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A PEN-PICTURE OF THE MODIFIED (2+1)-DIMENSIONAL KdV-CALOGERO BOGOYAVLENSKII-SCHIFF EQUATION IN SHALLOW WATER WAVES: LIE SYMMETRY ANALYSIS AND CONSERVATION LAWS

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This work investigates the KdV-Calogero Bogoyavlenkskii-Schiff equation in modified (2+1) dimensions. The Lie symmetries are created using Lie group analysis, and the similarity solutions are then found using the symmetries. Using the multiple exp-function approach, numerous wave solutions are obtained. Furthermore, we use the multiplier approach to express the conserved currents, which is essential for comprehending the nature of non-linear equations, particularly in engineering and physics.

Keywords: *KdV-Calogero Bogoyavlenkskii-Schiff Equation; Conserved currents; Lie group analysis; Multiple exp-function method*

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1. INTRODUCTION

In domains such as oceanography, superfluids, hydrodynamics, plasma physics, optical systems, and other relevant nonlinear science [1–8] topics, nonlinear evolution equations (NLEEs) exhibit a variety of fascinating nonlinear dynamic phenomena. The physical insights offered by the solutions to NLEEs can help us better understand the underlying phenomena and possibly pave the way for novel applications. Therefore, it is essential to look for precise answers to these equations. But finding precise answers for NLEEs is frequently difficult, and their answers can only be simplified in rare circumstances. Numerous techniques have been used to address NLEEs [9–25].

The illustrious Korteweg-de Vries (KdV) equation

$$\omega_\tau + 6\omega\omega_\chi + \omega_{\chi\chi\chi} = 0 \tag{1.1}$$

is an example of a nonlinear evolution equation (NLEE) [26–44]. It describes the dynamics of solitary waves. It was first developed to characterize long-wavelength, small-amplitude waves in shallow water. Its multiple-soliton solutions, unlimited number of conservation laws, and numerous other physical characteristics make it an important equation in the study of integrable systems.

We study one such NLEE namely a modified (2+1)-dimensional KdV-Calogero Bogoyavlenkskii-Schiff (KdV-CBS) equation [45, 46]:

$$4\omega_\tau - h_1\left(4\omega\omega_\mu + 2\omega_\chi\partial_\chi^{-1}\omega_\mu + \omega_{\chi\chi\mu}\right) - h_2\left(6\omega\omega_\chi + \omega_{\chi\chi\chi}\right) = 0, \tag{1.2}$$

where h_1 and h_2 are arbitrary constant. It should be noted that when $h_1 \neq 0$, $h_2 = 0$, equation (1.2) becomes to Calogero Bogoyavlenkskii-Schiff equation [47].

$$4\omega_\tau - h_1\left(4\omega\omega_\mu + 2\omega_\chi\partial_\chi^{-1}\omega_\mu + \omega_{\chi\chi\mu}\right) = 0. \tag{1.3}$$

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Furthermore, when $h_1 = 0$ and $h_2 \neq 0$, equation (1.2) is reduced to the well known KdV equation

$$4\omega_\tau - h_2\left(6\omega\omega_\chi + \omega_{\chi\chi\chi}\right) = 0. \tag{1.4}$$

The transformation $\omega = \nu_\chi$, casts(1.2) into

$$4\nu_{\tau\chi} - h_1\left(4\nu_\chi\nu_{\chi\mu} + 2\nu_\mu\nu_{\chi\chi} + \nu_{\chi\chi\chi\mu}\right) - h_2\left(6\nu_\chi\nu_{\chi\chi} + \nu_{\chi\chi\chi\chi}\right) = 0. \tag{1.5}$$

There are five sections in this study. The calculation of exact solutions and similarity reductions using Lie point symmetry analysis is shown in Section 2. In Section 3, we use the multiple exp-function method, an extension of Hirota's perturbation technique, to investigate a number of physically relevant waves with novel general wave frequencies and phase shifts. The multiplier approach is used to discuss the conservation laws in Section 4. Lastly, the closing remarks are given in Section 5.

2. LIE SYMMETRIES AND REDUCTION (1.5)

The Norwegian mathematician Sophus Lie (1842–1899) created the Lie symmetry method, a potent mathematical instrument for evaluating and resolving differential equations. This approach focuses on detecting differential equation symmetries, which can be utilized to simplify the solution-seeking process by lowering the number of independent variables. Closed-form solutions are frequently the outcome of this. Numerous disciplines, including physics, fluid dynamics, electromagnetic, and engineering, heavily rely on the Lie symmetry technique [48–51].

We now compute the symmetry group of the transformed (2+1)-dimensional KdV-Calogero Bogoyavlenkskii-Schiff equation (1.5). The vector field of the form

$$\mathbf{\Gamma} = \xi^1(\tau,\chi,\mu,\nu)\frac{\partial}{\partial\tau} + \xi^2(\tau,\chi,\mu,\nu)\frac{\partial}{\partial\chi} + \xi^3(\tau,\chi,\mu,\nu)\frac{\partial}{\partial\mu} + \eta(\tau,\chi,\mu,\nu)\frac{\partial}{\partial\nu}.$$

would generate all the desired Lie point symmetries of (1.5). It should be noted that this would be initiated by applying the fourth prolongation $\mathrm{pr}^{(4)}\mathbf{\Gamma}$ to (1.5). This algorithmic procedure yields an overdetermined system of linear partial differential equations of the following form:

$$\begin{aligned}
&\xi^3_\chi = 0,\ \xi^1_\chi = 0,\ \xi^1_\mu = 0,\ \xi^2_\nu = 0,\ \xi^3_\nu = 0,\ \xi^1_\nu = 0,\\
&\eta_{\tau\chi} = 0,\ \eta_{\chi\chi} = 0,\ \xi^2_{\chi\chi} = 0,\ \eta_{\chi\nu} = 0,\ \eta_{\nu\nu} = 0,\\
&\eta_\nu + \xi^2_\chi = 0,\ h_1\eta_\chi - \xi^3_\tau = 0,\ 3\xi^2_{\chi\mu} - \eta_{\nu\mu} = 0,\\
&\xi^2_{\chi\mu} - \eta_{\nu\mu} = 0,\ h_1\eta_\mu + 3h_2\eta_\chi + 2\xi^2_\tau = 0,\\
&h_1\eta_{\chi\mu} - \eta_{\tau\nu} - \xi^2_{\tau\chi} = 0,\ h_1\xi^2_\mu - h_2\xi^3_\mu + h_2\xi^2_\chi = 0,\\
&\xi^1_\tau - \xi^3_\mu + \eta_\nu - \xi^2_\chi = 0,\ h_1\xi^2_\mu - h_2\xi^3_\mu + h_2\eta_\nu + 2h_2\xi^2_\chi = 0.
\end{aligned}$$

Hence the above system yields the following Lie point symmetries:

$$\begin{aligned}
&\mathbf{\Gamma_1} = -2h_1\tau\frac{\partial}{\partial\tau} + (h_2\mu - h_1\chi)\frac{\partial}{\partial\chi} + h_1\nu\frac{\partial}{\partial\nu},\ \text{dilation \& space \& time} - \text{dependent shift}\\
&\mathbf{\Gamma_2} = h_1\tau\frac{\partial}{\partial\tau} + h_2\mu\frac{\partial}{\partial\chi} + h_1\mu\frac{\partial}{\partial\mu},\ \text{space \& time} - \text{dependent shift}\\
&\mathbf{\Gamma_3} = -2h_1^2\tau^2\frac{\partial}{\partial\tau} - \left(h_1^2\tau\chi + h_1h_2\tau\mu\right)\frac{\partial}{\partial\chi} - 2h_1^2\tau\mu\frac{\partial}{\partial\mu} + \left(h_1^2\nu\tau + 2h_1\chi\mu - 2h_2\mu^2\right)\frac{\partial}{\partial\nu},\\
&\text{dilation \& space \& time} - \text{dependent shift}\\
&\mathbf{\Gamma_4} = \frac{h_2}{(h_1+h_2)}\frac{\partial}{\partial\chi} + \frac{h_1}{(h_1+h_2)}\frac{\partial}{\partial\mu},\ \text{space translation}\\
&\mathbf{\Gamma_5} = h_1h_2\tau\frac{\partial}{\partial\chi} + h_1^2\tau\frac{\partial}{\partial\mu} + (h_2\mu - h_1\chi)\frac{\partial}{\partial\nu},\ \text{rotational \& space} - \text{dependent shift}\\
&\mathbf{\Gamma_6} = \frac{\partial}{\partial\tau},\ \text{time translation}\\
&\mathbf{\Gamma_{F(\tau)}} = h_1F(\tau)\frac{\partial}{\partial\chi} + 2F'(\tau)\mu\frac{\partial}{\partial\nu},\ \text{space \& time} - \text{dependent shift}\\
&\mathbf{\Gamma_{G(\tau)}} = G(\tau)\frac{\partial}{\partial\nu}.\ \text{space \& time} - \text{dependent shift}
\end{aligned}$$

2.1. Symmetry reduction and invariant solutions of (1.5)

Our goal is to use the symmetries we discovered above to achieve symmetry reduction of (1.5). The related Lagrange equations must be solved in order to achieve symmetry reduction.

$$\frac{d\tau}{\xi^1(\tau,\chi,\mu,\nu)} = \frac{d\chi}{\xi^2(\tau,\chi,\mu,\nu)} = \frac{d\mu}{\xi^3(\tau,\chi,\mu,\nu)} = \frac{d\nu}{\eta(\tau,\chi,\mu,\nu)}.$$

We examine the three cases listed below.

