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QUANTUM INFORMATION AND ENERGY SPECTRUM OF THE ULTRA GENERALIZED EXPONENTIAL–HYPERBOLIC POTENTIAL VIA THE NIKIFOROV–UVAROV METHOD

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This study presents a comprehensive investigation of quantum information measures for the one-dimensional Ultra Generalized Exponential–Hyperbolic Potential (UGEHP) within the framework of the Schrödinger equation. Analytical solutions for the energy eigenvalues and corresponding wave functions are obtained using the Nikiforov–Uvarov method. These solutions serve as the basis for evaluating Shannon entropy and Fisher information in both position and momentum spaces. The numerical results show that the total Shannon entropy satisfies the Beckner–Białynicki-Birula–Mycielski (BBM) inequality, while the Fisher information adheres to the Stam–Cramér–Rao bounds, thereby confirming consistency with fundamental quantum uncertainty principles. In addition, the energy spectrum exhibits a pronounced dependence on the screening parameter and quantum numbers, indicating the tunable confinement properties of the potential. The combined analysis establishes a clear connection between energy quantization, wavefunction localization, and information-theoretic measures, offering deeper insight into the effectiveness of the UGEHP model in describing complex quantum systems.
Keywords: *Probability density; Information-theoretic measures; Ultra Generalized Exponential–Hyperbolic Potential; Energy spectrum; Bound-state solutions*

INTRODUCTION

In recent years, quantum information theory has attracted considerable interest due to its deep impact on quantum computing, quantum communication, and fundamental physics. Among the different methods used to investigate quantum information measures, potential models have emerged as effective tools for gaining insights into the behavior of quantum systems [1,2]. Quantum information measures (QIM), including Shannon entropy, Fisher information, Rényi entropy, and Tsallis entropy, are essential for examining the localization and delocalization characteristics of wavefunctions (WF) in confined quantum systems [3,4]. These measures provide valuable insights into quantum uncertainty, correlations, and entanglement, which are essential for applications in quantum computing and information processing. While these techniques have been extensively applied to different potential models [5-7], studies specifically focusing on the Ultra Generalized Exponential–Hyperbolic Potential (UGEHP) remain limited. Due to its flexibility and tunability, the UGEHP serves as an ideal model for exploring complex quantum systems under external constraints. This potential encompasses a broad class of models, generalizing well-known potentials such as the Morse, Pöschl-Teller, and Eckart potentials [8]. It has been particularly useful in describing diatomic molecular interactions [9,10] and quarkonium systems [11,12] in both relativistic and non-relativistic frameworks. The functional form of the UGEHP includes multiple adjustable parameters, allowing for a detailed investigation of quantum information measures in relation to system confinement and external influences. Notably, this potential has emerged as an effective model for examining the interplay between quantum information measures and potential-driven quantum dynamics. This study aims to systematically analyze the information-theoretic properties and bound state energy associated with the UGEHP, providing new insights into its role in Quantum information measures.

The potential is expressed in the form [9].

$$V\left(r_o\right)=\frac{A_o e^{-4\alpha_o r_o}+B_o e^{-2\alpha_o r_o}}{r_o^{\,2}}+\frac{C_0 e^{-2\alpha_o r_o}-d_o\cosh\left(\delta_o\alpha_o r_o\right)e^{-\alpha_o r_o}+G_0\cos ech\ \alpha_o r_o\ e^{-\alpha_o r_o}}{r_o}+J_o, \tag{1}$$

where α_o is the screening parameter. The parameters $A_o, B_o, C_o, d_0, J_0, G_0, \delta_0$ are potential strengths.
when $\delta_o=1$

$$\cosh\alpha_o r_o=\frac{e^{\alpha_o r_o}+e^{-\alpha_o r_o}}{2},\cos ech\alpha_o r_o=\frac{2}{e^{\alpha_o r_o}-e^{-\alpha_o r_o}}. \tag{2}$$

The dependence of the potential on distance is illustrated in Figure 1.

Recent advancements in quantum information theory have highlighted the crucial role of entropic measures in understanding the uncertainty and information content of quantum states [13]. Shannon entropy serves as a global

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indicator of the uncertainty and randomness associated with electron localization and delocalization [14]. In contrast, Fisher information measures the sensitivity of a WF to external perturbations, making it essential in quantum metrology and estimation theory [15].

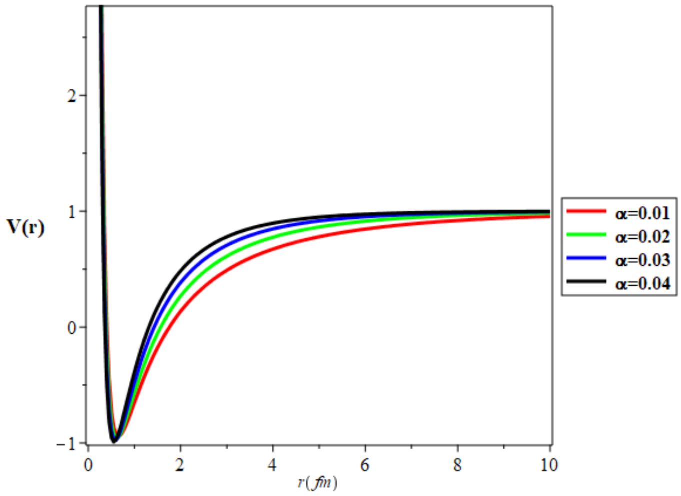

Figure 1. Variation of the Ultra Generalized Exponential–Hyperbolic Potential with position for different screening parameters and the arbitrary parameters $A_0 = 0.1, J_0 = -0.5, B_0 = 2, G_0 = 0.1, d_0 = 4, \delta_0 = 1, C_0 = 4$

In recent years, Shannon entropy has garnered considerable attention among quantum physics researchers due to its broad applications in modern quantum communication systems [16,17]. Its significance lies in its function as a generalized form of Heisenberg's uncertainty principle, providing a quantitative measure of uncertainty in quantum systems while also characterizing various physical properties [17]. Beckner-Białynicki-Birula and Mycielski (BBM) [18] introduced a novel formulation of Heisenberg's uncertainty principle in quantum mechanics, expressed as:

$$S_T = S_{r_o} + S_p \geq D\left(1 + 2\ln\pi\right), \tag{3}$$

In this context, D signifies the spatial dimension, while S_{r_o} *and* S_p represent the Shannon entropies in coordinate and momentum spaces, respectively. The total Shannon entropy, denoted as S_T, is the sum of these two components. The expressions for S_{r_o} *and* S_p are provided in references [19–21].

$$\begin{aligned} S_{r_o} &= -\int \left|\Psi\left(r_o, \theta_o, \varphi_o\right)\right|^2 \ln\left|\Psi\left(r_o, \theta_o, \varphi_o\right)\right|^2 d^3 r_o, \\ S_p &= -\int \left|\varphi_o\left(p\right)\right|^2 \ln\left|\varphi_0\left(p\right)\right|^2 d^3 p, \end{aligned} \tag{4}$$

where $d^3 r_o = r_o{}^2 dr d\Omega, d^3 p = p^2 dp d\Omega, and\ d\Omega = \sin\theta_o d\theta_o d\varphi$ is the solid angle with $\Psi\left(r_o, \theta_o, \varphi\right)$ being the normalized wave function in the special coordinate. Also, $\rho\left(r_o\right) = \left|\Psi\left(r_o, \theta_0, \varphi\right)\right|^2$ is the probability density in the spatial co-ordinate, $\rho\left(p\right) = \left|\phi\left(p\right)\right|^2$ is the probability density in the momentum space and $\phi\left(p\right)$ is the associated normalized Fourier transform [21]. It should be noted that the entropy measures evaluated in this work correspond solely to the radial component of the wavefunction associated with the Ultra Generalized Exponential–Hyperbolic Potential. In the general three-dimensional framework, the total Shannon entropy comprises both radial and angular contributions. However, since the present study focuses on the radial solutions of the Schrödinger equation and the influence of the potential parameters on the radial probability density, only the radial contribution has been considered. Therefore, the reported entropy values represent radial Shannon entropies rather than the complete three-dimensional Shannon entropy [22]. The global measure is important because it enables researchers to analyze the uncertainty associated with the probability distribution. In contrast, Fisher information quantifies the efficiency of a measurement procedure in estimating quantum limits [23]. This information-theoretic tool primarily captures local variations in probability density [24]. Fisher information plays a crucial role in density functional theory [25], statistical estimation, and molecular chemical reactivity [26]. It is also strongly correlated with physical properties such as magnetic susceptibility and kinetic energy [27], providing valuable insights into electron correlation and atomic structure in both coordinate and momentum spaces [28].

As it is written,

$$\left.\begin{aligned} I_{r_0} &= \int \frac{|\rho'(r_0)|^2}{\rho(r_0)} dr_0 = 4\int |\psi'(r_0)|^2 \, dr_0 = 4\langle p^2 \rangle - 2(2l+1)|m|\langle r_0^{-2} \rangle \\ I_p &= \int \frac{|\rho'(p)|^2}{\rho(p)} dp = 4\int |\psi'(p)|^2 \, dp = 4\langle r_0^2 \rangle - 2(2l+1)|m|\langle p^{-2} \rangle \end{aligned}\right\}. \tag{5}$$

Unlike, the Shannon entropy that satisfied the BBM inequality, the Fisher information fulfill the Stam inequalities $I_{r_o} \le 4\left\langle r_o^2 \right\rangle; I_p \le 4\left\langle p^2 \right\rangle$ [29], and the Cramer–Rao inequalities $I_{r_o} \ge \frac{9}{\left\langle r_o^2 \right\rangle}; I_p \ge \frac{9}{\left\langle p^2 \right\rangle}$ [30]. Generally, for an arbitrary angular momentum quantum number of any central potential model, the two products of the Fisher information must satisfy the relation [30]

$$I_{r_o} I_p \ge 4\left\langle r_o^2 \right\rangle \left\langle p^2 \right\rangle \left[2 - \frac{2l+1}{l(l+1)}|m| \right]^2, \tag{6}$$

Extensive studies have explored Shannon entropy and Fisher information using well-motivated potential models. Udoh et al. [31] applied the Nikiforov–Uvarov functional analysis (NUFA) approach to solve the radial Schrödinger equation (SE) for the shifted Morse potential, deriving analytical energy eigenvalues and eigenfunctions, and analyzing quantum information measures. Nath and Roy [32] obtained exact solutions to the non-central Makarov potential and studied information-theoretic parameters for low-lying states. Inyang et al. [33] investigated the QIM with the screened modified Kratzer-Yukawa potential using the Nikiforov-Uvarov (NU) method, while Ikot et al. [34] applied NUFA to the Möbius square potential, both analyzing Shannon entropy and Fisher information. Onate et al. [35] employed supersymmetric techniques for the Morse potential to evaluate QIM and thermodynamic properties. Bloch and Cohen [36] linked Fisher information to an enhanced uncertainty principle. Omugbe et al. [37] used supersymmetric WKB for the QIM assessments, Boumali and Labidi [38] studied the Dirac oscillator under energy-dependent potentials, and Onate et al. [39] solved the SE for the Eckart-Manning-Rosen potential. In this work, the SE, an essential tool in quantum mechanics, is solved using the NU method [40] for the UGEHP. The derived energy equation and WF will serve as the foundation for analyzing QIM, including Shannon entropy, Fisher information, and bound-state energy. To the best of our knowledge, this has not been presented in the literature.

