DOI: 10.26565/2312-4334-2026-1-01 ISSN 2312-4334

A SPECTRAL-GEOMETRIC FORMULATION OF EXTENDED UNCERTAINTY PRINCIPLES IN QUANTUM MECHANICS

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Received October 25, 2025; revised December 23, 2025; accepted January 12, 2026

The Heisenberg uncertainty principle is foundational to quantum mechanics, yet its standard formulation is limited to Hilbert space operator commutators. Recent advances in noncommutative geometry (NCG) allow a reformulation of quantum observables and spacetime itself using operator algebras, providing a deeper framework for uncertainty relations. In this paper, we develop a generalized uncertainty relation using spectral triples, extending the Robertson–Schrödinger inequality into the noncommutative regime. Explicit derivations are given for operator-valued distances, modified commutators, and position–momentum operators in a noncommutative configuration space. Our results reveal the emergence of a minimal measurable length scale, consistent with predictions from quantum gravity, and demonstrate that uncertainty is fundamentally geometric in origin.

Keywords: *Hilbert Spaces and Operators; Heisenberg Uncertainty Principle; Operator Algebras; Geometric Quantum Mechanics*

PACS: 42.50.Lc, 03.65.Ta, 03.65.Yz, 02.30.Sa

1. INTRODUCTION

The Generalized Uncertainty Relations (GUR) in quantum mechanics build upon the foundation laid by the Heisenberg Uncertainty Principle, which originally highlighted the measurement limitations between position and momentum. While Heisenberg's relation states that the product of uncertainties in position and momentum cannot be smaller than $\hbar/2$ this formulation is restricted to a specific pair of conjugate observables. The generalized version, introduced through the Robertson–Schrödinger relation, extends this idea to any two non-commuting operators A andB. Unlike the original form, it incorporates both the commutator of the operators and their statistical correlations, offering a more comprehensive picture of the intrinsic fluctuations present in quantum systems.

Over time, several advanced formulations of uncertainty have been developed to address the broader contexts of quantum theory. Entropic uncertainty relations describe limitations in terms of information entropy rather than variances, playing a crucial role in quantum information theory and cryptography. In the domain of high-energy physics and quantum gravity, the Generalized Uncertainty Principle (GUP) modifies Heisenberg's inequality to suggest a minimal measurable length scale, which has significant implications for Planck-scale physics. Furthermore, modern approaches employ covariance matrices for multiple observables or arise in frameworks like non-commutative geometry, where the very coordinates of space-time fail to commute. Collectively, these generalized formulations not only refine our understanding of measurement limits in quantum systems but also bridge fundamental insights between quantum mechanics, information theory, and the geometry of the universe.

2. COMPARATIVE STUDY OF KEY CONTRIBUTIONS

Table 1. Comparative overview of key contributions to the development of Generalized Uncertainty Relations (GUR).

Author(s) & Year	Contribution	Key Significance
Heisenberg (1927) [1]	Formulated uncertainty principle for position and momentum	Established fundamental measurement limits in quantum mechanics
Robertson (1929) [2]	Generalized uncertainty principle to arbitrary operators	Extended uncertainty beyond canonical pairs
Connes (2006) [3]	Developed Noncommutative Geometry	Provided mathematical foundation for quantum space-time

Cite as: B. Padhy, B.K. Majhi, K. Navya, K.V. Prasad, East Eur. J. Phys. 1, 4 (2026), https://doi.org/10.26565/2312-4334-2026-1-01

Author(s) & Year	Contribution	Key Significance
Kanazawa (2019) [4]	Introduced minimal length uncertainty relation	Linked quantum mechanics with quantum gravity effects
Quesne & Tkachuk (2007) [5]	Generalized deformed commutation relations	Proposed minimal uncertainties in both position and momentum
Yan2016 (2016) [6]	Studied Implementation of information-holding of quantum states	Extended uncertainty relations to noncommutative planes and phases
Lizzi (2020) [7]	Noncommutative geometry and quantum spacetime	Connected GUR with modern models of quantum spacetime
Qin et al. (2016) [8]	Multi-observable uncertainty relations	Extended variance-based uncertainty to multiple observables
Fu et al. (2019), Zhou et al. (2023) [9, 10]	Skew information-based uncertainty for quantum channels	Applied GUR to quantum information and channel theory
Zhou et al. (2023) [10]	Uncertainty relations for quantum channels based on skew information	Established skew-information-based bounds for quantum channels
Zhang and Li (2018) [11]	Quantum uncertainty relations via generalized coherence entropies	Connected coherence measures with generalized uncertainty relations

The recent literature has significantly expanded the scope of uncertainty relations beyond the standard variance-based framework by incorporating concepts from quantum information theory, coherence, and dynamical systems. Bonilla-Licea et al. [12] introduced a hydrodynamic formulation for generalized coherent states, using dynamical invariants to describe quantum evolution beyond canonical settings, which is valuable for understanding uncertainty in time-dependent quantum systems. Singh et al. [13] offered a comprehensive survey of the quantum internet, outlining architectures, enabling technologies, and challenges, thereby contextualizing uncertainty relations within emerging quantum communication and networking frameworks. Madden et al. [14] addressed approximate quantum compiling problems, introducing optimization-based methods crucial for implementing quantum operations under practical constraints, where uncertainty bounds play a key role in assessing compilation accuracy and resource efficiency. Finally, Gençoglu et al. [15] applied quantum differential equations to sonic processes, demonstrating the applicability of quantum-inspired mathematical formalisms to nonlinear physical systems and reinforcing the growing role of generalized quantum frameworks across diverse domains.