Case 1. First, we look at infinitesimal generator $\mathbf{\Gamma_1}$

$$\frac{d\tau}{-2h_1\tau} = \frac{d\chi}{(h_2\mu - h_1\chi)} = \frac{d\mu}{0} = \frac{d\nu}{h_1\nu}$$

which gives rise to the group invariant

$$\Psi = \mu,\ \gamma = \frac{\tau}{(h_1\chi - h_2\mu)^2},\ \Theta = -\nu\,(h_1\chi - h_2\mu)\,.$$

Equation (1.5) reduces to the nonlinear partial differential equation when Ψ and γ are taken into consideration as the new independent variables and Θ as the new dependent variable.

$$\begin{aligned}&4\gamma^3h_1^3\Theta_{\gamma\gamma\gamma\Psi} + 8\gamma^2h_1^2\Theta_\gamma\Theta_{\gamma\Psi} + 24\gamma^2h_1^3\Theta_{\gamma\gamma\Psi} + 4\gamma^2h_1^2\Theta_\Psi\Theta_{\gamma\gamma} + 4\gamma h_1^2\Theta\Theta_{\gamma\Psi} + 27\gamma h_1^3\Theta_{\gamma\Psi} + 14\gamma h_1^2\Theta_\gamma\Theta_\Psi\\ &+4h_1^2\Theta\Theta_\Psi + 3h_1^3\Theta_\Psi - 4\gamma\Theta_{\gamma\gamma} - 6\Theta_\gamma = 0.\end{aligned} \tag{2.6}$$

We now further reduce (2.6) by using its symmetries, it can be seen that the corresponding vector of (2.6) is as follows:

$$\begin{aligned}S_1 &= \Psi\frac{\partial}{\partial\Psi} + \gamma\frac{\partial}{\partial\gamma},\\ S_2 &= 2h_1^2\gamma^{\frac{3}{2}}\frac{\partial}{\partial r} + \left(-\sqrt{\gamma}\Theta h_1^2 - \frac{\Psi}{\sqrt{\gamma}}\right)\frac{\partial}{\partial\Theta},\\ S_3 &= \frac{\partial}{\partial\Psi},\\ S_4 &= \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial\Theta}.\end{aligned}$$

Taking symmetry $S_3 + S_4$, yields the following invariants:

$$\kappa = \gamma,\ \Phi = \frac{\Theta\sqrt{\gamma} - \Psi}{\sqrt{\gamma}},$$

and making use of them, we obtain a second order ordinary differential equation

$$2\kappa^2h_1^2\Phi''(\kappa) + 5\kappa h_1^2\Phi'(\kappa) - 2\Phi''(\kappa) + h_1^2\Phi(\kappa) - 3\sqrt{\kappa}\Phi'(\kappa) = 0,$$

which gives the general solution of the form

$$\Phi(\kappa) = \frac{\sqrt{\kappa}C_1 + C_2}{\sqrt{\kappa}\left(h_1^2\sqrt{\kappa} - 1\right)}, \tag{2.7}$$

where the arbitrary integration constants C_1 and C_2 are used. The invariant solution of (1.5), when rewritten in our original variables, looks like this:

$$\begin{aligned}\nu\,(\tau,\chi,\mu) \quad = \quad & -\frac{1}{h_1\,\chi - h_2\,\mu}\left(\sqrt{\frac{\tau}{(h_1\,\chi - h_2\,\mu)^2}}{h_1}^2\mu + C_1\sqrt{\frac{t}{(h_1\,\chi - h_2\,\mu)^2}} + C_2 - \mu\right)\\ &\left(\frac{1}{\sqrt{\frac{\tau}{(h_1\,\chi - h_2\,\mu)^2}}}\left({h_1}^2\sqrt{\frac{\tau}{(h_1\,\chi - h_2\,\mu)^2}} - 1\right)\right)^{-1}.\end{aligned} \tag{2.8}$$

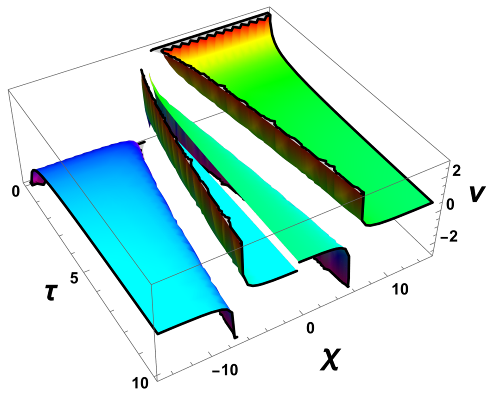

Figure 1. A 3D wave solution (2.8)

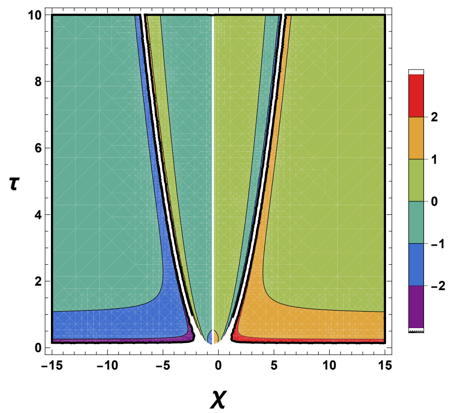

Figure 2. A density plot of solution (2.8)

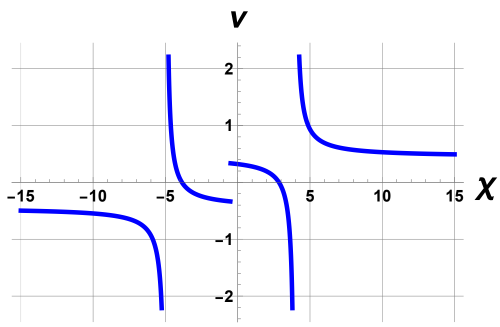

Figure 3. A 2D plot of (2.8)

Case 2. We consider infinitesimal generator $\mathbf{\Gamma_2}$

$$\frac{d\tau}{h_1\tau} = \frac{d\chi}{h_2\mu} = \frac{d\mu}{h_1\mu} = \frac{d\nu}{0}$$

we obtained the group invariant

$$\Psi = \frac{h_1\chi - h_2\mu}{h_2},\ \gamma = \frac{\tau}{h_2\tau},\ \ \Theta = \nu.$$

Equation (1.5) changes to Θ as the new dependent variable and Ψ and γ as new independent variables.

$$4\gamma h_1^2 h_2 \Theta_\Psi \Theta_{\Psi\gamma} + 2\gamma h_2 h_1^2 \Theta_\gamma \Theta_{\Psi\Psi} - \gamma h_1^3 \Theta_{\gamma\Psi\Psi\Psi} - 4h_2\Theta_{\gamma\Psi} = 0. \tag{2.9}$$

After employing its symmetries to reduce (2.9), we can observe that the equivalent vector of (2.9) is as follows:

$$\mathbf{S_1} = \frac{\partial}{\partial \Theta},$$
$$\mathbf{S_2} = h_1^2\gamma^2\frac{\partial}{\partial \gamma} - \Psi\frac{\partial}{\partial \Psi},$$
$$\mathbf{S_3} = \frac{\partial}{\partial \Psi},$$
$$\mathbf{S_4} = \Psi\frac{\partial}{\partial f} + 2\gamma\frac{\partial}{\partial \gamma} - \Theta\frac{\partial}{\partial \Theta}.$$

Taking symmetry $\mathbf{S_2}$, yields the following invariants:

$$\kappa = \Psi,\ \ \Phi = \frac{\Theta\gamma h_1^2 - \Psi}{h_1^2\gamma},$$

and making use of them, we obtain a second order ordinary differential equation

$$\kappa\Phi''(\kappa) + 2\Phi'(\kappa) = 0,$$

which yields the general solution of the form

$$\Phi(\kappa) = C_1 + \frac{1}{\kappa}C_2, \tag{2.10}$$

where C_1 and C_2 are arbitrary constant of integration. Reverting back to our original variables, the invariant solution of (1.5) takes the form

$$v(\tau,\chi,\mu) = \frac{C_1\, h_1{}^3\tau\chi - C_1\, h_1{}^2 h_2\,\tau\mu - C_2\, h_1{}^2 h_2\,\tau - h_1{}^2\chi^2\mu + 2\,h_1\,h_2\,\chi\mu^2 - h_2{}^2\mu^3}{(h_1\,\chi - h_2\,\mu)\,h_1{}^2\tau}. \tag{2.11}$$

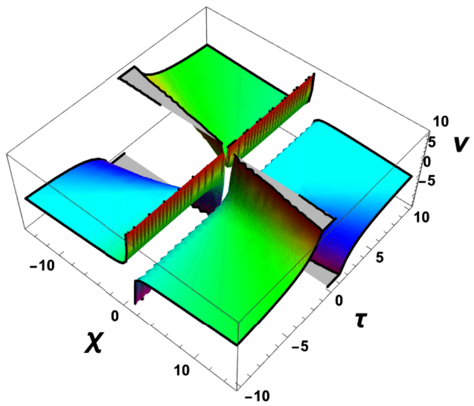

Figure 4. The profile structure of solution (2.11)

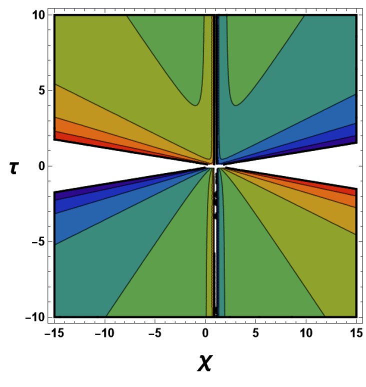

Figure 5. A density plot of (2.11)

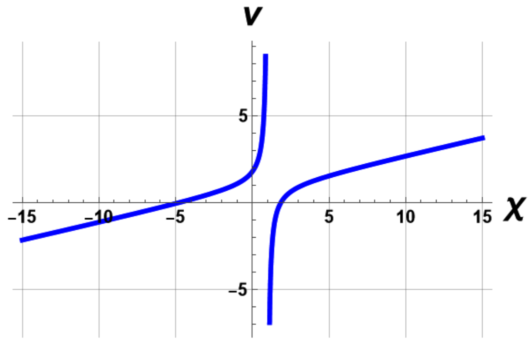

Figure 6. A 2D side view of (2.11)

Case 3. The symmetry $\mathbf{\Gamma_5}$ leads to

$$v(\tau,\chi,\mu) = \frac{1}{\tau^{3/2}{h_1}^2}\left(g(\tau)\tau^{3/2}{h_1}^2 - \sqrt{\tau}h_1\,\chi\mu + \sqrt{\tau}h_2\,\mu^2 + z\left(\frac{h_1\,\chi - h_2\,\mu}{h_1\,\sqrt{\tau}}\right){h_1}^2\tau\right) \tag{2.12}$$

with invariants

$$\Psi = \tau,\ \ \gamma = \frac{h_1\chi - h_2\mu}{h_1},\ \ \Theta = \frac{h_1^2\tau v + h_1\chi\mu - h_2\mu^2}{h_1^2\tau},$$

and $\Theta(\gamma,\Psi)$ satisfies the nonlinear partial differential equation

$$2\Psi\Theta_{\Psi\gamma} + \gamma\Theta_{\gamma\gamma} + 2\Theta = 0. \tag{2.13}$$

The integration of (2.13) leads to

$$\Theta(\Psi,\gamma) = g(\Psi) + \frac{z\left(\frac{\gamma}{\sqrt{\tau}}\right)}{\sqrt{\Psi}}, \tag{2.14}$$

where g and z are arbitrary functions of f and $\gamma/\sqrt{\tau}$ respectively. Finally, (2.14) accomplishes the group invariant solution (2.12).

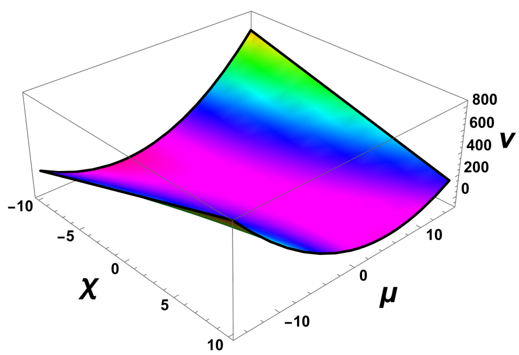

Figure 7. A 3D of solution (2.12)

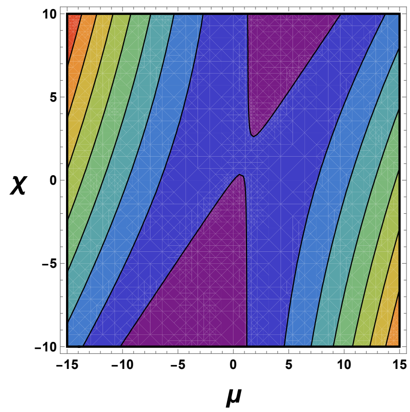

Figure 8. A density plot of solution (2.12)

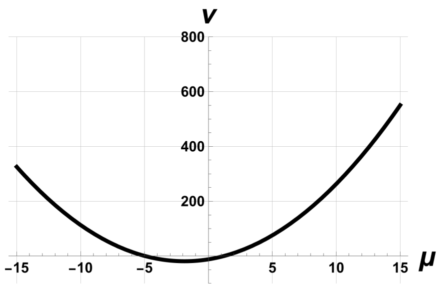

Figure 9. A 2D side view of (2.12)

The limiting behavior of problems that are distant from their beginning or boundary conditions is often captured by group invariant solutions.

3. MULTIPLE EXP-FUNCTION METHOD

The multiple exp-function technique operates independently in constructing bilinear forms. It begins with the notion that polynomials of exponential functions can be used to define multi-soliton solutions. In essence, Hirota's perturbation approach is expanded upon by the multiple exp-function algorithm. Furthermore, general wave frequencies and phase shifts are included in the resultant solutions. The following is a summary of the multiple exp-function method's main steps. [52–56]:

Step 1. Let us consider the following (1 + 1)-dimensional NLEE:

$$P(\chi, \tau, v_\chi, v_\tau, \cdots) = 0. \tag{3.15}$$

Step 2. Suppose the solution of above NLEE can be expressed as

$$v(\tau, \chi) = \frac{p(\eta_1, \eta_2, \cdots, \eta_n)}{q(\eta_1, \eta_2, \cdots, \eta_n)}, \quad p = \sum_{r,s=1}^{n} \sum_{i,j=0}^{M} p_{rs,ij} \eta_r^i \eta_s^j,$$
$$q = \sum_{r,s=1}^{n} \sum_{i,j=0}^{N} q_{rs,ij} \eta_r^i \eta_s^j, \tag{3.16}$$

in which $p_{rs,ij}$ and $q_{rs,ij}$ are unknowns to be determined and

$$\eta_i = c_i e^{\xi_i}, \quad \xi_i = k_i \chi - \rho_i \tau, \quad 1 \leqslant i \leqslant n. \tag{3.17}$$

Step 3. Substituting (3.16) and its derivatives into (3.15) yields the following transformed equation

$$Q(\chi, \tau, \eta_1, \eta_2, \cdots, \eta_n) = 0. \tag{3.18}$$

Step 4. The multiple wave solution of (3.15) can be obtained by solving an algebraic system whose numerator is set to zero for the function $Q(\chi, \tau, \eta_1, \eta_2, \cdots, \eta_n)$,

$$v(\tau, \chi) = \frac{p(c_1 e^{k_1 \chi - \rho_1 \tau}, \cdots, c_n e^{k_n \chi - \rho_n \tau})}{q(c_1 e^{k_1 \chi - \rho_1 \tau}, \cdots, c_n e^{k_n \chi - \rho_n t})}. \tag{3.19}$$

3.1. Application of the multiple exp-function algorithm in (1.5)

The multiple exp-function approach will be used in this subsection to get the one- and two-wave solutions of (1.5).

3.1.1. One-wave solution of (1.5) We start with one-wave function

$$v(\chi,\tau,\mu) = \frac{p}{q}, \quad p = A_1 e^{k_1\chi + l_1\mu - \rho_1\tau}, \quad q = 1 + e^{k_1\chi + l_1\mu - \rho_1\tau} \tag{3.20}$$

where A_1 is a constant. By applying the multiple exp-function algorithm, we obtain with the aid of Maple:

$$A_1 = 2k_1, \quad \rho_1 = -\frac{1}{4}k_1^2\left(h_1 l_1 + h_2 k_1\right). \tag{3.21}$$

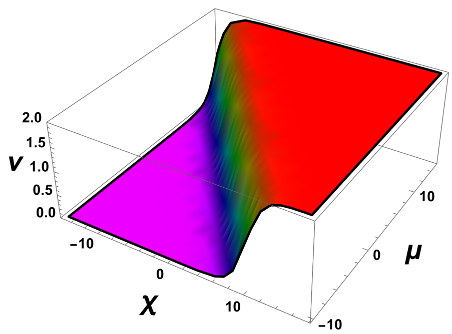

Figure 10. A 3D plot of solution (3.20).

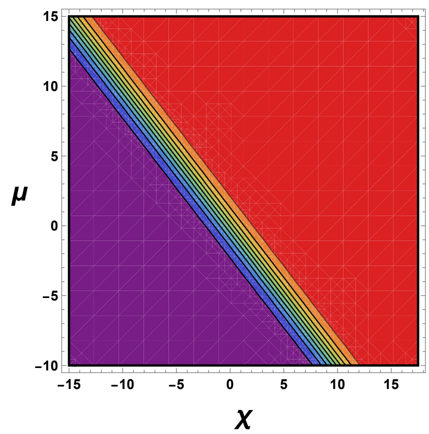

Figure 11. A density plot of solution (3.20)

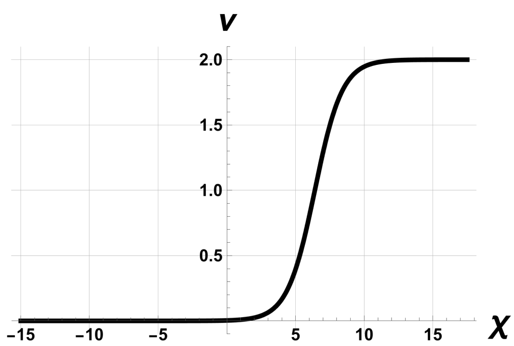

Figure 12. A 2D side view of solution (3.20).