2. THE SOLUTION OF THE SCHRÖDINGER EQUATION WITH THE UGEHP

In this study, the NU method is employed to solve the SE, with detailed formulations available in Reference [40]. The radial Schrödinger equation for a spherically symmetric potential is given by [41]

$$\frac{d^2R(r_o)}{dr_o^2} + \left[\frac{2\mu}{\hbar^2}\left(E_{nl} - V(r_o)\right) - \frac{l(l+1)}{r_o^2} \right] R(r_o) = 0, \tag{7}$$

where E_{nl} is the eigenvalue, $R(r_o)$ is the eigenfunction, $\hbar$ is the reduced Planck's constant, r_o is the radial distance and μ is the reduced mass.

To address the centrifugal barrier in Equation (7), we employ the Greene-Aldrich approximation scheme (GAAS) [42]. This method provides an effective approximation for the centrifugal term, ensuring accuracy within a specific validity range $\alpha_o << 1$, . Under these conditions, the centrifugal term transforms into the following expression:

$$\frac{1}{r_o^2} \approx \frac{4\alpha_o^2}{\left(1 - e^{-\alpha_o r_o}\right)^2} \Rightarrow \frac{1}{r_o} \approx \frac{2\alpha_o}{\left(1 - e^{-2\alpha_o r_o}\right)}. \tag{8}$$

Substituting Eqs. (1), (2), and (8) into Eq. (7) and introducing an appropriate coordinate transformation $q_o = e^{-2\alpha_o r_o}$ gives

$$\frac{d^2\psi(q_o)}{dq_o^2} + \frac{1-q_o}{q_o(1-q_o)}\frac{d\psi(q_o)}{dq_o} + \frac{1}{\left[q_o(1-q_o)\right]^2}\begin{bmatrix} -\left(\varepsilon^2 + \alpha_{o1}\right)q_o^2 + \\ \left(2\varepsilon^2 - \alpha_{o2}\right)q_o - \left(\varepsilon^2 - \alpha_{o3}\right) \end{bmatrix}\psi(q_o) = 0, \tag{9}$$

where:

$$\left.\begin{aligned}
-\varepsilon^2 &= \frac{\mu}{2{\alpha_o}^2\hbar^2}\left(E_{nl}-J_o\right),\\
\alpha_{o1} &= \frac{2\mu}{4{\alpha_o}^2\hbar^2}\left(4{\alpha_o}^2A_o-2\alpha_oC_0+\alpha_od_o\right),\\
\alpha_{o2} &= \frac{2\mu}{4{\alpha_o}^2\hbar^2}\left(4{\alpha_o}^2B_o+2\alpha_oC_o\right),\\
\alpha_{o3} &= \frac{2\mu}{4{\alpha_o}^2\hbar^2}\left(4\alpha_oG_o-\alpha_od_o\right)+l\left(l+1\right)
\end{aligned}\right\}. \tag{10}$$

For this research, the NU method is employed, as outlined in Ref. [40]. This approach yields the eigenvalue and eigenfunction equations, which are expressed as follows:

$$E_{nl} = -\frac{2{\alpha_o}^2\hbar^2}{\mu}\left[\frac{\frac{\mu}{2{\alpha_o}^2\hbar^2}\left(4\alpha_oG_o+2\alpha_oC_o-4{\alpha_o}^2A_o-2\alpha_od_o+\frac{2{\alpha_o}^2\hbar^2}{\mu}l\left(l+1\right)\right)}{2\left(n+\chi_0\right)}+\frac{\left(n+\chi_0\right)}{2}\right]^2 \\ +4\alpha_oG_o-\alpha_od_o+\frac{2{\alpha_o}^2\hbar^2}{\mu}\left[l\left(l+1\right)\right]+J_o \tag{11}$$

where:

$$\chi_0 = \frac{1}{2}+\sqrt{\frac{2\mu}{\hbar^2}\left(A_o+B_0+\frac{G_0}{\alpha_0}\right)+\frac{1}{4}+l\left(l+1\right)} \tag{12}$$

