plot of Energy density vs cosmic time $\boldsymbol{t}$ with $\boldsymbol{\alpha} = \boldsymbol{\beta} = -\mathbf{0.1}, \boldsymbol{c_1} = \mathbf{1}$

Figure 1 depicts the behavior of pressure over cosmic time reflects the dynamic character of dark energy within the universe. Pressure remains negative throughout cosmic evolution, gradually increasing from a highly negative value and vanishes at infinite time. Again, supporting additional evidence for the universe's accelerated expansion. This trend is typical of certain dark energy models, where dark energy's impact diminishes over time, resulting in a more stable universe. A negative pressure is thought to be necessary for generating an anti-gravity effect that drives this acceleration. Figure 2 depicts the energy density for the power law model. Energy density is positive throughout the time evolution and decreases over time. The initial density is extremely large and drops swiftly to a constant value, as parameter m becomes larger. For small values of m, density settles close to zero, signifying lasting steady condition where the effect on the universe's expansion remains constant.
The equation of state parameter for this model is

$$\omega = \left(\frac{2}{2\pi+\beta}\right)\left\{\frac{\left[\frac{1}{2}(4m-3)\beta^2+16(2m-1)\pi^2+(20m-11)\pi\beta\right]\frac{4m}{t^2}+(\beta^2+8\pi^2+6\pi\beta)c_1^2 t^{-8m}}{(8\pi+7\beta)c_1^2 t^{-8m}-\left[\beta\left(\frac{3}{2m}+1\right)+8\pi\right]\frac{16m^2}{t^2}}\right\} \tag{46}$$

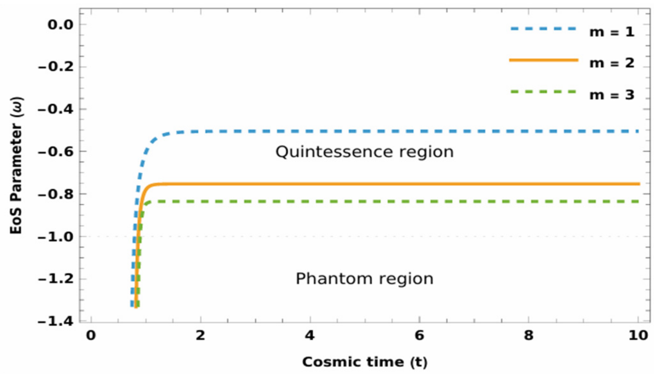

Figure 3. Plot of EoS parameter vs cosmic time $\boldsymbol{t}$ with $\boldsymbol{\alpha} = \boldsymbol{\beta} = -\mathbf{0.1}, \boldsymbol{c_1} = \mathbf{1}$

Figure 3 represents the graphical representation of EoS parameter (ω) and cosmic time. Initially, the universe evolves in the phantom region ($\omega < -1$), crossing the divide line $\omega = -1$, suddenly enters into quintessence region $-1 < \omega < -{}^{1}/_{3}$ and ω remains fixed with advancing time. A similar behavior of EoS parameter has been observed by some researcher [38].

6. EXPONENTIAL EXPANSION MODEL

The exponential expansion of volume factor

$$V = e^{4nt} \tag{47}$$

typically results in a de Sitter kind of universe. Here n is constant. The Hubble parameter is a constant quantity throughout the universe's expansion. For this choice, the directional Hubble parameters will be

$$H_x = H_y = H_z = n - \frac{c_1}{4}e^{-4nt} \tag{48}$$

$$H_\psi = n + \frac{3c_1}{4} e^{-4nt} \tag{49}$$

It's noteworthy that although the mean Hubble parameter is a constant quantity, the directional Hubble values exhibit time dependence. The expansion rate along the conventional axes increases from $n - \frac{c_1}{4}$ in the initial stage to the constant value n at the later stage. Whereas, the expansion rate along the extra dimension diminishes from $n + \frac{3c_1}{4}$ in the initial stage to the constant value n at the later stage. The deceleration parameter for this model is $q = -1$. The obtained model matches the current scenario of accelerating expansion of the universe, as the sign of deceleration parameter is negative.
For this model, the metric potentials A and B using the Eq. (47) are obtained as

$$A = c_2^{-\frac{1}{4}} exp\left[nt + \frac{c_1}{16n} e^{-4nt}\right] \tag{50}$$

$$B = c_2^{\frac{3}{4}} exp\left[nt - \frac{3c_1}{16n} e^{-4nt}\right] \tag{51}$$

Initially the model has no singularity as the metric potentials A and B are constant at $t = 0$.
The various cosmological parameters such as Hubble parameter H, expansion scalar θ, the shear scalar σ^2 and the mean anisotropic parameter A_m are found to be

$$H = n \tag{52}$$

$$\theta = 4n \tag{53}$$

$$\sigma^2 = \frac{9}{32} c_1^2 e^{-8nt} \tag{54}$$

$$A_m = \frac{3c_1^2 e^{-8nt}}{16n^2} \tag{55}$$

From the Eqs. (52) and (53), it is observed that Hubble parameter H, expansion scalar θ are constant throughout the evolution. Shear scalar σ^2 is decreasing function of time t. It can be observed that the shear scalar $\sigma^2 \to 0$ as $t \to \infty$. The ratio of shear scalar to expansion scalar was non-zero initially, but decreased to zero as time progressed, indicating the universe was initially anisotropic and at a later time it approaches isotropy. The mean anisotropic parameter A_m remains fixed in an initial phase i.e. the universe was anisotropic in the early stage and evolves isotropically in later times.
Pressure p, energy density ε for the model (9) can be obtained from Eqs. (32)-(33) and Eqs. (48)-(49) as

$$p = \frac{\delta}{(8\pi+4\beta)} \{8(\beta^2 + 16\pi^2 + 10\pi\beta)n^2 + (\beta^2 + 8\pi^2 + 6\pi\beta)c_1^2 e^{-8nt}\} \tag{56}$$

$$\varepsilon = \delta\left\{-2(8\pi + \beta)n^2 + \frac{1}{8}(8\pi + 7\beta)c_1^2 e^{-8nt}\right\} \tag{57}$$

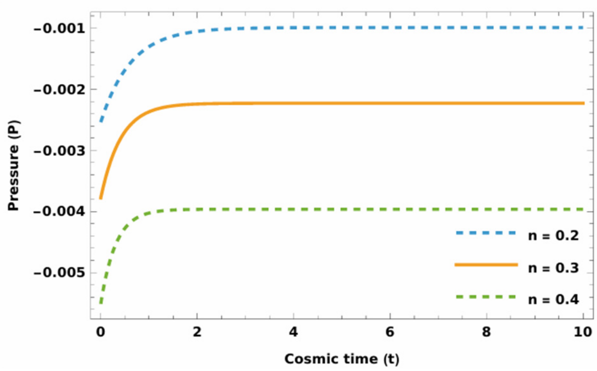

Figure 4. Plot of Pressure vs cosmic time $\boldsymbol{t}$ with $\boldsymbol{\alpha = \beta = -0.1, c_1 = 1}$

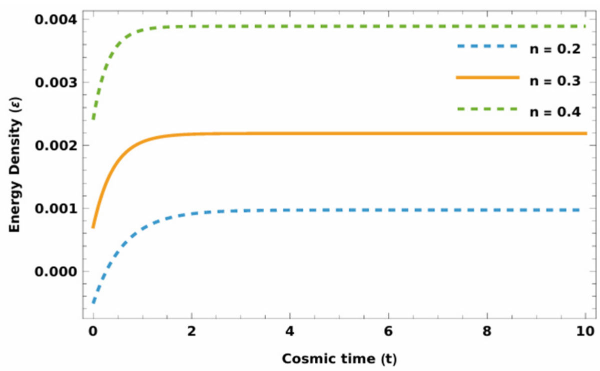

Figure 5. Plot of Energy density vs cosmic time $\boldsymbol{t}$ with $\boldsymbol{\alpha = \beta = -0.1, c_1 = 1}$