3.1.2. Two-wave solution of (1.5) Based on the assertion made in Step 2, we examine two-wave solutions and suppose that equation (1.5) has the rational function of two-wave solutions, which is represented as follows:

$$v(\chi,\tau,\mu)=\frac{p}{q}, \tag{3.22}$$

with p and q being defined by

$$\begin{aligned} p &= 2\,k_1 e^{k_1\chi+l_1\mu-\rho_1\tau}+2\,k_2 e^{k_2\chi+l_2\mu-\rho_2\tau} \\ &\quad +2\,A_{12}\,(k_1+k_2)\,e^{k_1\chi+l_1\mu-\rho_1\tau}e^{k_2\chi+l_2\mu-\rho_2\tau}, && (3.23)\\ q &= 1+e^{k_1\chi+l_1\mu-\rho_1\tau}+e^{k_2\chi+l_2\mu-\rho_2\tau}+A_{12}e^{k_1\chi+l_1\mu-\rho_1\tau}e^{k_2\chi+l_2\mu-\rho_2\tau}, && (3.24)\end{aligned}$$

where

$$\lambda_i=k_i\chi+l_i\mu-\rho_i\tau,\quad i=1,2 \tag{3.25}$$

and A_{12} is a constant. Applying the multiple exp-function algorithm, with the aid of Maple leads to

$$A_{12}=\frac{k_1^2-2k_1k_2+k_2^2}{(k_1+k_2)^2},\ \ \rho_1=-\frac{1}{4}k_1^2\,(h_1l_1+h_2k_1)\,,\ \ \rho_2=-\frac{1}{4}k_2^2\,(h_1l_2+h_2k_2)\,. \tag{3.26}$$

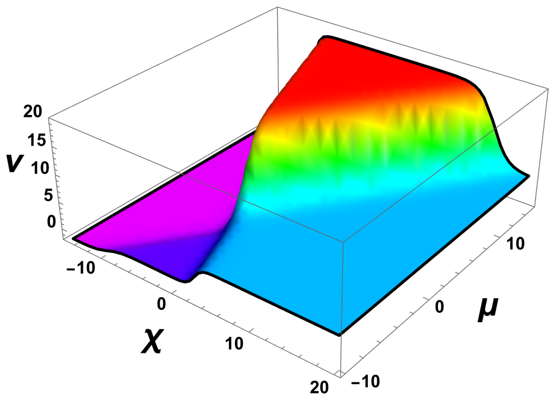

Figure 13. A 3 dimensional simulation of solution (3.22).

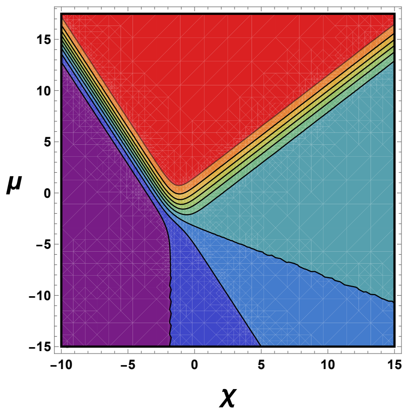

Figure 14. A density plot of solution (3.22).

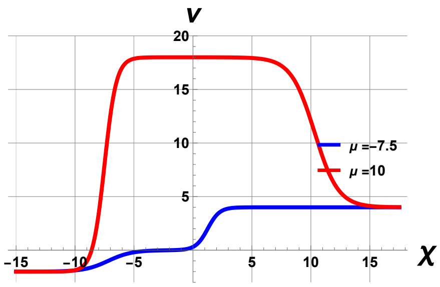

Figure 15. A 2D side view of solution (3.22).

4. ANALYSIS AND DISCUSSION OF THE RESULTS

The first solution in Fig.1, Fig.2 and Fig.3 portrays a multi-singular wave structure, with spatial variables restricted to the intervals of $-15 \leq \chi \leq 15$ and $0 \leq \tau \leq 10$, with $h_2 = 1$, $\mu = -1$, $h_1 = 1$, $C_1 = 1$, $C_2 = 0$. Conversely, the second illustration in Fig.4, Fig.5 and Fig.5, presents a singular wave structure , with spatial variables restricted to $-15 \leq \chi \leq 15$ and $-10 \leq \tau \leq 10$, with $h_2 = 1$, $\tau = 1$, $h_1 = 1$, $C_2 = 1$ and $C_1 = 1$. Singular periodic waves are employed in the study of wave dynamics across different optical systems and fibers. Additionally, they are relevant in the examination of chaotic phenomena, providing essential insights into the behavior of these systems, as they facilitate the interactions of particles in the fields of physics and chemistry. In Fig.7, Fig.8 and Fig.9 , the graphical representation shows a concave-up structure which is analyzed within the spatial variables constrained to the range of $-10 \leq \tau \leq 10$ and $-15 \leq \chi \leq 15$ $h_2 = 1$, $\tau = 1$, $h_1 = 1$, $g(\tau) = 0$, $z\left(\frac{h_1\chi - h_2\mu}{h_1\sqrt{\tau}}\right) = \operatorname{sech}^2(\chi - \mu)$. Furthermore, Fig.10, Fig.11 and Fig.12, engenders a one-wave solution portraying a kink profile defined within the intervals of $-15 \leq \chi \leq 20$ and $-10 \leq \mu \leq 15$ and the rest of the elements equated to unity. Finally, Fig.13, Fig.14 and Fig.15, presents a two-wave kink profile structure with spatial variables restricted to the intervals of$-15 \leq \chi \leq 15$ and $-15 \leq \chi \leq 20$, with the parameter with $h_1 = 5$, $h_2 = 2.5$, $k_1 = 2$ and $t = 0.5$ and the rest set one. Kink solitary waves have been extensively utilized in optics and various fields of nonlinear science. They serve as logic units or polarization switches connecting two separate points. Additionally, they are applied in the analysis and comprehension of the propagation of extremely short light pulses within a cubic-quintic nonlinear medium.

5. CONSERVATION LAWS

Key physical quantities like mass, energy, momentum, electric charge, and other constants of motion can be understood through the framework provided by conservation laws [56]. Because they define conserved quantities and shed light on the characteristics and behavior of partial differential equation solutions, conservation laws are essential to the study of differential equations. They play a crucial role in establishing whether solutions to nonlinear partial differential equations exist, are unique, and are stable. Conservation laws also aid in the evaluation of linearization, integrability, and the validity of numerical techniques for resolving these equations.

Our goal in this section is to use the multiplier approach to construct the conservation laws of (1.5). We start by reviewing some fundamental findings that will be applied later in this section. Examine a system of partial differential equations of pth order with n independent variables $\chi = (\chi^1, \chi^2, \dots, \chi^n)$ and m dependent variables $\nu = (\nu^1, \nu^2, \dots, \nu^m)$, i.e.,

$$H_\alpha(\chi, \nu, \nu_{(1)}, ..., \nu_{(p)}) = 0, \quad \alpha = 1, ...m, \tag{5.27}$$

where $\nu_{(1)}, \nu_{(2)}, \dots, \nu_{(\zeta)}$ denote the collections of all first, second, ..., ζth-order partial derivatives, that is, $\nu_i^\alpha = D_i(\nu^\alpha), \nu_{ij}^\alpha = D_j D_i(\nu^\alpha), \dots$ respectively, with the *total derivative operator* with respect to χ^i is given by

$$D_i = \frac{\partial}{\partial \chi^i} + \nu_i^\alpha \frac{\partial}{\partial \nu^\alpha} + \nu_{ij}^\alpha \frac{\partial}{\partial \nu_j^\alpha} + \dots, \quad i = 1, \dots, n, \tag{5.28}$$

where the summation convention is used whenever appropriate.

The *Euler-Lagrange operator*, for each α, is given by

$$\frac{\delta}{\delta \nu^\alpha} = \frac{\partial}{\partial \nu^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial \nu^\alpha_{i_1 i_2 \dots i_s}}, \qquad \alpha = 1, \dots, m. \tag{5.29}$$

The n-tuple vector $T = (T^1, T^2, \dots, T^n)$, $T^j \in \mathcal{A}$, $j = 1, \dots, n$, is a *conserved vector* of (5.27) if T^i satisfies

$$D_i T^i|_{(5.27)} = 0. \tag{5.30}$$

The equation (5.30) defines a *local conservation law* of system (5.27).

