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# VISCOUS STRING COSMOLOGICAL MODEL WITH ELECTROMAGNETIC FIELD IN SAEZ-BALLESTER THEORY

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The framework of Saez-Ballester formalism is employed to study Bianchi Type III metrics, which is homogeneous and anisotropic, wherein the energy-momentum tensor is derived from a bulk viscous fluid containing one-dimensional strings embedded within an electromagnetic field. To obtain exact solutions, we utilized the proportionality relation between the expansion and shear scalars, as well as the average scale factor of a special form. In addition to the energy conditions, several dynamical and physical parameters were calculated, and their physical implications on cosmology were examined.
**Keywords:** *Bianchi type III Metric; Bulk Viscous Fluid; Cosmic Strings; Electromagnetic Field; Saez-Ballester theory*

## 1. INTRODUCTION

There has been increasing evidence over the past few decades that the universe is in accelerating expansion. This accelerating expansion has been confirmed by recent findings [1], [2], [3]. This accelerated expansion is believed to be driven by a substance known as 'Dark Energy. In modern cosmology, the mystery of its origin remains unresolved; nevertheless, it is indirectly viewed as a consequence of 'negative pressure' within the universe. Caldwell and Kamionkowski [4] provided the physics of cosmic acceleration. Silvestri and Trodden [5] presented an approach to understanding cosmic acceleration. Astier and Pain [6] gave observational evidence of the accelerated expansion of the universe. Perivolaropoulos [7] provided accelerating Universe's observational status. However, to be more precise, two different approaches have been proposed to resolve this problem: one involves developing viable 'dark energy' models, and the other involves modifying Einstein's theory of gravity. In this context, Turner and Huterer [8] investigated cosmic acceleration, dark energy and fundamental physics. Dolgov and Kawasaki [9] discussed whether modified gravity explains accelerated cosmic expansion. Nojiri and Odintsov [10] studied modified gravity and its reconstruction from the universe expansion history. Again, Nojiri and Odintsov [11] consider modifying GR theory - termed as the modified gravity approach. Again, Nojiri et al. [12], [13], [14] discussed some recent developments in modified gravity and unified cosmic history in addition to covering a number of common challenges.

The scalar tensor theories put forth by Lyra [15], Sen [16], Brans and Dicke [17], Sen and Dunn [18], Barber [19], Lau and Prokhovnik [20], Saez and Ballester [21] etc., have garnered a lot of attention in the past few decades. In the theory of Saez and Ballester [21], the metric is coupled to a dimensionless scalar field in a remarkably simple manner. This coupling provides a satisfactory description of weak fields; in such fields, despite the dimensionless nature of the scalar field's behaviour, a distinct regime of 'anti-gravity' is observed. This theory proposes a potential solution to the problem of 'missing matter' in non-FRW models. In this context, higher-dimensional Bianchi type-III string cosmological models with dark energy in Saez–Ballester theory is investigated by Dabgar and Bhabor [22]. Chetia et al. [23] studied particle creation and bulk viscosity in Bianchi-I universe in Saez–Ballester theory with different deceleration parameters. Kumawat et al. [24] studied an anisotropic Bianchi type $VI_0$ Spacetime with Barotropic Fluid in Sáez - Ballester Theory of Gravitation. Ram et al. [25] examined Bianchi type-V cosmological models with perfect fluid and heat flow in Saez-Ballester theory. Rao et al. [26] discussed Bianchi type-I and -III modified holographic Ricci Dark energy models in Saez-Ballester theory.

Bulk viscosity is important in cosmology and describes the universe's fast expansion, often known as the Big Bang epoch. Using a bulk-viscous fluid as the energy-momentum tensor (EMT), several cosmologists have investigated the theoretical evolution of the universe as well as the impact of bulk viscosity on cosmic history. Behera et al. [27] studied bulk viscous Bianchi-III models with time-dependent G and Λ. Bali and Pradhan [28] discussed Bianchi type III string cosmological models with time-dependent bulk viscosity. Constraining a bulk viscous Bianchi type I dark energy dominated universe with recent observational data observed by Yadav et al. [29]. Goswami et al. [30] studied modeling of an accelerating universe with bulk viscous fluid in Bianchi V spacetime. Bulk viscous cosmological model with varying deceleration parameter in modified gravity examined by Tiwari et al. [31]. Pawar and Dabre discussed [32] a bulk viscous string cosmological model with power law volumetric expansion in teleparallel gravity. Shukla et al. [33] studied viscous fluid dynamics in Bianchi type-I universe with decaying cosmological term.