Figures 4 and 5 depicted the graphical behavior of pressure and energy density. The pressure is negative throughout the time of evolution which indicates that the matter behaves like dark energy. Pressure and energy density are increasing function of cosmic time. Pressure increases from a large negative value at the initial epoch and approaches to a constant value. Energy density was constant near $t = 0$, then increased slightly to its maximum value and remains constant over time. So, in this model, a steady state for pressure and energy density is rapidly achieved.
The equation of state parameter for this model is found to be

$$\omega = \left(\frac{1}{8\pi+4\beta}\right)\left\{\frac{8(\beta^2+16\pi^2+10\pi\beta)n^2+(\beta^2+8\pi^2+6\pi\beta)c_1^2e^{-8nt}}{-2(8\pi+\beta)n^2+\frac{1}{8}(8\pi+7\beta)c_1^2e^{-8nt}}\right\} \tag{58}$$

The evolution trajectory of EoS parameter (ω) depicted in Fig. 6. It is clear that, ω lies in the phantom region. As time grows, it increases from a high negative value, approaches to -1.02 at late times.

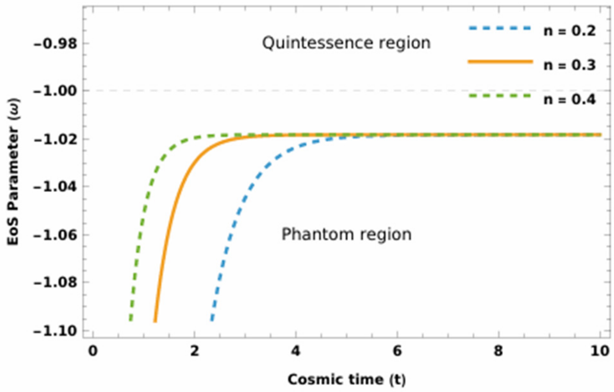

Figure 6. Plot of EoS parameter vs cosmic time $\boldsymbol{t}$ with $\boldsymbol{\alpha = \beta = -0.1, c_1 = 1}$

7. STATEFINDER DIAGNOSTIC, ENERGY CONDITIONS AND STABILITY ANALYSIS

7.1. Statefinder Diagnostic

Statefinder analysis is a cosmological method used to study the dynamics of the universe. It is a key approach to understand various dark energy models. A statefinder pair (r, s) consists of two dimensionless parameters employed in cosmology to distinguish between various cosmological models by investigating the evolution of the cosmic scale factor a and its temporal derivatives. Sahni et al. [47] were the first to introduce these parameters as an extension of the EoS parameter (ω) to analyze the dark energy and the expansion history of the universe. The statefinder parameters are defined as follows:

$$r = \frac{\dddot{a}}{aH^3} = 2q^2 + q - \frac{\dot{q}}{H} \quad \text{and} \quad s = \frac{(r-1)}{3(q-{}^1\!/_2)} \tag{59}$$

The statefinder parameter r measures how much cosmic expansion deviates from a pure-law behavior, while parameter s describes its acceleration. It is a model-dependent approach to distinguish between different dark energy scenarios such as the ^CDM (Lambda Cold Dark Matter), SCDM (Standard Cold Dark Matter), HDE (Holographic Dark Energy), CG (Chaplygin Gas) and quintessence. For corresponding values of r and s parameters, here are some DE scenarios:

- $(r, s) = (1, 0)$ represents ^CDM scenario.
- $(r, s) = (1, 1)$ represents SCDM scenario.
- $(r, s) = (1, {}^2\!/_3)$ represents HDE scenario.
- $r > 1, s < 0$ represents CG scenario.
- $r < 1, s > 0$ represents Quintessence scenario.

In power law model, the relationship between the two statefinder parameter is

$$r = \frac{9s^2-9s+2}{2} \tag{60}$$

The graphical plot of the statefinder parameter r versus s is presented in Fig. 7. Its evolution moves through the dark energy zone of Chaplygin gas and quintessence in the r-s plane and tends to approach the ^CDM model in the upcoming epoch, which is consistence with the findings discussed by Shaikh [40,48].

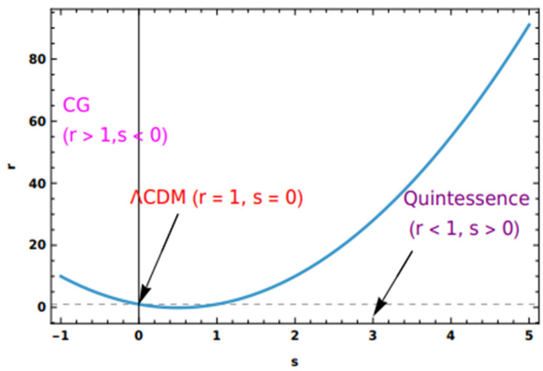

Figure 7. Plot of $\boldsymbol{r}$ versus $\boldsymbol{s}$ in power law model

In exponential model, the statefinder parameters are found as $r = 1$ and $s = 0$. Hence the model behaves like ΛCDM model currently and in the future cosmic evolution which is consistent with the investigations of Katore et al. [38] and Shaikh [40].

7.2. Energy Conditions

The energy conditions are the expressions that describe how energy density and pressure relate linearly. It plays a crucial role in understanding the singularity theorem, describing the nature of geodesics and studying black holes properties. The energy conditions are also intrinsic for comprehending the geodesics of the universe from the cosmological point of view. The energy conditions are constraints on the energy-momentum tensor which are obtained by linking Raychaudhuri equation [49] and the Einstein field equation. There are four widely used fundamental energy conditions. These energy conditions are described in $f(Q,T)$ gravity, specifically in terms of energy density ε and pressure p as follows:

- SEC (Strong Energy condition): $\varepsilon + 3p \geq 0$
- NEC (Null Energy condition): $\varepsilon + p \geq 0$
- WEC (Weak Energy condition): $\varepsilon \geq 0$, $\varepsilon + p \geq 0$
- DEC (Dominant Energy condition): $\varepsilon \geq |p|$, $\varepsilon \geq 0$

Figures 8 and 9 illustrates graphical representation of SEC, NEC and DEC versus cosmic time for power law and exponential law respectively. In power law model, we have observed a consistent fulfillment of WEC($\varepsilon \geq 0$), NEC($\varepsilon + p \geq 0$), DEC($\varepsilon - p \geq 0$) throughtout the cosmic evolution. However SEC is found to be violated in this model. Further, the violation of SEC results in accelerating expansion of the universe. It should be noted that our investigation agrees with the investigation of Koussour et al. [50]. In exponential law model, we observed that NEC and SEC are violated while DEC is satisfied as shown in Fig. 9, which is consistent with the investigations of Koussour [51]. As a result, violation of SEC and NEC causes the universe to accelerate and behave like a phantom model.

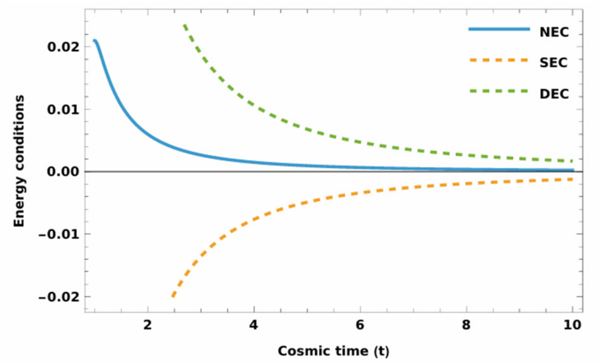

Figure 8. Plot of Energy conditions vs cosmic time $\boldsymbol{t}$ in power law model

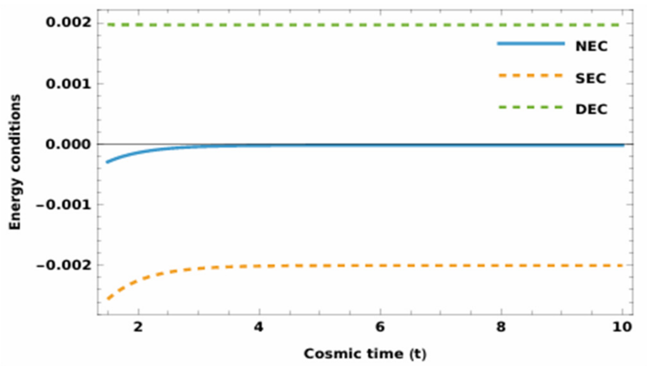

Figure 9. Plot of Energy conditions vs cosmic time $\boldsymbol{t}$ in exponential law model