A multiplier $\Lambda_\alpha(\chi, \nu, \nu_{(1)}, \ldots)$ has the property that

$$\Lambda_\alpha H_\alpha = D_i T^i \tag{5.31}$$

holds identically. Here we will consider multiplier of the third order, i.e.,
$\Lambda_\alpha = \Lambda_\alpha(\tau, \chi, \mu, \nu, \nu_\tau, \nu_\chi, \nu_\mu, \nu_{\tau\chi}, \nu_{\tau\mu}, \nu_{\chi\mu}, \nu_{\tau\tau}, \nu_{\chi\chi}, \nu_{\mu\mu}, \nu_{\tau\tau\tau}, \nu_{\tau\tau\chi}, \nu_{\tau\chi\chi}, \nu_{\tau\tau\mu}, \nu_{\tau\mu\mu}, \nu_{\chi\chi\chi}, \nu_{\chi\chi\mu}, \nu_{\chi\mu\mu}, \nu_{\mu\mu\mu})$. The right hand side of (5.31) is a divergence expression. The determining equation for the multiplier Λ_α is

$$\frac{\delta(\Lambda_\alpha H_\alpha)}{\delta \nu^\alpha} = 0. \tag{5.32}$$

5.1. Conservation Laws of (1.5)

The (2+1) dimensional KdV-Calogero-Bogoyavlenkskii-Schiff equation (1.5) has a multiplier of the form

$$\begin{aligned}
\Lambda \;=\; & \left(\frac{1}{2}h_1\tau\nu + h_1\tau\mu\nu_\mu + h_1\tau^2\nu_\tau + \frac{1}{2}h_2\tau\mu\nu_\chi + \frac{1}{2}h_1\tau\chi\nu_\chi - \frac{h_2\mu^2}{h_1} + \chi\mu\right)C_1 + \nu_\mu C_4 \\
& + \left(h_1\tau\nu_\mu - \frac{3h_2\mu}{h_1} + \chi\right)C_2 + \left(\frac{h_2\mu\nu_\chi}{h_1} + 4\tau\nu_\tau + \nu + \chi\nu_\chi + 2\mu\nu_\mu\right)C_3 + \nu_\tau C_5 \\
& + \frac{4\mu^2 f''(\tau)}{h_1^2} + \frac{4\mu f'(\tau)\nu_\chi}{h_1} + f(\tau)\nu_{\chi\chi\chi} + 3f(\tau){\nu_\chi}^2 + \frac{2\mu r'(\tau)}{h_1} + r(\tau)\nu_\chi + g(\tau),
\end{aligned} \tag{5.33}$$

where C_1, C_2, C_3, C_4 and C_5 are arbitrary constant and $f(\tau), r(\tau)$ and $g(\tau)$ are arbitrary functions of τ. Corresponding to the above multiplier we have the following conserved vectors of (1.2):

$$\begin{aligned}
T_1^\tau \;=\; & \frac{1}{3h_1}\Big(-16\tau\nu\nu_\chi\nu_{\chi\mu}h_1^2 - 8\tau\nu\nu_\mu\nu_{\chi\chi}h_1^2 - 6\tau\nu\nu_{\chi\chi\chi\mu}h_1^2 + 3\chi{\nu_\chi}^2 h_1 - 3\nu\nu_\chi h_1 + 6\mu\nu_\mu\nu_\chi h_1 \\
& -3\chi\nu\nu_{\chi\chi}h_1 - 24\tau h_2\nu\nu_\chi\nu_{\chi\chi}h_1 - 6\tau h_2\nu\nu_{\chi\chi\chi\chi}h_1 + 12\tau\nu_\chi\nu_\tau h_1 + 12\tau\nu\nu_{\tau\chi}h_1 + 3\mu h_2{\nu_\chi}^2 \\
& -6\mu\nu\nu_{\chi\mu}h_1 - 3\mu h_2\nu\nu_{\chi\chi}\Big),
\end{aligned}$$

$$\begin{aligned}
T_1^\chi \;=\; & \frac{1}{24h_1}\Big(-48\mu h_2^2{\nu_\chi}^3 - 48\chi h_1h_2{\nu_\chi}^3 + 64h_1h_2\nu{\nu_\chi}^2 - 32\chi h_1^2\nu_\mu{\nu_\chi}^2 - 128\mu h_1h_2\nu_\mu{\nu_\chi}^2 \\
& -64\mu h_1^2{\nu_\mu}^2\nu_\chi + 48h_1^2\nu\nu_\mu\nu_\chi + 32\mu h_1^2\nu\nu_{\mu\mu}\nu_\chi - 12h_1^2\nu_{\chi\mu}\nu_\chi - 16\chi h_1^2\nu\nu_{\chi\mu}\nu_\chi + 80\mu h_1h_2\nu\nu_{\chi\mu}\nu_\chi \\
& -15h_1h_2\nu_{\chi\chi}\nu_\chi - 15\chi h_1^2\nu_{\chi\chi\mu}\nu_\chi - 39\mu h_1h_2\nu_{\chi\chi\mu}\nu_\chi - 24\mu h_2^2\nu_{\chi\chi\chi}\nu_\chi - 24\chi h_1h_2\nu_{\chi\chi\chi}\nu_\chi \\
& -128\tau h_1^2\nu_\mu\nu_\tau\nu_\chi + 64\tau h_1^2\nu\nu_{\tau\mu}\nu_\chi + 192\tau h_1h_2\nu\nu_{\tau\chi}\nu_\chi - 24\tau h_1^2\nu_{\tau\chi\mu}\nu_\chi - 48\tau h_1h_2\nu_{\tau\chi\chi}\nu_\chi \\
& +96\tau h_1{\nu_\tau}^2 + 32\mu h_1^2\nu\nu_\mu\nu_{\chi\mu} + 6\mu h_1^2\nu_{\mu\mu}\nu_{\chi\chi} + 9\chi h_1^2\nu_{\chi\mu}\nu_{\chi\chi} + 33\mu h_1h_2\nu_{\chi\mu}\nu_{\chi\chi} + 36h_1^2\nu\nu_{\chi\chi\mu} \\
& +18\mu h_1^2\nu\nu_{\chi\chi\mu\mu} + 45h_1h_2\nu\nu_{\chi\chi\chi} - 3\chi h_1^2\nu_\mu\nu_{\chi\chi\chi} - 27\mu h_1h_2\nu_\mu\nu_{\chi\chi\chi} - 3\chi h_1^2\nu\nu_{\chi\chi\chi\mu} \\
& +48\mu h_1\nu_\mu\nu_\tau - 36\tau h_1^2\nu_{\chi\chi\mu}\nu_\tau - 48\tau h_1h_2\nu_{\chi\chi\chi}\nu_\tau - 48\mu h_1\nu\nu_{\tau\mu} + 12\tau h_1^2\nu_{\chi\chi}\nu_{\tau\mu} + 24\chi h_1\nu\nu_{\tau\chi} \\
& +64\tau h_1^2\nu\nu_\mu\nu_{\tau\chi} + 24\tau h_1^2\nu_{\chi\mu}\nu_{\tau\chi} + 48\tau h_1h_2\nu_{\chi\chi}\nu_{\tau\chi} - 12\tau h_1^2\nu_\mu\nu_{\tau\chi\chi} + 36\tau h_1^2\nu\nu_{\tau\chi\chi\mu} \\
& -24\mu h_1^2\nu_\mu\nu_{\chi\chi\mu} - 12\mu h_1^2\nu_{\chi\mu\mu}\nu_\chi + 24\mu h_2\nu_\tau\nu_\chi + 12\mu h_2^2{\nu_{\chi\chi}}^2 - 96h_1\nu\nu_\tau - 96\tau h_1\nu\nu_{\tau\tau} \\
& -192\tau h_1h_2\nu_\tau{\nu_\chi}^2 + 12\chi h_1h_2{\nu_{\chi\chi}}^2 + 48\tau h_1h_2\nu\nu_{\tau\chi\chi\chi} + 24\chi h_1\nu_\tau\nu_\chi + 12\mu h_1^2{\nu_{\chi\mu}}^2 \\
& +24\mu h_2\nu\nu_{\tau\chi} + 21\mu h_1h_2\nu\nu_{\chi\chi\chi\mu}\Big),
\end{aligned}$$

$$\begin{aligned}
T_1^\mu \;=\; & \frac{1}{24}\Big(-16\chi h_1{\nu_\chi}^3 - 16\mu h_2{\nu_\chi}^3 + 16h_1\nu{\nu_\chi}^2 - 32\mu h_1\nu_\mu{\nu_\chi}^2 - 64\tau h_1\nu_\tau{\nu_\chi}^2 - 32\mu h_1\nu\nu_{\chi\mu}\nu_\chi \\
& +16\chi h_1\nu\nu_{\chi\chi}\nu_\chi - 80\mu h_2\nu\nu_{\chi\chi}\nu_\chi - 6\mu h_1\nu_{\chi\chi\mu}\nu_\chi - 6\chi h_1\nu_{\chi\chi\chi}\nu_\chi - 6\mu h_2\nu_{\chi\chi\chi}\nu_{,\chi} + 64\tau h_1\nu\nu_{\tau\chi}\nu_\chi \\
& -12\tau h_1\nu_{\tau\chi\chi}\nu_\chi + 3\chi h_1{\nu_{\chi\chi}}^2 + 3\mu h_2{\nu_{\chi\chi}}^2 - 32\mu h_1\nu\nu_\mu\nu_{\chi\chi} + 6\mu h_1\nu_{\chi\mu}\nu_{\chi\chi} + 9h_1\nu\nu_{\chi\chi\chi} \\
& -18\mu h_1\nu\nu_{\chi\chi\chi\mu} + 3\chi h_1\nu\nu_{\chi\chi\chi\chi} - 21\mu h_2\nu\nu_{\chi\chi\chi\chi} - 12\tau h_1\nu_{\chi\chi\chi}\nu_\tau + 96\mu\nu\nu_{\tau\chi} - 6\mu h_1\nu_\mu\nu_{\chi\chi\chi} \\
& +12\tau h_1\nu_{\chi\chi}\nu_{\tau\chi} + 12\tau h_1\nu\nu_{\tau\chi\chi\chi} - 3h_1\nu_{\chi\chi}\nu_\chi\Big);
\end{aligned}$$