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Cosmic strings are one-dimensional topological defects that might have evolved during symmetry-breaking phase transitions in the early Universe. They should be separated from the basic strings found in string theory, since the two ideas have distinct physical origins and meanings. In this paper, the word "string" refers only to cosmology cosmic strings represented by a string-cloud energy-momentum tensor, not the basic strings of string theory. Cosmic strings may have a major role in the dynamics of the early Universe [34], contributing to density perturbations and the development of large-scale structures [35, 36]. Their stress-energy may also affect the anisotropic development of spacetime. Vilenkin and Shellard [37] provide a full study of cosmic strings and associated topological flaws. In this framework, Singh [38] examined a string LRS Bianchi type-I universe with a functional relationship for the Hubble parameter. Rao et al. [39] investigated a Bianchi type-III magnetised string model with a constant deceleration parameter in general relativity. Pawar et al. [40] investigated a plane-symmetric string cosmology model with a zero-mass scalar field in modified gravity theory, while Mete et al. [41] investigated the dynamics of cosmic strings and domain barriers using a specific version of the deceleration parameter.

Cosmic magnetic fields, which are parts of the electromagnetic field, may have a substantial impact on the distribution of energy and the anisotropic development of the Universe because of the existence of ionised matter. Magnetic fields may be amplified by adiabatic compression within astrophysical objects, and the stress-energy of large-scale electromagnetic fields may contribute to spacetime anisotropy. Thus, the electromagnetic energy-momentum tensor is used to reliably integrate the magnetic contribution in the current task. Wang [42] investigated the Bianchi type-III string cosmological model with bulk viscosity and magnetic field. Tripathy et al. [43] studied string cloud cosmologies for Bianchi type-III models with electromagnetic field. Brahma and Dewri [44] discussed the role of $f(R,T)$ gravity in Bianchi Type-V dark energy model with electromagnetic field based on Lyra geometry. Rehman et al. [45] studied electromagnetic field effects on anisotropic cylindrically symmetric compact objects within the framework of $f(R,L_m,T)$ gravity. Five-dimensional Bianchi type-I anisotropic cloud string cosmological model with electromagnetic field in Sáez-Ballester theory, examined by Daimary and Baruah [46]. Ramprasad [47] investigated a study on cosmological model of anisotropic universe with electromagnetic field and cloud strings in the framework of general relativity.

A variety of topics related to anisotropic cosmology, cosmic strings, bulk viscosity, electromagnetic fields, and Saez-Ballester gravity are covered in the works discussed above. However, further research is need to fully understand the simultaneous impact of these components in a Bianchi type-III cosmological scenario. For instance, Dabgar and Bhabor [22] examines a higher-dimensional Bianchi type-III string cosmology with dark energy in Saez–Ballester theory, whereas Chetia et al [23] studies bulk viscosity and particle formation in a Bianchi type-I universe. In contrast, a five-dimensional Bianchi type-I anisotropic cloud-string cosmological model with an electromagnetic field in Saez–Ballester theory is studied by Daimary and Baruah [45]. Motivated by these findings, the current study examines a four-dimensional Bianchi type-III spacetime including a bulk-viscous fluid and one-dimensional cosmic strings contained in an electromagnetic field inside the Saez-Ballester gravitational framework. The goal is to acquire accurate solutions and look into the combined effects of anisotropy, bulk viscosity, string tension, the electromagnetic field, and the Saez-Ballester scalar field on cosmic development. We next examine the kinematical parameters, matter variables, effective pressure, and energy conditions to evaluate the model's physical behaviour.

## 2. METRIC & FIELD EQUATIONS

We consider a homogeneous and anisotropic space-time described by a Bianchi type-III metric as

$$ds^2 = dt^2 - A^2dx^2 - B^2e^{-2mx}dy^2 - C^2dz^2, \tag{1}$$

where *A, B* and *C* are the metric potentials considered as functions of cosmic time *t* only.
The field equations given by Saez and Ballester [21] for the combined scalar $\omega$ and tensor fields are

$$G_i^j - \omega\varphi^n\left(\varphi_{,i}\varphi^{,j} - \frac{1}{2}\delta_i^j\varphi_{,k}\varphi^{,k}\right) = -T_i^j\,, \tag{2}$$

The scalar field $\varphi$ satisfies the equation

$$2\varphi^n\varphi^i_{;i} + n\varphi^{n-1}\varphi_{,k}\varphi^{,k} = 0\,, \tag{3}$$