7.3. Stability Analysis

Depending on the sign of the squared sound speed υ_s^2, it becomes simpler to check the compatibility of dark energy models. The model's stability is examined by the positive sign of υ_s^2, while instability is determined by a

negative υ_s^2. Also, to ensure mechanical stability, υ_s^2 should not exceed 1. Therefore, the region bounded by $0 < \upsilon_s^2 < 1$ gives stable solutions.
The stability parameter of the model is defined as

$$\upsilon_s^2 = \frac{dp}{d\varepsilon} \tag{61}$$

For power expansion law model, the stability parameter υ_s^2 is found to be

$$\upsilon_s^2 = \left(\frac{2}{2\pi+\beta}\right)\left\{\frac{\left[\frac{1}{2}(4m-3)\beta^2+16(2m-1)\pi^2+(20m-11)\pi\beta\right]\frac{1}{t^3}+(\beta^2+8\pi^2+6\pi\beta)c_1^2t^{-8m-1}}{(8\pi+7\beta)c_1^2t^{-8m-1}-\left[\beta\left(\frac{3}{2m}+1\right)+8\pi\right]\frac{4m}{t^3}}\right\} \tag{62}$$

Figure 10 plots the squared sound speed versus cosmic time in power law model. It is observed that in power law model, $\upsilon_s^2 < 0$ for $m \geq 0.5$, hence the model is unstable in the present and late-time phases of cosmic evolution. For $m < 0.5$, the squared sound speed υ_s^2 is negative at the early stages of the cosmic evolution and tends to positive at late-time. Thus for $m < 0.5$, the model exhibits instability in early cosmic evolution and become stable as the universe evolves.

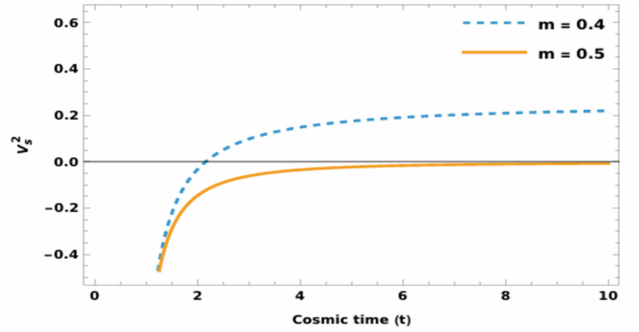

Figure 10. Plot of Squared speed sound vs cosmic time t in power law model

For Exponential expansion law model, the stability parameter υ_s^2 is found to be

$$\upsilon_s^2 = \frac{8(\beta^2+8\pi^2+6\pi\beta)}{(8\pi+4\beta)(8\pi+7\beta)} \tag{63}$$

Thus, the model is stable as $\upsilon_s^2 > 0$.

8. CONCLUSION

In this work, the dynamics of a five-dimensional Kaluza-Klein metric in presence of matter distribution is studied in the framework of $f(Q,T)$ gravity by considering two different kinds of volumetric expansion namely power law and exponential volumetric expansion.
In power law model:

- The metric potentials vanish initially and eventually diverge to infinity as $t \longrightarrow \infty$. Thus, the model consistent with a big-bang model and has an initial singularity.
- Hubble parameter and expansion scalar diverges initially but approaches zero monotonically as time tends to infinity.
- Mean anisotropy parameter decreases with time and approaches to zero as $t \longrightarrow \infty$. This rapid decline indicates that the universe is initially anisotropic but evolves towards isotropy at later times.
- For the value of parameter $m > 1$, the deceleration parameter is negative which shows that universe undergoes accelerated expansion.
- The initial density is extremely large and drops swiftly to a constant value signifying steady condition.
- The dynamics of the universe is studied using equation of state parameter. In this model, initially the universe evolves in the phantom region, crossing the phantom divide line, suddenly enters into quintessence region and ω remains fixed with advancing time.
- From the statefinder diagnosis, we observed that r-s trajectory moves through the dark energy zone of Chaplygin gas and quintessence and tends to approach the ΛCDM model in the upcoming epoch as shown in Fig. 7.