The WF is expressed in terms of the Jacobi polynomial as follows:

$$\psi_{nl}\left(q_o\right) = \mathrm{N}_{nl}\, {q_o}^{\sqrt{\varepsilon^2+\alpha_{o3}}}\left(1-q_o\right)^{\left(\frac{1}{2}+\frac{1}{2}\sqrt{\alpha_{o1}+\alpha_{o2}+\alpha_{o3}+\frac{1}{4}}\right)} P_n^{\left(2\sqrt{\varepsilon^2+\alpha_{o3}},\,2\sqrt{\alpha_{o1}+\alpha_{o2}+\alpha_{o3}+\frac{1}{4}}\right)}\left(1-2q_o\right), \tag{13}$$

where N_{nl} represents the normalization constant, which can be determined by applying the normalization condition:

$$\int_0^\infty R_{nl}(r_o)R_{nl}^*(r_o)dr_o = 1 \tag{14}$$

3. RESULTS AND DISCUSSION

The wave functions and energy eigenvalues of the Schrödinger equation with the Ultra Generalized Exponential–Hyperbolic Potential (UGEHP) are obtained analytically using the Nikiforov–Uvarov method. Based on these solutions, Shannon entropy and Fisher information are evaluated in both position and momentum spaces to investigate the information-theoretic properties of the system.

Tables 1 and 2 present the calculated Shannon entropies and Fisher information, respectively, for the ground state at different values of the screening potential parameter (SPP), ranging from 0.01 to 0.10. The results in Table 1 show that the total Shannon entropy satisfies the Beckner–Białynicki-Birula–Mycielski (BBM) inequality, which establishes a fundamental lower bound on the sum of the position- and momentum-space entropies. This confirms that the uncertainty associated with the quantum system is accurately captured. Similarly, the Fisher information values reported in Table 2 satisfy the Stam–Cramér–Rao (SCR) inequality, demonstrating consistency with the fundamental limits of statistical estimation and measurement precision in quantum mechanics. These findings validate the theoretical framework employed and further emphasize the intimate connection between quantum uncertainty and information-theoretic measures.

Table 3 presents the variation of the energy eigenvalues of the UGEHP with the principal quantum number (n), angular momentum quantum number (l), and screening parameter (α_o). The results display the characteristic behavior expected for quantum systems governed by short-range interaction potentials. For a fixed value of α_o, the energy eigenvalues increase with increasing (n), corresponding to transitions from lower to higher excited states. As the particle occupies larger spatial regions in these excited states, it becomes less tightly bound within the potential well, resulting in a reduction in the magnitude of the binding energy.

For a given value of (n), the energy eigenvalues also increase with increasing angular momentum quantum number (l). This behavior arises from the centrifugal contribution to the effective potential, which introduces a repulsive term proportional to $(l(l+1)/r^2)$. As l increases, the particle is pushed away from the central region of the potential, leading to reduced localization and higher energy levels. Consequently, the degeneracy commonly associated with ideal central potentials is lifted, reflecting the non-Coulombic nature of the UGEHP.

The influence of the screening parameter (α_o) is equally significant. As α_o increases, the energy eigenvalues shift toward higher values, indicating weaker binding of the system. Physically, stronger screening reduces the effective range of the interaction potential and weakens its confining capability, thereby producing less localized quantum states. Conversely, smaller values of α_o correspond to weaker screening and stronger confinement, resulting in lower energy eigenvalues and enhanced localization. These results demonstrate that the UGEHP provides a highly tunable framework for controlling the spectral properties of quantum systems. Furthermore, the dependence of the energy spectrum on n, l, and α_o directly influences wave-function localization and consequently affects the behavior of Shannon entropy and Fisher information, establishing a clear relationship between spectral characteristics and quantum information measures.

To further illustrate the localization properties of the system, Figure 2 depicts the ground-state wave function in position space for various values of the screening parameter. It is observed that decreasing the screening parameter leads to a more sharply peaked wave function that shifts toward shorter distances, indicating stronger localization and enhanced confinement of the particle. In contrast, increasing the screening parameter broadens the wave function, reflecting weaker confinement and greater spatial delocalization.

A similar trend is observed in Figure 3, which shows the corresponding probability density distributions. As the screening parameter decreases, the probability density becomes more concentrated and exhibits a higher peak, indicating an increased likelihood of finding the particle within a localized spatial region. Conversely, increasing the screening parameter results in a broader probability distribution, consistent with reduced confinement. These graphical results support the trends observed in the energy spectrum and information-theoretic quantities, further confirming the strong influence of the screening parameter on the localization and quantum behavior of the system.

Table 1. Shannon entropies for the ground state for the Ultra Generalized Exponential–Hyperbolic Potential

α_o	S_{r_o}	S_p	$S_T \geq 2.14473$
0.01	2.457340	0.123150	2.58049
0.02	2.218196	0.190375	2.40857
0.03	2.107611	0.244707	2.35232
0.04	2.047531	0.288308	2.33584
0.05	2.014564	0.322102	2.33667
0.06	1.998876	0.338815	2.33769
0.07	1.995648	0.343301	2.33895
0.08	2.002399	0.338093	2.34049
0.09	2.017946	0.324438	2.34238
0.10	2.041994	0.302725	2.34472

Table 2. Fisher Information entropies for the ground state for the Ultra Generalized Exponential–Hyperbolic Potential

α_o	I_{r_o}	I_p	$I_{r_o} I_p \geq 4$
0.01	0.552196	8.333156	4.601897
0.02	0.574371	8.347331	4.794750
0.03	0.591231	8.355159	4.939883
0.04	0.670146	8.391088	5.623137
0.05	0.682951	8.404442	5.739414
0.06	0.699872	8.417453	5.891126
0.07	0.700436	8.426681	5.902263
0.08	0.712074	8.432596	6.004092
0.09	0.735457	8.467918	6.227006
0.10	0.740081	8.601521	6.365387

Table 3 Energy eigenvalues (in eV) of the Ultra Generalized Exponential–Hyperbolic Potential with the parameters $\{A_0 = 0.1, J_0 = -0.5, B_0 = 2, G_0 = 0.1, d_0 = 4, \hbar = \mu = 1, C_0 = 4\}$ for various values of α_o = 0.01, 0.02, 0.03 and 0.04

n	l	α_o = 0.01	α_o = 0.02	α_o = 0.03	α_o = 0.04
0	0	−0.540123	−0.580658	−0.621715	−0.663342
1	0	−0.540528	−0.582224	−0.625141	−0.669298
	1	−0.540524	−0.582195	−0.625054	−0.669112
2	0	−0.541104	−0.584345	−0.629689	−0.677129
	1	−0.541107	−0.584363	−0.629744	−0.677243
	2	−0.541112	−0.584394	−0.629831	−0.677419

n	l	$\alpha_o = 0.01$	$\alpha_o = 0.02$	$\alpha_o = 0.03$	$\alpha_o = 0.04$
3	0	−0.541820	−0.586942	−0.635240	−0.686681
	1	−0.541833	−0.587025	−0.635479	−0.687169
	2	−0.541855	−0.587166	−0.635866	−0.687941
	3	−0.541883	−0.587332	−0.636303	−0.688790
4	0	−0.542659	−0.589981	−0.641744	−0.697894
	1	−0.542684	−0.590140	−0.642191	−0.698798
	2	−0.542729	−0.590413	−0.642928	−0.700253
	3	−0.542787	−0.590739	−0.643778	−0.701894
	4	−0.542850	−0.591076	−0.644630	−0.703506
5	0	−0.543614	−0.593444	−0.649180	−0.710742
	1	−0.543653	−0.593687	−0.649850	−0.712086
	2	−0.543724	−0.594105	−0.650967	−0.714278
	3	−0.543815	−0.594613	−0.652278	−0.716796
	4	−0.543916	−0.595146	−0.653615	−0.719316
	5	−0.544021	−0.595667	−0.654892	−0.721692
6	0	−0.544678	−0.597324	−0.657534	−0.725211
	1	−0.544733	−0.597655	−0.658436	−0.727010
	2	−0.544832	−0.598228	−0.659955	−0.729976
	3	−0.544960	−0.598932	−0.661761	−0.733428
	4	−0.545105	−0.599681	−0.663628	−0.736937
	5	−0.545255	−0.600421	−0.665435	−0.740292
	6	−0.545403	−0.601127	−0.667130	−0.743410

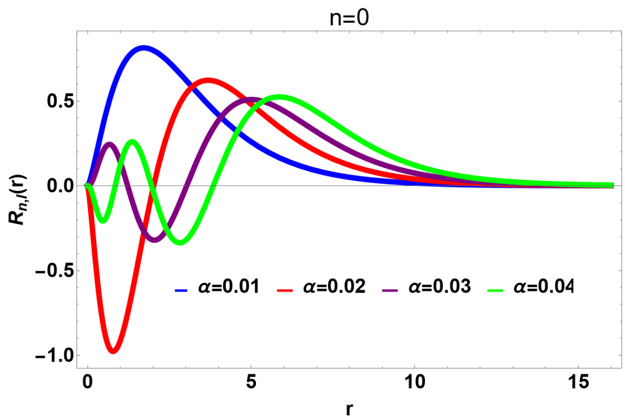

Figure 2. The ground state wave function in position space is analyzed for different values of the screening potential parameter

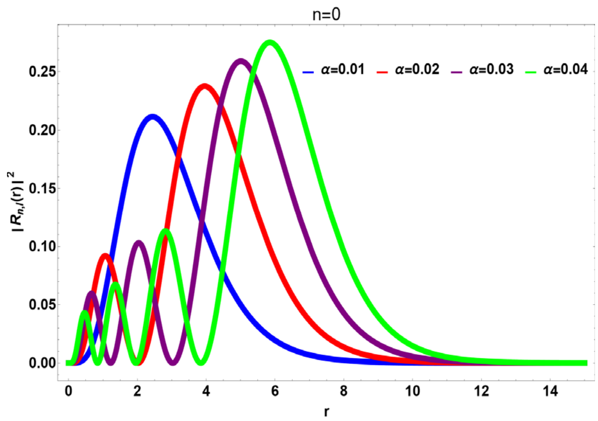

Figure 3. The probability density of the ground state in position space is examined for different values of the screening potential parameter

4. CONCLUSIONS

In this work, the Schrödinger equation for the Ultra Generalized Exponential–Hyperbolic Potential has been solved using the Nikiforov–Uvarov method, yielding an expression for the energy eigenvalues and corresponding wave functions. The resulting energy spectrum demonstrates a clear dependence on the principal and angular momentum quantum numbers, as well as on the screening parameter, confirming the tunable nature of the potential in controlling quantum confinement. The analysis of Shannon entropy and Fisher information in both position and momentum spaces reveals that the system satisfies the BBM and Stam–Cramér–Rao inequalities, thereby reinforcing the consistency of the results with fundamental quantum mechanical uncertainty relations. Moreover, the interplay between energy quantization and information-theoretic measures highlights the connection between spectral properties and wavefunction localization. Overall, the findings establish the UGEHP as a robust and versatile model for investigating quantum information measures in bound systems, with potential applications in molecular physics and quantum information science.

Availability of Data and Materials

All data presented in this study are derived from the analytical equations provided within the article.

Competing Interests

The authors declare that they have no competing interests.

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Authors' Contributions
E. P. Inyang: Conceptualization, resources, investigation, and funding acquisition. **J. E. Osang:** Validation and computation.
S. E. Mopta: Writing—review and editing.
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APPENDIX A: Review of Nikiforov-Uvarov (NU) method

The NU method was proposed by Nikiforov and Uvarov [40] to transform Schrödinger-like equations into a second-order differential equation via a coordinate transformation $x_c = x_c(r)$, of the form

$$\psi''(x_c)+\frac{\tilde{\tau}(x_c)}{\sigma(x_c)}\psi'(s)+\frac{\tilde{\sigma}(x_c)}{\sigma^2(x_c)}\psi(x_c)=0 \tag{A1}$$

where $\tilde{\sigma}(x_c)$, and $\sigma(x_c)$ are polynomials, at most second degree and $\tilde{\tau}(x_c)$ is a first-degree polynomial. The exact solution of Eq.(A1) can be obtain by using the transformation.