$$T_2^\tau \;=\; \frac{\tau\nu_\mu\nu_\chi h_1^2 - \tau\nu\nu_{\chi\mu}h_1^2 - 2\nu h_1 + 2\chi\nu_\chi h_1 - 6\mu h_2\nu_\chi}{h_1},$$

$$
\begin{aligned}
T_2^{\chi} &= \frac{1}{24h_1}\Big(6\tau v_{\chi\mu}{}^2h_1^3 - 32\tau v_\mu{}^2v_\chi h_1^3 + 16\tau vv_{\mu\mu}v_\chi h_1^3 + 16\tau vv_\mu v_{\chi\mu}h_1^3 - 6\tau v_\chi v_{\chi\mu\mu}h_1^3\\
&+3\tau v_{\mu\mu}v_{\chi\chi}h_1^3 - 12\tau v_\mu v_{\chi\chi\mu}h_1^3 + 9\tau vv_{\chi\chi\mu\mu}h_1^3 - 48\tau h_2v_\mu v_\chi{}^2h_1^2 - 48\chi v_\mu v_\chi h_1^2\\
&+48\tau h_2vv_\chi v_{\chi\mu}h_1^2 + 12v_{\chi\mu}h_1^2 + 12\tau h_2v_{\chi\mu}v_{\chi\chi}h_1^2 - 18\chi v_{\chi\chi\mu}h_1^2 - 12\tau h_2v_\chi v_{\chi\chi\mu}h_1^2\\
&+12\tau h_2vv_{\chi\chi\chi\mu}h_1^2 + 24\tau v_\mu v_\tau h_1^2 - 24\tau vv_{\tau\mu}h_1^2 - 72\chi h_2v_\chi{}^2h_1 + 144\mu h_2v_\mu v_\chi h_1 + 6h_2v_{\chi\chi}h_1\\
&+54\mu h_2v_{\chi\chi\mu}h_1 - 24\chi h_2v_{\chi\chi\chi}h_1 + 48\chi v_\tau h_1 + 216\mu h_2^2v_\chi{}^2 + 72\mu h_2^2v_{\chi\chi\chi} - 144\mu h_2v_\tau\\
&-12\tau h_2v_\mu v_{\chi\chi\chi}h_1^2\Big),\\
T_2^{\mu} &= \frac{1}{24}\Big(-16\tau v_\mu v_\chi{}^2h_1^2 - 16\tau vv_\chi v_{\chi\mu}h_1^2 - 16\tau vv_\mu v_{\chi\chi}h_1^2 + 3\tau v_{\chi\mu}v_{\chi\chi}h_1^2 - 3\tau v_\chi v_{\chi\chi\mu}h_1^2\\
&-3\tau v_\mu v_{\chi\chi\chi}h_1^2 - 9\tau vv_{\chi\chi\chi\mu}h_1^2 - 24\chi v_\chi{}^2h_1 + 24vv_\chi h_1 - 48\tau h_2vv_\chi v_{\chi\chi}h_1 + 6v_{\chi\chi}h_1\\
&-6\chi v_{\chi\chi\chi}h_1 - 12\tau h_2vv_{\chi\chi\chi\chi}h_1 + 48\tau vv_{\tau\chi}h_1 + 72\mu h_2v_\chi{}^2 + 18\mu h_2v_{\chi\chi\chi}\Big);
\end{aligned}
$$

$$
\begin{aligned}
T_3^{\tau} &= \frac{1}{6h_1}\Big(-8\tau^2vv_\chi v_{\chi\mu}h_1^3 - 4\tau^2vv_\mu v_{\chi\chi}h_1^3 - 3\tau^2vv_{\chi\chi\chi\mu}h_1^3 + 3\tau\chi v_\chi{}^2h_1^2 - 3\tau vv_\chi h_1^2 + 6\tau\mu v_\mu v_\chi h_1^2\\
&-6\tau\mu vv_{\chi\mu}h_1^2 - 3\tau\chi vv_{\chi\chi}h_1^2 - 12\tau^2h_2vv_\chi v_{\chi\chi}h_1^2 - 3\tau^2h_2vv_{\chi\chi\chi\chi}h_1^2 + 6\tau^2v_\chi v_\tau h_1^2 + 6\tau^2vv_{\tau\chi}h_1^2\\
&+3\tau\mu h_2v_\chi{}^2h_1 - 12\mu vh_1 + 12\chi\mu v_\chi h_1 - 3\tau\mu h_2vv_{\chi\chi}h_1 - 12\mu^2h_2v_\chi\Big),\\
T_3^{\chi} &= \frac{1}{48h_1}\Big(-32\tau\chi v_\mu v_\chi{}^2h_1^3 + 12\tau\mu v_{\chi\mu}{}^2h_1^3 - 64\tau\mu v_\mu{}^2v_\chi h_1^3 + 48\tau vv_\mu v_\chi h_1^3 + 32\tau\mu vv_{\mu\mu}v_\chi h_1^3\\
&+32\tau\mu vv_\mu v_{\chi\mu}h_1^3 - 12\tau v_\chi v_{\chi\mu}h_1^3 - 16\tau\chi vv_\chi v_{\chi\mu}h_1^3 - 12\tau\mu v_\chi v_{\chi\mu\mu}h_1^3 + 6\tau\mu v_{\mu\mu}v_{\chi\chi}h_1^3\\
&+36\tau vv_{\chi\chi\mu}h_1^3 - 24\tau\mu v_\mu v_{\chi\chi\mu}h_1^3 - 15\tau\chi v_\chi v_{\chi\chi\mu}h_1^3 + 18\tau\mu vv_{\chi\chi\mu\mu}h_1^3 - 3\tau\chi v_\mu v_{\chi\chi\chi}h_1^3\\
&-64\tau^2v_\mu v_\chi v_\tau h_1^3 - 18\tau^2v_{\chi\chi\mu}v_\tau h_1^3 + 32\tau^2vv_\chi v_{\tau\mu}h_1^3 + 6\tau^2v_{\chi\chi}v_{\tau\mu}h_1^3 + 32\tau^2vv_\mu v_{\tau\chi}h_1^3\\
&-12\tau^2v_\chi v_{\tau\chi\mu}h_1^3 - 6\tau^2v_\mu v_{\tau\chi\chi}h_1^3 + 18\tau^2vv_{\tau\chi\chi\mu}h_1^3 - 48\tau\chi h_2v_\chi{}^3h_1^2 - 24v^2h_1^2 + 64\tau h_2vv_\chi{}^2h_1^2\\
&-128\tau\mu h_2v_\mu v_\chi{}^2h_1^2 + 12\tau\chi h_2v_{\chi\chi}{}^2h_1^2 + 48\tau^2v_\tau{}^2h_1^2 + 24\chi vv_\chi h_1^2 - 96\chi\mu v_\mu v_\chi h_1^2 - 24v_\chi h_1^2\\
&+80\tau\mu h_2vv_\chi v_{\chi\mu}h_1^2 + 12\chi v_{\chi\chi}h_1^2 - 15\tau h_2v_\chi v_{\chi\chi}h_1^2 + 33\tau\mu h_2v_{\chi\mu}v_{\chi\chi}h_1^2 - 36\chi\mu v_{\chi\chi\mu}h_1^2\\
&+45\tau h_2vv_{\chi\chi\chi}h_1^2 - 27\tau\mu h_2v_\mu v_{\chi\chi\chi}h_1^2 - 24\tau\chi h_2v_\chi v_{\chi\chi\chi}h_1^2 + 21\tau\mu h_2vv_{\chi\chi\chi\mu}h_1^2 - 96\tau^2h_2v_\chi{}^2v_\tau h_1^2\\
&-96\tau vv_\tau h_1^2 + 48\tau\mu v_\mu v_\tau h_1^2 + 24\tau\chi v_\chi v_\tau h_1^2 - 24\tau^2h_2v_{\chi\chi\chi}v_\tau h_1^2 - 48\tau\mu vv_{\tau\mu}h_1^2 + 24\tau\chi vv_{\tau\chi}h_1^2\\
&+24\tau^2h_2v_{\chi\chi}v_{\tau\chi}h_1^2 - 24\tau^2h_2v_\chi v_{\tau\chi\chi}h_1^2 + 24\tau^2h_2vv_{\tau\chi\chi\chi}h_1^2 - 48\tau^2vv_{\tau\tau}h_1^2 - 48\tau\mu h_2^2v_\chi{}^3h_1\\
&+12\tau\mu h_2^2v_{\chi\chi}{}^2h_1 + 24\mu h_2vv_\chi h_1 + 96\mu^2h_2v_\mu v_\chi h_1 + 24\mu h_2v_{\chi\chi}h_1 + 36\mu^2h_2v_{\chi\chi\mu}h_1\\
&-24\tau\mu h_2^2v_\chi v_{\chi\chi\chi}h_1 + 96\chi\mu v_\tau h_1 + 24\tau\mu h_2v_\chi v_\tau h_1 + 24\tau\mu h_2vv_{\tau\chi}h_1 + 144\mu^2h_2^2v_\chi{}^2\\
&-96\mu^2h_2v_\tau - 48\chi\mu h_2v_{\chi\chi\chi}h_1 + 96\tau^2h_2vv_\chi v_{\tau\chi}h_1^2 - 144\chi\mu h_2v_\chi{}^2h_1 + 48\mu^2h_2^2v_{\chi\chi\chi}\\
&+9\tau\chi v_{\chi\mu}v_{\chi\chi}h_1^3 - 3\tau\chi vv_{\chi\chi\chi\mu}h_1^3 + 12\tau^2v_{\chi\mu}v_{\tau\chi}h_1^3 + 24\mu v_{\chi\mu}h_1^2 - 39\tau\mu h_2v_\chi v_{\chi\chi\mu}h_1^2\Big),\\
T_3^{\mu} &= \frac{1}{48}\Big(-16\tau\chi h_1^2v_\chi{}^3 - 16\tau\mu h_1h_2v_\chi{}^3 - 48\chi\mu h_1v_\chi{}^2 + 48\mu^2h_2v_\chi{}^2 + 16\tau h_1^2vv_\chi{}^2\\
&+48\mu h_1vv_\chi - 32\tau\mu h_1^2vv_{\chi\mu}v_\chi - 3\tau h_1^2v_{\chi\chi}v_\chi + 16\tau\chi h_1^2vv_{\chi\chi}v_\chi - 80\tau\mu h_1h_2vv_{\chi\chi}v_\chi\\
&-6\tau\chi h_1^2v_{\chi\chi\chi}v_\chi - 6\tau\mu h_1h_2v_{\chi\chi\chi}v_\chi + 32\tau^2h_1^2vv_{\tau\chi}v_\chi - 6\tau^2h_1^2v_{\tau\chi\chi}v_\chi + 3\tau\chi h_1^2v_{\chi\chi}{}^2\\
&-32\tau\mu h_1^2vv_\mu v_{\chi\chi} + 6\tau\mu h_1^2v_{\chi\mu}v_{\chi\chi} - 12\chi\mu h_1v_{\chi\chi\chi} + 12\mu^2h_2v_{\chi\chi\chi} + 9\tau h_1^2vv_{\chi\chi\chi}\\
&+3\tau\chi h_1^2vv_{\chi\chi\chi\chi} - 21\tau\mu h_1h_2vv_{\chi\chi\chi\chi} - 6\tau^2h_1^2v_{\chi\chi\chi}v_\tau + 96\tau\mu h_1vv_{\tau\chi} + 6\tau^2h_1^2v_{\chi\chi}v_{\tau\chi}\\
&-32\tau^2h_1^2v_\tau v_\chi{}^2 + 12\mu h_1v_{\chi\chi} - 18\tau\mu h_1^2vv_{\chi\chi\chi\mu} + 6\tau^2h_1^2vv_{\tau\chi\chi\chi} - 6\tau\mu h_1^2v_{\chi\chi\mu}v_\chi\\
&+3\tau\mu h_1h_2v_{\chi\chi}{}^2 - 6\tau\mu h_1^2v_\mu v_{\chi\chi\chi} - 32\tau\mu h_1^2v_\mu v_\chi{}^2\Big);
\end{aligned}
$$