Where $G_j^i = R_j^i - \frac{1}{2}\delta_j^i R$ is Einstein's equation and *n* are constants, $T_i^j$ is the stress energy-momentum tensor, comma and semicolon represent partial and covariant differentiation, respectively.
A bulk viscous fluid with one-dimensional strings immersed in an electromagnetic field, as an EMT, is calculated as

$$T_\mu^\nu = (\rho + \overline{p})u_\mu u^\nu - \overline{p}g_\mu^\nu - \lambda x_\mu x^\nu + E_\mu^\nu, \tag{4}$$

And

$$\bar{p} = p - 3\xi H, \tag{5}$$

where $\rho = \rho_p + \lambda$ is the proper string energy density with particles attached to them, and $\rho_p$ is the particle energy density, $\lambda$ is the string's tension density, $3\xi H$ is bulk viscous pressure, $\xi(t)$ is the coefficient of bulk viscosity, $H$ is Hubble's parameter, $x^{\nu}$ denotes a unit space-like vector for the cloud string and $u^{\nu}$ denotes a four-velocity vector satisfying the conditions, $u^{\nu}u_{\nu} = 1 = -x^{\nu}x_{\nu}$ and $u_{\nu}x^{\nu} = 0$.
In a co-moving coordinate system, we have

$$u^{\nu} = (0,0,0,1), \;\; x^{\nu} = (0,0,C^{-1},0). \tag{6}$$

Also, $E_{\mu\nu}$ is a part of EMT given as

$$E_{\mu\nu} = \frac{1}{4\pi}\left[F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta} - \frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right], \tag{7}$$

where $F_{\mu\nu}$ is electromagnetic field tensor that satisfies Maxwell equations,

$$F_{[\mu\nu,k]} = 0. \tag{8}$$

In a comoving coordinate system, the incident magnetic field is taken along z-axes with the help of Maxwell equation (8), the components of $F_{\mu\nu}$ except $F_{12}$ vanish, i.e. $F_{12}$ is the only non-vanishing component along z-axes.

$$F_{12} = \text{Constant} = M \;\; (\text{Say})$$

From equation (7), we have

$$E_1^1 = E_2^2 = \frac{M^2}{8\pi A^4} \;\; \text{and} \;\; E_3^3 = E_4^4 = -\frac{M^2}{8\pi A^4}. \tag{9}$$

Now, the field equations (2) & (3) from metric (1) using (4), we get,

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} - \frac{\omega}{2}\varphi^n\dot{\varphi}^2 = -(p - 3\xi H) + \frac{M^2}{8\pi A^4}, \tag{10}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}}{A}\frac{\dot{C}}{C} - \frac{\omega}{2}\varphi^n\dot{\varphi}^2 = -(p - 3\xi H) + \frac{M^2}{8\pi A^4}, \tag{11}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} - \frac{m^2}{A^2} - \frac{\omega}{2}\varphi^n\dot{\varphi}^2 = -(p - 3\xi H) + \lambda - \frac{M^2}{8\pi A^4}, \tag{12}$$

$$\frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} + \frac{\dot{A}}{A}\frac{\dot{C}}{C} - \frac{m^2}{A^2} + \frac{\omega}{2}\varphi^n\dot{\varphi}^2 = \rho - \frac{M^2}{8\pi A^4}, \tag{13}$$

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = 0, \tag{14}$$

$$\ddot{\varphi} + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\dot{\varphi} + \frac{n}{2}\frac{\dot{\varphi}^2}{\varphi} = 0, \tag{15}$$

where the overhead dot ( $\cdot$ ) denotes the derivative with respect to cosmic time $t$. By solving (14), one can get $A = kB$, where $k$ is an integrating constant. Without loss of generality, we take $k = 1$. Then (10) - (13) reduces to

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} - \frac{\omega}{2}\varphi^n\dot{\varphi}^2 = -(p - 3\xi H) + \frac{M^2}{8\pi B^4}, \tag{16}$$

$$2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} - \frac{m^2}{A^2} - \frac{\omega}{2}\varphi^n\dot{\varphi}^2 = -(p - 3\xi H) + \lambda - \frac{M^2}{8\pi B^4}, \tag{17}$$

$$\frac{\dot{B}^2}{B^2} + 2\frac{\dot{B}}{B}\frac{\dot{C}}{C} - \frac{m^2}{A^2} + \frac{\omega}{2}\varphi^n\dot{\varphi}^2 = \rho - \frac{M^2}{8\pi B^4}, \tag{18}$$

Here we have three non-linear differential field equations with seven unknowns, namely $B$, $C$, $p$, $\xi$, $\lambda$, $\rho$ and $\varphi$. The solution of these unknowns is discussed in the next section.