$$\psi(x_c)=\phi(x_c)y(x_c) \tag{A2}$$

This transformation reduces Eq.(A1) into a hypergeometric-type equation of the form

$$\sigma(x_c)y''(x_c)+\tau(x_c)y'(x_c)+\lambda y(x_c)=0 \tag{A3}$$

The function $\phi(x_c)$ can be defined as the logarithm derivative

$$\frac{\phi'(x_c)}{\phi(x_c)} = \frac{\pi(x_c)}{\sigma(x_c)} \tag{A4}$$

With $\pi(x_c)$ being at most a first-degree polynomial. The second part of $\psi(x_c)$ being $y(x_c)$ in Eq. (A2) is the hypergeometric function with its polynomial solution given by Rodrigues relation as

$$y(x_c) = \frac{B_{nl}}{\rho(x_c)} \frac{d^n}{dx_c{}^n}\left[\sigma^n(x_c)\rho(x_c)\right] \tag{A5}$$

where B_{nl} is the normalization constant and $\rho(x_c)$ the weight function which satisfies the condition below;

$$\left(\sigma(x_c)\rho(x_c)\right)' = \tau(x_c)\rho(x_c) \tag{A6}$$

where also

$$\tau(x_c) = \tilde{\tau}(x_c) + 2\pi(x_c) \tag{A7}$$

For bound solutions, it is required that

$$\tau^{/}(x_c) < 0 \tag{A8}$$

The eigenfunctions and eigenvalues can be obtained using the definition of the following function $\pi(s)$ and parameter λ, respectively:

$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \tilde{\tau}(s)}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)} \tag{A9}$$

and

$$\lambda = k_- + \pi'_-(x_c) \tag{A10}$$

The value of k can be obtained by setting the discriminant in the square root in Eq. (A9) equal to zero. As such, the new eigenvalues equation can be given as

$$\lambda + n\tau^{'}(x_c) + \frac{n(n-1)}{2}\sigma^{''}(x_c) = 0, (n = 0,1,2,...) \tag{A11}$$

КВАНТОВИЙ ІНФОРМАЦІЙНИЙ ТА ЕНЕРГЕТИЧНИЙ СПЕКТР УЛЬТРАУЗАГАЛЬНЕНОГО ЕКСПОНЕНЦІАЛЬНО-ГІПЕРБОЛІЧНОГО ПОТЕНЦІАЛУ ЗА ДОПОМОГОЮ МЕТОДУ НІКІФОРОВА-УВАРОВА

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Це комплексне дослідження квантових інформаційних мір для одновимірного ультраузагальненого експоненціально-гіперболічного потенціалу (УУГГП) в рамках рівняння Шредінгера. Аналітичні розв'язки для власних значень енергії та відповідних хвильових функцій отримані за допомогою методу Нікіфорова-Уварова. Ці розв'язки слугують основою для оцінки ентропії Шеннона та інформації Фішера як у просторі положення, так і в просторі імпульсу. Числові результати показують, що повна ентропія Шеннона задовольняє нерівність Бекнера-Бялиницького-Бірули-Міцельського (BBM), тоді як інформація Фішера дотримується меж Стама-Крамера-Рао, що підтверджує узгодженість з фундаментальними принципами квантової невизначеності. Крім того, енергетичний спектр демонструє виражену залежність від параметра екранування та квантових чисел, що вказує на властивості налаштованого обмеження потенціалу. Комбінований аналіз встановлює чіткий зв'язок між квантуванням енергії, локалізацією хвильової функції та інформаційно-теоретичними заходами, пропонуючи глибше розуміння ефективності моделі UGEHP в описі складних квантових систем.

Ключові слова: *щільність ймовірності; інформаційно-теоретичні заходи; ультраузагальнений експоненціально-гіперболічний потенціал; енергетичний спектр; розв'язки зв'язаних станів*