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# HYBRID SOLITARY WAVES AND SHOCK WAVES FOR DOUBLE–LAYERED FLUID FLOW WITH DISPERSION TRIPLET: ZAREMAOGHADDAM AND GEAR–GRIMSHAW MODELS (mKdV EQUATION)

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The current paper recovers hybrid solitary waves for double–layered shallow water waves with the basic platform being the mKdV equation. The selected models are the Zaremaoghaddam equation and the Gear–Grimshaw equation. The integration algorithm adopted is the generalized exponential differential function method. This yields hybrid waves that emerge from solitary waves, shock waves and the singular solitary waves. The existence criteria for such waves are also presented as parameter constraints.

**Keywords:** *mKdV equation; Hybrid; Integrability; Parameter constraints; Gear–Grimshaw equation*

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## 1. INTRODUCTION

The concept of double–layered shallow water waves existed in the literature of Fluid Dynamics for a few decades. There are several aspects to this that has been successfully addressed and reported in the past. The current paper revisits the double–layered shallow water wave dynamics but from a fresher perspective. The concept of dispersion triplet was first introduced in 1977 and was later implemented during 2022 to study the standard fundamental models from shallow water wave dynamics [1, 2]. After a long hiatus, the 2022 implementation incorporated dispersion–triplet into KdV and mKdV–type settings and, within the traveling–wave framework, retrieved solitary and shock structures together with the associated conservation laws [1]. The current paper recovers the hybrid of solitary waves and shock waves for double–layered fluid flow with modified Korteweg–de Vries equation (mKdV) equation as its platform. The two governing models that were considered in this paper were Zaremaoghaddam model and the Gear–Grimshaw model. It must be noted that the same study has been previously carried out with KdV equation as its platform [3]. Thus, the present contribution constitutes the first dispersion–triplet double–layered revisit in which mKdV (rather than KdV) serves as the platform, thereby extending the double–layered dispersion–triplet framework to cubic nonlinearity. Meanwhile, double–layered flow itself remains a steady research theme; see, e.g., layered shallow-water approximations and stable or energy-consistent schemes [4, 5], and related solitary-wave dynamics in near-integrable regimes [6], alongside coupled-model developments such as Zaremaoghaddam–type formulations [7]. There are several additional double–layered models that exist in the literature such as the Bona–Chen equation, the current paper focuses on the aforementioned couple of models only. Additional models (e.g., Bona–Chen [8]) will be pursued in forthcoming work.

The adopted integration algorithm in this work is the generalized exponential differential rational function method which is an efficient one as compared to the pre–existing similar such schemes [9–14]. This integration scheme retrieves hybrid solitary waves and shock waves to the two models that were constructed with mKdV equation as their platform. One of the several shortcomings of this adopted integration approach is its failure to recover the solitary wave radiation as well as the phonons' component of shock waves. Such are only recoverable with the usage of inverse scattering transform

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that is outside the scope of this paper. The details are exhibited in the rest of the paper after a quick display of the two governing models.

From the modeling perspective, modern two-layer shallow-water theory is often developed as a systematic long-wave reduction of the full water-wave problem, with the Green–Naghdi framework providing a canonical fully nonlinear, weakly dispersive baseline [15,16]. In particular, refined two-layer Green–Naghdi and Boussinesq-type closures have been constructed to improve frequency-dispersion properties and to better represent interfacial mode interactions [17,18]. Such formulations connect naturally with classical internal-wave models in two-fluid systems and support coherent interfacial structures of practical interest [19,20]. In parallel, the coupling between surface and internal modes, including barotropic–baroclinic transfer and resonance mechanisms, has been studied in several contexts, ranging from analytical mode-coupling descriptions to topographically mediated resonant exchange [21,22].

At the level of reduced evolution equations, modified Korteweg–de Vries (mKdV) dynamics occupy a central role in nonlinear dispersive-wave theory, with a well-developed analytical foundation for well-posedness and global dynamics in both the line and periodic settings [23]. Beyond the classical scalar model, integrable and near-integrable generalizations continue to be developed, providing structured coupled dynamics and enlarged families of coherent waves [24]. Moreover, coupled and multi-component mKdV systems exhibit rich asymptotic behavior and long-time dynamics, complementing the study of higher-order coherent structures such as breather families in higher-order mKdV settings [25, 26]. These advances motivate revisiting double-layered hydrodynamic models on an mKdV platform, where coupling and dispersive extensions may yield hybrid morphologies beyond the classical single-field picture.

Alongside these modeling developments, there has been sustained progress in constructive methods for obtaining exact solutions of nonlinear evolution equations. Representative strands include the extended $F$-expansion methodology [27], transformed rational-function approaches capable of generating complexiton-type structures [28], and rational-exponential constructions for fractional or generalized equal-width models [29]. More recently, generalized Exp-function frameworks have been successfully applied to dispersive hydrodynamic equations motivated by wave phenomena, illustrating the flexibility of exponential-ansatz technology in producing closed-form wave families [30].

### 1.1. Governing Models

There are several models to address double-layered shallow water waves. With the basic platform at mKdV equation, the two models that ate commonly addressed are going to be taken uop in this paper. They are the Zaremaoghaddam model and the Gear-Grimshaw equation. One model that has been tacitly omitted is the Bona—Chen equation. These two models are first introduced in the next subsection.

**1.1.1. Zaremaoghaddam Model.** The interesting dynamics of two-layered water waves are described by Zareomaghoddam mKdV model, in which $q(x,t)$ and $r(x,t)$ represent the wave variables of the two layers. This study consist of examination of the following structure of Zareomaghoddam mKdV model, in order to provide exact solutions for the very first and foremost time in the literature:

$$\begin{aligned} q_t + a_1 q^2 q_x + a_2 r^2 q_x + a_3 q_{xxx} + a_4 q_{xxt} + a_5 q_{xtt} &= 0, \\ r_t + b_1 r^2 r_x + b_2 q^2 r_x + b_3 r_{xxx} + b_4 r_{xxt} + b_5 r_{xtt} &= 0, \end{aligned} \tag{1}$$

such that $a_i,\ b_i,\ i = 1, 2, 3, 4, 5$ display the physical parameters related to model (1).

**1.1.2. Gear–Grimshaw Model.** Gear-Grimshaw (GG) mKdV model, with spatial and time variables $x$ and $t$, and dependent variables as $q(x,t)$ and $r(x,t)$, is mentioned as below:

$$\begin{aligned} q_t + a_1 q^2 q_x + a_2 q_{xxx} + a_3 q_{xxt} + a_4 q_{xtt} &= a_5 r r_x + a_{10}\left(q^2 r\right)_x + a_6\left(q r^2\right)_x + a_7 r_{xxx} + a_8 r_{xxt} + a_9 r_{xtt}, \\ r_t + b_1 r^2 r_x + b_2 r_{xxx} + b_3 r_{xxt} + b_4 r_{xtt} &= b_5 q q_x + b_6 (r^2 q)_x + b_{10}\left(r q^2\right)_x + b_7 q_{xxx} + b_8 q_{xxt} + b_9 q_{xtt}. \end{aligned} \tag{2}$$

The first term in (2) represents the temporal evolution, and $a_1,\ a_2,\ a_3,\ a_4$ represents the coefficients of dispersion terms, with similar while the coefficients of $b_1,\ b_2,\ b_3,\ b_4$ accountfor nonlinearity. For $j = 5,\ 6,\ 7,\ 8,\ 9,\ 10$, the coupling terms are represented by the coefficients $a_j,\ b_j$. This model provides a good description of a resonant interaction between traverse internal gravity wave modes in a stratified liquid.

The structure of this study is described as: An overview of the employed technique, referred as generalized exponential differential rational function technique, is given in Section 2. A variety of exact solutions are found in Section 3, respectively, by applying the stated procedure to the models (1) and (2). Finally, a thorough, succinct words of conclusion is appended to complement a few simulations of these research findings.

## 2. THE GENERALIZED EXPONENTIAL DIFFERENTIAL RATIONAL FUNCTION METHOD (GEDRF): AN OVERVIEW

The algorithmic perspective of the generalized exponential rational function (GEDRF) technique is presented in this section, and we have briefly described each stage in this procedure. It offers a variety of analytical solutions for various physical phenomena, being expressed in terms of NLPDEs, by combining exponential and rational functions. The following detailed explanations can be used to develop a number of exact solutions for nonlinear evolution equations (NLEEs):

**Step–1**: The general form of NLEE with variable $\Lambda(x,t)$ is considered as follows:

$$\rho(\Lambda, \Lambda_x, \Lambda_t, \Lambda_{xx}, \Lambda_{tt}, \Lambda_{xt}, \Lambda_{xxt}, \ldots) = 0. \tag{3}$$

The subsequent transformation is applied to (3) for seeking reduced ordinary differential equation (ODE) in terms of new variables as $\sigma,\ \tau$:

$$\Lambda(x,t) = \sigma(\tau), \quad \text{with} \quad \tau = x - p_1 t. \tag{4}$$

The expression (4) is plugged into equation (3) and hence resulting ODE is produced as follows:

$$\mathcal{M}(\sigma, \sigma', \sigma'', \sigma''' \ldots) = 0. \tag{5}$$

**Step–2**: Then after, ODE (5) is assumed to have following solution structure in terms of new dependent variable $\mathcal{T}(\tau)$ as:

$$\mathcal{M}(\zeta) = A_0 + \sum_{j=1}^{N} A_j \left(\frac{d^j}{d\tau^j}\mathcal{T}(\tau)\right)^j + \sum_{i=1}^{N} B_i \left(\frac{d^i}{d\tau^i}\mathcal{T}(\tau)\right)^{-i}, \tag{6}$$

also, $\mathcal{T}(\tau)$ is characterized as:

$$\mathcal{T}(\tau) = \frac{R_1 \exp(S_1\tau) + R_2 \exp(S_2\tau)}{R_3 \exp(S_3\tau) + R_4 \exp(S_4\tau)}. \tag{7}$$

**Step–3**: Balancing principle is used in ODE (5) for calculating the corresponding value of $N$ for expression (6). One can choose the parameters $R_1, R_2, R_3, R_4, S_1, S_2, S_3$, and $S_4$ so that the function $\mathcal{T}(\tau)$ can be well described in terms of hyperbolic functions for seeking new exact solutions for considered model. Additionally, $A_0$, $A_j$, and $B_i$, $1 \le i, j \le N$ are determined in main procedure of computations.