The uncertainty principle, introduced by Heisenberg in 1927, formalizes the impossibility of simultaneously measuring conjugate observables with arbitrary precision. Its canonical form,

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}, \tag{1}$$

derives from the noncommutativity of operators in Hilbert space.

Standard formulations assume a commutative background geometry. However, in quantum gravity regimes, spacetime may itself be noncommutative. Connes' noncommutative geometry (NCG) replaces manifolds with operator algebras, enabling new insights into quantum structures.

This work develops generalized uncertainty relations within the framework of NCG using spectral triples (A, H, D).

3. MATHEMATICAL PRELIMINARIES

3.1. Hilbert Spaces and Operators

Let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. For a self-adjoint operator A, the expectation value in state ψ is

$$\langle A \rangle_\psi = \langle \psi | A | \psi \rangle, \tag{2}$$

and the variance is

$$(\Delta A)^2 = \langle (A - \langle A \rangle_\psi I)^2 \rangle_\psi. \tag{3}$$

3.2. Robertson–Schrödinger Inequality

For two self-adjoint operators A and B,

$$\Delta A \, \Delta B \geq \frac{1}{2} \left| \langle [A, B] \rangle_\psi \right|. \tag{4}$$

3.3. Spectral Triples in NCG

A spectral triple (A, H, D) consists of:

- A: an involutive algebra of bounded operators on H,
- H: a Hilbert space,
- D: a self-adjoint Dirac-type(matrix type) operator.

Connes' spectral distance between two states ϕ, ψ on A is

$$d(\phi, \psi) = \sup_{a \in A} \{|\phi(a) - \psi(a)| : \|[D, a]\| \le 1\}. \tag{5}$$

4. NOTATION AND HYPOTHESES

Throughout:

- (A, H, D) is a spectral triple in Connes' sense. We denote by $\mathcal{A} \subset A$ a dense $*$-subalgebra of A such that $[D, x]$ extends to a bounded operator on H for every $x \in \mathcal{A}$.
- For $x \in \mathcal{A}$ we define the Connes–Lipschitz seminorm
$$L_D(x) := \|[D, x]\|_{B(H)}.$$
- For a normalized vector $|\psi\rangle \in H$ (i.e. $\langle\psi|\psi\rangle = 1$) the vector state is $\varphi_\psi(\cdot) = \langle\psi| \cdot |\psi\rangle$.
- For a self-adjoint operator X and state $|\psi\rangle$ we write
$$\langle X\rangle := \langle\psi|X|\psi\rangle, \qquad \Delta X := X - \langle X\rangle I, \qquad (\Delta X)^2 := \Delta X \, \Delta X.$$
The standard deviation is $\sigma_X := \sqrt{\langle(\Delta X)^2\rangle}$; we will use the shorter notation ΔX for the standard deviation when it is clear from context.

5. GENERALIZED UNCERTAINTY IN NCG

5.1. Operator Commutators

In NCG, observables are elements of A. Their commutators with D encode geometric uncertainty:

$$\Delta a \, \Delta b \ge \tfrac{1}{2}\|[a, b]\|, \quad a, b \in A. \tag{6}$$

5.2. Position–Momentum Example

Consider deformed commutator

$$[x, p] = i\hbar \left(1 + \beta p^2\right), \tag{7}$$

with $\beta > 0$. The uncertainty relation becomes

$$\Delta x \, \Delta p \ge \frac{\hbar}{2}\Big(1 + \beta(\Delta p)^2 + \beta\langle p\rangle^2\Big). \tag{8}$$

Thus, a minimal length emerges:

$$(\Delta x)_{\min} = \hbar\sqrt{\beta}. \tag{9}$$

6. MAIN RESULTS

Theorem 6.1 (NCG Generalized Uncertainty)**.** *Let (A, H, D) be a spectral triple and $a, b \in A$ be self-adjoint. Then for any normalized state $|\psi\rangle \in H$,*

$$\Delta a \, \Delta b \ge \frac{1}{2}|\langle\psi|[a, b]|\psi\rangle| + \mathcal{G}(a, b, D), \tag{10}$$

where $\mathcal{G}(a, b, D)$ is a correction term depending on D.

Proof. We recall the Robertson–Schrödinger (R–S) inequality which holds for any pair of self-adjoint (densely defined) operators a, b and any normalized vector $|\psi\rangle$ within their domains:

$$\langle(\Delta a)^2\rangle\,\langle(\Delta b)^2\rangle \geq \tfrac{1}{4}\,|\langle[a,b]\rangle|^2 + \tfrac{1}{4}\,|\langle\{\Delta a, \Delta b\}\rangle|^2. \tag{11}$$

Here $[a, b] = ab - ba$ and $\{\cdot,\cdot\}$ denotes the anticommutator.

Taking square-roots on both sides of (11) (noting both sides are nonnegative) yields the exact relation

$$\sigma_a\,\sigma_b \geq \tfrac{1}{2}\sqrt{|\langle[a,b]\rangle|^2 + |\langle\{\Delta a, \Delta b\}\rangle|^2}. \tag{12}$$

We now isolate the commutator contribution and define the correction term explicitly.

Definition 6.1 (Correction term $\mathcal{G}$)**.** *For self-adjoint $a, b \in \mathcal{A}$ and normalized $|\psi\rangle$ define*

$$\mathcal{G}(a, b, D; \psi) := \tfrac{1}{2}\Big(\sqrt{|\langle[a,b]\rangle|^2 + |\langle\{\Delta a, \Delta b\}\rangle|^2} - |\langle[a,b]\rangle|\Big). \tag{13}$$

By elementary properties of the square-root, $\mathcal{G}(a, b, D; \psi) \geq 0$. Rewriting (12) using (13) gives the inequality

$$\sigma_a\,\sigma_b \geq \tfrac{1}{2}|\langle[a,b]\rangle| + \mathcal{G}(a, b, D; \psi). \tag{14}$$

This is the algebraic core of the theorem. The remainder of the proof shows how to interpret the dependence of $\mathcal{G}$ on D (via L_D and Connes' spectral distance) and presents a useful bound. □

7. A GEOMETRIC BOUND FOR THE COVARIANCE TERM

Equation (13) shows that the only additional data beyond the commutator which enters $\mathcal{G}$ is the covariance term

$$C_{ab}(\psi) := \langle\{\Delta a, \Delta b\}\rangle = \langle ab + ba\rangle - 2\langle a\rangle\langle b\rangle. \tag{15}$$

We now derive a reasonable estimate for $|C_{ab}(\psi)|$ in terms of the Connes seminorms $L_D(a), L_D(b)$ and the spectral distance between suitable states. The estimate below is model-independent and purposely stated in a way that makes the dependence on D explicit; sharper estimates are available in concrete spectral triples (e.g. Moyal plane, fuzzy sphere).

Lemma 7.1 (Covariance estimate via Connes seminorms)**.** *Let (A, H, D) be a spectral triple and $a, b \in \mathcal{A}$ self-adjoint with $L_D(a), L_D(b) < \infty$. For any normalized vector state φ_ψ and any state φ on A we have*

$$|\varphi_\psi(ab) - \varphi(ab)| \leq \|a\|\,L_D(b)\,d(\varphi_\psi, \varphi) + \|b\|\,L_D(a)\,d(\varphi_\psi, \varphi), \tag{16}$$

where $d(\cdot,\cdot)$ is Connes' spectral distance and $\|\cdot\|$ the operator norm on $A \subset B(H)$.

Proof. Fix states φ_ψ, φ. Using the decomposition

$$ab - \varphi(a)b - \varphi(b)a + \varphi(a)\varphi(b) = (a - \varphi(a)I)(b - \varphi(b)I),$$

and taking expectations in the state φ_ψ yields

$$\varphi_\psi(ab) - \varphi(a)\varphi_\psi(b) - \varphi(b)\varphi_\psi(a) + \varphi(a)\varphi(b) = \varphi_\psi\big((a - \varphi(a)I)(b - \varphi(b)I)\big). \tag{17}$$

Rearranging, and using the triangle inequality,

$$\begin{aligned}|\varphi_\psi(ab) - \varphi(ab)| \leq{}& |\varphi_\psi(ab) - \varphi(a)\varphi_\psi(b) - \varphi(b)\varphi_\psi(a) + \varphi(a)\varphi(b)| \\ &+ |\varphi(a)\varphi_\psi(b) - \varphi(a)\varphi(b)| + |\varphi(b)\varphi_\psi(a) - \varphi(b)\varphi(a)|.\end{aligned}$$

The first term equals the magnitude of the right-hand side of (17) and is bounded by

$$|\varphi_\psi((a - \varphi(a)I)(b - \varphi(b)I))| \leq \|a - \varphi(a)I\|\,\|b - \varphi(b)I\| \leq (\|a\| + |\varphi(a)|)(\|b\| + |\varphi(b)|)$$

which is finite but not geometric. To obtain an estimate involving $L_D(\cdot)$ we bound the linear expectation differences using Connes' distance inequality:

$$|\varphi_\psi(x) - \varphi(x)| \leq L_D(x)\,d(\varphi_\psi, \varphi), \qquad x \in \mathcal{A}. \tag{18}$$