Now, we find some kinematical space-time quantities of physical interest in cosmology, as follows:

The average scale factor $a$ and the spatial volume $V$ are respectively defined as

$$a = \sqrt[3]{B^2C}, \text{ and } V = a^3. \tag{19}$$

The volumetric expansion rate of the universe is described by the generalised mean Hubble's parameter $H$ & given by

$$H = \frac{1}{3}\sum_{i=1}^{3} H_i = \frac{1}{3}\left(H_1 + H_2 + H_3\right), \tag{20}$$

in which $H_1 = H_2 = \frac{\dot{B}}{B}$, and $H_3 = \frac{\dot{C}}{C}$ denotes the directional Hubble's parameters.

Using (19) and (20), we have obtained the expansion scalar $\theta$, mean anisotropy parameter $\Delta$, shear scalar $\sigma$, and deceleration parameter $q$, respectively, as

$$\theta = 3H, \tag{21}$$

$$\Delta = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2, \tag{22}$$

$$\sigma^2 = \frac{1}{2}\left(\sum_{i=1}^{3} H_i^2 - \theta^2\right), \tag{23}$$

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -1 + \frac{d}{dt}\left(\frac{1}{H}\right). \tag{24}$$

## 3. SOLUTION OF FIELD EQUATIONS

The three nonlinear differential field equations (16) - (18) and seven unknowns are solved by taking into account the following physically possible possibilities.
Initially, we took the mean scale factor's spatial form as

$$a = \left(ct + d\right)^{\frac{1}{m}}, \tag{25}$$

where, $c$, $d$ and $m$ are non-negative constants.

Next, we argued that the shear scalar is proportional to the expansion scalar, providing the analytical relation.

$$B = C^{\zeta}, \tag{26}$$

Using (19), (25), & (26), we obtained

$$A = B = \left(ct + d\right)^{\frac{3\zeta}{m(1+2\zeta)}} \text{ and } C = \left(ct + d\right)^{\frac{3}{m(1+2\zeta)}}. \tag{27}$$

From (27) in (1), we have

$$ds^2 = dt^2 - \left(ct + d\right)^{\frac{6\zeta}{m(1+2\zeta)}} dx^2 - \left(ct + d\right)^{\frac{6\zeta}{m(1+2\zeta)}} e^{-2mx} dy^2 - \left(ct + d\right)^{\frac{6}{m(1+2\zeta)}} dz^2. \tag{28}$$

The model generated in expression (27) is devoid of any form of singularity until infinite cosmic time.
The Volume

$$V = a^3 = \left(ct + d\right)^{\frac{3}{m}}. \tag{29}$$

It is evident from expression (29) that this formulation is in power law form, which establishes that volume grows with time.
The Hubble parameter & expansion scalar

$$H = \frac{1}{3}\sum_{i=1}^{3} H_i = \frac{c}{m\left(ct + d\right)}, \tag{30}$$

$$\theta = 3H = \frac{3c}{m(ct+d)}. \tag{31}$$

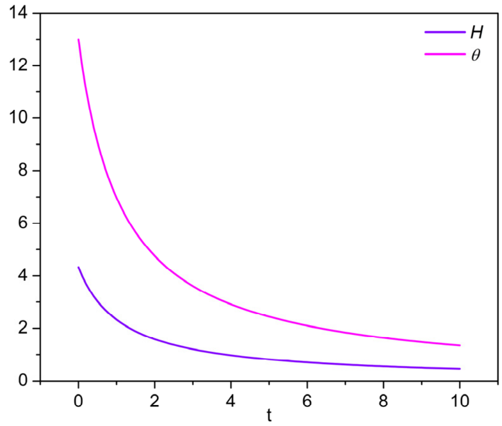


**Figure 1.** Plot of the expansion scalar & Hubble parameter vs. time for $c = 1.3,\ d = 1.5,\ m = 0.2$.

Figure 1 shows a diminishing graphical representation of the Hubble parameter and expansion scalar over time, implying that the universe's expansion rate is slowing.