**Step–4**: In this step, equation (6) is clubbed with equation (5), resulting into polynomial equation and then all of the coefficients in this polynomial are set to zero, creating a system of algebraic equations.

**Step–5**: In order to determine exact solutions for equation (3), the system of equations produced in the previous stage is investigated to explore the appropriate values of the involved parameters. These parameter values are then inserted into equations (6) and (7), for procuring the final closed form solutions of considered model.

The GEDRF scheme used here can be viewed as an ansatz-based traveling-wave approach. In this sense, it is related in spirit to classical methods such as the tanh-method, $F$-expansion, and exp-function approaches. The distinguishing feature is the choice of generating function $\mathcal{T}(\tau)$ as a rational combination of exponentials and the resulting solution structure, which admits both "direct" and "inverse/reciprocal" derivative contributions (cf. the trial form and the rational-exponential definition of $\mathcal{T}(\tau)$). By suitable selections of parameters in the rational-exponential representation, $\mathcal{T}(\tau)$ reduces to standard hyperbolic building blocks (e.g., tanh, coth, sech, csch), so many individual solutions can, in principle, be reproduced using traditional ansätze.

However, the practical advantage of GEDRF is that it provides a single unified template from which multiple waveform families—regular solitary profiles as well as singular/hybrid profiles—are generated by parameter choices within the same algebraic framework, together with the corresponding parameter constraints ("existence conditions"). In contrast, reproducing the entire catalogue typically requires multiple standard ansätze (polynomial-in-tanh, polynomial-in-exp, singular coth/csch ansätze, etc.), each treated separately with its own balancing and coefficient matching. The contribution of this work is therefore not to claim that ansatz methods are new, but to provide a systematic, family-wise construction of exact solutions and parameter regimes for the considered coupled mKdV-based double-layer models using this unified rational-exponential ansatz.

## 3. APPLICATION TO DOUBLE–LAYERED FLUID FLOW

### 3.1. Zaremaoghaddam Model

Generalized exponential differential function (GEDRF) method has been implemented to system of NLEEs, termed as Zareomaghoddam mKdV model (1) in this phase of the study. The primary objective is to investigate multiple novel exact soliton solutions to the governing model (1), which could greatly progress pertinent physical circumstances related to two layer liquid waves dynamics.

In accordance with GEDRF method, we have rewritten equation (1) in the form of an ordinary differential equation by using the wave translation

$$q(x,t) = K(\tau), \quad r(x,t) = L(\tau) \quad \text{where} \quad \tau = x - p_1 t, \tag{8}$$

with $p_1$ as free parameter, representing wave velocity, which could be evaluated in mean implementation procedure.

Then, the resultant ODEs are procured as follows, utilizing (8) in equation (1) :

$$\begin{aligned} &\left(a_5 {p_1}^2 - a_4 p_1 + a_3\right) K'''(\tau) + \left((K(\tau))^2 a_1 + (L(\tau))^2 a_2 - p_1\right) K'(\tau) = 0, \\ &\left(b_5 {p_1}^2 - b_4 p_1 + b_3\right) L'''(\tau) + \left((K(\tau))^2 b_2 + (L(\tau))^2 b_1 - p_1\right) L'(\tau) = 0. \end{aligned} \tag{9}$$

After applying the balancing principle on the highest order derivative terms and a nonlinear term of equation (9), we have obtained that $N = 1$. Thus, from equation (6), the following trial solution structure for equation (9) is proposed:

$$\begin{aligned} K(\tau) &= \theta_1 \mathcal{T}'(\tau) + \frac{\omega_1}{\mathcal{T}'(\tau)} + \theta_0, \\ L(\tau) &= \theta_2 \mathcal{T}'(\tau) + \frac{\omega_2}{\mathcal{T}'(\tau)} + \theta_3, \end{aligned} \tag{10}$$

Here, the parameters $\theta_i, i = 0,1,2,3,$ and $\omega_j, j = 1,2$ are to be assessed in all calculation process. Next, using (10), the system (9) is converted into another system of ODEs with the new dependent variable $\mathcal{T}(\tau)$. Following that, work is being carried out to obtain exact solitary wave solutions for the system of ODEs provided in equation (9), as follows:

**Family–I**:

Putting up $[R_1, R_2, R_3, R_4] = [1, -1, 1, 1]$ and $[S_1, S_2, S_3, S_4] = [1, -1, 0, 0]$ in equation (7), we have retrieved hyperbolic function as follows:

$$\mathcal{T}(\tau) = \sinh(\tau). \tag{11}$$

Setting equation (11) in equation (10) for equation (9), the following coefficient values are acquired:

$$\begin{aligned} &a_1 = b_2, \ a_2 = -\frac{b_2 {\theta_1}^2}{{\theta_2}^2}, \ a_3 = -16\, a_5 {b_2}^2 {\omega_1}^2 {\theta_1}^2 + 4\, a_4 b_2 \omega_1 \theta_1, \\ &b_1 = -\frac{b_2 {\theta_1}^2}{{\theta_2}^2}, \ b_3 = -4\, b_2 \omega_1 \theta_1 \left(4\, b_2 b_5 \omega_1 \theta_1 - b_4\right), \\ &\omega_2 = -\frac{\omega_1 \theta_2}{\theta_1}, p_1 = 4\, b_2 \omega_1 \theta_1, \ \theta_0 = 0, \ \theta_3 = 0, \end{aligned} \tag{12}$$

along with $a_4,\ a_5,\ b_4,\ b_5,\ b_2,\ \omega_1,\ \theta_2,\ \theta_1$ as free parameters.

Hence, the solution of system (9) is represented with help of parameter values (12) as

$$\begin{aligned} K(\tau) &= \theta_1 \cosh(\tau) + \frac{\omega_1}{\cosh(\tau)}, \\ L(\tau) &= \theta_2 \cosh(\tau) - \frac{\omega_1 \theta_2}{\theta_1 \cosh(\tau)}. \end{aligned} \tag{13}$$

Finally, the newly constructed solution of model (1) is depicted in terms of original variables $x,\ t$ as

$$\begin{aligned} q(x,t) &= \theta_1 \cosh(x - p_1 t) + \frac{\omega_1}{\cosh(x - p_1 t)}, \\ r(x,t) &= \theta_2 \cosh(x - p_1 t) - \frac{\omega_1 \theta_2}{\theta_1 \cosh(x - p_1 t)}. \end{aligned} \tag{14}$$

**Family–II**:

Letting $[R_1, R_2, R_3, R_4] = [1, 1, 1, 1]$ and $[S_1, S_2, S_3, S_4] = [1, -1, 0, 0]$ in equation (7), leads to subsequent function:

$$\mathcal{T}(\tau) = \cosh(\tau). \tag{15}$$

The following coefficient values are generated after setting equation (15) in equation (10) for equation (9):

$$\begin{aligned} &a_1 = b_2,\ a_2 = -\frac{b_2{\theta_1}^2}{{\theta_2}^2}, a_3 = -\frac{4b_2\omega_2{\theta_1}^2\left(4\,a_5b_2\omega_2{\theta_1}^2 + a_4\theta_2\right)}{{\theta_2}^2},\ b_1 = -\frac{b_2{\theta_1}^2}{{\theta_2}^2},\\ &b_3 = -\frac{4b_2\omega_2{\theta_1}^2\left(4\,b_2b_5\omega_2{\theta_1}^2 + b_4\theta_2\right)}{{\theta_2}^2},\ \omega_1 = -\frac{\omega_2\theta_1}{\theta_2},\\ &p_1 = -\frac{4b_2\omega_2{\theta_1}^2}{\theta_2},\ \theta_0 = 0,\ \theta_3 = 0, \end{aligned} \tag{16}$$

along with $a_4,\ a_5,\ b_4,\ b_5,\ b_2,\ \omega_2,\ \theta_2,\ \theta_1$ as free parameters.

Consequently, exact solution of system (1) is depicted with the help of (16) and (15) and reverting back into original variables $x,\ t$ as follows:

$$\begin{aligned} q(x,t) &= \theta_1 \sinh\left(x - p_1t\right) - \frac{\omega_2\theta_1}{\theta_2 \sinh\left(x - p_1t\right)},\\ r(x,t) &= \theta_2 \sinh\left(x - p_1t\right) + \frac{\omega_2}{\sinh\left(x - p_1t\right)}. \end{aligned} \tag{17}$$

**Family–III**:

Regarding $[R_1, R_2, R_3, R_4] = [1, -1, 1, 1]$ and $[S_1, S_2, S_3, S_4] = [1, -1, 1, -1]$ in equation (7), we have found

$$\mathcal{T}(\tau) = \tanh(\tau). \tag{18}$$

The following set of coefficients are obtained from equation (9) by substituting (10) and (18):

$$\begin{aligned} &a_1 = b_2,\ a_2 = -\frac{b_2{\theta_1}^2}{{\theta_2}^2},\ a_3 = 4\,b_2\omega_1\theta_1\left(-4\,a_5b_2\omega_1\theta_1 + a_4\right),\\ &b_1 = -\frac{b_2{\theta_1}^2}{{\theta_2}^2},\ b_3 = -4\,b_2\omega_1\theta_1\left(4\,b_2b_5\omega_1\theta_1 - b_4\right),\ \omega_2 = -\frac{\omega_1\theta_2}{\theta_1},\\ &p_1 = 4\,b_2\omega_1\theta_1,\ \theta_0 = 0,\ \theta_3 = 0, \end{aligned} \tag{19}$$

with $a_4,\ a_5,\ b_4,\ b_5,\ b_2,\ \omega_1,\ \theta_2,\ \theta_1$ as free parameters.