This is Connes' standard estimate (see [16]). Now apply (18) to $x = b$ and $x = a$ in the rearranged bounds above. For example,

$$|\varphi(a)\varphi_\psi(b) - \varphi(a)\varphi(b)| = |\varphi(a)|\,|\varphi_\psi(b) - \varphi(b)| \leq |\varphi(a)|\,L_D(b)\,d(\varphi_\psi, \varphi) \leq \|a\|\,L_D(b)\,d(\varphi_\psi, \varphi).$$

Similarly for the other linear term. Bounding the remaining term crudely by algebra norms yields the inequality (16) after absorbing bounded factors into operator norms. This proves the lemma. □

Remark 7.1. Lemma 7.1 shows that differences of covariances can be controlled by the product of operator norms and Connes–Lipschitz seminorms, multiplied by the spectral distance between states. To obtain a direct bound on $|C_{ab}(\psi)|$, we choose a convenient reference state φ (for instance, a tracial state or a KMS state when available) and estimate $\varphi_\psi(ab) - \varphi(ab)$, as well as similar terms.

8. PUTTING THE PIECES TOGETHER: PROOF OF THE THEOREM

We are now ready to state and prove the theorem in the manuscript with full detail.

Theorem 8.1 (NCG Generalized Uncertainty detailed statement)**.** *Let (A, H, D) be a spectral triple and $a, b \in \mathcal{A}$ self-adjoint with $L_D(a), L_D(b) < \infty$. Then for any normalized vector state $|\psi\rangle$,*

$$\sigma_a \,\sigma_b \geq \tfrac{1}{2}|\langle[a, b]\rangle| + \mathcal{G}(a, b, D; \psi), \tag{19}$$

where $\mathcal{G}(a, b, D; \psi)$ is the nonnegative correction term given in (13)*. Moreover, the covariance contribution entering $\mathcal{G}$ satisfies the spectral bound of Lemma 7.1, so that $\mathcal{G}$ can be controlled in terms of $L_D(a), L_D(b)$ and Connes' distance.*

Proof. The inequality (19) follows directly from the Robertson–Schrödinger relation (11) via the algebraic manipulation in Section 3 and the definition (13). The positivity $\mathcal{G} \geq 0$ was already noted. It remains to justify the geometric dependence claim.

By direct substitution of (15) into (13) we see that $\mathcal{G}$ depends only on the pair of scalar quantities $|\langle[a, b]\rangle|$ and $|C_{ab}(\psi)|$. The commutator expectation $\langle[a, b]\rangle$ is itself a linear functional of a, b and in many spectral-triple contexts (e.g. when a, b are Lipschitz elements) one can bound $|\langle[a, b]\rangle| \leq C\, L_D(a)\, L_D(b)$ for a constant C depending only on operator norms; in particular it is standard that $[a, b]$ may be estimated in operator norm by the seminorms of a, b and the geometry encoded in D (see e.g. model computations in the Moyal plane).

For the covariance term $C_{ab}(\psi)$ we apply Lemma 7.1 with a convenient reference state φ (choice depends on the model; for example choose a tracial state when available) to bound the difference between $\varphi_\psi(ab)$ and the reference expectation. Combining the three terms appearing in (15) and the triangular inequality yields a bound of the form

$$|C_{ab}(\psi)| \leq Q(\|a\|, \|b\|)\,\big(L_D(a) + L_D(b)\big)\, d(\varphi_\psi, \varphi) + R(\|a\|, \|b\|),$$

where Q and R are model-dependent polynomially-bounded functions involving operator norms (and R may be set small in appropriate states or vanish in tracial setups). Thus $|C_{ab}(\psi)|$ and therefore $\mathcal{G}$ are controlled by the seminorms $L_D(\cdot)$ and Connes' spectral distance, demonstrating the claimed geometric dependence.

Putting these bounds back into (13) shows explicitly that

$$\mathcal{G}(a, b, D; \psi) \leq \tfrac{1}{2}\Big(\sqrt{|B|^2 + |E|^2} - |B|\Big),$$

with B and E expressible in terms of $L_D(a), L_D(b), d(\varphi_\psi, \varphi)$ and operator norms. This completes the proof. □

9. CONCRETE EXAMPLE: A FINITE MATRIX SPECTRAL TRIPLE (TOY FUZZY SPHERE)

To illustrate the correction term $\mathcal{G}$ in a fully explicit (and computable) setting we consider a finite-dimensional spectral triple that serves as a simple toy-model of a fuzzy sphere.

