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PREDICTING NEW B_c MESONS' EXCITED STATES: THE TRIDIAGONAL MATRIX-NUMEROV APPROACH

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The properties of Bottom-charmed meson states were extensively investigated using Numerov's tridiagonal matrix approach to predicting the radial wavefunctions. Based on the resulted values, we predicted the β anharmonicity values and root mean square radii for different excited states of B_c mesons. A comprehensive comparison between the estimated results and recently published theoretical and experimental data was conducted, exhibiting a comparable product with a high degree of accuracy.
Keywords: *Bottom-charmed meson; Numerov's tridiagonal matrix; β coefficient; Root-mean-square radii*
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1. INTRODUCTION

The primary motivation for studying the spectra and characteristics of heavy mesons is to better understand the subatomic particles and the forces that govern matter's behavior. A few of the primary goals of these investigations include, but are not limited to, the following [1-4]: studying heavy mesons is essential for examining and improving models such as the Quark Model, Lattice QCD, and effective field theories, as heavy mesons challenge our current understanding of the strong force. Experimental results and theoretical predictions can be cross-checked to validate these models and reveal inconsistencies that may lead to new insights [4]. Finally, the study of heavy mesons can uncover novel physics by identifying particles and events that do not conform to the Standard Model. It is possible to find exotic states, make unexpected observations, or deviate from theoretical predictions, all of which might indicate the presence of new particles or interactions that are beyond our present knowledge [5-6]. Venturing beyond the conventional Standard Model of particle physics becomes imperative when unraveling the enigmatic traits of the elusive Bottom-charmed (B_c) mesons, despite the Model's hint at their existence [7]. Here we shall look at B_c mesons and their behavior using a nonrelativistic quark model. The nonrelativistic quark model provides a simpler explanation of the quarks' activities and interactions in heavy mesons. Based on this model, we will study B_c meson properties such as quark composition, quantum properties, mass spectra, and finally the essential coefficients that are needed to study decay dynamics.

Heavy meson decay characteristics may be used to derive standard model parameters such as CKM matrix elements and quark mixing angles. Understanding the genesis of CP violation requires knowledge of the CKM matrix, which explains quark mixing in weak interactions. Precise observations of heavy meson decays constrain CKM matrix elements and quark mixing angles, aiding Standard Model accuracy tests [8, 9].

As a result, we are highly motivated to determine the root mean square radius (r_{ms}) of various B_c states and the numerical values of the β coefficient. These values may then be utilized to compute decay widths and differential cross-sections for quarkonium states [8, 10]. Furthermore, one of our supplementary goals is to study the mass-radius dependency of the B_c states within the framework of the nonrelativistic quark model.

Notably, the nonrelativistic quark model may explain several properties of B_c meson decay. The decay of B_c mesons is triggered by weak interactions, and the W boson plays a crucial role in facilitating these decays. Lighter mesons, leptons, and neutrinos are the ultimate states that determine the decay routes and velocities. Using the quark model, one can determine the branching percentages and decay widths of B_c mesons [11].

Our investigation uses numerical-based approaches for the qualitative analysis of heavy mesons, with results that highlight their potential to enhance future experimental studies while reducing the large-scale costs that currently define such research [12, 13].

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To address the challenging QCD mathematical framework, researchers can employ numerical approaches such as matrix method, the Shooting method [14, 15], the four-step exponentially fitted method [16, 17], and Numerov's method simulations [12]. The strong interaction between quarks and gluons describes heavy meson behavior, which can only be explained in terms of QCD [17]. It is feasible to estimate quantities such as meson masses, decay rates, and form factors, values that are difficult to determine analytically through numerical calculations [18, 19]. Experimental investigations rely heavily on the properties of heavy meson, and numerical methods enable precise predictions and provide valuable insights into these characteristics.

Recent publications have demonstrated that Numerov's tridiagonal matrix technique is a highly efficient and quick method for achieving our objective [12, 20-21]. We anticipate that this approximation will provide reliable characteristics for investigating heavy mesons. Furthermore, the wave functions of heavy mesons, investigated in this study, may be utilized to generate predictions regarding additional characteristics such as the root mean square radius ($r_{\rm ms}$) of various states for B_c mesons, as well as the numerical values of the β coefficient and differential cross-sections for B_c states.

This work is organized as follows: The unique characteristics of B_c mesons are outlined in section 2. The main problem and thorough analytical solutions are addressed in section 3. Section 4 is devoted to discussing resulting data in detail. Finally, we briefly summarize our main discoveries and conclusions in section 5.

2. Characteristics of Bottom-Charmed Mesons

2.1. The Potential Model

Applying an adequate potential model to solve the non-relativistic Schrodinger equation for quark-anti quark states is widely regarded as one of the most efficient approaches for modeling the heavy meson system [22-25].

The effective quark-antiquark potential may be expressed as the sum of two terms, one of which is spin-independent, and the other is spin-dependent. The linear confinement and standard color Coulomb interaction are included in the first term, whereas the spin-dependent component consists of the tensor potential, spin-orbit interaction potential, and the hyperfine potential between spin-spin interaction. As a result, the following is the final form of the potential model that was employed in this work [24-27]:

$$V(r) = -\frac{4\alpha_s}{3\,r} + br + \frac{32\pi\alpha_s}{9m_b m_{\bar{c}}}\left(\frac{\sigma}{\sqrt{\pi}}\right)^3 e^{-\sigma^2 r^2}\, S_b S_{\bar{c}} + \frac{1}{m_b m_{\bar{c}}}\left(\frac{2\alpha_s}{r^3} - \frac{b}{2r}\right)\vec{l}.\vec{S} + \frac{4\alpha_s}{r^3}T, \quad (1)$$

$$S_b.S_{\bar{c}} = \frac{s(s+1)}{2} - \frac{3}{4}, \quad (2)$$

where l is the orbital momentum, r represents the distance between the quarks, α_s is the strong running coupling constant, −4/3 is the color factor, b is a potential parameter, σ is the string tension and $S_b.S_{\bar{c}}$ is the spin–spin contact hyperfine interaction, whereas m_b and $m_{\bar{c}}$ depict the masses of the bottom and anti-charm quarks. $\vec{S}$ denotes the overall spin quantum number of the meson.

The parameters for the B_c mesons are detailed in Table 1. T represents the tensor operator.

Table 1. The parameters to fit the theoretical masses to get the best theoretical spectra of B_c states

Parameters	Theo. (NR) Potential
$m_{\bar{c}}$ [GeV]	1.4794
m_b [GeV]	4.825
α_s	0.48
b [GeV^2]	0.137
σ [GeV]	1.0946

The spin-orbit operator is diagonal when expressed in the $|J, L, S>$ basis with the matrix components:

$$\langle\vec{L}.\vec{S}\rangle = \frac{[J(J+1) - (L(L+1) - S(S+1))]}{2}$$

The tensor operator T exhibits non-zero diagonal matrix components exclusively among spin-triplet states with L>0.

$$T = \begin{cases} -\dfrac{L}{6(2L+3)}, J = L+1 \\ +\dfrac{1}{6}, J = L \\ -\dfrac{(L+1)}{6(2L-1)}, J = L-1 \end{cases}$$

2.2. Wave Functions of Bottom-Charmed Mesons

By using the potential from equation (1), B_c mesons may be modeled by the wave function of the bound quark-antiquark state that satisfies the Schrödinger equation. The radial Schrödinger equation is defined as [1, 2]:

$$-\frac{\hbar^2}{2\mu}\frac{d^2R_{nl}(r)}{dr^2}+\left(V(r)+\frac{l(l+1)}{2\mu r^2}\right)R_{nl}(r)=E_{nl}R_{nl}(r), \tag{3}$$

where n is the principal quantum number and μ represents the reduced mass of the quark and anti-quark.

The radial wave function is denoted as $R_{nl}(r)$. The total energy of the quark-antiquark system is denoted by E. The tridiagonal matrix Numerov's (TMN) approach is employed to solve equation (3) and get the mass spectra of B_c mesons. Further information on this method may be found in Reference [12]. In the subsequent sections, we utilize this approach to derive the wave functions of B_c bound states.

2.3. The tridiagonal matrix Numerov's approach

Our approach is based on the numerical solution of Eq. (3) as a matrix eigenvalue problem. The radial second-derivative finite difference approximation can be simplified by converting it into tridiagonal matrix form. Therefore, we solve this equation numerically using the Numerov technique to derive the eigenvalue and eigenfunction equations for heavy quarkonium spectrum and wave functions.

We can rewrite the equation (3) in a slightly different way to understand the probable use of Numerov's technique more clearly:

$$f(r)=2\mu\big(E-V(r)\big) \qquad ; \hbar=1, \tag{4}$$

with a distance d between each point on the lattice and x_i, which are equally spaced, we can derive the integration formula,

$$\psi_{i+1}=\frac{\psi_{i-1}(12-d^2f_{i-1})-2\psi_i(5d^2f_i+12)}{d^2f_{i+1}-12}. \tag{5}$$

Hence

$$d^2f_{i+1}\psi_{i+1}-12\psi_{i+1}=12\psi_{i-1}-d^2f_{i-1}\psi_{i-1}-10d^2f_i\psi_i-24\psi_i. \tag{6}$$

Applying Eq. (4), we obtain:

$$-2\mu d^2/\hbar^2\left[(E\psi_{i-1}-V_{i-1}\psi_{i-1})+(10E\psi_i-10V_i\psi_i)+(E\psi_{i+1}-V_{i+1}\psi_{i+1})\right]=12(\psi_{i-1}-2\psi_i+\psi_{i+1}). \tag{7}$$

After rearranging the equation above, we get:

$$\frac{-1}{2\mu}\frac{(\psi_{i-1}-2\psi_i+\psi_{i+1})}{d^2}+\frac{(V_{i-1}\psi_{i-1}+10V_i\psi_i+V_{i+1}\psi_{i+1})}{12}=E\frac{(\psi_{i+1}+10\psi_i+\psi_{i\_1})}{12}. \tag{8}$$

We will convert the well-known Numerov's approach into a matrix representation on a discrete lattice. To accomplish that, ψ will be defined as a matrix and represented by a column vector $(\ldots.\psi_{i-1},\psi_i,\psi_{i+1}\ \ldots.)$

$$A_{N,N}=\frac{(I_{-1}-2I_0+I_1)}{d^2},\ B_{N,N}=\frac{(I_{-1}+10I_0+I_1)}{12}, V_N=diag(\ldots.,V_{i-1},V_i,V_{i+1})$$

where an N-point grid and the unit matrices I_{-1}, I_0, and I_1 stand for the sub-, main-, and up-diagonals, respectively. The matrix version of Eq. (8) could be created as follows.

$$\frac{-1}{2\mu}A_{N,N}\psi_i+B_{N,N}V_N\psi_i=E_iB_{N,N}\psi_i. \tag{9}$$

Multiplying Eq. (9) by $B_{N,N}^{-1}$ yields

$$\frac{-1}{2\mu}B_{N,N}^{-1}A_{N,N}\psi_{nl}+V_N\psi_i=E_i\psi_{nl}. \tag{10}$$

This numerical technique allows us to solve the eigenvalue problem for any possible hadron-hadron bound states.

2.4. Root Mean Square Radii and β Coefficient

The afterward reasoning shall be employed to offer an explanation or description of the root mean square (r_{ms}) radius of B_c that is both clear and succinct. Among the key characteristics of this particle system is the root mean square (r_{ms}) of B_c mesons. Assuming that the distance between the quark and anti-quark in bottom-charmed mesons is represented by the symbol r (*fm*), it is possible to deduce that the radius of B_c is equal to $(r/2)$ *fm*. With the use of the meson wave function, one can derive the root mean square (r_{ms}), which may be stated accordingly [12, 26, 28]:

$$r^2{}_{ms}=\int_0^\infty\{\psi^2(r)r^2dr\}. \tag{10}$$

One aspect that characterizes the momentum width of a meson wave function is the beta coefficient. The symbol β is commonly used to represent it, and it is connected to the root mean square (r_{ms}) distance between the quark and antiquark in the meson. In addition to affecting the decay rates and form factors of the meson, the beta coefficient reveals information on the spatial distribution of the quark-antiquark pair inside the meson. Mesons' internal structure and behavior can be better understood by analyzing and describing the beta coefficient [29]:

$$\beta = \sqrt{2(n-1) + (L) + \frac{3}{2}}\ \ \frac{1}{r_{ms}} \tag{11}$$

The parameter β is commonly regarded as a model parameter. However, as we aim to explain the deterioration of heavy quark states, it is more desirable to replicate the β coefficient of the quark model states. Therefore, we recommend utilizing it for the computation of the decay width of B_c states.

3. RESULTS AND DISCUSSION

We have conducted a theoretical examination of the mass spectrum of both the ground and excited states of B_c mesons using a non-relativistic component quark model. The model parameters were adjusted to accurately replicate the empirically determined 1S ground state. