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ANISOTROPIC DARK ENERGY COSMOLOGY IN THE FRAMEWORK OF $f(R, L_m)$ GRAVITY

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In this paper, we investigated the Locally Rotationally Symmetric (LRS) Bianchi Type-I cosmological model with dark energy in the framework of $f(R, L_m)$ gravity theory, where R is the Ricci scalar and L_m is the matter Lagrangian. Using the functional form $f(R, L_m) = \frac{R}{2} + L_m^{\alpha} + \beta$ with $L_m = \rho$,and applying the special law of variation for the Hubble parameter, we derived exact solutions to the field equations and analyzed the physical and dynamical properties of the universe. Our results show that the model exhibits accelerated expansion consistent with the observational data, with the energy density decreasing and the deceleration parameter transitioning from positive to negative values. The anisotropy parameter initially approaches zero but increases with time for $n > 0.5$, indicating the evolution from isotropy to anisotropy. These findings provide insights into dark energy behavior within modified gravity frameworks and offer testable predictions for cosmological observations.

Keywords: *LRS Bianchi Type-I; Dark Energy; Cosmic Time*

PACS: specify the PACS code(s) here

1. INTRODUCTION

The General Theory of Relativity (GTR) forms the fundamental framework of modern cosmology, explaining gravitation as a manifestation of spacetime curvature generated by the presence of matter and energy. A comprehensive formulation of GTR and its cosmological applications can be found in the classical reference by Weinberg [1].Although GTR has been remarkably successful in describing various gravitational phenomena, astronomical observations of distant Type Ia supernovae have revealed that the present universe is expanding at an accelerated rate [2, 3, 4, 5, 6]. This unexpected result suggests the existence of an unknown form of energy, commonly referred to as dark energy, which is estimated to account for nearly 68% of the total energy density of the universe [7, 8, 9, 10]. Such observations indicate that the standard cosmological model based solely on GTR may require refinement or modification to consistently explain the observed late-time cosmic acceleration. To address this challenge, various modified theories of gravity have been developed as alternatives to dark energy. One of the promising approaches is the f(R,Lm) gravity theory proposed by Harko and Lobo [11], which generalizes the conventional f(R) gravity by introducing an explicit coupling between the Ricci scalar R and the matter Lagrangian density Lm .This framework provides a richer geometric structure and allows new possibilities for understanding the interaction between matter and curvature in explaining the accelerated expansion of the universe.

Research on anisotropic cosmological models with dark energy in the framework of $f(R, L_m)$ gravity has emerged as a critical area of inquiry due to its potential to address the accelerating expansion of the universe and the limitations of standard general relativity (GR) in explaining dark energy phenomena [12, 13]. Since the introduction of $f(R)$ gravity as a modification of the Einstein-Hilbert action [14], the field has evolved to incorporate more general couplings between curvature and matter, notably through the $f(R, L_m)$ gravity theory [15]. This framework allows for non-minimal interactions between geometry and matter, which can lead to novel cosmological dynamics, including anisotropic effects relevant to early universe conditions [16]. Observational data from type Ia supernovae, cosmic microwave background, and baryon acoustic oscillations have reinforced the significance of exploring such models to better understand cosmic acceleration and anisotropy [17].

The specific problem addressed involves constructing and constraining anisotropic cosmological models within $f(R, L_m)$ gravity that can accommodate dark energy effects and observational data [18, 19]. Despite progress, a knowledge gap persists regarding the precise role of anisotropy and bulk viscosity in these models, as well as the impact of different functional forms of $f(R, L_m)$ on cosmic evolution [20, 21]. Competing perspectives exist on whether dark energy should be modeled as a quintessence-like field or phantom energy within this framework, with some studies favoring quintessence behavior and others indicating phantom-like characteristics.This review aims to clarify the theoretical underpinnings and observational viability of these models, addressing the identified gaps and providing a comprehensive understanding of their cosmological implications [22, 23]. The consequences of this gap include uncertainties in predicting the universe's late time behavior and reconciling theoretical models with high-precision observational constraints [24].

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Recently Shambel et al. [25] have focused on the late-time accelerated expansion of the universe and cosmic structure evolution within the $f(R, L_m)$ gravity model, rather than the early universe. Romanshu Garg and G.P. Singh [26] analyzed cosmological parameters and the present age of the universe, providing insights into the model's implications for cosmic evolution. Y.D. Devi et al [27] have studied on an accelerating cosmological model, deriving the Hubble parameter and analyzing parameters such as the deceleration parameter, jerk, and snap at present times. It derives Friedmann-like equations for two non-linear models, analyzing the effects of model parameters on the equation of state, pressure, and energy density. V. Patil et al. [28] have done on the late-time acceleration of the Universe within the $f(R, L_m)$ gravity framework, rather than the early universe. Shukla et al. [29] primarily focused on the Friedmann-Lemaitre-Robertson-Walker (FLRW) model in $f(R, L_m)$ gravity, analyzing the transition from deceleration to acceleration in the universe. J.K. Singh and Shaily [30] have studied on the late-time behavior of the universe in the context of $f(R, L_m)$ gravity, rather than the early universe. It investigates the transition from deceleration to acceleration phases, showing that the model behaves like ΛCDM at late times. Pawar et al.[31] have focused on the expansion of the universe in $f(T)$ gravity and derived the Hubble parameter in terms of redshift and examines the equation of state parameter, energy density, and pressure. Young Jin Suh et al. [32] have investigated protectively flat perfect fluid spacetime solutions in $f(R, L_m)$ gravity, focusing on energy conditions and their relation to the Ricci scalar.