Thus, utilizing (19) and (18) in equation (10), the following solution of equation (9) is found:

$$\begin{aligned} K(\tau) &= \theta_1\left(\operatorname{sech}(\tau)\right)^2 + \frac{\omega_1}{\left(\operatorname{sech}(\tau)\right)^2},\\ L(\tau) &= \theta_2\left(\operatorname{sech}(\tau)\right)^2 - \frac{\omega_1\theta_2}{\theta_1\left(\operatorname{sech}(\tau)\right)^2}. \end{aligned} \tag{20}$$

Then after the exact solution of equation (1) is written in terms of original independent and dependent variables $x,\ t$ with the assistance of equation (20) and equation (19) as follows:

$$\begin{aligned} q\left(x,t\right) &= \theta_1\left(\operatorname{sech}\left(x - p_1t\right)\right)^2 + \frac{\omega_1}{\left(\operatorname{sech}\left(x - p_1t\right)\right)^2},\\ r\left(x,t\right) &= \theta_2\left(\operatorname{sech}\left(x - p_1t\right)\right)^2 - \frac{\omega_1\theta_2}{\theta_1\left(\operatorname{sech}\left(x - p_1t\right)\right)^2}. \end{aligned} \tag{21}$$

### 3.2. Gear–Grimshaw Model

This portion of current study involves the execution of GEDRF method to the system of NLEEs (2), describing the dynamics of resonant interaction between the internal gravity wave modes in a thin stratified liquid. It lead to curation of several novel soliton solutions to the governing model (2), equipped with a number of free parameters, which have considerable ability to illustrate crucial physical situations.

The subsequent variable transformation is executed to reduce equations (2) to system of ODEs:

$$q(x,t) = U(\tau), \quad r(x,t) = V(\tau) \quad \text{where} \quad \tau = x - p_1 t, \tag{22}$$

with $p_1$ as free parameter, representing wave velocity, which is evaluated in computational procedure.
Then, resultant system of ODEs is obtained by considering (22) in (2) as follows:

$$\begin{aligned}
&\left(-a_{10}\,(U(\tau))^2 - 2\,a_6 V(\tau)\,U(\tau) - a_5 V(\tau)\right) V'(\tau) + \left(a_1\,(U(\tau))^2 - 2\,a_{10} U(\tau)\,V(\tau) - a_6\,(V(\tau))^2 - p_1\right) U'(\tau)\\
&+ \left({p_1}^2 a_4 - p_1\,a_3 + a_2\right) U'''(\tau) + \left(-{p_1}^2 a_9 + p_1\,a_8 - a_7\right) V'''(\tau) = 0,\\
&\left(-b_{10}\,(U(\tau))^2 - 2\,b_6 V(\tau)\,U(\tau) + b_1\,(V(\tau))^2 - p_1\right) V'(\tau) + \left(-2\,b_{10} U(\tau)\,V(\tau) - b_6\,(V(\tau))^2 - b_5 U(\tau)\right) U'(\tau)\\
&+ \left({p_1}^2 b_4 - p_1\,b_3 + b_2\right) V'''(\tau) + \left(-{p_1}^2 b_9 + p_1\,b_8 - b_7\right) U'''(\tau) = 0.
\end{aligned} \tag{23}$$

As stated in Step–3 of section 2, we have calculated $N = 1$ using the balancing principle in equation (23). Equation (23) yielded the following solution structure:

$$\begin{aligned}
U(\tau) &= \theta_1 \mathcal{T}'(\tau) + \frac{\omega_1}{\mathcal{T}'(\tau)} + \theta_0,\\
V(\tau) &= \theta_4 \mathcal{T}'(\tau) + \frac{\omega_4}{\mathcal{T}'(\tau)} + \theta_5,
\end{aligned} \tag{24}$$

The parameters $\theta_i, i = 0,1,4,5$ and $\omega_j, j = 1,4$ can be assessed in the mean way of the entire computations. Next, equation (24) is used to convert the system (23) into another system of ODEs, with a new dependent variable $\mathcal{T}(\tau)$. After then, investigation is being done to obtain exact solitary wave solutions for the system of ODEs provided in equation (23), as follows:

**Family–I**:
The equation (7) is appraised with $[R_1, R_2, R_3, R_4] = [1, -1, 1, 1]$ and $[S_1, S_2, S_3, S_4] = [1, -1, 0, 0]$ and its recovered that

$$\mathcal{T}(\tau) = \sinh(\tau). \tag{25}$$

Plugging this value in (24) for equation (23) along with substituting trail solution, we achieved the system of algebraic equations. By setting coefficients of each function to zero, the following coefficient values are recovered:

$$\begin{aligned}
a_1 &= \frac{a_{10}\omega_4 + 2\,b_6\omega_4 + 3\,b_{10}\omega_1}{\omega_1}, \; a_2 = \frac{Z_1}{{\omega_4}^4{\omega_1}^3},\\
a_5 &= \frac{2}{3}\,\frac{\omega_1\left(a_{10}\omega_1\omega_4\theta_5 - a_{10}{\omega_4}^2\theta_0 - 4\,b_6\omega_1\omega_4\theta_5 + 4\,b_6{\omega_4}^2\theta_0 - 6\,b_{10}{\omega_1}^2\theta_5 + 6\,b_{10}\omega_1\omega_4\theta_0\right)}{{\omega_4}^3},\\
a_6 &= -\frac{\omega_1\,(2\,a_{10}\omega_4 - 2\,b_6\omega_4 - 3\,b_{10}\omega_1)}{3{\omega_4}^2},\\
b_1 &= \frac{3\omega_1\,(b_6\omega_4 + b_{10}\omega_1)}{{\omega_4}^2}, \; b_5 = \frac{2\left(b_6\omega_1\omega_4\theta_5 - b_6{\omega_4}^2\theta_0 + 2\,b_{10}{\omega_1}^2\theta_5 - 2\,b_{10}\omega_1\omega_4\theta_0\right)}{{\omega_1}^2},\\
b_7 &= \frac{Z_2}{{\omega_4}^4{\omega_1}^3}, \;\; \theta_1 = \frac{\omega_1\theta_4}{\omega_4}\\
p_1 &= \frac{2\,b_6{\omega_1}^2\omega_4{\theta_5}^2 - 4\,b_6\omega_1{\omega_4}^2\theta_0\theta_5 + 2\,b_6{\omega_4}^3{\theta_0}^2 + 3\,b_{10}{\omega_1}^3{\theta_5}^2 - 6\,b_{10}{\omega_1}^2\omega_4\theta_0\theta_5 + 3\,b_{10}\omega_1{\omega_4}^2{\theta_0}^2}{\omega_1{\omega_4}^2},
\end{aligned} \tag{26}$$

with
$Z_1 = -4\,a_4{b_6}^2{\omega_1}^5{\omega_4}^2{\theta_5}^4 + 16\,a_4{b_6}^2{\omega_1}^4{\omega_4}^3\theta_0{\theta_5}^3 - 24\,a_4{b_6}^2{\omega_1}^3{\omega_4}^4{\theta_0}^2{\theta_5}^2 + 16\,a_4{b_6}^2{\omega_1}^2{\omega_4}^5{\theta_0}^3\theta_5 - 4\,a_4{b_6}^2\omega_1{\omega_4}^6{\theta_0}^4 - 12\,a_4b_6b_{10}{\omega_1}^6\omega_4{\theta_5}^4 + 48\,a_4b_6b_{10}{\omega_1}^5{\omega_4}^2\theta_0{\theta_5}^3 - 72\,a_4b_6b_{10}{\omega_1}^4{\omega_4}^3{\theta_0}^2{\theta_5}^2 + 48\,a_4b_6b_{10}{\omega_1}^3{\omega_4}^4{\theta_0}^3\theta_5 - 12\,a_4b_6b_{10}{\omega_1}^2{\omega_4}^5{\theta_0}^4 - 9\,a_4{b_{10}}^2{\omega_1}^7{\theta_5}^4 + 36\,a_4{b_{10}}^2{\omega_1}^6\omega_4\theta_0{\theta_5}^3 - 54\,a_4{b_{10}}^2{\omega_1}^5{\omega_4}^2{\theta_0}^2{\theta_5}^2 + 36\,a_4{b_{10}}^2{\omega_1}^4{\omega_4}^3{\theta_0}^3\theta_5 - 9\,a_4{b_{10}}^2{\omega_1}^3{\omega_4}^4{\theta_0}^4 + 4\,a_9{b_6}^2{\omega_1}^4{\omega_4}^3{\theta_5}^4 - 16\,a_9{b_6}^2{\omega_1}^3{\omega_4}^4\theta_0{\theta_5}^3 + 24\,a_9{b_6}^2{\omega_1}^2{\omega_4}^5{\theta_0}^2{\theta_5}^2 - 16\,a_9{b_6}^2\omega_1{\omega_4}^6{\theta_0}^3\theta_5 + 4\,a_9{b_6}^2{\omega_4}^7{\theta_0}^4 + 12\,a_9b_6b_{10}{\omega_1}^5{\omega_4}^2{\theta_5}^4 - 48\,a_9b_6b_{10}{\omega_1}^4{\omega_4}^3\theta_0{\theta_5}^3 + 72\,a_9b_6b_{10}{\omega_1}^3{\omega_4}^4{\theta_0}^2{\theta_5}^2 - 48\,a_9b_6b_{10}{\omega_1}^2{\omega_4}^5{\theta_0}^3\theta_5 + 12\,a_9b_6b_{10}\omega_1{\omega_4}^6{\theta_0}^4 + 9\,a_9{b_{10}}^2{\omega_1}^6\omega_4{\theta_5}^4 - 36\,a_9{b_{10}}^2{\omega_1}^5{\omega_4}^2\theta_0{\theta_5}^3 + 54\,a_9{b_{10}}^2{\omega_1}^4{\omega_4}^3{\theta_0}^2{\theta_5}^2 - 36\,a_9{b_{10}}^2{\omega_1}^3{\omega_4}^4{\theta_0}^3\theta_5 + 9\,a_9{b_{10}}^2{\omega_1}^2{\omega_4}^5{\theta_0}^4 + 2\,a_3b_6{\omega_1}^4{\omega_4}^3{\theta_5}^2 - 4\,a_3b_6{\omega_1}^3{\omega_4}^4\theta_0\theta_5 + 2\,a_3b_6{\omega_1}^2{\omega_4}^5{\theta_0}^2 + 3\,a_3b_{10}{\omega_1}^5{\omega_4}^2{\theta_5}^2 - 6\,a_3b_{10}{\omega_1}^4{\omega_4}^3\theta_0\theta_5 + 3\,a_3b_{10}{\omega_1}^3{\omega_4}^4{\theta_0}^2 - 2\,a_8b_6{\omega_1}^3{\omega_4}^4{\theta_5}^2 + 4\,a_8b_6{\omega_1}^2{\omega_4}^5\theta_0\theta_5 - 2\,a_8b_6\omega_1{\omega_4}^6{\theta_0}^2 - 3\,a_8b_{10}{\omega_1}^4{\omega_4}^3{\theta_5}^2 + 6\,a_8b_{10}{\omega_1}^3{\omega_4}^4\theta_0\theta_5 - 3\,a_8b_{10}{\omega_1}^2{\omega_4}^5{\theta_0}^2 + a_7{\omega_1}^2{\omega_4}^5,$