$$
T_4^{\tau} = v_\chi v_\mu - v_{\chi\mu}v,
$$

$$
\begin{aligned}
T_4^{\chi} &= \frac{1}{24}\Big(16h_1 v_\mu v_{\chi\mu} v + 16h_1 v_\chi v_{\mu\mu} v + 48h_2 v_\chi v_{\chi\mu} v + 9h_1 v_{\chi\chi\mu\mu} v + 12h_2 v_{\chi\chi\chi\mu} v - 24 v_{\tau\mu} v \\
&\quad -32h_1 v_\chi {v_\mu}^2 - 48h_2 {v_\chi}^2 v_\mu - 12h_1 v_\mu v_{\chi\chi\mu} - 12h_2 v_{\chi\chi\chi} v_\mu + 6h_1 {v_{\chi\mu}}^2 - 6h_1 v_\chi v_{\chi\mu\mu} \\
&\quad +3h_1 v_{\chi\chi} v_{\mu\mu} + 12h_2 v_{\chi\chi} v_{\chi\mu} - 12h_2 v_\chi v_{\chi\chi\mu} + 24 v_\tau v_\mu\Big), \\
T_4^{\mu} &= \frac{1}{24}\Big(- 16h_1 v_\chi v_{\chi\mu} v - 48h_2 v_{\chi\chi} v_\chi v - 16h_1 v_{\chi\chi} v_\mu v - 9h_1 v_{\chi\chi\chi\mu} v - 12h_2 v_{\chi\chi\chi\chi} v \\
&\quad +48 v_{\tau\chi} v - 16h_1 {v_\chi}^2 v_\mu - 3h_1 v_\chi v_{\chi\chi\mu} + 3h_1 v_{\chi\chi} v_{\chi\mu} - 3h_1 v_{\chi\chi\chi} v_\mu\Big);
\end{aligned}
$$

$$
\begin{aligned}
T_5^{\tau} &= \frac{1}{6}\Big(- 8h_1 v_\chi v_{\chi\mu} v - 4h_1 v_{\chi\chi} v_\mu v - 12h_2 v_\chi v_{\chi\chi} v - 3h_1 v_{\chi\chi\chi\mu} v - 3h_2 v_{\chi\chi\chi\chi} v + 6 v_{\tau\chi} v + 6 v_\tau v_\chi\Big), \\
T_5^{\chi} &= \frac{1}{24}\Big(16h_1 v_\chi v_{\tau\mu} v + 48h_2 v_\chi v_{\tau\chi} v + 16h_1 v_\mu v_{\tau\chi} v + 9h_1 v v_{\tau\chi\chi\mu} + 12h_2 v_{,\tau\chi\chi\chi} v - 24 v_{\tau\tau} v \\
&\quad -6h_1 v_\chi v_{\tau\chi\mu} - 9h_1 v_\tau v_{\chi\chi\mu} + 3h_1 v_{\chi\chi} v_{\tau\mu} + 6h_1 v_{\tau\chi} v_{\chi\mu} - 3h_1 v_\mu v_{\tau\chi\chi} - 48h_2 v_\tau {v_\chi}^2 \\
&\quad -12h_2 v_\tau v_{\chi\chi\chi} + 12h_2 v_{\chi\chi} v_{\tau\chi} + 24 {v_\tau}^2 - 32h_1 v_\tau v_\chi v_\mu - 12h_2 v_{\ \chi} v_{\tau\chi\chi}\Big), \\
T_5^{\mu} &= \frac{1}{24}\Big(16h_1 v_\chi v_{\tau\chi} v + 3h_1 v_{\tau\chi\chi\chi} v - 16h_1 v_\tau {v_\chi}^2 - 3h_1 v_\chi v_{\tau\chi\chi} - 3h_1 v_\tau v_{,\chi\chi\chi} + 3h_1 v_{\chi\chi} v_{\tau\chi}\Big);
\end{aligned}
$$