- The model satisfies NEC and DEC but SEC is violated as shown in Fig. 8. The violation of SEC results in accelerating expansion of the universe.
- The model is unstable for $m \geq 0.5$ as shown in Fig. 10. However, for $m < 0.5$, the model is unstable at the early phases of cosmic evolution and is stable at late-time.

In Exponential volumetric expansion law:

- The metric potentials accept constant value at initial time and after which they eventually diverge to infinity. Thus, it is free from any type of singularity.
- Hubble parameter and expansion scalar exhibits constant value throughout the evolution indicating uniform exponential expansion i.e. as time t increases, the universe expands homogeneously from initial epoch to infinity.
- Mean anisotropy parameter shows a constant value at initial epoch, indicating that universe approaches isotropy.
- The deceleration parameter is $q = -1$ which indicates that universe accelerating expansion matches current observations.
- Energy density was constant near $t = 0$ and then increased slightly and remains constant as time increases achieving steady state.
- EoS parameter ω lies in the phantom region. As time progresses, it increases from a high negative value and approaches to value -1.2 at late times.
- From the statefinder diagnosis, we observed that the exponential law model behaves more like a ΛCDM model having statefinder pair $(r, s) = (1,0)$.
- NEC and SEC are violated while DEC is satisfied as shown in Fig. 9.
- The model is stable.

It is important to note that for both models, the pressure is negative throughout the time of evolution gradually increasing from a highly negative value and vanishes at infinite time which indicates that the matter behave like dark energy.

Overall, the proposed model effectively describes cosmic evolution and offers a flexible framework for understanding the late-time acceleration of the universe.

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REFERENCES

[1] A. G. Riess, *et al.*, Astron. J. **116**, 1009 (1998). https://doi.org/10.1086/300499
[2] A. G. Riess, *et al.*, Astrophys. J. **560**, 49 (2001). https://doi.org/10.1086/322348
[3] S. Perlmutter, *et al.,* Astrophys. J. **517**, 565 (1999). https://doi.org/10.1086/307221
[4] C. L. Bennett, *et al.*, Astrophys. J. **583**, 1 (2003). https://doi.org/10.1086/345346
[5] D. N. Spergel, *et al.,* Astrophys. J. Suppl. Ser. **148**, 175 (2003). https://doi.org/10.1086/377226
[6] S. I. Nojiri, S. D. Odintsov and P. V. Tretyakov, Phys. Lett. B **651**, 224 (2007). https://doi.org/10.1016/j.physletb.2007.06.029
[7] H. A. Buchdahl, Mon. Not. R. Astron. Soc. **150**, 1 (1970). https://doi.org/10.1093/mnras/150.1.1
[8] T. Harko, F. S. Lobo, S. I. Nojiri and S. D. Odintsov, Phys. Rev. D **84**, 024020 (2011). https://doi.org/10.1103/PhysRevD.84.024020
[9] Y. F. Cai, S. Capozziello, M. De. Laurentis and E. N. Saridakis, Rep. Prog. Phys. **79**, 106901 (2016). https://doi.org/10.1088/0034-4885/79/10/106901
[10] M. Sharif and A. Ikram, Eur. Phys. J. C **76**, 640 (2016). http://doi.org/10.1140/epjc/s10052-016-4502-1
[11] I. Ayuso, J. Beltrán Jiménez and Á. de la Cruz-Dombriz, Phys. Rev. D **91**, 104003 (2015). https://doi.org/10.1103/PhysRevD.91.104003
[12] J. M. Nester and H. J. Yo, arXiv preprint gr-qc/9809049 (1998). https://doi.org/10.48550/arXiv.gr-qc/9809049
[13] Y. Xu, G. Li, T. Harko and S. D. Liang, Eur. Phys. J. *C* **79**, 708 (2019). https://doi.org/10.1140/epjc/s10052-019-7207-4
[14] Y. Xu, T. Harko, S. Shahidi and S. D. Liang, Eur. Phys. J. C **80**, 449 (2020). https://doi.org/10.1140/epjc/s10052-020-8023-6
[15] V. G. Mete, M. T. Sarode, J. S. Wath, Sch J Phys Math Stat **13***,* 49 (2026). https://doi.org/10.36347/sjpms.2026.v13i01.006
[16] N. Godani and G. C. Samanta, Int. J. Geom. Methods Mod. Phys. **18**, 2150134 (2021). https://doi.org/10.1142/S0219887821501346
[17] R. Zia, D. C. Maurya and A. K. Shukla, Int. J. Geom. Methods Mod. Phys. **18**, 2150051 (2021). https://doi.org/10.1142/S0219887821500511
[18] M. Narzary and M. Dewri, J. Sci. Res. **18**, 53 (2026). https://dx.doi.org/10.3329/jsr.v18i1.81063

[19] S. Arora and P. K. Sahoo, Phys. Scr. **95**, 095003 (2020). https://doi.org/10.1088/1402-4896/abaddc
[20] M. Shiravand, S. Fakhry and M. Farhoudi, Phy. Dark Univ. **37**, 101106 (2022). https://doi.org/10.1016/j.dark.2022.101106
[21] K. El Bourakadi, M. Koussour, G. Otalora, M. Bennai and T. Ouali, Phy. Dark Univ. **41**, 101246 (2023). https://doi.org/10.1016/j.dark.2023.101246
[22] M. Tayde, Z. Hassan and P. K. Sahoo, Nucl. Phys. B **1000**, 116478 (2024). https://doi.org/10.1016/j.nuclphysb.2024.116478
[23] M. Tayde, J. R. Santos, J. N. Araujo and P. K. Sahoo, Eur. Phys. J. Plus **138**, 539 (2023). https://doi.org/10.1140/epjp/s13360-023-04172-1
[24] M. Tayde, S. Ghosh and P. K. Sahoo, Chin. Phys. C **47**, 075102 (2023). https://doi.org/10.1088/1674-1137/acd2b7
[25] R. Bhagat, S. V. Lohakare and B. Mishra, Phy. Dark Univ. **49**, 102048 (2025). https://doi.org/10.1016/j.dark.2025.102048
[26] A. Nájera and A. Fajardo, Phy. Dark Univ. **34**, 100889 (2021). https://doi.org/10.1016/j.dark.2021.100889
[27] S. Bhattacharjee and P. K. Sahoo, Eur. Phys. J. C **80**, 289 (2020). https://doi.org/10.1140/epjc/s10052-020-7844-7
[28] A. S. Agrawal, L. Pati, S. K. Tripathy and B. Mishra, Phy. Dark Univ. **33**, 100863 (2021). https://doi.org/10.1016/j.dark.2021.100863
[29] A. Zhadyranova, M. Koussour and S. Bekkhozhayev, Chin. J. Phys. **89**, 1483 (2024). https://doi.org/10.1016/j.cjph.2024.04.023
[30] M. Narzary and M. Dewri, Edelweiss Appl. Sci. Technol. **9**, 2623 (2025). https://doi.org/10.55214/25768484.v9i5.7526
[31] L. Pati, S. A. Kadam, S. K. Tripathy and B. Mishra, Phy. Dark Univ. **35**, 100925 (2022). https://doi.org/10.1016/j.dark.2021.100925
[32] T. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1921**, 966 (1921). https://doi.org/10.1142/S0218271818700017
[33] O. Klein, Zeitschrift für Physik **37**, 895 (1926). https://doi.org/10.1007/BF01397481
[34] V. G. Mete, V. S. Deshmukh and D. V. Kapse, J. Sci. Res. **16**, 479 (2024). https://doi.org/10.3329/jsr.v16i2.68364
[35] N. I. Jain, Int. J. Math. Phys. **14**, 95 (2023). https://doi.org/10.26577/ijmph.2023.v14.i1.011
[36] P. K. Ray, R. Roy Baruah, East Eur. J. Phys. **2**, 452 (2025). https://doi.org/10.26565/2312-4334-2025-2-56
[37] D. D. Pawar, B. L. Jakore and V. J. Dagwal, Int. J. Geom. Methods Mod. Phys. **20**, 2350079 (2023). https://doi.org/10.1142/S0219887823500792
[38] S. D. Katore, S. P. Hatkar and D. P. Tadas, Astrophys. **66**, 98 (2023). https://doi.org/10.1007/s10511-023-09773-3
[39] S. C. Wankhade, A.S. Nimkar, Eur. Phys. J. C **85**, 769 (2025). https://doi.org/10.1140/epjc/s10052-025-14464-8
[40] A. Y. Shaikh, Bulg. J. Phys. **49**, 190 (2022). https://doi.org/10.55318/bgjp.2022.49.2.190
[41] P. K. Sahoo, B. Mishra and S. K. Tripathy, Indian J. Phys. **90**, 485 (2016). https://doi.org/10.1007/s12648-015-0759-8
[42] M. R. Ugale, H. A. Nimkar, A. O. Dhore, Sch J Phys Math Stat **12**, 357 (2025). https://doi.org/10.36347/sjpms.2025.v12i08.003
[43] J. S. Wath and A. S. Nimkar, Jnanabha **54**, 1 (2024). https://doi.org/10.58250/jnanabha.2024.54101
[44] A. S. Nimkar and S. R. Hadole, J. Sci. Res. **15**, 55 (2023). https://doi.org/10.3329/jsr.v15i1.59676
[45] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, Fourth Revised English Edition (Pergamon Press ,1994).
[46] S. Thakre, R. Mapari and V. Thakare, East Eur. J. Phys. **2**, 21 (2024). https://doi.org/10.26565/2312-4334-2024-2-02
[47] V. Sahni, T. D. Saini, A. A. Starobinsky and U. Alam, JETP Letters **77**, 201 (2003). https://doi.org/10.1134/1.1574831
[48] A. Shaikh, A. Jenekar and S. Shingne, East Eur. J. Phys. **3**, 4 (2025). https://doi.org/10.26565/2312-4334-2025-3-01
[49] A. Raychaudhuri, Phys. Rev. **98**, 1123 (1955). https://doi.org/10.1103/PhysRev.98.1123
[50] M. Koussour, N. Myrzakulov, J. Rayimbaev, A. Errehymy and O. Donmez, Chin. J. Phys. **90**, 108 (2024). https://doi.org/10.1016/j.cjph.2024.04.024
[51] M. Koussour, Chin. J. Phys. **83**, 454 (2023). https://doi.org/10.1016/j.cjph.2023.04.003