$Z_2 = 4\,b_4 b_6^2 \omega_1^4 \omega_4^3 \theta_5^4 - 16\,b_4 b_6^2 \omega_1^3 \omega_4^4 \theta_0 \theta_5^3 + 24\,b_4 b_6^2 \omega_1^2 \omega_4^5 \theta_0^2 \theta_5^2 - 16\,b_4 b_6^2 \omega_1 \omega_4^6 \theta_0^3 \theta_5 + 4\,b_4 b_6^2 \omega_4^7 \theta_0^4 + 12\,b_4 b_6 b_{10} \omega_1^5 \omega_4^2 \theta_5^4 - 48\,b_4 b_6 b_{10} \omega_1^4 \omega_4^3 \theta_0 \theta_5^3 + 72\,b_4 b_6 b_{10} \omega_1^3 \omega_4^4 \theta_0^2 \theta_5^2 - 48\,b_4 b_6 b_{10} \omega_1^2 \omega_4^5 \theta_0^3 \theta_5 + 12\,b_4 b_6 b_{10} \omega_1 \omega_4^6 \theta_0^4 + 9\,b_4 b_{10}^2 \omega_1^6 \omega_4 \theta_5^4 - 36\,b_4 b_{10}^2 \omega_1^5 \omega_4^2 \theta_0 \theta_5^3 + 54\,b_4 b_{10}^2 \omega_1^4 \omega_4^3 \theta_0^2 \theta_5^2 - 36\,b_4 b_{10}^2 \omega_1^3 \omega_4^4 \theta_0^3 \theta_5 + 9\,b_4 b_{10}^2 \omega_1^2 \omega_4^5 \theta_0^4 - 4\,b_6^2 b_9 \omega_1^5 \omega_4^2 \theta_5^4 + 16\,b_6^2 b_9 \omega_1^4 \omega_4^3 \theta_0 \theta_5^3 - 24\,b_6^2 b_9 \omega_1^3 \omega_4^4 \theta_0^2 \theta_5^2 + 16\,b_6^2 b_9 \omega_1^2 \omega_4^5 \theta_0^3 \theta_5 - 4\,b_6^2 b_9 \omega_1 \omega_4^6 \theta_0^4 - 12\,b_6 b_9 b_{10} \omega_1^6 \omega_4 \theta_5^4 + 48\,b_6 b_9 b_{10} \omega_1^5 \omega_4^2 \theta_0 \theta_5^3 - 72\,b_6 b_9 b_{10} \omega_1^4 \omega_4^3 \theta_0^2 \theta_5^2 + 48\,b_6 b_9 b_{10} \omega_1^3 \omega_4^4 \theta_0^3 \theta_5 - 12\,b_6 b_9 b_{10} \omega_1^2 \omega_4^5 \theta_0^4 - 9\,b_9 b_{10}^2 \omega_1^7 \theta_5^4 + 36\,b_9 b_{10}^2 \omega_1^6 \omega_4 \theta_0 \theta_5^3 - 54\,b_9 b_{10}^2 \omega_1^5 \omega_4^2 \theta_0^2 \theta_5^2 + 36\,b_9 b_{10}^2 \omega_1^4 \omega_4^3 \theta_0^3 \theta_5 - 9\,b_9 b_{10}^2 \omega_1^3 \omega_4^4 \theta_0^4 - 2\,b_3 b_6 \omega_1^3 \omega_4^4 \theta_5^2 + 4\,b_3 b_6 \omega_1^2 \omega_4^5 \theta_0 \theta_5 - 2\,b_3 b_6 \omega_1 \omega_4^6 \theta_0^2 - 3\,b_3 b_{10} \omega_1^4 \omega_4^3 \theta_5^2 + 6\,b_3 b_{10} \omega_1^3 \omega_4^4 \theta_0 \theta_5 - 3\,b_3 b_{10} \omega_1^2 \omega_4^5 \theta_0^2 + 2\,b_6 b_8 \omega_1^4 \omega_4^3 \theta_5^2 - 4\,b_6 b_8 \omega_1^3 \omega_4^4 \theta_0 \theta_5 + 2\,b_6 b_8 \omega_1^2 \omega_4^5 \theta_0^2 + 3\,b_8 b_{10} \omega_1^5 \omega_4^2 \theta_5^2 - 6\,b_8 b_{10} \omega_1^4 \omega_4^3 \theta_0 \theta_5 + 3\,b_8 b_{10} \omega_1^3 \omega_4^4 \theta_0^2 + b_2 \omega_1^2 \omega_4^5$,
and equipped with $\theta_0, \theta_4, \theta_5, b_8, b_9, b_{10}, b_6, \omega_1, \omega_4, b_2, b_3, b_4, a_7, a_8, a_9, a_{10}, a_3, a_4$ as arbitrary parameters.

Hence, the solution of system (23) is represented as, with help of (26) in terms of parent variables $x,\ t$ as mentioned below:

$$\begin{aligned} U(\tau) &= \frac{\omega_1 \theta_4 \cosh(\tau)}{\omega_4} + \frac{\omega_1}{\cosh(\tau)} + \theta_0, \\ V(\tau) &= \theta_4 \cosh(\tau) + \frac{\omega_4}{\cosh(\tau)} + \theta_5. \end{aligned} \tag{27}$$

Finally, the novel soliton solution for model (2) is formulated as

$$\begin{aligned} q(x,t) &= \frac{\omega_1 \theta_4 \cosh(x - p_1 t)}{\omega_4} + \frac{\omega_1}{\cosh(x - p_1 t)} + \theta_0, \\ r(x,t) &= \theta_4 \cosh(x - p_1 t) + \frac{\omega_4}{\cosh(x - p_1 t)} + \theta_5. \end{aligned} \tag{28}$$

**Family–II**:

Letting $[R_1, R_2, R_3, R_4] = [1, 1, 1, 1]$ and $[S_1, S_2, S_3, S_4] = [1, -1, 0, 0]$ in equation (7), we found

$$\mathcal{T}(\tau) = \cosh(\tau). \tag{29}$$

The equation (24) is reformatted with (29) for equation (23), and then collecting the coefficients of like powers leads to subsequent coefficient values:

$$\begin{aligned} a_1 &= \frac{a_{10}\omega_4 + 2\,b_6\omega_4 + 3\,b_{10}\omega_1}{\omega_1},\ a_2 = \frac{Z_3}{\omega_4^4 \omega_1^3}, \\ a_5 &= \frac{2}{3}\,\frac{\omega_1\left(a_{10}\omega_1\omega_4\theta_5 - a_{10}\omega_4^2\theta_0 - 4\,b_6\omega_1\omega_4\theta_5 + 4\,b_6\omega_4^2\theta_0 - 6\,b_{10}\omega_1^2\theta_5 + 6\,b_{10}\omega_1\omega_4\theta_0\right)}{\omega_4^3}, \\ a_6 &= -\frac{\omega_1\left(2\,a_{10}\omega_4 - 2\,b_6\omega_4 - 3\,b_{10}\omega_1\right)}{3\omega_4^2},\ b_1 = \frac{3\omega_1\left(b_6\omega_4 + b_{10}\omega_1\right)}{\omega_4^2}, \\ b_5 &= \frac{2\left(b_6\omega_1\omega_4\theta_5 - b_6\omega_4^2\theta_0 + 2\,b_{10}\omega_1^2\theta_5 - 2\,b_{10}\omega_1\omega_4\theta_0\right)}{\omega_1^2}, \\ b_7 &= \frac{Z_4}{\omega_4^4\omega_1^3},\ \theta_1 = \frac{\omega_1\theta_4}{\omega_4}, \\ p_1 &= \frac{2\,b_6\omega_1^2\omega_4\theta_5^2 - 4\,b_6\omega_1\omega_4^2\theta_0\theta_5 + 2\,b_6\omega_4^3\theta_0^2 + 3\,b_{10}\omega_1^3\theta_5^2 - 6\,b_{10}\omega_1^2\omega_4\theta_0\theta_5 + 3\,b_{10}\omega_1\omega_4^2\theta_0^2}{\omega_1\omega_4^2}, \end{aligned} \tag{30}$$