The mean anisotropy parameter & shear scalar is

$$\Delta = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 = \frac{2(\zeta-1)^2}{(1+2\zeta)^2}. \tag{32}$$

$$\sigma^2 = \frac{1}{2}\left(\sum_{i=1}^{3} H_i^2 - \theta^2\right) = \frac{3(\zeta-1)^2 c^2}{(1+2\zeta)^2 m^2 (ct+d)^2}. \tag{33}$$

Expressions (32) and (33) show that, with the exception of $\varsigma = 1$, the universe never approaches isotropy and maintains shear.

The deceleration parameter (DP)

$$q = -1 + \frac{d}{dt}\left(\frac{1}{H}\right) = m - 1. \tag{34}$$

The model's acceleration is indicated by the sign of $q$. A decelerating phase of the universe is indicated by a positive sign of DP $q$ for $m > 1$, an acceleration phase is indicated by $-1 \le q < 0$, and a stable rate of evolution is indicated by $q = 0$. The accelerated phase of the cosmos is supported by recent observations [48], [49], [50]. Additionally, for the selected choice of constants, i.e. $m = 0.2$, the value of DP is $q = -0.8$, showing accelerated expansion.

From (15), the scalar field can be obtained as

$$\varphi = \left(\frac{m(2+n)(ct+d)(ct+d)^{-\frac{3}{m}} + 2c(m-3)c_1}{2c(m-3)}\right)^{\frac{2}{2+n}}. \tag{35}$$

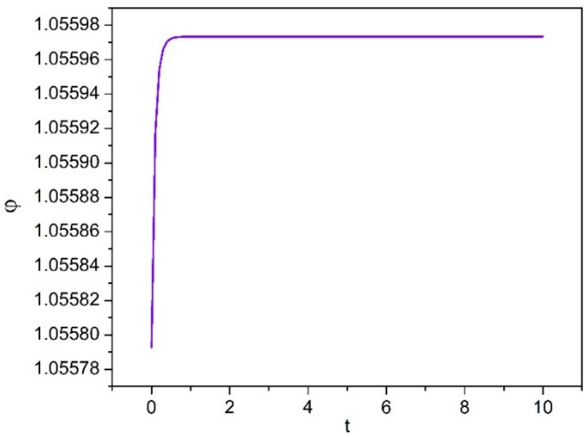


**Figure 2.** Plot of the scalar field vs. time for $c = 1.3,\ d = 1.5,\ m = 0.2,\ n = 1.5,\ c_1 = 1.1$.

As seen in Figure 2, the scalar field graph first climbs rapidly throughout the range of $t = 0$ to $0.4$ up to $\varphi = 1.055973$, and it remains at this constant value for the rest of the evolution.
From (18), the energy density is

$$\rho = \frac{M^2 (ct+d)^{-\frac{12\zeta}{m(1+2\zeta)}} m^2 (1+2\zeta)^2 (ct+d)^2 - 4\pi \begin{bmatrix} 2m^4 (ct+d)^{-\frac{6\zeta}{m(1+2\zeta)}} (1+2\zeta)^2 (ct+d)^2 \\ -\omega (ct+d)^{-\frac{6}{m}} m^2 (1+2\zeta)^2 (ct+d)^2 - 18\zeta c^2 (\zeta+2) \end{bmatrix}}{8(1+2\zeta)^2 (ct+d)^2 m^2 \pi}. \quad (36)$$

Solving (16) & (17), the tension density is

$$\lambda = \frac{(1+2\zeta) M^2 (ct+d)^{-\frac{12\zeta}{m(1+2\zeta)}} m^2 (ct+d)^2 - 4\pi \left[ m^4 (1+2\zeta)(ct+d)^2 (ct+d)^{-\frac{6\zeta}{m(1+2\zeta)}} + 3c^2 (\zeta-1)(m-3) \right]}{4m^2 \pi (1+2\zeta)(ct+d)^2}. \quad (37)$$

The particle density is

$$\rho_p = \frac{\begin{array}{c} 4\pi \left\{ m^2 \omega (1+2\zeta)^2 (ct+d)^{\frac{2m-6}{m}} + 6c^2 \left[ (2\zeta^2 - \zeta - 1)m - 3(\zeta^2 + 3\zeta + 1) \right] \right\} \\ -m^2 (1+2\zeta)^2 M^2 (ct+d)^{\frac{(2+4\zeta)m - 12\zeta}{m(1+2\zeta)}} \end{array}}{8m^2 \pi (1+2\zeta)^2 (ct+d)^2}. \quad (38)$$

Letelier [51] examined the prospect that strings would vanish during the universe's history, leaving only particles, and noted that the string tension density could be either positive or negative. The physical behaviour of particles, tension, and energy density is viewed in the context of cosmic time, which is illustrated in Figure 3; this figure demonstrates that as cosmic time progresses, both of these densities decrease. In our model, the condition λ > 0 during the evolution of the universe signifies the existence of strings; however, particles ultimately dominate the universe, as the contribution of strings is negligible compared to that of particles.

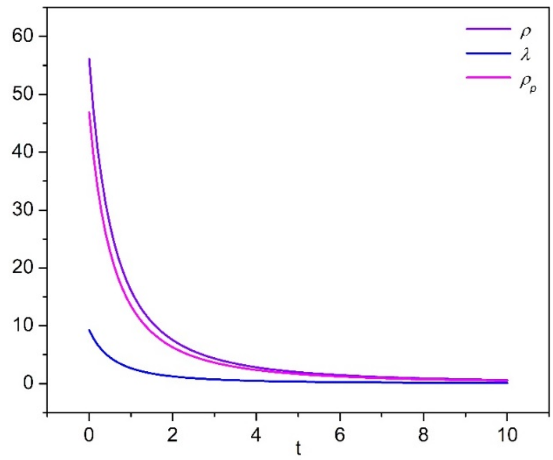


**Figure 3.** Plot of the energy, tension & particle density vs. time for $M = 2,\ \zeta = 1.2,\ c = 1.3,\ d = 1.5,\ m = 0.2,\ \omega = 0.1$.

We assume that the coefficient of viscosity should vary with the expansion scalar in such a way that

$$\xi\theta = \xi_0 = \text{Constant}. \quad (39)$$

Therefore, we obtain

$$\xi = \frac{m\xi_0 (ct+d)}{3c}. \quad (40)$$

The bulk viscosity coefficient is a strictly rising function of cosmic time, as seen in Figure 4.

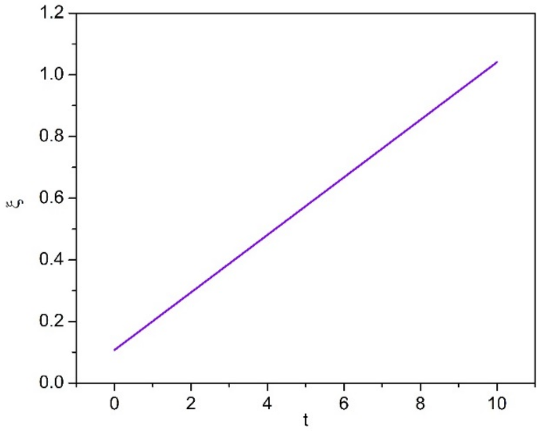


**Figure 4.** Plot of the viscosity coefficient vs. time for $c = 1.3,\ d = 1.5,\ m = 0.2,\ \xi_0 = 1.4.$

From (16), the pressure is

$$p = \frac{M^2 (ct+d)^{-\frac{12\zeta}{m(1+2\zeta)}} m^2 (1+2\zeta)^2 (ct+d)^2 + 4\pi \left[ \begin{array}{l} \omega (ct+d)^{-\frac{6}{m}} m^2 (1+2\zeta)^2 (ct+d)^2 \\ +2(1+2\zeta)^2 (ct+d)^2 \xi_0 m^2 \\ +6(1+2\zeta)(\zeta+1)c^2 m - 18c^2 (\zeta^2+\zeta+1) \end{array} \right]}{8m^2\pi (1+2\zeta)^2 (ct+d)^2}. \tag{41}$$

From (5), the effective pressure is

$$\bar{p} = \frac{m^2 (1+2\zeta)^2 M^2 (ct+d)^{\frac{(2+4\zeta)m-12\zeta}{m(1+2\zeta)}} + 4\pi \left\{ m^2 \omega (1+2\zeta)^2 (ct+d)^{\frac{2m-6}{m}} + 6c^2 \left[ \left(2\zeta^2+3\zeta+1\right)m - 3\left(\zeta^2-\zeta-1\right) \right] \right\}}{8m^2\pi (1+2\zeta)^2 (ct+d)^2}. \tag{42}$$

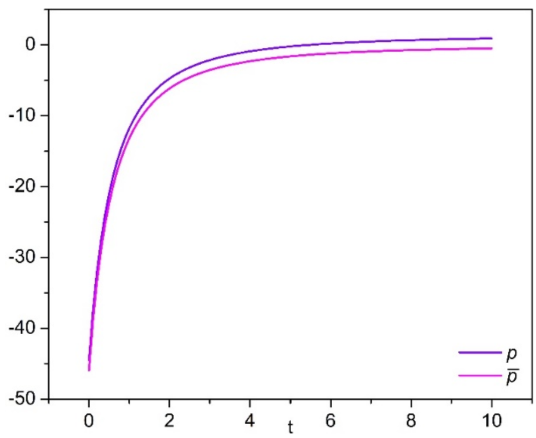


**Figure 5.** Plot of the pressure & effective pressure vs. time for $M = 2,\ \zeta = 1.2,\ c = 1.3,\ d = 1.5,\ m = 0.2,\ \omega = 0.1,\ \xi_0 = 1.4.$

A graph illustrating pressure and effective pressure is presented in Figure 5. It is evident from Figure 5 that both these parameters commence with negative values and increase over the course of cosmic time; however, in subsequent epochs, the cosmic pressure becomes slightly positive, whereas the effective pressure remains negative. It is a well-established fact that negative cosmic pressure serves as an indicator of cosmic acceleration.

***The Energy Conditions***

The energy conditions are respectively defined as [52]

i. Null Energy Condition (NEC) : $\rho + p \geq 0$

ii. Weak Energy Condition (WEC) : $\rho \geq 0,\ \rho + p \geq 0$

iii. Strong Energy Condition (SEC) : $\rho + 3p \geq 0$

iv. Dominant Energy Condition (DEC) : $\rho \geq 0,\ \rho \pm p \geq 0$

From (32) and (37), we can express the conditions as

$$\rho+p=\frac{m^2(1+2\zeta)^2M^2(ct+d)^{\frac{(2+4\zeta)m-12\zeta}{m(1+2\zeta)}}+4\pi\left[\begin{array}{l}m^2\omega(1+2\zeta)^2(ct+d)^{\frac{2m-6}{m}}-(ct+d)^{\frac{(2+4\zeta)m-6\zeta}{m(1+2\zeta)}}(1+2\zeta)^2m^4\\+(1+2\zeta)^2(ct+d)^2\xi_0m^2+3(1+2\zeta)(\zeta+1)c^2m+9c^2(\zeta-1)\end{array}\right]}{4m^2\pi(1+2\zeta)^2(ct+d)^2},\tag{43}$$

$$\rho+3p=\frac{4m^2(1+2\zeta)^2M^2(ct+d)^{\frac{(2+4\zeta)m-12\zeta}{m(1+2\zeta)}}-8\pi\left\{\begin{array}{l}(1+2\zeta)^2\left[\begin{array}{l}(ct+d)^{\frac{(2+4\zeta)m-6\zeta}{m(1+2\zeta)}}m^4\\-2m^2\omega(ct+d)^{\frac{2m-6}{m}}-3(ct+d)^2\xi_0m^2\end{array}\right]\\-9c^2\left[2(m-1)\zeta^2+(3m-1)\zeta+m-3\right]\end{array}\right\}}{8m^2\pi(1+2\zeta)^2(ct+d)^2},\tag{44}$$

$$\rho-p=-\left\{\frac{(1+2\zeta)\left[(ct+d)^{\frac{(2+4\zeta)m-6\zeta}{m(1+2\zeta)}}m^4+(ct+d)^2\xi_0m^2\right]+3c^2(\zeta+1)[m-3]}{(1+2\zeta)m^2(ct+d)^2}\right\}.\tag{45}$$

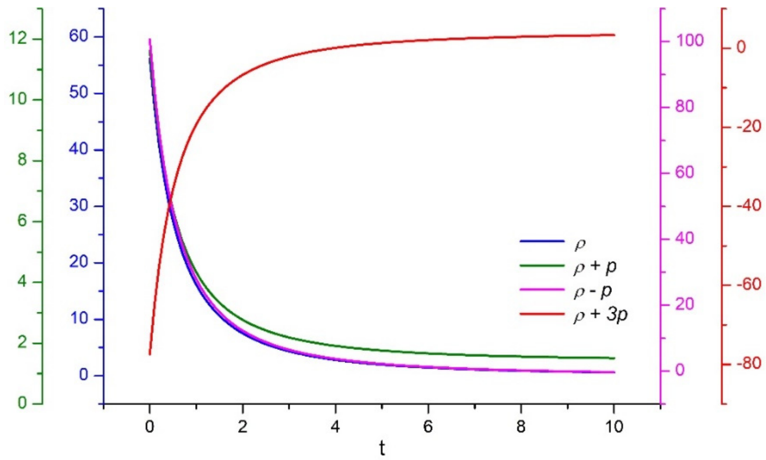


**Figure 6.** Plot of the energy conditions vs. time for $M=2,\ \zeta=1.2,\ c=1.3,\ d=1.5,\ m=0.2,\ \omega=0.1,\ \xi_0=1.4$.