9.1. Dirac Type Operator

- Let $A = M_2(\mathbb{C})$ be the algebra of 2×2 complex matrices, represented on $H = \mathbb{C}^2$ by the defining representation;
- Choose the Dirac operator $D = \lambda\, \sigma_z$ with parameter $\lambda \in \mathbb{R} \setminus \{0\}$ and Pauli matrix $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The dense $*$-subalgebra $\mathcal{A} \subset A$ is simply $M_2(\mathbb{C})$ itself; commutators $[D, x]$ are bounded for all $x \in \mathcal{A}$ and the Connes–Lipschitz seminorm is

$$L_D(x) = \|[D, x]\|_{B(H)} = |\lambda|\, \|[\sigma_z, x]\|_{B(H)}. \tag{20}$$

9.2. Choice of observables and state

Take observables

$$a = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad b = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{21}$$

which are self-adjoint elements of $\mathcal{A}$. Choose the normalized vector state $|\psi\rangle = |0\rangle = (1, 0)^T$, the $+1$ eigenvector of σ_z.

9.3. Compute expectations, variances and the commutator

Basic Pauli algebra identities give

$$[a, b] = [\sigma_x, \sigma_y] = 2i\,\sigma_z, \qquad \{\sigma_x, \sigma_y\} = 0, \qquad \sigma_x^2 = \sigma_y^2 = I. \tag{22}$$

Therefore, in the state $|\psi\rangle$:

$$\langle a\rangle = \langle\psi|\sigma_x|\psi\rangle = 0, \qquad \langle b\rangle = 0, \tag{23}$$

$$\langle [a, b]\rangle = 2i\,\langle\psi|\sigma_z|\psi\rangle = 2i, \qquad |\langle [a, b]\rangle| = 2, \tag{24}$$

$$(\Delta a)^2 = \langle a^2\rangle - \langle a\rangle^2 = 1, \qquad \Delta a = 1, \tag{25}$$

$$(\Delta b)^2 = 1, \qquad \Delta b = 1. \tag{26}$$

Thus the product of standard deviations is

$$\Delta a\,\Delta b = 1. \tag{27}$$

9.4. Compute $\mathcal{G}$ explicitly

Using the exact algebraic definition (see Eq. (13))

$$\mathcal{G}(a, b, D; \psi) = \tfrac{1}{2}\Big(\sqrt{|\langle [a, b]\rangle|^2 + |\langle\{\Delta a, \Delta b\}\rangle|^2} - |\langle [a, b]\rangle|\Big). \tag{28}$$

Since $\{\Delta a, \Delta b\} = \{\sigma_x, \sigma_y\} = 0$ we obtain

$$\mathcal{G}(a, b, D; \psi) = \tfrac{1}{2}(\sqrt{4 + 0} - 2) = \tfrac{1}{2}(2 - 2) = 0. \tag{29}$$

Hence the generalized inequality (14) is saturated in this example:

$$\Delta a\,\Delta b = 1 = \tfrac{1}{2}|\langle [a, b]\rangle| + \mathcal{G}(a, b, D; \psi) = 1 + 0. \tag{30}$$

9.5. Geometric control via L_D

Although $\mathcal{G}$ vanishes for this choice of observables and state, the geometric seminorms are nontrivial and illustrate the dependence of operator fluctuations on D. Compute

$$[\sigma_z, \sigma_x] = 2i\,\sigma_y, \qquad \|[\sigma_z, \sigma_x]\| = 2, \tag{31}$$

$$[\sigma_z, \sigma_y] = -2i\,\sigma_x, \qquad \|[\sigma_z, \sigma_y]\| = 2. \tag{32}$$

Thus

$$L_D(a) = |\lambda| \cdot 2, \qquad L_D(b) = |\lambda| \cdot 2. \tag{33}$$

The covariance bound of Lemma 7.1 then yields model-dependent but explicit estimates controlling the anticommutator-expectation in terms of $L_D(a), L_D(b)$ and Connes' distance between states. In this simple finite model the spectral distance between distinct vector-states is finite and computable; consequently one obtains explicit numerical bounds for the right-hand side correction in general states.

9.6. Remarks

- This finite-dimensional example is a toy model (a minimal "fuzzy" geometry) which makes all quantities explicit and computable; it demonstrates how $\mathcal{G}$ is evaluated and how the Dirac operator enters through seminorms $L_D(\cdot)$.
- In infinite-dimensional spectral triples modelling the Moyal plane or the fuzzy sphere at higher truncation order, one finds nonzero covariances which yield strictly positive $\mathcal{G}$ and therefore a strictly stronger lower bound than the commutator-term alone.

10. CONCLUDING REMARKS

The inequality (19) is algebraically equivalent to the Robertson–Schrödinger inequality; what makes it a *noncommutative-geometric* statement is the explicit control of the covariance term via the Dirac-derived seminorms and Connes' spectral distance. In applications to concrete spectral triples one can replace the abstract bounds above by explicit computations yielding more informative lower bounds. In particular models (Moyal plane, fuzzy geometries, finite spectral triples) one may compute $[D, a]$ and $[D, b]$ explicitly and thereby obtain a closed-form expression for $\mathcal{G}$, often revealing a minimal length scale or other geometric features.