with
$Z_3 = -4\,a_4 b_6^2 \omega_1^5 \omega_4^2 \theta_5^4 + 16\,a_4 b_6^2 \omega_1^4 \omega_4^3 \theta_0 \theta_5^3 - 24\,a_4 b_6^2 \omega_1^3 \omega_4^4 \theta_0^2 \theta_5^2 + 16\,a_4 b_6^2 \omega_1^2 \omega_4^5 \theta_0^3 \theta_5 - 4\,a_4 b_6^2 \omega_1 \omega_4^6 \theta_0^4 - 12\,a_4 b_6 b_{10} \omega_1^6 \omega_4 \theta_5^4 + 48\,a_4 b_6 b_{10} \omega_1^5 \omega_4^2 \theta_0 \theta_5^3 - 72\,a_4 b_6 b_{10} \omega_1^4 \omega_4^3 \theta_0^2 \theta_5^2 + 48\,a_4 b_6 b_{10} \omega_1^3 \omega_4^4 \theta_0^3 \theta_5 - 12\,a_4 b_6 b_{10} \omega_1^2 \omega_4^5 \theta_0^4 - 9\,a_4 b_{10}^2 \omega_1^7 \theta_5^4 + 36\,a_4 b_{10}^2 \omega_1^6 \omega_4 \theta_0 \theta_5^3 - 54\,a_4 b_{10}^2 \omega_1^5 \omega_4^2 \theta_0^2 \theta_5^2 + 36\,a_4 b_{10}^2 \omega_1^4 \omega_4^3 \theta_0^3 \theta_5 - 9\,a_4 b_{10}^2 \omega_1^3 \omega_4^4 \theta_0^4 + 4\,a_9 b_6^2 \omega_1^4 \omega_4^3 \theta_5^4 - 16\,a_9 b_6^2 \omega_1^3 \omega_4^4 \theta_0 \theta_5^3 + 24\,a_9 b_6^2 \omega_1^2 \omega_4^5 \theta_0^2 \theta_5^2 - 16\,a_9 b_6^2 \omega_1 \omega_4^6 \theta_0^3 \theta_5 + 4\,a_9 b_6^2 \omega_4^7 \theta_0^4 + 12\,a_9 b_6 b_{10} \omega_1^5 \omega_4^2 \theta_5^4 - 48\,a_9 b_6 b_{10} \omega_1^4 \omega_4^3 \theta_0 \theta_5^3 + 72\,a_9 b_6 b_{10} \omega_1^3 \omega_4^4 \theta_0^2 \theta_5^2 - 48\,a_9 b_6 b_{10} \omega_1^2 \omega_4^5 \theta_0^3 \theta_5 + 12\,a_9 b_6 b_{10} \omega_1 \omega_4^6 \theta_0^4 + 9\,a_9 b_{10}^2 \omega_1^6 \omega_4 \theta_5^4 - 36\,a_9 b_{10}^2 \omega_1^5 \omega_4^2 \theta_0 \theta_5^3 + 54\,a_9 b_{10}^2 \omega_1^4 \omega_4^3 \theta_0^2 \theta_5^2 - 36\,a_9 b_{10}^2 \omega_1^3 \omega_4^4 \theta_0^3 \theta_5 + 9\,a_9 b_{10}^2 \omega_1^2 \omega_4^5 \theta_0^4 + 2\,a_3 b_6 \omega_1^4 \omega_4^3 \theta_5^2 - 4\,a_3 b_6 \omega_1^3 \omega_4^4 \theta_0 \theta_5 + 2\,a_3 b_6 \omega_1^2 \omega_4^5 \theta_0^2 + 3\,a_3 b_{10} \omega_1^5 \omega_4^2 \theta_5^2 - 6\,a_3 b_{10} \omega_1^4 \omega_4^3 \theta_0 \theta_5 + 3\,a_3 b_{10} \omega_1^3 \omega_4^4 \theta_0^2 - 2\,a_8 b_6 \omega_1^3 \omega_4^4 \theta_5^2 + 4\,a_8 b_6 \omega_1^2 \omega_4^5 \theta_0 \theta_5 - 2\,a_8 b_6 \omega_1 \omega_4^6 \theta_0^2 - 3\,a_8 b_{10} \omega_1^4 \omega_4^3 \theta_5^2 + 6\,a_8 b_{10} \omega_1^3 \omega_4^4 \theta_0 \theta_5 - 3\,a_8 b_{10} \omega_1^2 \omega_4^5 \theta_0^2 + a_7 \omega_1^2 \omega_4^5$,

$Z_4 = 4\,b_4 b_6^2 \omega_1^4 \omega_4^3 \theta_5^4 - 16\,b_4 b_6^2 \omega_1^3 \omega_4^4 \theta_0 \theta_5^3 + 24\,b_4 b_6^2 \omega_1^2 \omega_4^5 \theta_0^2 \theta_5^2 - 16\,b_4 b_6^2 \omega_1 \omega_4^6 \theta_0^3 \theta_5 + 4\,b_4 b_6^2 \omega_4^7 \theta_0^4 + 12\,b_4 b_6 b_{10} \omega_1^5 \omega_4^2 \theta_5^4 - 48\,b_4 b_6 b_{10} \omega_1^4 \omega_4^3 \theta_0 \theta_5^3 + 72\,b_4 b_6 b_{10} \omega_1^3 \omega_4^4 \theta_0^2 \theta_5^2 - 48\,b_4 b_6 b_{10} \omega_1^2 \omega_4^5 \theta_0^3 \theta_5 + 12\,b_4 b_6 b_{10} \omega_1 \omega_4^6 \theta_0^4 +$

$9\,b_4b_{10}{}^2\omega_1{}^6\omega_4\theta_5{}^4 - 36\,b_4b_{10}{}^2\omega_1{}^5\omega_4{}^2\theta_0\theta_5{}^3 + 54\,b_4b_{10}{}^2\omega_1{}^4\omega_4{}^3\theta_0{}^2\theta_5{}^2 - 36\,b_4b_{10}{}^2\omega_1{}^3\omega_4{}^4\theta_0{}^3\theta_5 + 9\,b_4b_{10}{}^2\omega_1{}^2\omega_4{}^5\theta_0{}^4 - 4\,b_6{}^2b_9\omega_1{}^5\omega_4{}^2\theta_5{}^4 + 16\,b_6{}^2b_9\omega_1{}^4\omega_4{}^3\theta_0\theta_5{}^3 - 24\,b_6{}^2b_9\omega_1{}^3\omega_4{}^4\theta_0{}^2\theta_5{}^2 + 16\,b_6{}^2b_9\omega_1{}^2\omega_4{}^5\theta_0{}^3\theta_5 - 4\,b_6{}^2b_9\omega_1\omega_4{}^6\theta_0{}^4 - 12\,b_6b_9b_{10}\omega_1{}^6\omega_4\theta_5{}^4 + 48\,b_6b_9b_{10}\omega_1{}^5\omega_4{}^2\theta_0\theta_5{}^3 - 72\,b_6b_9b_{10}\omega_1{}^4\omega_4{}^3\theta_0{}^2\theta_5{}^2 + 48\,b_6b_9b_{10}\omega_1{}^3\omega_4{}^4\theta_0{}^3\theta_5 - 12\,b_6b_9b_{10}\omega_1{}^2\omega_4{}^5\theta_0{}^4 - 9\,b_9b_{10}{}^2\omega_1{}^7\theta_5{}^4 + 36\,b_9b_{10}{}^2\omega_1{}^6\omega_4\theta_0\theta_5{}^3 - 54\,b_9b_{10}{}^2\omega_1{}^5\omega_4{}^2\theta_0{}^2\theta_5{}^2 + 36\,b_9b_{10}{}^2\omega_1{}^4\omega_4{}^3\theta_0{}^3\theta_5 - 9\,b_9b_{10}{}^2\omega_1{}^3\omega_4{}^4\theta_0{}^4 - 2\,b_3b_6\omega_1{}^3\omega_4{}^4\theta_5{}^2 + 4\,b_3b_6\omega_1{}^2\omega_4{}^5\theta_0\theta_5 - 2\,b_3b_6\omega_1\omega_4{}^6\theta_0{}^2 - 3\,b_3b_{10}\omega_1{}^4\omega_4{}^3\theta_5{}^2 + 6\,b_3b_{10}\omega_1{}^3\omega_4{}^4\theta_0\theta_5 - 3\,b_3b_{10}\omega_1{}^2\omega_4{}^5\theta_0{}^2 + 2\,b_6b_8\omega_1{}^4\omega_4{}^3\theta_5{}^2 - 4\,b_6b_8\omega_1{}^3\omega_4{}^4\theta_0\theta_5 + 2\,b_6b_8\omega_1{}^2\omega_4{}^5\theta_0{}^2 + 3\,b_8b_{10}\omega_1{}^5\omega_4{}^2\theta_5{}^2 - 6\,b_8b_{10}\omega_1{}^4\omega_4{}^3\theta_0\theta_5 + 3\,b_8b_{10}\omega_1{}^3\omega_4{}^4\theta_0{}^2 + b_2\omega_1{}^2\omega_4{}^5$,
along with $\theta_0, \theta_4, \theta_5, b_8, b_9, b_{10}, b_6, \omega_1, \omega_4, b_2, b_3, b_4, a_7, a_8, a_9, a_{10}, a_3, a_4$ as free parameters.

Consequently, exact soliton solution for system (2) is depicted with the help of (30) and (29) and followed by reverting back into original variables $x,\ t$ as follows:

$$\begin{aligned} q(x,t) &= \frac{\omega_1\theta_4\sinh\left(x-p_1t\right)}{\omega_4} + \frac{\omega_1}{\sinh\left(x-p_1t\right)} + \theta_0, \\ r(x,t) &= \theta_4\sinh\left(x-p_1t\right) + \frac{\omega_4}{\sinh\left(x-p_1t\right)} + \theta_5. \end{aligned} \tag{31}$$

**Important Remark**:

Following the same procedure outlined in the previous sections, additional solutions of the considered model (2) can be derived with parameter values as (26). For brevity, the detailed computations are omitted, and hence resulting hybrid solitary waves are enlisted below for enhanced comprehension:

**Family–III**:

$$\begin{aligned} q(x,t) &= \frac{\omega_1\theta_4\left(\operatorname{sech}\left(x-p_1t\right)\right)^2}{\omega_4} + \frac{\omega_1}{\left(\operatorname{sech}\left(x-p_1t\right)\right)^2} + \theta_0, \\ r(x,t) &= \theta_4\left(\operatorname{sech}\left(x-p_1t\right)\right)^2 + \frac{\omega_4}{\left(\operatorname{sech}\left(x-p_1t\right)\right)^2} + \theta_5. \end{aligned} \tag{32}$$

Figure 1 shows the 3D and 2D profile of a traveling wave solution for the Gear–Grimshaw model which corresponds to the solutions given in equations (28) and for the parameters indicated in the same figure.

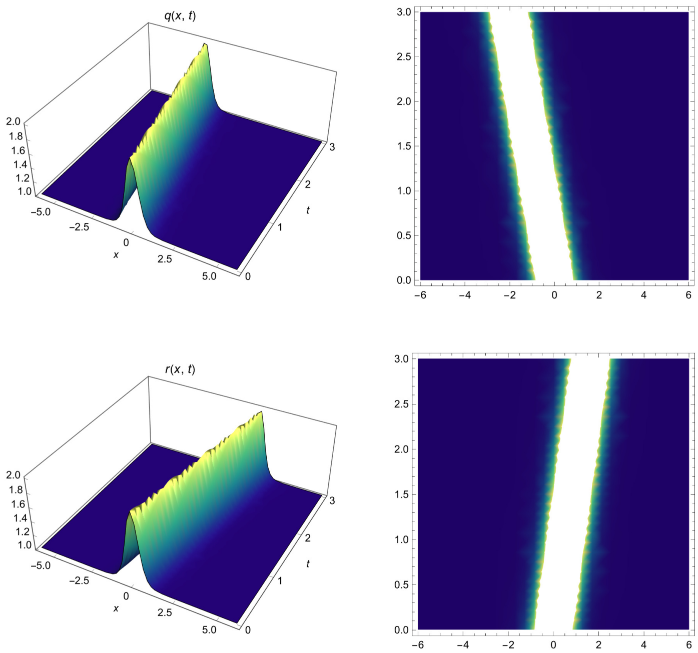


**Figure 1.** 3D and 2D profile of a traveling wave solution for the Gear–Grimshaw model which corresponds to the solutions given in equations (28). The parameter selection is: $a_1 = 2.12$, $a_2 = 0.78$, $a_3 = 0.95$, $a_4 = 3.22$, $a_5 = 2.45$, $a_6 = 1.67$, $a_7 = 2.66$, $a_8 = 0.55$, $a_9 = 0.33$, $b_1 = 1.88$, $b_2 = 0.45$, $b_3 = -5.22$, $b_4 = 0.68$, $b_5 = 6.04$, $b_6 = 3.76$, $b_7 = 1.88$, $b_8 = -3.50$ and $b_9 = 3.65$.

Figure 2 shows the 3D and 2D singular traveling wave solution for the Zaremaoghaddam model which corresponds to the solutions given in equations (53) and for the parameters indicated in the same figure. The left side of the waveform decreases abruptly, while the right side grows abruptly along a straight line.

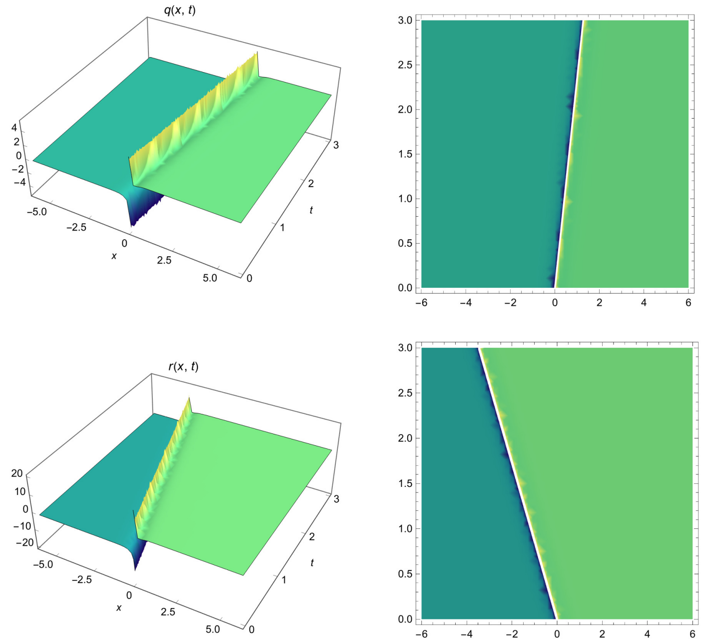


**Figure 2.** 3D and 2D singular traveling wave solution for the Zaremaoghaddam model which corresponds to the solutions given in equations (53). The parameter selection is: $a_1 = 0.50$, $a_2 = 2.18$, $a_3 = 2.05$, $a_4 = 1.05$, $a_5 = 0.55$, $b_1 = 1.05$, $b_2 = 0.65$, $b_3 = 5.05$, $b_4 = 5.12$, and $b_5 = 2.45$.

## 4. RESULTS AND DISCUSSION

The generalized exponential differential rational function (GEDRF) construction yields multiple closed-form traveling-wave families for the coupled double-layer models, including: (i) localized solitary profiles (e.g., $\text{sech}^2$-type), (ii) singular solitary profiles (e.g., $\text{csch}^2$-type and csch coth-type), and (iii) non-decaying hyperbolic growth profiles (e.g., cosh- and sinh-type). From a physical standpoint for stratified shallow-water internal modes, only the localized (finite-energy) profiles are directly compatible with the usual interpretation of solitary waves, whereas the singular and non-decaying profiles require additional discussion (Sections 4.2–4.4).

A key structural feature of several families is that the two layer variables share the *same* basic shape function of the comoving coordinate $\xi = x - p_1 t$ (up to amplitude scaling and possible additive offsets). This “shape locking” is consistent with a resonant two-mode interaction in which the interfacial displacement and the associated velocity potential components (or modal amplitudes) propagate coherently as a composite entity.

### 4.1. Dependence of wave shape on parameters

**4.1.1. Amplitude, polarity, and background level.** For each family, the solution can be written schematically as

$$q(x,t) = Q(\xi), \qquad r(x,t) = R(\xi), \qquad \xi = x - p_1 t,$$

where the *polarity* and *peak amplitude* are controlled by algebraic combinations of the free parameters appearing in the GEDRF ansatz.

**Localized** $\text{sech}^2$ **family (prototype).** For the Zaremaoghaddam model, a representative localized family has the form

$$q(x,t) = A_q \ \text{sech}^2(\xi), \qquad r(x,t) = A_r \ \text{sech}^2(\xi),$$

where $A_q$ and $A_r$ are explicit algebraic combinations of the free parameters (e.g., combinations of $\theta$’s and $\omega$’s in the reported families). In this case:

- **Amplitude scaling:** $|A_q|$ and $|A_r|$ scale linearly with the corresponding free amplitude parameters; changing the sign of $A_q$ (resp. $A_r$) flips the polarity of the pulse in that layer.
- **Relative layer strength:** the ratio $\Gamma := A_r/A_q$ determines whether the two layers exhibit in-phase ($\Gamma > 0$) or out-of-phase ($\Gamma < 0$) excursions.

**Offset solitary waves in the Gear–Grimshaw model.** Some Gear–Grimshaw families include additive constants,

$$q(x,t) = \tilde{Q}(\xi) + q_\infty, \qquad r(x,t) = \tilde{R}(\xi) + r_\infty,$$

so that the traveling structure sits on a nonzero background state. In that setting, $(q_\infty, r_\infty)$ controls the far-field equilibrium, while the localized part $(\tilde{Q}, \tilde{R})$ controls the pulse/shock component. This is important for stability because the continuous spectrum is determined by the linearization about the far-field state.

**4.1.2. Width, steepness, and wave speed.** In the explicit families presented, the basic width is set by the scaling chosen in the generating function (e.g., $\tanh(\xi)$, $\operatorname{sech}(\xi)$, $\operatorname{csch}(\xi)$). More generally, if one uses a scaled argument $\kappa\xi$ in the generating function, then $\kappa > 0$ controls the width: larger $\kappa$ produces narrower (steeper) pulses.

The **speed** $p_1$ is not arbitrary; it is typically determined by the algebraic constraints that enforce exact solvability. Consequently:

- changing amplitude parameters can *also* change $p_1$;
- in many families, the speed is proportional to an amplitude product, producing an amplitude–velocity relation (qualitatively consistent with classical soliton theory).

This coupling between speed and amplitude is central in any stability discussion, since the derivative of an appropriate conserved quantity with respect to $p_1$ often enters stability criteria (see Section 4.4).

### 4.2. Physical realizability of singular solutions

**4.2.1. Nature of the singularity.** Singular families (e.g., $\operatorname{csch}^2(\xi)$, $\operatorname{csch}(\xi)\coth(\xi)$) blow up at $\xi = 0$, i.e., along the characteristic line $x = p_1 t$. In the context of two-layer shallow-water dynamics, such divergences correspond to an *unbounded* modal amplitude or slope at the interface, and therefore lie outside the strict validity range of weakly nonlinear, weakly dispersive asymptotic models. Practically, these solutions are best interpreted as:

1. **local asymptotics** describing the tendency toward gradient catastrophe (incipient breaking) in an unregularized model; or
2. **idealized limits** of sharply peaked structures that would be regularized by physics not included here (viscosity, higher-order dispersion, surface tension, finite-depth corrections, or nonhydrostatic effects).

**4.2.2. Regularization viewpoints.** There are three standard ways to give singular profiles a controlled physical meaning:

**(i) Exclusion of the singular core:** treat the solution as valid only for $|\xi| \geq \delta$ for some small $\delta > 0$, matching to a different inner solution near $\xi = 0$.

**(ii) Weak/distributional interpretation:** interpret the singular family as a weak solution capturing a steep-front/shock-like transition, with the understanding that additional entropy/selection mechanisms would be required to pick the physically relevant branch.

**(iii) Model regularization:** add physically motivated higher-order terms (e.g., fifth-order dispersion, dissipative terms) that smooth the singularity into a finite-amplitude sharply localized pulse.

In the reported numerical visualizations of singular profiles, one indeed observes extremely steep behavior near $x = p_1 t$, consistent with the above interpretation.

### 4.3. Comparison with classical mKdV solitons

**4.3.1. Classical scalar mKdV solitary waves.** Consider the classical scalar mKdV equation

$$u_t + \alpha u^2 u_x + \beta u_{xxx} = 0, \qquad \alpha\beta \neq 0, \tag{33}$$

which admits the well-known one-soliton

$$u(x,t) = \pm\sqrt{\frac{2c}{\alpha}}\,\operatorname{sech}\left(\sqrt{\frac{c}{\beta}}\,(x - ct - x_0)\right), \qquad c > 0 \text{ if } \beta > 0, \tag{34}$$

with amplitude proportional to $\sqrt{c}$ and width proportional to $1/\sqrt{c}$.

**4.3.2. Differences induced by coupling and dispersion-triplet structure.** The coupled double-layer models studied here extend the classical setting in two major ways:

1. **Two-component coupling:** energy can be exchanged between layers, producing locked two-field pulses $(q, r)$ and allowing out-of-phase structures not present in scalar mKdV.
2. **Extended dispersive operator:** the mixed-derivative terms (e.g., $q_{xxt}$ and $q_{xtt}$) modify the linear dispersion and may allow non-classical localized profiles (including hybrid and singular shapes) beyond the standard sech soliton in (34).

Therefore, while some families mimic classical solitons through a single-hump localized structure and amplitude–speed coupling, the catalog also contains non-classical profiles (e.g., $\text{sech}^2$-type or singular $\text{csch}^2$-type) that should be regarded as *hybrid solitary/shock-like* structures enabled by the extended ansatz and the higher-order dispersive framework.

### 4.4. Stability analysis

This subsection provides a linear (spectral) stability framework for the obtained traveling waves and derives explicit, checkable stability conditions for the far-field spectrum. The analysis is presented for the Zaremaoghaddam system; the Gear–Grimshaw system follows analogously.