A graph of the energy conditions is presented in Figure 6. The figure reveals that, during the evolution of the universe, since only $\rho+3p<0$, all energy conditions with the exception of the SEC are satisfied. The SEC has significant implications for cosmology. Negative pressure is sufficient to violate the SEC; that is, a violation of the SEC accelerates cosmic expansion, and therefore, it supports the inflation and dark energy models. Furthermore, a violation of the SEC implies that this model can avoid a singularity. These findings are fully consistent with [53], [54].

## 4. CONCLUSIONS

Within the framework of Saez-Ballester theory, in this research, the authors have investigated a homogeneous and anisotropic Bianchi Type III metric, wherein the EMT consists of a viscous fluid coupled with a string of clouds embedded in an electromagnetic field. To obtain exact solutions to the field equations, a specific form of the average scale factor and a shear-expansion scalar relationship are utilized. Some kinematic and physical parameters were calculated in addition to energy conditions, and their physical implications in cosmology were examined.

The created model behaves like a conventional cosmological model that is free of singularities until infinite duration, is accelerating, expanding with a decreasing rate of expansion over time, and anisotropic with holding shear. The scalar field of the model increases rapidly from $t=0$ to 0.4, reaching a value of $\varphi=1.055973$, and remains constant for the remainder of its evolution. Both the energy, tension, and particle density are positive and exhibit a decreasing trend. In our model, we have detected the presence of strings; however, particles ultimately dominate the universe, as the contribution of strings is negligible compared to that of particles. The bulk viscosity coefficient is a strictly increasing function of cosmic time. The values for both pressure and effective pressure are negative and increase with cosmic time; however, in subsequent epochs, only the cosmic pressure becomes slightly positive. Furthermore, except for the SEC, all energy conditions are satisfied. The SEC has significant implications in cosmology. Negative pressure is sufficient to violate the SEC; that is, a violation of the SEC accelerates cosmic expansion and, consequently, lends support to inflationary and dark energy models. Moreover, a violation of the SEC suggests that this model may be able to avoid a singularity. The findings obtained in this research are fully consistent with recent observational data.

Compared to the previous research presented in the Introduction, the current analysis contains Bianchi type-III anisotropy, a bulk-viscous cosmic-string fluid, an electromagnetic field, and the Saez-Ballester scalar degree of freedom. The resultant model shows rapid expansion for the chosen model parameters, whereas the Hubble parameter and expansion scalar decrease with cosmic time. The matter, string-tension, and particle densities decline throughout evolution, whereas the effective pressure stays negative throughout the relevant era. The model remains anisotropic except for the exceptional isotropic limit. These characteristics define the cosmic behaviour of the matter-gravity system under consideration. The Saez-Ballester theory introduces a dimensionless scalar field that is connected to the gravitational sector. In this study, it is viewed as an effective gravitational field and is not associated with the Higgs field or any empirically discovered basic scalar particle. Thus, its function in the model is phenomenological under the Saez-Ballester gravitational framework.

Future research might expand on the current work by investigating the model's dynamical stability, cosmic disturbances, other forms of the bulk-viscosity coefficient, and the impact of various electromagnetic-field configurations. It would also be interesting to restrict the free model parameters using empirical datasets and look into the approach to the isotropic limit at late cosmic periods.

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**КОСМОЛОГІЧНА МОДЕЛЬ В'ЯЗКИХ СТРУН З ЕЛЕКТРОМАГНІТНИМ ПОЛЕМ У ТЕОРІЇ САЕЗА-БАЛЛЕСТЕРА**

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Для вивчення метрик Б'янкі III типу, які є однорідними та анізотропними, де тензор енергії-імпульсу виводиться з об'ємної в'язкої рідини, що містить одновимірні струни, вбудовані в електромагнітне поле. Для отримання точних розв'язків ми використовували співвідношення пропорційності між скалярами розширення та зсуву, а також середній масштабний коефіцієнт спеціальної форми. Окрім енергетичних умов, було розраховано кілька динамічних та фізичних параметрів, а також досліджено їхній фізичний вплив на космологію.

**Ключові слова:** *метрика Біанкі III типу; об'ємна в'язка рідина; космічні струни; електромагнітне поле; Теорія Саеза-Баллестера*