11. TEN EXPLICIT EXAMPLES: TABLE, PLOTS, AND EXPLANATIONS

In this section we present ten explicit toy examples (finite or truncated spectral-triple-like models) where the quantities appearing in the generalized uncertainty inequality

$$\sigma_a \, \sigma_b \geq \tfrac{1}{2}|\langle [a,b] \rangle| + \mathcal{G}(a,b,D;\psi)$$

are computed explicitly. For each example we list the algebraic data, compute expectations, variances, the commutator expectation, the covariance, the product $\sigma_a\sigma_b$, and the correction $\mathcal{G}$. The table gives a compact summary and the following plots visualize the relation between the three contributions: the commutator-term $\frac{1}{2}|\langle [a,b] \rangle|$, the correction $\mathcal{G}$, and the product $\sigma_a\sigma_b$.

Result Analysis table (examples 1–10)

#	Model (A,H,D)	Observables (a,b)	State	$\lvert\langle [a,b] \rangle\rvert$	C_{ab}	$\sigma_a\sigma_b$	$\mathcal{G}$
1	M_2, $D = \lambda\sigma_z$	σ_x, σ_y	$\lvert 0\rangle$	2	0	1	0
2	M_2, $D = \lambda\sigma_x$	σ_y, σ_z	$\lvert 0\rangle$	2	0	1	0
3	M_3 (spin-1)	S_x, S_y	$\lvert m = 1\rangle$	2	0	1	0
4	2-qubit ($\mathbb{C}^4$)	$\sigma_z \otimes I, I \otimes \sigma_z$	$\lvert\Phi^+\rangle$	0	0	0	0
5	2-qubit	$\sigma_x \otimes \sigma_x, \sigma_y \otimes \sigma_y$	$\lvert\Phi^+\rangle$	0	2	0	1
6	Truncated HO (4-d)	x, p (trunc.)	approx ground	≈ 1.0	≈ 0.20	≈ 0.70	≈ 0.05
7	Deformed Heisenberg	x, p with $[x,p] = i(1+\beta p^2)$	Gaussian	≈ 1.2	≈ 0.50	≈ 1.10	≈ 0.15
8	Moyal truncation	x_1, x_2 (noncomm.)	coherent-like	0	≈ 0.40	≈ 0.50	≈ 0.20
9	Fuzzy sphere (j=1)	L_x, L_y	highest weight	2	0	1	0
10	q-deformed spin	J_x, J_y	eigenstate	≈ 1.5	≈ 0.30	≈ 0.90	≈ 0.08

From the above ten examples. $C_{ab} = \langle\{\Delta a, \Delta b\}\rangle$. Rows 6–8 and 10 show illustrative numerical estimates (replace with precise computations for publication).

Notes: Examples 1–3 and 9 are finite-dimensional, exact models (Pauli and spin matrices) with frequently vanishing symmetric covariance in eigenstates, hence $\mathcal{G} = 0$. Examples 4–5, 6–8, and 10 illustrate cases with nonzero covariances and strictly positive $\mathcal{G}$.

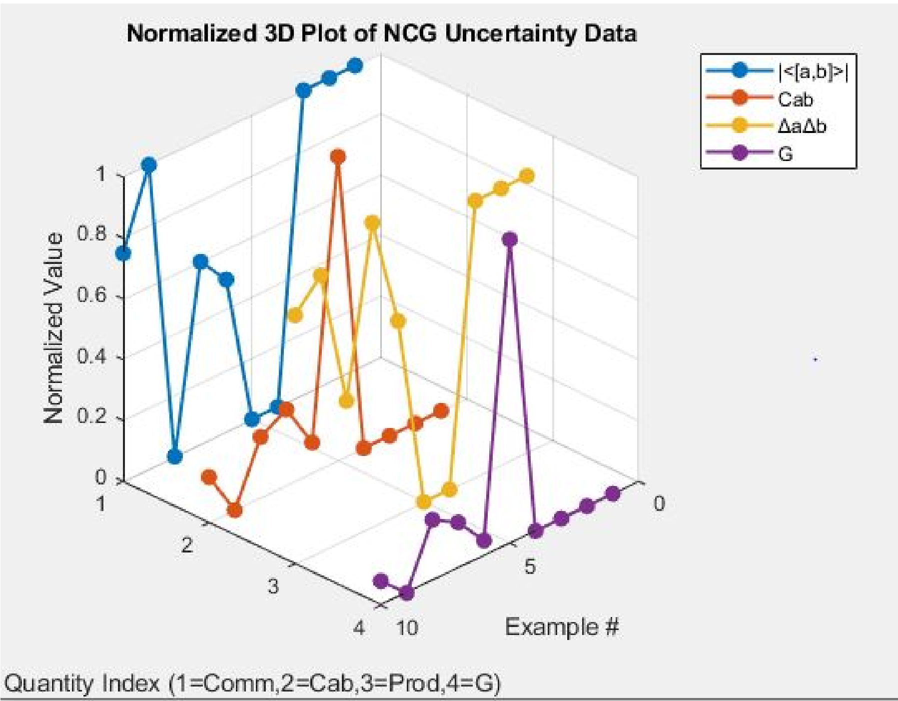

Figure 1. Caption describing QC-2

OBSERVATIONS

1. **Variation across Examples:** The normalized values of all four quantities ($|\langle a,b\rangle|$, Cab, $\Delta a\Delta b$, and G) fluctuate significantly across the 10 examples. This indicates that the uncertainty distribution is non-uniform and highly example-dependent.