**4.4.1. Linear dispersion and well-posedness about a uniform state.** Let $(q_\infty, r_\infty)$ be a constant background. Linearizing the Zaremaoghaddam system about that state yields (to first order in perturbations) a pair of *uncoupled* linear dispersive equations because the nonlinear coupling terms are quadratic. In particular, about the rest state $(0, 0)$ one has

$$q_t + a_3 q_{xxx} + a_4 q_{xxt} + a_5 q_{xtt} = 0, \qquad r_t + b_3 r_{xxx} + b_4 r_{xxt} + b_5 r_{xtt} = 0.$$

Using plane waves $q \sim e^{i(kx-\omega t)}$ gives the quadratic dispersion relation

$$a_5 k\,\omega^2 + (1 - a_4 k^2)\omega - a_3 k^3 = 0, \tag{35}$$

and similarly

$$b_5 k\,\omega^2 + (1 - b_4 k^2)\omega - b_3 k^3 = 0. \tag{36}$$

A sufficient condition for *real* $\omega(k)$ for all $k \in \mathbb{R}$ (hence absence of exponential growth in the linearized constant-coefficient problem) is

$$a_3 a_5 \geq 0, \qquad b_3 b_5 \geq 0, \tag{37}$$

since then the discriminants

$$\Delta_q(k) = (1 - a_4 k^2)^2 + 4a_3 a_5 k^4, \qquad \Delta_r(k) = (1 - b_4 k^2)^2 + 4b_3 b_5 k^4$$

are nonnegative for all $k$. Condition (37) is a practical *baseline* requirement before discussing the stability of any nonlinear coherent structure.

**4.4.2. Spectral problem for perturbations of a traveling wave.** Let $(Q(\xi), R(\xi))$ be an exact traveling wave with $\xi = x - p_1 t$. Consider perturbed solutions

$$q(x,t) = Q(\xi) + \varepsilon u(\xi) e^{\lambda t}, \qquad r(x,t) = R(\xi) + \varepsilon v(\xi) e^{\lambda t}, \qquad 0 < \varepsilon \ll 1.$$

In the moving frame, the time derivative acts as $\partial_t \mapsto \lambda - p_1 \partial_\xi$, and one computes

$$q_t \mapsto (\lambda - p_1 \partial_\xi)u, \quad q_{xxx} \mapsto \partial_\xi^3 u, \quad q_{xxt} \mapsto \partial_\xi^2 (\lambda - p_1 \partial_\xi)u, \quad q_{xtt} \mapsto \partial_\xi (\lambda - p_1 \partial_\xi)^2 u,$$

and similarly for $r$.

Linearizing the nonlinear terms gives

$$q^2 q_x \mapsto Q^2 u' + 2QQ'u, \qquad r^2 q_x \mapsto R^2 u' + 2RR'v,$$
$$r^2 r_x \mapsto R^2 v' + 2RR'v, \qquad q^2 r_x \mapsto Q^2 v' + 2QQ'u,$$

where $' = \partial_\xi$. Thus the eigenvalue problem takes the compact quadratic-pencil form

$$\left(\mathcal{A}_2 \lambda^2 + \mathcal{A}_1 \lambda + \mathcal{A}_0\right)\begin{pmatrix} u \\ v \end{pmatrix} = 0, \tag{38}$$

where $\mathcal{A}_2, \mathcal{A}_1, \mathcal{A}_0$ are matrix differential operators in $\xi$ determined by $(Q, R)$ and the model coefficients. Explicitly, the $q$-equation component reads

$$0 = (\lambda - p_1 \partial_\xi)u + a_3 u''' + a_4 \partial_\xi^2 (\lambda - p_1 \partial_\xi)u + a_5 \partial_\xi (\lambda - p_1 \partial_\xi)^2 u$$

$$+ a_1\big(Q^2u' + 2QQ'u\big) + a_2\big(R^2u' + 2RR'v\big), \tag{39}$$

and the $r$-equation component is

$$\begin{aligned} 0 &= (\lambda - p_1\partial_\xi)v + b_3v''' + b_4\partial_\xi^2(\lambda - p_1\partial_\xi)v + b_5\partial_\xi(\lambda - p_1\partial_\xi)^2v \\ &\quad + b_1\big(R^2v' + 2RR'v\big) + b_2\big(Q^2v' + 2QQ'u\big). \end{aligned} \tag{40}$$

A traveling wave is **spectrally stable** if the spectrum of (38) satisfies $\Re(\lambda) \le 0$.

**Neutral translation mode.** Because the governing PDEs are translation invariant in $x$, the derivative $(u, v) = (Q', R')$ always produces a neutral eigenfunction at $\lambda = 0$:

$$\Big(\mathcal{A}_0\Big)\begin{pmatrix} Q' \\ R' \end{pmatrix} = 0.$$

Hence $\lambda = 0$ is generically present in the point spectrum (simple under nondegeneracy conditions), and stability means there are *no* eigenvalues with $\Re(\lambda) > 0$.

**4.4.3. Essential spectrum and a far-field stability criterion.** Assume the traveling wave is localized about a constant state, i.e.,

$$(Q(\xi), R(\xi)) \longrightarrow (q_\infty, r_\infty) \quad \text{as} \quad |\xi| \to \infty.$$

Then the coefficients in (39)–(40) tend to constants, and the far-field eigenvalue problem is diagonal to leading order. Substituting Fourier modes $(u, v) \sim e^{ik\xi}$ yields the far-field (essential-spectrum) relations

$$(\lambda + ip_1k)\,(1 + a_4k^2) + ia_3k^3 + ia_5k\,(\lambda + ip_1k)^2 = 0, \tag{41}$$

$$(\lambda + ip_1k)\,(1 + b_4k^2) + ib_3k^3 + ib_5k\,(\lambda + ip_1k)^2 = 0. \tag{42}$$

Equations (41)–(42) provide a *checkable* necessary condition for spectral stability: for each real $k$, all roots $\lambda(k)$ must satisfy $\Re(\lambda(k)) \le 0$. A sufficient practical condition ensuring that the essential spectrum remains on (or to the left of) the imaginary axis is:

$$a_3a_5 \ge 0, \quad b_3b_5 \ge 0, \quad 1 + a_4k^2 > 0, \quad 1 + b_4k^2 > 0 \quad \forall k \in \mathbb{R}, \tag{43}$$

which prevents ill-posedness and avoids sign changes in the effective far-field “mass” factors $(1 + a_4k^2)$ and $(1 + b_4k^2)$ that can trigger high-wavenumber instabilities.

**4.4.4. Point spectrum and stability of localized pulses: energy-index viewpoint.** The remaining (discrete) spectrum of (38) depends on the localized potentials $QQ'$, $RR'$, $Q^2$, $R^2$ and requires an index argument.

For coherent structures generated as solitary critical points of an augmented functional

$$\mathcal{F}[q, r] = \mathcal{H}[q, r] + p_1\,\mathcal{P}[q, r],$$

(where $\mathcal{H}$ is an energy/Hamiltonian and $\mathcal{P}$ is the momentum associated with translation invariance), the Grillakis–Shatah–Strauss philosophy predicts orbital stability when the second variation $\delta^2\mathcal{F}$ has exactly one negative direction and a simple kernel spanned by $(Q', R')$, together with a *slope condition* (a generalized Vakhitov–Kolokolov condition)

$$\frac{d}{dp_1}\mathcal{P}[Q_{p_1}, R_{p_1}] \ne 0, \qquad \text{with sign determining stability branch.} \tag{44}$$

In practice, for the present double-layer setting, (44) is evaluated on the explicit families by computing $\mathcal{P}$ as a function of the parameters through $(Q, R)$ and $p_1$ (recall $p_1$ is itself constrained algebraically). This provides an analytically tractable stability diagnostic because the integrals of $\text{sech}^m$ profiles are closed form.

**Concrete diagnostic for** $\text{sech}^2$**-type localized pulses.** For a representative localized pulse of the form

$$Q(\xi) = A_q\,\text{sech}^2(\kappa\xi), \qquad R(\xi) = A_r\,\text{sech}^2(\kappa\xi),$$

one may use the quadratic momentum proxy

$$\mathcal{P}_2 = \frac{1}{2}\int_{-\infty}^{\infty}\big(Q^2 + R^2\big)\,d\xi = \frac{1}{2}(A_q^2 + A_r^2)\int_{-\infty}^{\infty}\text{sech}^4(\kappa\xi)\,d\xi = \frac{(A_q^2 + A_r^2)}{3\kappa}.$$

Thus, if $\kappa$ is fixed by the family scaling, the monotonicity of $\mathcal{P}_2$ with respect to $p_1$ reduces to the monotonicity of $A_q^2 + A_r^2$ with respect to $p_1$ through the algebraic constraints. A positive slope $\frac{d}{dp_1}\mathcal{P}_2 > 0$ is typically associated with the stable branch in Hamiltonian dispersive systems, while a negative slope signals an instability exchange.

**4.4.5. Instability mechanisms for singular and non-decaying families.** Finally, two generic instability mechanisms are expected for the non-regular families:

- **Singular families:** the blow-up at $\xi = 0$ implies infinite $L^2$ norm and breakdown of standard energy methods; linearization about a singular profile typically yields non-self-adjoint operators with spectrum extending into $\Re(\lambda) > 0$, so these waves are best viewed as *nonlinearly unstable* idealizations unless regularized.
- **Non-decaying** cosh/sinh **families:** these do not approach a finite far-field equilibrium, so the essential spectrum is not defined in the usual solitary-wave sense; physically, such profiles correspond to nonlocalized excitations or boundary-driven responses rather than freely propagating solitary waves.

### 4.5. Summary of implications

Overall, the explicit solution catalog demonstrates that (i) coherent two-layer traveling structures exist in multiple morphologies, (ii) only the localized finite-energy families are directly consistent with solitary-wave propagation in stratified fluids, (iii) singular solutions should be interpreted as limiting/shock-like objects requiring regularization, and (iv) a rigorous stability assessment reduces to the spectral analysis of the quadratic pencil (38), with immediate far-field admissibility conditions given by (43) and point-spectrum diagnostics accessible via the explicit parameter dependence of integral invariants such as $\mathcal{P}_2$.

## 5. CONCLUSIONS

The paper recovered solitary waves, shock waves and their combination thereof for double layered shallow water flow. Two of the models out of very many such models, have been addressed in this paper. They are Zaremaoghaddam model and the Gear–Grimshaw model. The basic platform is the mKdV equation and therefore the current work is a continuation or a follow–up of the previously reported work that has KdV equation as its basic platform. The adopted integration algorithm was the generalized exponential differential rational function approach. This algorithm revealed various hybrid forms of solitary waves, shock waves and singular solitary waves. An obvious drawback is that the integration scheme failed to reveal single–standing waves such as solitary waves or singular solitary waves or shock waves. For retrieving such single–standing waves one must resort to additional integration approaches. The results from such research activities will be disseminated in future.

## A. ADDITIONAL SOLUTION FAMILIES FOR MODEL (1)

### Family–IV

Upon embedding $[R_1, R_2, R_3, R_4] = [1, 1, 1, -1]$ and $[S_1, S_2, S_3, S_4] = [1, -1, 1, -1]$ in equation (7), we obtain

$$\mathcal{T}(\tau) = \coth(\tau). \tag{45}$$

By making substitution of (45) in (10) for equation (9), coefficient values are procured as:

$$\begin{aligned} &a_1 = b_2,\ a_2 = -\frac{b_2{\theta_1}^2}{{\theta_2}^2}, a_3 = -\frac{4b_2{\theta_1}^2\omega_2\left(4\,a_5b_2\omega_2{\theta_1}^2 + a_4\theta_2\right)}{{\theta_2}^2},\\ &b_1 = -\frac{b_2{\theta_1}^2}{{\theta_2}^2}, b_3 = -\frac{4b_2\omega_2{\theta_1}^2\left(4\,b_2b_5\omega_2{\theta_1}^2 + b_4\theta_2\right)}{{\theta_2}^2},\\ &\omega_1 = -\frac{\theta_1\omega_2}{\theta_2},\ p_1 = -\frac{4b_2{\theta_1}^2\omega_2}{\theta_2},\ \theta_0 = 0,\ \theta_3 = 0. \end{aligned} \tag{46}$$

having $a_4,\ a_5,\ b_4,\ b_5,\ b_2,\ \omega_2,\ \theta_2,\ \theta_1$ as free parameters.

Accordingly, newly generated solution of equation (1) is presented with the help of equation (45) and (46) as follows:

$$\begin{aligned} q(x,t) &= -\theta_1\big(\operatorname{csch}(x - p_1t)\big)^2 + \frac{\theta_1\omega_2}{\theta_2\big(\operatorname{csch}(x - p_1t)\big)^2},\\ r(x,t) &= -\theta_2\big(\operatorname{csch}(x - p_1t)\big)^2 - \frac{\omega_2}{\big(\operatorname{csch}(x - p_1t)\big)^2}. \end{aligned} \tag{47}$$

### Family–V

Upon embedding $[R_1, R_2, R_3, R_4] = [1, 1, 1, -1]$ and $[S_1, S_2, S_3, S_4] = [0, 0, 1, -1]$ in equation (7), we obtain

$$\mathcal{T}(\tau) = \operatorname{csch}(\tau). \tag{48}$$

Putting (48) in (10) for equation (9), following parameter values are obtained:

$$
\begin{aligned}
&a_1 = b_2,\ a_2 = -\frac{b_2{\theta_1}^2}{{\theta_2}^2}, a_3 = -\frac{4b_2\omega_2{\theta_1}^2\left(4\,a_5b_2\omega_2{\theta_1}^2 + a_4\theta_2\right)}{{\theta_2}^2},\\
&b_1 = -\frac{b_2{\theta_1}^2}{{\theta_2}^2},\ b_3 = -\frac{4b_2\omega_2{\theta_1}^2\left(4\,b_2b_5\omega_2{\theta_1}^2 + b_4\theta_2\right)}{{\theta_2}^2},\\
&\omega_1 = -\frac{\omega_2\theta_1}{\theta_2},\ p_1 = -\frac{4b_2\omega_2{\theta_1}^2}{\theta_2},\ \theta_0 = 0,\ \theta_3 = 0.
\end{aligned}
\tag{49}
$$

having $a_4,\ a_5,\ b_4,\ b_5,\ b_2,\ \omega_2,\ \theta_2,\ \theta_1$ as arbitrary constants.

Thus, using equations (48) and (49), the newly generated solution of equation (1) is shown as follows:

$$
\begin{aligned}
q(x,t) &= -\theta_1\ \operatorname{csch}(x - p_1t)\coth(x - p_1t) + \frac{\omega_2\theta_1}{\theta_2\ \operatorname{csch}(x - p_1t)\coth(x - p_1t)},\\
r(x,t) &= -\theta_2\ \operatorname{csch}(x - p_1t)\coth(x - p_1t) - \frac{\omega_2}{\operatorname{csch}(x - p_1t)\coth(x - p_1t)}.
\end{aligned}
\tag{50}
$$

**Family–VI**

Upon embedding $[R_1, R_2, R_3, R_4] = [1, 1, 1, 1]$ and $[S_1, S_2, S_3, S_4] = [0, 0, 1, -1]$ in equation (7), we obtain

$$
\mathcal{T}(\tau) = \operatorname{sech}(\tau).
\tag{51}
$$

Putting (51) in (10) for equation (9), following parameter values are obtained:

$$
\begin{aligned}
&a_1 = b_2,\ a_2 = -\frac{b_2{\theta_1}^2}{{\theta_2}^2},\ a_3 = -\frac{4b_2\omega_2{\theta_1}^2\left(4\,a_5b_2\omega_2{\theta_1}^2 + a_4\theta_2\right)}{{\theta_2}^2},\\
&b_1 = -\frac{b_2{\theta_1}^2}{{\theta_2}^2},\ b_3 = -\frac{4b_2\omega_2{\theta_1}^2\left(4\,b_2b_5\omega_2{\theta_1}^2 + b_4\theta_2\right)}{{\theta_2}^2},\\
&\omega_1 = -\frac{\omega_2\theta_1}{\theta_2},\ p_1 = -\frac{4b_2\omega_2{\theta_1}^2}{\theta_2},\ \theta_0 = 0,\ \theta_3 = 0.
\end{aligned}
\tag{52}
$$

having $a_4,\ a_5,\ b_4,\ b_5,\ b_2,\ \omega_2,\ \theta_2,\ \theta_1$ as arbitrary parameters.

As a result, using (51) and (52), the new soliton solution of equation (1) is written as:

$$
\begin{aligned}
q(x,t) &= -\theta_1\ \operatorname{sech}(x - p_1t)\tanh(x - p_1t) + \frac{\omega_2\theta_1}{\theta_2\ \operatorname{sech}(x - p_1t)\tanh(x - p_1t)},\\
r(x,t) &= -\theta_2\ \operatorname{sech}(x - p_1t)\tanh(x - p_1t) - \frac{\omega_2}{\operatorname{sech}(x - p_1t)\tanh(x - p_1t)}.
\end{aligned}
\tag{53}
$$

## A. ADDITIONAL SOLUTION FAMILIES FOR MODEL (2)

**Family–IV**

$$
\begin{aligned}
q(x,t) &= -\frac{\omega_1\theta_4\left(\operatorname{csch}(x - p_1t)\right)^2}{\omega_4} - \frac{\omega_1}{\left(\operatorname{csch}(x - p_1t)\right)^2} + \theta_0,\\
r(x,t) &= -\theta_4\left(\operatorname{csch}(x - p_1t)\right)^2 - \frac{\omega_4}{\left(\operatorname{csch}(x - p_1t)\right)^2} + \theta_5.
\end{aligned}
\tag{54}
$$

**Family–V**

$$
\begin{aligned}
q(x,t) &= -\frac{\omega_1\theta_4\operatorname{csch}(x - p_1t)\coth(x - p_1t)}{\omega_4} - \frac{\omega_1}{\operatorname{csch}(x - p_1t)\coth(x - p_1t)} + \theta_0,\\
r(x,t) &= -\theta_4\operatorname{csch}(x - p_1t)\coth(x - p_1t) - \frac{\omega_4}{\operatorname{csch}(x - p_1t)\coth(x - p_1t)} + \theta_5.
\end{aligned}
\tag{55}
$$

**Family–VI**

$$
\begin{aligned}
q(x,t) &= -\frac{\omega_1\theta_4 \operatorname{sech}(x-p_1t)\tanh(x-p_1t)}{\omega_4} - \frac{\omega_1}{\operatorname{sech}(x-p_1t)\tanh(x-p_1t)} + \theta_0,\\
r(x,t) &= -\theta_4 \operatorname{sech}(x-p_1t)\tanh(x-p_1t) - \frac{\omega_4}{\operatorname{sech}(x-p_1t)\tanh(x-p_1t)} + \theta_5.
\end{aligned}
\tag{56}
$$

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**Author contributions**

All authors contributed to the study conception and design. Material preparation, data collection, review & editing, analysis and result interpretation were performed by [O. González-Gaxiola], [Husham M. Ahmed ], [Haitham Alqahtani], and [Ahmed H. Arnous]. The first draft of the manuscript was written by [Lakhveer Kaur] and [Anjan Biswas]. All authors commented on previous versions of the manuscript. The final manuscript was read and approved by all writers.

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**Data availability**

The article contains all the information required to comprehend the results of this investigation.

## Declarations

**Conflict of interest**

We are ignorant of any financial conflicts of interest or personal relationships that could have potentially impacted the research described in this article.

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## ГІБРИДНІ ОДИНОКІ ХВИЛІ ТА УДАРНІ ХВИЛІ ДЛЯ ДВОСТАРОВОГО ПОТОКУ РІДИНИ З ТРИПЛЕТОМ ДИСПЕРСІЇ: МОДЕЛІ ЗАРЕМАОГАДДАМА ТА ГІРА-ГРІМШОУ (РІВНЯННЯ mKdV)

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У цій статті відновлено гібридні поодинокі хвилі для двошарових хвиль на мілководді, базовою платформою яких є рівняння mKdV. Вибраними моделями є рівняння Заремаогаддама та рівняння Гіра-Грімшоу. Використаний алгоритм інтегрування — узагальнений метод експоненціальної диференціальної функції. Це призводить до гібридних хвиль, що виникають з одиночних хвиль, ударних хвиль та одиночних одиночних хвиль. Критерії існування таких хвиль також представлені як обмеження параметрів.
**Ключові слова:** *рівняння mKdV; гібрид; інтегрованість; обмеження параметрів; рівняння Гіра–Грімшоу*