$$
\begin{aligned}
T_6^{\tau} &= \frac{1}{h_1^2}\Big(2f(\tau)h_1^2 {v_{,\chi}}^3 + 4\mu h_1 f'(\tau){v_\chi}^2 + 8\mu^2 f''(\tau)v_\chi - 4f(\tau)h_1^2 v v_{\chi\chi} v_\chi + f(\tau)h_1^2 v_{\chi\chi\chi} v_\chi \\
&\quad -f(\tau)h_1^2 v v_{\chi\chi\chi\chi} - 4\mu h_1 v f'(\tau) v_{\chi\chi}\Big), \\
T_6^{\chi} &= \frac{1}{24h_1^2}\Big(- 108f(\tau)h_1^2 h_2 {v_\chi}^4 - 192\mu h_1 h_2 f'(\tau){v_\chi}^3 - 72f(\tau)h_1^3 v_\mu {v_\chi}^3 + 16h_1^2 v f'(\tau){v_\chi}^2 \\
&\quad -128\mu h_1^2 f'(\tau)v_\mu {v_\chi}^2 - 72f(\tau)h_1^3 v v_{\chi\mu} {v_\chi}^2 - 10f(\tau)h_1^3 v_{\chi\chi\mu} {v_\chi}^2 - 72f(\tau)h_1^2 h_2 v_{\chi\chi\chi} {v_\chi}^2 \\
&\quad -192\mu^2 h_1 f''(\tau)v_\mu v_\chi - 64\mu h_1^2 v f'(\tau)v_{\chi\mu} v_\chi - 12h_1^2 f'(\tau)v_{\chi\chi} v_\chi + 28f(\tau)h_1^3 v_{\chi\mu} v_{\chi\chi} v_\chi \\
&\quad -28f(\tau)h_1^3 v_\mu v_{\chi\chi\chi} v_\chi - 28f(\tau)h_1^3 v v_{\chi\chi\chi\mu} v_\chi + 6f(\tau)h_1^3 v_{\chi\chi\chi\chi\mu} v_\chi + 96\mu h_1 f'(\tau)v_\tau v_\chi \\
&\quad -48f(\tau)h_1^2 v_{\tau\chi\chi} v_\chi + 48\mu h_1 h_2 f'(\tau){v_{\chi\chi}}^2 - 28f(\tau)h_1^3 v_\mu {v_{\chi\chi}}^2 - 12f(\tau)h_1^2 h_2 {v_{\chi\chi\chi}}^2 \\
&\quad +36\mu h_1^2 f'(\tau)v_{\chi\mu} v_{\chi\chi} - 72\mu^2 h_1 f''(\tau)v_{\chi\chi\mu} - 8f(\tau)h_1^3 v v_{\chi\chi} v_{\chi\chi\mu} + 12h_1^2 v f'(\tau)v_{\chi\chi\chi} \\
&\quad -28f(\tau)h_1^3 v v_{\chi\mu} v_{\chi\chi\chi} - 9f(\tau)h_1^3 v_{\chi\chi\mu} v_{\chi\chi\chi} - 12\mu h_1^2 v f'(\tau)v_{\chi\chi\chi\mu} - 9f(\tau)h_1^3 v_{\chi\chi} v_{\chi\chi\chi\mu} \\
&\quad -3f(\tau)h_1^3 v_\mu v_{\chi\chi\chi\chi\chi} - 3f(\tau)h_1^3 v v_{\chi\chi\chi\chi\chi\mu} + 192\mu^2 f''(\tau)v_\tau + 24f(\tau)h_1^2 v_{\chi\chi\chi} v_\tau \\
&\quad +24f(\tau)h_1^2 v v_{\tau\chi\chi\chi} - 96\mu h_1 h_2 f'(\tau)v_{\chi\chi\chi} v_\chi + 96\mu h_1 v f''(\tau)v_\chi + 48f(\tau)h_1^2 v_{\chi\chi} v_{\tau\chi} \\
&\quad +48\mu h_1 f''(\tau)v_{\chi\chi} - 288\mu^2 h_2 f''(\tau){v_\chi}^2 + 48f(\tau)h_1^2 v_\tau {v_\chi}^2 - 60\mu h_1^2 f'(\tau)v_{\chi\chi\mu} v_\chi \\
&\quad -12\mu h_1^2 f'(\tau)v_\mu v_{\chi\chi\chi} + 96\mu h_1 v f'(\tau)v_{\tau\chi} + 96f(\tau)h_1^2 v v_{\tau\chi} v_\chi - 192\mu^2 v f'''(\tau) \\
&\quad -96\mu^2 h_2 f''(\tau)v_{\ \chi\chi\chi} + 6f(\tau)h_1^3 v_{\chi\mu} v_{\chi\chi\chi\chi}\Big), \\
T_6^{\mu} &= \frac{1}{24h_1}\Big(- 36f(\tau)h_1^2 {v_\chi}^4 - 64\mu h_1 f'(\tau){v_\chi}^3 - 96\mu^2 f''(\tau){v_\chi}^2 + 72f(\tau)h_1^2 v v_{\chi\chi} {v_\chi}^2 \\
&\quad -34f(\tau)h_1^2 v_{\chi\chi\chi} {v_\chi}^2 - 24\mu h_1 f'(\tau)v_{\chi\chi\chi} v_\chi + 28f(\tau)h_1^2 v v_{\chi\chi\chi\chi} v_\chi - 3f(\tau)h_1^2 v_{\chi\chi\chi\chi\chi} v_{,\chi} \\
&\quad +12\mu h_1 f'(\tau){v_{\chi\chi}}^2 - 3f(\tau)h_1^2 {v_{\chi\chi\chi}}^2 + 64\mu h_1 v f'(\tau)v_{\chi\chi} v_\chi + 3f(\tau)h_1^2 v v_{\chi\chi\chi\chi\chi\chi} \\
&\quad -24\mu^2 f''(\tau)v_{\chi\chi\chi} + 36f(\tau)h_1^2 v v_{\chi\chi} v_{\chi\chi\chi} + 12\mu h_1 v f'(\tau)v_{\chi\chi\chi\chi} + 3f(\tau)h_1^2 v_{\chi\chi} v_{\chi\chi\chi\chi}\Big);
\end{aligned}
$$

$$
T_7^{\tau} = \frac{-h_1 r(\tau)v_{\chi\chi} v + h_1 r(\tau){v_\chi}^2 + 4\mu r'(\tau)v_\chi}{h_1},
$$

$$\begin{aligned} T_7^{\chi} &= \frac{1}{24h_1}\Big(24h_1r'(\tau)v_\chi v - 16h_1^2r(\tau)v_\chi v_{\chi\mu}v - 3h_1^2r(\tau)v_{\chi\chi\chi\mu}v + 24h_1r(\tau)v_{\tau\chi}v - 96\mu r''(\tau)v \\ &-96h_1\mu r'(\tau)v_\chi v_\mu - 36h_1\mu r'(\tau)v_{\chi\chi\mu} - 48h_2\mu r'(\tau)v_{\chi\chi\chi} + 12h_1r'(tau)v_{\chi\chi} - 32h_1^2r(\tau){v_\chi}^2v_\mu \\ &-48h_1h_2r(\tau){v_\chi}^3 - 24h_1h_2r(\tau)v_{\chi\chi\chi}v_\chi + 24h_1r(\tau)v_\tau v_\chi + 12h_1h_2r(\tau){v_{\chi\chi}}^2 + 96\mu r'(\tau)v_\tau \\ &+9h_1^2r(\tau)v_{\chi\chi}v_{\chi\mu} - 144h_2\mu r'(\tau){v_\chi}^2 - 15h_1^2r(\tau)v_\chi v_{\chi\chi\mu} - 3h_1^2r(\tau)v_{\chi\chi\chi}v_\mu\Big), \end{aligned}$$

$$\begin{aligned} T_7^{\mu} &= \frac{1}{24}\Big(16h_1r(\tau)v_{\chi\chi}v_\chi v + 3h_1r(\tau)v_{\chi\chi\chi\chi}v - 16h_1r(\tau){v_\chi}^3 - 6h_1r(\tau)v_{\chi\chi\chi}v_\chi + 3h_1r(\tau){v_{\chi\chi}}^2 \\ &-48\mu r'(\tau){v_\chi}^2 - 12\mu r'(\tau)v_{\chi\chi\chi}\Big); \end{aligned}$$

$$\begin{aligned} T_8^{\tau} &= 2g(\tau)v_\chi, \\ T_8^{\chi} &= \frac{1}{4}\left(-8g'(\tau)v - 8h_1g(\tau)v_\chi v_\mu - 3h_1g(\tau)v_{\chi\chi\mu} - 12h_2g(\tau){v_\chi}^2 - 4h_2g(\tau)v_{\chi\chi\chi} + 8g(\tau)v_\tau\right), \\ T_8^{\mu} &= \frac{1}{4}\left(-4h_1g(\tau){v_\chi}^2 - h_1g(\tau)v_{\chi\chi\chi}\right); \end{aligned}$$

Physical principles such as the conservation of energy, mass, and momentum are mathematically formulated as conservation laws. These laws are fundamental for the analysis and simplification of partial differential equations. The laws of conservation of momentum, energy, and angular momentum are originally derived from classical mechanics. However, these conservation laws hold true and are preserved in both relativistic mechanics and quantum mechanics. As a result, they transcend classical mechanics and stand as some of the most fundamental and universally valid principles in physics. Conservation laws have been extensively employed in investigating the existence, uniqueness, and stability of solutions to nonlinear partial differential equations, as well as in the development of numerical methods for these equations. It is important to recognize that an infinite number of conservation laws can be derived, given that the multiplier involves an arbitrary function.

6. CONCLUDING REMARKS

Eight symmetries were found when we calculated the Lie point symmetries of the modified (2+1)-dimensional KdV-Calogero Bogoyavlenkskii-Schiff equation. We performed group invariant solutions and symmetry reductions. After that, we obtained one-wave and two-wave solutions using the multiple exp-function approach. Lastly, we used the multiplier approach to derive the conserved currents. In the future, the modified (2+1)-dimensional KdV-Calogero Bogoyavlenkskii-Schiff equation can be solved more precisely using conserved currents.

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ОПИС МОДИФІКОВАНОГО (2+1)-ВИМІРНОГО РІВНЯННЯ KdV-КАЛОДЖЕРО БОГОЯВЛЕНСЬКОГО-ШИФФА У ХВИЛЯХ НА МІЛКОВОДДІ: АНАЛІЗ СИМЕТРІЇ ЛІ ТА ЗАКОНИ ЗБЕРЕЖЕННЯ

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У цій роботі досліджується рівняння KdV-Калоджеро Богоявленського-Шиффа у модифікованих вимірах (2+1). Симетрії Лі створюються за допомогою аналізу групи Лі, а потім за допомогою цих симетрій знаходяться розв'язки подібності. Використовуючи метод множинних експоненційних функцій, отримано численні хвильові розв'язки. Крім того, ми використовуємо метод множників для вираження збережених струмів, що є важливим для розуміння природи нелінійних рівнянь, зокрема в інженерії та фізиці.

Ключові слова: *рівняння KdV-Калоджеро Богоявленського-Шиффа; збережені струми; аналіз методом групи Лі; метод множинних експоненційних функцій*