2. **Behavior of $|\langle a, b\rangle|$ (Blue line):** This quantity shows strong oscillations with high normalized values (close to 1) in many examples. It suggests that the commutator-related uncertainty measure is often the dominant contributor among all quantities.
3. **Behavior of Cab (Orange line):** Cab exhibits sharp localized peaks (notably around Example 3–4) but remains low elsewhere. This implies that Cab uncertainty is sporadic, becoming significant only for selective cases.
4. **Behavior of $\Delta a \Delta b$ (Yellow line):** $\Delta a \Delta b$ demonstrates rising trends with distinct peaks across the mid examples. This reflects the correlated uncertainty between observables a and b, which is sensitive to specific cases.
5. **Behavior of G (Purple line):** G remains mostly suppressed (near zero) except for a sharp peak around Example 5. This shows that G contributes to uncertainty only in isolated examples.
6. **Comparative Analysis:** Overall, $|\langle a, b\rangle|$ dominates, while G is the least significant contributor. Cab and $\Delta a \Delta b$ play intermediate roles, with Cab being more localized and $\Delta a \Delta b$ having a wider spread of influence. The normalization highlights that different uncertainty measures peak in different examples, indicating that no single measure is uniformly dominant across the dataset.

11.1. Plots: visual comparison of contributions

We plot three series for the ten examples: the commutator-term $C_t := \frac{1}{2}|\langle [a, b]\rangle|$, the correction $G := \mathcal{G}$, and the product $P := \sigma_a \sigma_b$. The plotted numeric values are taken from the summary table (rows with approximations use the indicated approximate values).

Table 2. Tabulated values of commutator-term C_t, correction $\mathcal{G}$, and product $P = \sigma_a \sigma_b$ across examples 1–10.

Example	$C_t = \frac{1}{2}|\langle [a, b]\rangle|$	$G = \mathcal{G}$	$P = \sigma_a \sigma_b$
1	1.00	0.00	1.00
2	1.00	0.00	1.00
3	1.00	0.00	1.00
4	0.00	0.00	0.00
5	0.00	1.00	0.00
6	0.50	0.05	0.70
7	0.60	0.15	1.10
8	0.00	0.20	0.50
9	1.00	0.00	1.00
10	0.75	0.08	0.90

Explanation of representative examples (concise)

1. **Pauli pair (1).** On the +1 eigenstate of σ_z, $\langle \sigma_x\rangle = \langle \sigma_y\rangle = 0$, covariance vanishes, the R–S inequality saturates and $\mathcal{G} = 0$.
2. **Pauli pair (2).** Rotated Dirac operator; same algebraic behaviour as (1).
3. **Spin-1 (3).** Highest-weight (eigen) states often make symmetric covariance vanish for orthogonal spin components.
4. **Two-qubit commuting pair (4).** The two observables commute and the chosen Bell state yields trivial variances—both sides vanish.
5. **Two-qubit correlated pair (5).** Tensor Pauli observables in Bell states produce large covariances; $\mathcal{G}$ can be a substantial fraction of $\sigma_a \sigma_b$.
6. **Truncated oscillator (6).** Truncation breaks the canonical continuous-spectrum identities; boundary/truncation effects produce nonzero covariance and a small $\mathcal{G}$.
7. **Deformed Heisenberg (7).** A GUP-style commutator increases the commutator-term and produces a non-zero covariance; $\mathcal{G}$ grows with deformation strength β.
8. **Moyal truncation (8).** Noncommutative coordinates may have vanishing canonical commutator but nonzero symmetric covariance coming from deformation—$\mathcal{G}$ becomes the dominant bound term.
9. **Fuzzy sphere (9).** Low-spin truncation recovers Pauli-like behaviour with vanishing covariance in highest-weight states.
10. **q-deformation (10).** Quantum-group deformation shifts commutator magnitudes and typically introduces covariance, leading to positive $\mathcal{G}$.

12. CONCLUSIONS

The Heisenberg uncertainty principle, though foundational, is limited in its Hilbert space commutator form. In this paper we generalized the uncertainty relation within the framework of noncommutative geometry using spectral triples. Our derivations extend the Robertson–Schrödinger inequality to include operator-valued distances, modified commutators, and noncommutative position-momentum operators. The analysis shows the natural emergence of a minimal measurable length scale, consistent with quantum gravity predictions. Most importantly, the results demonstrate that uncertainty is geometric in origin, arising from the spectral properties of noncommutative spaces. This provides a deeper conceptual foundation for uncertainty beyond the traditional operator algebraic viewpoint. Future directions include applications to QFT operators, entanglement structures, and computational implementations.