The functional form of $f(R, Lm)$ gravity has been extended to related theories such as $f(R, T)$ and $f(Q)$ gravity, incorporating trace of energy-momentum tensor or non-metricity scalar, broadening the functional diversity [33]. Furthermore, alternative formulations inspired by logarithmic corrections, Born–Infeld structures, and holographic dark energy have also been explored, highlighting the continuous efforts to enhance the physical viability and observational consistency of such models [34, 35]. Several studies have investigated cosmological models with variable anisotropy parameters or within different Bianchi classifications, such as Bianchi type-III and type-VI_0, revealing diverse evolutionary patterns of anisotropy in the universe [36, 37]. While many theoretical and reconstruction-based works provide valuable insights into the role of anisotropy and modified gravity, some lack direct confrontation with observational datasets. However, these studies establish a strong foundation for future comparisons with astrophysical data [38, 39]. In most models, the focus remains confined to the contributions of modified gravity terms and conventional matter fields, primarily analyzing geometric features and dark energy dynamics [40].

In summary, the body of literature underscores that matter-curvature coupling in $f(R, Lm)$ gravity significantly influences anisotropic cosmological evolution and dark energy dynamics, yielding models capable of describing late-time acceleration with evolving equations of state. The inclusion of bulk viscosity further enriches the phenomenology, enhancing the viability of anisotropic models. However, challenges persist in capturing fully dynamical anisotropy, achieving consistent observational integration of anisotropic effects, and expanding the functional diversity of $f(R, L_m)$ to encompass broader physical scenarios. Future research addressing these gaps could sharpen understanding of anisotropic dark energy models in modified gravity and their cosmological implications

This work is structured as follows: In Section 2 we give a brief account of the $f(R, L_m)$ gravity formalism and the field equations. Anisotropic cosmology using LRS Bianchi I metric is discussed in Section 3. Section 4 discusses the type of models and solutions in which this study is centered. In Section 5 we present the results with particular emphasis on the physical consequences and observational predictions. Last but not least; Section 6 gives the conclusion and research recommendations.

2. $f(R, L_m)$ GRAVITY AND FIELD EQUATION

The action integral for the framework of f (R,Lm) interpreted with the matter Lagrangian density Lm and the Ricci scalar R is given as

$$S = \int f(R, Lm)\sqrt{-g}dx^4 \tag{1}$$

where $f(R, Lm)$ is arbitrary function of Ricci scalar R and matter Lagrangian L_m. By contracting the Ricci tensor R_{mn}, one may get the Ricci scalar R,

$$R = g^{ij}R_{ij} \tag{2}$$

where, the Ricci tensor is defined by,

$$R_{ij} = -\partial_\lambda \Gamma^\lambda_{ij} + \partial_j \Gamma^\lambda_{i\lambda} - \Gamma^\lambda_{\lambda\sigma}\Gamma^\sigma_{ij} + \Gamma^\lambda_{j\sigma}\,\Gamma^\sigma_{i\lambda} \tag{3}$$

Here, $\Gamma^\alpha_{\beta\gamma}$ represents the components of well-known Levi-Civita connection defined by

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\lambda}\left(\frac{\partial g_{\lambda\beta}}{\partial x^\gamma} + \frac{\partial g_{\lambda\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\lambda}\right) \tag{4}$$

The corresponding field equations of $f(R, L_m)$ gravity are obtained by varying the action (1) for metric g_{ij} is given by,

$$f_R(R, L_m)R_{ij} + \left(g_{ij}\nabla_i\nabla^i - \nabla_i\nabla_j\right) f_R(R, L_m) - \frac{1}{2}\left[f(R, L_m) - f_{L_m}(R, L_m)L_m\right] g_{ij} = \frac{1}{2} f_{L_m}(R, L_m)T_{ij} \tag{5}$$

Where, $f_R(R, L_m) = \frac{\delta f(R,L_m)}{\delta R}$, $f_{L_m}(R, L_m) = \frac{\delta f(R,L_m)}{\delta L_m}$.Here covariant derivative is represented by ∇_i and the energy momentum tensor T_{ij} can be expressed as,

$$T_{ij} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}L_m\right)}{\delta g^{ij}} = g_{ij}L_m - 2\frac{\delta L_m}{\delta g^{ij}} \tag{6}$$

Now, from the explicit form of the field equation (5), the covariant divergence of Energy momentum tensor T_{ij} can be obtained as,

$$\nabla^i T_{ij} = 2\nabla^i \ln\left[f_{L_m}(R, L_m)\right]\frac{\delta L_m}{\delta g_{ij}} \tag{7}$$

The relation between the trace of energy momentum-tensor T, Ricci scalar R, and the Lagrangian density of the matter L_m obtained by contracting the field equation (5)

$$f_R(R, L_m)R + 3\nabla_i\nabla^i f_R(R, L_m) - 2\left[f(R, L_m) - f_{L_m}(R, L_m)L_m\right] = \frac{1}{2}f_{L_m}(R, L_m)T \tag{8}$$

The relation between the trace of the energy momentum tensor $T = T_{ij}$, L_m, and R can be established by taking account of the previously mentioned equation.

3. METRIC AND FIELD EQUATION IN $f(R, L_m)$ GRAVITY

The spatially homogeneous and anisotropic LRS Bianchi type I spacetime can be written as,

$$ds^2 = -dt^2 + X^2dx^2 + Y^2(dy^2 + dz^2) \tag{9}$$

Where X and Y are the metric potential that are the functions of cosmic time t only.

The Ricci scalar for LRS Bianchi-I spacetime can be expressed as

$$R = -2\left[\frac{\ddot{X}}{X} + 2\frac{\ddot{Y}}{Y} + 2\frac{\dot{X}\dot{Y}}{XY} + \frac{\dot{Y}^2}{Y^2}\right] \tag{10}$$

The overhead dot (.) denotes the derivative with respect to time t. The spatial volume V of the universe is defined as

$$V = XY^2 \tag{11}$$

The average scale factor

$$a(t) = \left(XY^2\right)^{\frac{1}{3}} \tag{12}$$

The generalized mean Hubble parameter H, which describes the space-time expansion rate, can be stated as

$$H = \frac{1}{3}(H_1 + H_2 + H_3) = \frac{1}{3}\left(\frac{X}{\dot{X}} + \frac{2Y}{\dot{Y}}\right) \tag{13}$$

where H_1, H_2, H_3 are the directional Hubble's parameters in the direction of the $x-, y-, andz - axes$, respectively. In order to figure out whether the models approach isotropy or not, we define the expansion's anisotropy parameter as

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i}{H} - 1\right)^2 \tag{14}$$

The expansion scalar and shear scalar are defined as follows,

$$\theta = \frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y} \tag{15}$$

$$\sigma^2 = \frac{3}{2}A_m H^2 \tag{16}$$

The deceleration parameter (DP) is

$$q = -1 + \frac{d}{dt}\left(\frac{1}{H}\right) = -\frac{a\dot{a}}{a^2\ddot{a}} \tag{17}$$

Let us take the matter that contains the energy momentum tensor for dark energy, which is of the form

$$T_{ij} = (p + \rho)u_i u_j + pg_{ij} \tag{18}$$

Where ρ is the energy density and p is the pressure of the fluid. Where $u^i = (1, 0, 0, 0)$ is the four-velocity vector in co-moving coordinates which satisfying $u_i u^i = -1$. The EoS parameter for quark matter is defined as

$$p = \omega\rho \quad 0 \leq \omega \leq 1 \tag{19}$$

By using the equation (16), the field equation (5) can be written as,

$$\left(\frac{\ddot{X}}{X} + 2\frac{\dot{X}\dot{Y}}{XY}\right) f_R + \frac{1}{2}\left(f - f_{L_m} L_m\right) + 2\frac{\dot{Y}}{Y}\dot{f}_R + \ddot{f}_R = -\frac{1}{2} f_{L_m} p \tag{20}$$

$$\left(\frac{\ddot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2} + \frac{\dot{X}\dot{Y}}{XY}\right) f_R + \frac{1}{2}\left(f - f_{L_m} L_m\right) + \left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y}\right)\dot{f}_R + \ddot{f}_R = -\frac{1}{2} f_{L_m} p \tag{21}$$

$$\left(\frac{\ddot{X}}{X} + 2\frac{\ddot{Y}}{Y}\right) f_R + \frac{1}{2}\left(f - f_{L_m} L_m\right) + \left(\frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y}\right)\dot{f}_R = \frac{1}{2} f_{L_m} \rho \tag{22}$$

4. COSMOLOGICAL $f(R, L_m)$ MODEL

To examine the dynamics of Universe, we employ the functional form of $f(R, L_m)$ gravity as,

$$f(R, L_m) = \frac{R}{2} + L_m^\alpha + \beta \tag{23}$$

where α and β are free parameters and can be retained as GR for $\alpha = 1$ and β=0

For $f(R, L_m)$ model,the matter Lagrangian L_m is generalized, and the coupling with curvature R produces extra terms that can mimic dark energy behavior, we have to consider $L_m = \rho$ [41]

Using the above particular choice of L_m, the field equations (20), (21) and (22) becomes,

$$\frac{2\ddot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2} - (1-\alpha)\rho^\alpha - \beta = \alpha\rho^{\alpha-1} p \tag{24}$$

$$\frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} + \frac{\dot{X}\dot{Y}}{XY} - (1-\alpha)\rho^\alpha - \beta = \alpha\rho^{\alpha-1} p \tag{25}$$

$$\frac{\dot{Y}^2}{Y^2} + 2\frac{\dot{X}\dot{Y}}{XY} - \beta = (1-2\alpha)\rho^\alpha \tag{26}$$

The field equations (24), (25) and (26) are three independent differential equations with four unknowns: X, Y, ρ and p. Hence, to determine solutions, we have to use physically plausible conditions.

Berman and Gomide [42] indicate that there exists a connection between the Hubble parameter (H) and average scale factor (a) given as,

$$H = l a^{-n} = l\left(XY^2\right)^{-\frac{n}{3}} \tag{27}$$

Where $l > 0$ and $n \geq 0$ are constants.

Now, from equations (13) and (27), we get the following.

$$\dot{a} = l a^{-n+1} \tag{28}$$

$$\ddot{a} = -l^2(n-1)a^{-2n+1} \tag{29}$$

From equations (16) we obtain

$$q = n - 1 \tag{30}$$

Now using equations (27) and (30), the solution of equation (17) gives the law of variation of the average scale factor of the form,

$$a = (nlt)^{\frac{1}{n}}, \quad n \neq 0 \tag{31}$$

5. SOLUTION OF THE FIELD EQUATIONS

The field equations (24)-(26) reduce to a system of three non-linear equations in four unknown parameters X, Y, ρ and p. Hence to find the determination solution of the system , we used the law of variation [42]. Now from equations (24) and (25), we get

$$\frac{\ddot{Y}}{Y} - \frac{\ddot{X}}{X} + \frac{\dot{Y}^2}{Y^2} - \frac{\dot{X}\dot{Y}}{XY} = 0 \tag{32}$$

On integrating, we get

$$\frac{\dot{Y}}{Y} - \frac{\dot{X}}{X} = \frac{c_1}{XY^2} \tag{33}$$

Where c_1 is the constant of integration Using equation (12) in equation (33) and integrating again, we get

$$\frac{Y}{X} = c_2 \exp\left(\int \frac{c_1}{a^3}\, dt\right) \tag{34}$$

Multiplying and divide by Y^2,we get

$$\frac{Y^3}{XY^2} = c_2 \exp\left(\int \frac{c_1}{a^3}\, dt\right) \tag{35}$$

Using equation (12), $a^3 = XY^2$ in equation (35), we get

$$Y^3 = a^3 c_2 \exp\left(\int \frac{c_1}{a^3}\, dt\right) \tag{36}$$

The metric function X and Y in terms of the average scale factor $a(t)$ are given by

$$X(t) = c_2^{-\frac{2}{3}} a \exp\left(-\frac{2c_1}{3} \int a^{-3}\, dt\right) \tag{37}$$

$$Y(t) = c_2^{\frac{1}{3}} a \exp\left(\frac{c_1}{3} \int a^{-3}\, dt\right) \tag{38}$$

Now using equation (31) in equations (37) and (38), we get

$$X(t) = c_2^{-\frac{2}{3}} (nlt)^{\frac{1}{n}} \exp\left(\frac{-2c_1}{3l(n-3)} (nlt)^{\frac{n-3}{n}}\right) \tag{39}$$

$$Y(t) = c_2^{\frac{1}{3}} (nlt)^{\frac{1}{n}} \exp\left(\frac{c_1}{3l(n-3)} (nlt)^{\frac{n-3}{n}}\right) \tag{40}$$

Where $n \neq 3$

Putting the values of X and Y in eqn (9),we obtained the exact solution .

$$ds^2 = -dt^2 + c_2^{-\frac{1}{3}} (nlt)^{\frac{2}{n}} \left(\exp\left(\frac{-2c_1}{3l(n-3)} (nlt)^{\frac{n-3}{n}}\right)\right)^2 dx^2 + c_2^{\frac{2}{3}} (nlt)^{\frac{2}{n}} \left(\exp\left(\frac{c_1}{3l(n-3)} (nlt)^{\frac{n-3}{n}}\right)\right)^2 (dy^2 + dz^2) \tag{41}$$

which gives the desired cosmological model.

From the equations (25) and (26), with the help of metric potential, the energy density and pressure of dark energy are given by

$$\rho = \left[\frac{1}{1-2\alpha}\left(-\frac{c_1^2}{3}(nlt)^{-\frac{6}{n}} + \frac{3}{n^2t^2} - \beta\right)\right]^{\frac{1}{\alpha}} \tag{42}$$

$$p = -\frac{\dfrac{4c_1(nlt)^{-\frac{3}{n}}}{nt} - \dfrac{2c_1^2}{3}(nlt)^{-\frac{6}{n}} + \dfrac{2n+9}{n^2t^2} + \dfrac{\alpha}{1-2\alpha}\left(-\dfrac{c_1^2}{3}(nlt)^{-\frac{6}{n}} + \dfrac{3}{n^2t^2} - \beta\right)}{\alpha\left[\dfrac{1}{1-2\alpha}\left(-\dfrac{c_1^2}{3}(nlt)^{-\frac{6}{n}} + \dfrac{3}{n^2t^2} - \beta\right)\right]^{\frac{\alpha-1}{\alpha}}} \tag{43}$$

Using equations (42) and (43), the equation of state (EoS) for dark energy is given as

$$w = -\frac{\left((1-2\alpha)\left[4c_1 nt(nlt)^{-\frac{3}{n}} - \frac{2c_1^2}{3}(nt)^2(nlt)^{-\frac{6}{n}} + 2n + 9\right] + \alpha\left(-\frac{c_1^2}{3}(nt)^2(nlt)^{-\frac{6}{n}} - \beta(nt)^2 + 3\right)\right)}{\alpha\left(-\frac{c_1^2}{3}(nt)^2(nlt)^{-\frac{6}{n}} - \beta(nt)^2 + 3\right)} \tag{44}$$

6. SOME PHYSICAL PARAMETERS

The spatial volume V of the universe is given as

$$V = a^3(t) = (nlt)^{\frac{3}{n}} \tag{45}$$

The average Hubble parameter

$$H = (nt)^{-1} \tag{46}$$

The dynamical scalar expansion θ and shear scalar σ^2 are

$$\theta = 3(nt)^{-1}, \tag{47}$$

$$\sigma^2 = c_1^2 (nlt)^{\frac{-6}{n}} \tag{48}$$

The average anisotropic parameter

$$A_m = \frac{2c_1^2}{l^2}(nlt)^{\frac{2n-1}{n}} \tag{49}$$

7. FIGURES

In this section, to better understand the behavior of our cosmological model, we present plots of various physical and dynamical parameters as functions of cosmic time.

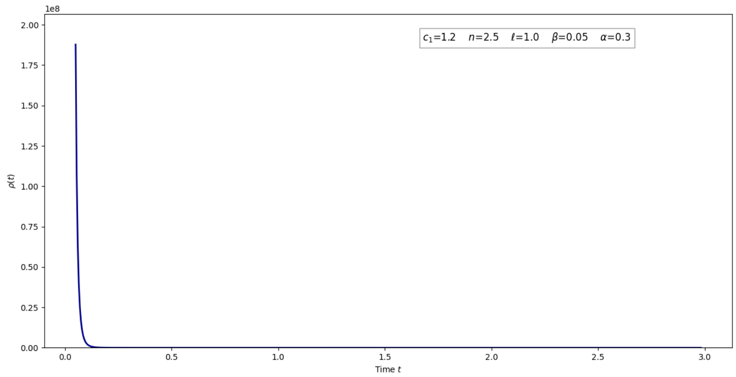

Figure 1. Variation of Energy Density

The graph indicates that the universe started with a very high density and then decreases rapidly, reflecting the expected expansion dynamics.

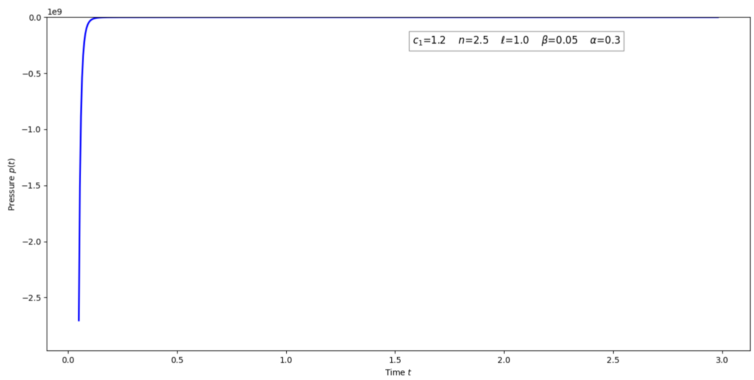

Figure 2. Variation of Pressure

The graph indicates that the universe begins with a strong negative pressure (accelerating expansion), but over time the pressure diminishes and tends toward zero, suggesting a transition toward a less repulsive, more stable cosmic state.

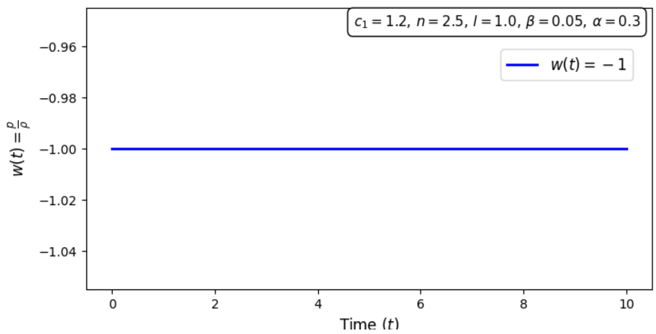

Figure 3. Equation of state parameter

The graph shows that the equation of state parameter $\omega = -1$ through out the cosmic time.This corresponds to vacuum energy or cosmological constant Λ.

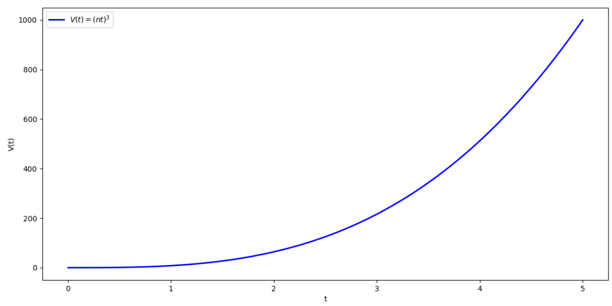

Figure 4. Variation of Spatial Volume V

The plot shows that the spatial volume of the universe expands rapidly as time increases. It begins at t=0 (Big Bang singularity) and indicates cosmic expansion.

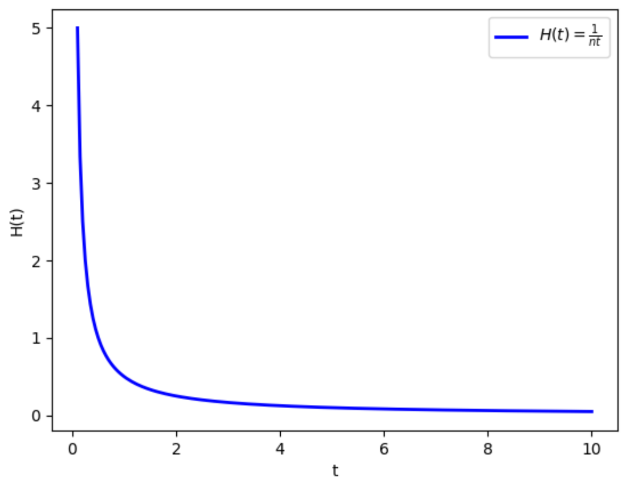

Figure 5. Variation of Hubble Parameter

Expansion rate of the universe is very high at early times ($t \longrightarrow 0$) but decreases as the universe grows late time.It never becomes negative, but approaches zero as $t \longrightarrow \infty$.

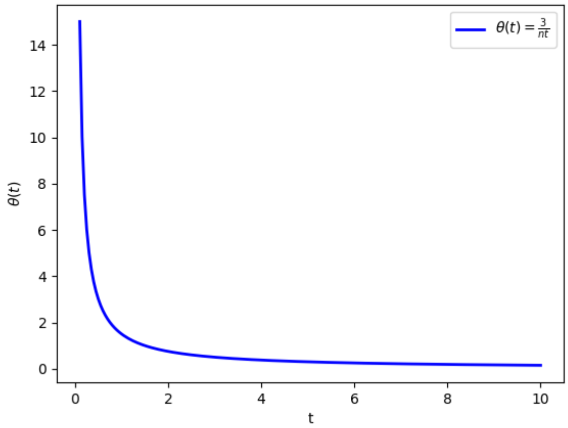

Figure 6. Variation of Scalar Expansion

The rate of volume expansion of the universe decreases with time. This supports the idea that the early universe expanded extremely fast but slowed down with cosmic time.

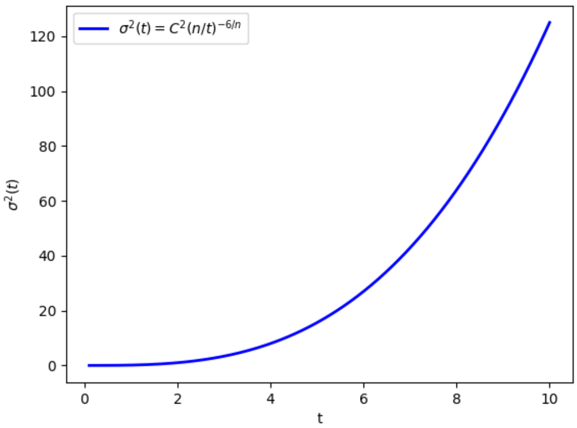

Figure 7. Variation of Shear Scalar

The plot represents the anisotropy (directional deformation) in the expansion. At early times ($t \longrightarrow 0$), anisotropy is large, but it decreases with time. This indicates that the universe tends to become isotropic as it evolves.

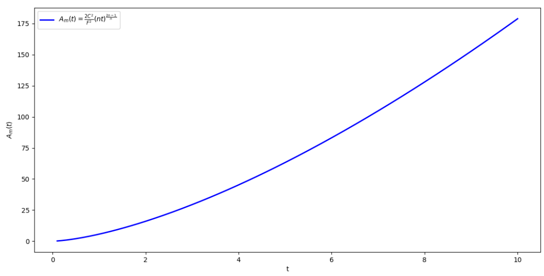

Figure 8. Variation of Anisotropic Parameter

Describes how strongly anisotropy affects cosmic expansion. If $n < 0.5$, the universe tends to become isotropic at late times.

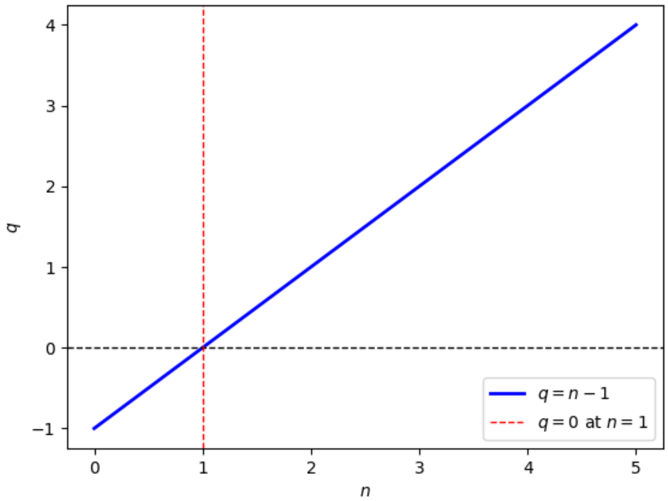

Figure 9. Variation of deceleration parameter

A graph represents a straight line in n, independent of time. For $n < 1$: $q < 0$, accelerated expansion (dark energy–like). For $n = 1$: $q = 0$, expansion at a uniform rate. For $n > 1$: $q > 0$, decelerated expansion (matter-dominated).

8. CONCLUSIONS

We have studied the anisotropic LRS Bianchi type-I cosmological model in the framework of $f(R, L_m)$ gravity, the contribution of dark energy in accelerating the cosmic evolution. Applying the functional form $f(R, L_m) = \frac{R}{2} + L_m^{\alpha} + \beta$ with $L_m = \rho$, and applying the special law of variation for the Hubble parameter, we obtained exact solutions to the modified field eq uations. Also, we have studied the physical and dynamical parameters such as Hubble parameter, spatial volume, deceleration parameter, scalar expansion, shear scalar, anisotropic parameter, pressure, density and EoS parameter and analyzed these parameters graphically from which we observed the following facts.

The energy density decreases rapidly with time and approaches near zero at late times. The pressure, initially highly negative, drives an accelerated expansion in the early universe but gradually stabilizes toward zero, reflecting a transition to a more balanced expansion regime. The equation of state parameter remains close to $\omega = -1$, suggesting that the dark energy component in this model mimics a cosmological constant.

The spatial volume expands unboundedly from a vanishing value at t=0, consistent with Big Bang cosmology, while the Hubble parameter and expansion scalar decrease over time, signifying a slowing but persistent expansion. The shear scalar and anisotropy parameter exhibit decreasing trends for $n < 0.5$, indicating isotropization of the universe, but for $n > 0.5$, anisotropy persists, showing that the universe evolves toward anisotropy in certain dynamical regimes. The deceleration parameter shows the different values of cosmic p hases: accelerated expansion for ($n < 1$), uniform expansion at ($n = 1$), and deceleration for ($n > 1$).

Overall, the model successfully reproduces an accelerated, expanding universe consistent with present-day observations, while also providing a framework to investigate anisotropy in cosmic evolution. In particular, it shows that matter curvature coupling in $f(R, L_m)$ gravity can explain the late time cosmic acceleration without using exotic scalar fields.

This work emphasizes that anisotropic cosmological models in $f(R, L_m)$ gravity are not only viable but also rich in dynamical structure, offering t estable p redictions f or d ark e nergy b ehavior. F uture i nvestigations m ay r efine this framework by incorporating bulk viscosity, observational constraints from supernovae and CMB data, and extensions to other Bianchi-type universes, thereby deepening our understanding of the interplay between anisotropy, dark energy, and modified gravity in shaping cosmic evolution.

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We would like to thank you for *following the instructions above* very closely in advance. It will definitely save us lot of time and expedite the process of your paper's publication.

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REFERENCES

[1] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, (John Wiley and Sons, 2013).

[2] I. Domínguez, E.B. Guil, L. Piersanti, O. Straniero, and A. Tornambé, "Supernovae and dark energy," in: *Cosmology Across Cultures*, (Astronomical Society of the Pacific, 2008), pp. 42-46. https://hdl.handle.net/2117/22952

[3] J.-M. Alimi, A. Füzfa, V. Boucher, Y. Rasera, J. Courtin, and P.-S.Corasaniti, "Imprints of dark energy on cosmic structure formation–I. Realistic quintessence models and the non-linear matter power spectrum," Monthly Notices of the Royal Astronomical Society, **401**(2), 775-790 (2010). https://doi.org/10.1111/j.1365-2966.2009.15712.x

[4] S. Perlmutter, M.S. Turner, and M. White, "Constraining dark energy with type Ia supernovae and large-scale structure," Physical Review Letters, **83**(4), 670 (1999). https://doi.org/10.1103/PhysRevLett.83.670

[5] S. Boughn, and R. Crittenden, "A correlation between the cosmic microwave background and large-scale structure in the Universe," Nature, 427(6969), 45-47 (2004). https://doi.org/10.1038/nature02139

[6] Montesano, Francesco, "The full shape of the large-scale galaxy power spectrum: modelling and cosmological implications," D.Sc. Thesis, Ludwig–Maximilians–Universit, Munich, 2011. https://doi.org/10.5282/edoc.13663

[7] J.-P. Dai, Y. Yang, and J.-Q. Xia, "Reconstruction of the dark energy equation of state from the latest observations," The Astrophysical Journal, **857**(1), 9 (2018). https://doi.org/10.3847/1538-4357/aab49a

[8] S. Tsujikawa, "Dark energy: Observational status and theoretical models," in: *Quantum Gravity and Quantum Cosmology*, edited by G. Calcagni, L. Papantonopoulos, G. Siopsis, and N. Tsamispp, (Springer Berlin, Heidelberg, 2013), pp. 289-331. https://doi.org/10.1007/978-3-642-33036-0_11

[9] J. Yoo, and Y. Watanabe, "Theoretical models of dark energy," International Journal of Modern Physics D, **21**(12), 1230002 (2012). https://doi.org/10.1142/S0218271812300029

[10] Tsujikawa, Shinji, "Dark energy: investigation and modeling," in: *Dark Matter and Dark Energy. Astrophysics and Space Science Library*, vol. 370. edited by S. Matarrese, M. Colpi, V. Gorini, and U. Moschella, (Springer, Dordrecht, 2011), pp. 331-402. https://doi.org/10.1007/978-90-481-8685-3_8

[11] T. Harko, and F.S.N. Lobo, "$f(R, L_m)$ gravity," The European Physical Journal C, **70**, 373–379 (2010). https://doi.org/10.1140/epjc/s10052-010-1467-3

[12] D.C. Maurya, "Bianchi-I dark energy cosmological model in f(R,Lm)-gravity," International Journal of Geometric Methods in Modern Physics, **21**(04), 2450072 (2024). https://doi.org/10.1142/S0219887824500725

[13] X. Liu, T. Harko, and S.D. Liang, "Cosmological implications of modified gravity induced by quantum metric fluctuations," The European Physical Journal C, **76**(8), 420 (2016). https://doi.org/10.1140/epjc/s10052-016-4275-6

[14] M.F. Shamir, "Dynamics of anisotropic power-law f (R) cosmology," Journal of Experimental and Theoretical Physics, **123**(6), 979-984 (2016). https://doi.org/10.1134/S1063776116150152

[15] M. Koussour, N. Myrzakulov, A.H.A. Alfedeel, and A. Abebe, "Constraining the cosmological model of modified f(Q) gravity: Phantom dark energy and observational insights," Progress of Theoretical and Experimental Physics, **2023**(1), 113E01 (2023). https://doi.org/10.1093/ptep/ptad133

[16] M. Zeyauddin, A. Dixit, and A. Pradhan, "Anisotropic dark energy models in f(R,Lm)-gravity," International Journal of Geometric Methods in Modern Physics, **21**(02), 2450038 (2024). https://doi.org/10.1142/S0219887824500385

[17] A. Pradhan, D.C. Maurya, Dinesh Chandra and G.K. Goswami, and A. Beesham, "Modeling transit dark energy in f(R,Lm)-gravity," International Journal of Geometric Methods in Modern Physics, **20**(06), 2350105 (2023). https://doi.org/10.1142/S0219887823501050

[18] D.C. Maurya, "Constrained transit cosmological models in $f(R, L\_\{m\}, T)$-gravity," arXiv:2409.14024, https://doi.org/10.48550/arXiv.2409.14024; International Journal of Geometric Methods in Modern Physics, **22**(07), 2550028 (2025). https://doi.org/10.1142/S0219887825500288

[19] A. Dixit, A. Pradhan, M. Zeyauddin, and J. Singh, "Dark energy nature of logarithmic f(R,Lm)-gravity models with observational constraints," International Journal of Modern Physics A, **39**(07n08), 2450043 (2024). https://doi.org/10.1142/S0217751X2450043X

[20] R. Solanki, B. Patel, L.V. Jaybhaye, and P.K. Sahoo, "Cosmic acceleration with bulk viscosity in an anisotropic f(R,Lm) background," Communications in Theoretical Physics, **75**(7), 075401 (2023). https://doi.org/10.1088/1572-9494/acd4aa

[21] L.V. Jaybhaye, R. Solanki, S. Mandal, P.K. Sahoo, "Constraining the Viscous Dark Energy Equation of State in f(R,Lm) Gravity, **9**(4), 163 (2023). https://doi.org/10.3390/universe9040163

[22] L.V. Jaybhaye, R. Solanki, and P.K. Sahoo, "Bouncing cosmological models in f(R,Lm) gravity," Physica Scripta, **99**(6), 065031 (2024). https://doi.org/10.1088/1402-4896/ad4838

[23] M. Koussour, N. Myrzakulov, J. Rayimbaev, A.H.A. Alfedeel, and H.M. Elkhair, "Bouncing behavior in $f(R, L_m)$ gravity: Phantom crossing and energy conditions," arXiv preprint arXiv:2403.15772, (2024). https://doi.org/10.48550/arXiv.2403.15772; International Journal of Geometric Methods in Modern Physics, **21**(11), 2450184 (2024). https://doi.org/10.1142/S0219887824501846

[24] D.C. Maurya, "Constraining on Dark Energy Models in Higher Derivative $F(R, LM)$-Gravity Theory with Hubble Function," http://dx.doi.org/10.2139/ssrn.4552229

[25] S. Sahlua, A.H.A. Alfedeelb, and A. Abebe, "The cosmology of $f(R, L\_m)$ gravity: constraining the background and perturbed dynamics," arXiv:2406.08303, (2024). https://doi.org/10.48550/arXiv.2406.08303

[26] R. Garg, G.P. Singh, A. Lalke, and S. Ray, "Cosmological model with linear equation of state parameter in f(R,Lm) gravity," Physics Letters A, **525**, 129937 (2024). https://doi.org/10.1016/j.physleta.2024.129937

[27] Y.K. Devi, S.A. Narawade, and B. Mishra, "Constraining parameters for the accelerating universe in f(R,Lm) gravity," hysics of the Dark Universe, **46**, 101640 (2024). https://doi.org/10.1016/j.dark.2024.101640

[28] V. Patil, J. Pawde, R. Mapari, S. Waghmare, "FLRW Cosmology with Hybrid Scale Factor in f(R,Lm) Gravity," East European Journal of Physics, (4), 8-17 (2023). https://doi.org/10.26565/2312-4334-2023-4-01

[29] B.K. Shukla, R.K. Tiwari, and A. Beesham, "FLRW universe in f(Q,T) gravity," International Journal of Geometric Methods in Modern Physics, **20**(14), 2350242 (2023). https://doi.org/10.1142/S0219887823502420

[30] J.K. Singh, R. Myrzakulov, and H. Balhara, *et al.*, "A constrained cosmological model in f(R,Lm) gravity," New Astronomy, **104**, 102070 (2023). https://doi.org/10.1016/j.newast.2023.102070

[31] K. Pawar, N.T. Katre, and A.K. Dabre, "Accelerating Expansion of the Universe with Dark Matter and Holographic Dark Energy in $f(\mathrm{T})$ Gravity," Int. J. Sci. Res. in Physics and Applied Sciences, **11**(2), 1-9 (2023). https://www.isroset.org/journal/IJSRPAS/full_paper_view.php?paper_id=3094

[32] Y.-J. Suh, K. De, and U.C. De, "On Pseudo B-symmetric spacetimes and $f(R)$ gravity," Int. J. Geom. Meth. Mod. Phys. **22**(02), 2450280 (2025). https://doi.org/10.1142/S0219887824502803

[33] C.R. Mahanta, S. Deka, and K. Pathak, "Anisotropic Cosmological Model in f (R, T) Theory of Gravity with a Quadratic Function of T," East European Journal of Physics, (3), 43-52 (2023). https://doi.org/10.26565/2312-4334-2023-3-02

[34] S. Kibaroglu, "Anisotropic Born Infeld f(R) cosmologies," Physics of the Dark Universe, **47**, 101784 (2025). https://doi.org/10.1016/j.dark.2024.101784

[35] R. Saleem, A. Ijaz, and S. Waheed, "Anisotropic Cosmic Expansion Inspired by Some Novel Holographic Dark Energy Models in f(Q) Theory," Fortschritte der Physik, **73**(3), 2300276 (2025). https://doi.org/10.1002/prop.202300276

[36] M.R. Ugale, and S.B. Deshmukh, "Anisotropic Bianchi type VI0 cosmological models in a modified f(R,T) gravity," Journal of Scientific Research, **16**(1), 17-30 (2024). https://doi.org/10.3329/jsr.v16i1.62830

[37] B. Mishra, F.MD. Esmeili, and S. Ray, "Cosmological models with variable anisotropic parameter in f(R,T) gravity," Indian Journal of Physics, **95**(10), 2245-2254 (2021). https://doi.org/10.1007/s12648-020-01877-2

[38] S. Chakraborty, "Reconstruction method of f(R) gravity for isotropic and anisotropic spacetimes," Physical Review D, **98**(2), 024009 (2018). https://doi.org/10.1103/PhysRevD.98.024009

[39] M.F. Shamir, and A. Komal, "Energy bounds in f(R,G) gravity with anisotropic background," International Journal of Geometric Methods in Modern Physics, **14**(12), 1750169 (2017). https://doi.org/10.1142/S0219887817501699

[40] P. Bolke, V. Patil, S. Waghmare, and N. Mahajan, "Anisotropic Cosmological Model with SQM in f(R,Lm) Gravity," East European Journal of Physics, (3), 45-55 (2024). https://doi.org/10.26565/2312-4334-2024-3-05

[41] M.S. Berman, "A special law of variation for Hubble's parameter," Nuovo Cimento B Serie, **74**, 182-186 (1983). http://dx.doi.org/10.1007/BF02721676

[42] M.S. Berman, and F. de Mello Gomide, "Cosmological models with constant deceleration parameter," General Relativity and Gravitation, **20**, 191-198 (1988). https://doi.org/10.1007/BF00759327

АНІЗОТРОПНА КОСМОЛОГІЯ ТЕМНОЇ ЕНЕРГІЇ В РАМКАХ $f(R, L_m)$ ГРАВІТАЦІЇ

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У цій статті ми досліджували локально обертально-симетричну (LRS) космологічну модель Біанкі I типу з темною енергією в рамках теорії гравітації $f(R, L_m)$, де R – скаляр Річчі, а L_m – лагранжіан матерії. Використовуючи функціональну форму $f(R, L_m) = \frac{R}{2} + L_m^\alpha + \beta$ з $L_m = \rho$, та застосовуючи спеціальний закон варіації для параметра Хаббла, ми отримали точні розв'язки рівнянь поля та проаналізували фізичні та динамічні властивості Всесвіту. Наші результати показують, що модель демонструє прискорене розширення, що узгоджується з даними спостережень, зі зменшенням густини енергії та переходом параметра уповільнення від позитивних до негативних значень. Параметр анізотропії спочатку наближається до нуля, але збільшується з часом для $n > 0.5$, що вказує на еволюцію від ізотропії до анізотропії. Ці результати дають уявлення про поведінку темної енергії в рамках модифікованої гравітації та пропонують перевірені прогнози для космологічних спостережень.

Ключові слова: *LRS Bianchi Type-I; темна енергія; космічний час*