КОСМІЧНЕ ПРИСКОРЕННЯ ТА СТАБІЛЬНІСТЬ ВСЕСВІТУ КАЛУЦИ-КЛЕЙНА В f(Q,T) ГРАВІТАЦІЇ

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Гравітація f(Q,T) була використана в цьому дослідженні для вивчення Всесвіту Калуци-Клейна за наявності розподілу матерії. У цій теорії гравітації дія містить довільну функцію f(Q,T), де Q та T відповідно позначають неметричність та слід тензора імпульсу енергії. У цій роботі враховується лінійна та адитивна форма гравітації f(Q,T), $f(Q,T)=\alpha Q+\beta T$, де α та β - довільні константи. Щоб досягти фізично життєздатного розв'язку рівнянь поля, ми розглянули степеневий закон та закон експоненціального розширення. Ми досліджуємо еволюціонуючий характер Всесвіту з фізичними та геометричними властивостями в рамках цієї теоретичної бази. Для кращої ясності досліджуються діагностика пошуку стану та параметр EoS, щоб охарактеризувати різні фази Всесвіту. Енергетичні умови моделей також аналізуються та виявляються такими, що узгоджуються з останніми космологічними спостереженнями. Крім того, одна з наших моделей за своєю суттю допускає фантомну область для темної енергії. Далі моделі, розглянуті в цій роботі, підтверджені за допомогою аналізу стійкості.
Ключові слова: *гравітація f(Q,T); простір-час Калуци-Клейна; рівняння стану; енергетичні умови*