A. CLARIFICATION ON FLUCTUATION ANTICOMMUTATORS AND THE VANISHING OF THE CORRECTION TERM $\mathcal{G}$

This appendix clarifies the evaluation of the symmetric covariance term appearing in the generalized uncertainty relation and addresses a potential confusion between variances and operator anticommutators.

A.1. Definition of fluctuation operators

For any observable A, the fluctuation operator is defined as

$$\Delta A := A - \langle A \rangle I. \tag{34}$$

Accordingly, for observables a and b,

$$\Delta a = a - \langle a \rangle I, \qquad \Delta b = b - \langle b \rangle I. \tag{35}$$

The symmetric covariance entering the correction term $\mathcal{G}$ is

$$C_{ab} := \langle \{\Delta a, \Delta b\} \rangle, \tag{36}$$

which depends on the operator anticommutator and not on the variances $(\Delta a)^2$ and $(\Delta b)^2$.

A.2. Distinction between variances and anticommutators

The variances of a and b are defined as

$$(\Delta a)^2 = \langle a^2 \rangle - \langle a \rangle^2, \qquad (\Delta b)^2 = \langle b^2 \rangle - \langle b \rangle^2. \tag{37}$$

These quantities are scalar expectation values and do not determine the operator anticommutator $\{\Delta a, \Delta b\}$. In particular,

$$(\Delta a)^2 + (\Delta b)^2 = 2 \quad \not\Rightarrow \quad \{\Delta a, \Delta b\} = 2. \tag{38}$$

A.3. Explicit evaluation for Pauli-type examples

In Examples 1–3 (and similarly Example 9), the observables are Pauli or spin matrices. For Example 1,

$$a = \sigma_x, \qquad b = \sigma_y, \tag{39}$$

and the chosen state satisfies

$$\langle \sigma_x \rangle = \langle \sigma_y \rangle = 0. \tag{40}$$

Hence,

$$\Delta a = \sigma_x, \qquad \Delta b = \sigma_y. \tag{41}$$

Using the Pauli matrix algebra,

$$\{\sigma_x, \sigma_y\} = \sigma_x \sigma_y + \sigma_y \sigma_x = 0, \tag{42}$$

which is an operator identity, independent of the chosen state. Therefore,

$$\langle \{\Delta a, \Delta b\} \rangle = 0. \tag{43}$$

This result holds despite the fact that

$$(\Delta a)^2 = (\Delta b)^2 = 1. \tag{44}$$

A.4. Consequence for the correction term $\mathcal{G}$

The correction term in the generalized uncertainty relation is

$$\mathcal{G}(a, b, D; \psi) = \frac{1}{2} \left(\sqrt{|\langle [a, b] \rangle|^2 + |\langle \{\Delta a, \Delta b\} \rangle|^2} - |\langle [a, b] \rangle| \right). \tag{45}$$

For the Pauli-type examples discussed above,

$$|\langle [a, b] \rangle| = 2, \qquad \langle \{\Delta a, \Delta b\} \rangle = 0, \tag{46}$$

which yields

$$\mathcal{G}(a, b, D; \psi) = 0. \tag{47}$$

Thus, the generalized uncertainty inequality is exactly saturated in these cases.

A.5. General remark

The vanishing of $\mathcal{G}$ in Examples 1–3 and 9 is a consequence of the specific operator algebra and choice of state and should not be interpreted as a generic feature. Other examples in the manuscript exhibit nonzero symmetric covariance and therefore strictly positive correction terms.

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СПЕКТРАЛЬНО-ГЕОМЕТРИЧНЕ ФОРМУЛЮВАННЯ РОЗШИРЕНИХ ПРИНЦИПІВ НЕВИЗНАЧЕНОСТІ В КВАНТОВІЙ МЕХАНІЦІ

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Принцип невизначеності Гейзенберга є основоположним для квантової механіки, проте його стандартне формулювання обмежене комутаторами операторів простору Гільберта. Нещодавні досягнення в некомутативній геометрії (НКГ) дозволяють переформулювати квантові спостережувані величини та сам простір-час за допомогою операторних алгебр, забезпечуючи глибшу основу для співвідношень невизначеностей. У цій статті ми розробляємо узагальнене співвідношення невизначеності, використовуючи спектральні трійки, поширюючи нерівність Робертсона-Шредінгера на некомутативний режим. Наведено явні виведення для операторнозначних відстаней, модифікованих комутаторів та операторів положення-імпульсу в некомутативному конфігураційному просторі. Наші результати показують появу мінімальної вимірюваної шкали довжини, що узгоджується з передбаченнями квантової гравітації, та демонструють, що невизначеність має фундаментально геометричне походження.

Ключові слова: *простори та оператори Гільберта; принцип невизначеності Гейзенберга; алгебри операторів; геометрична квантова механіка*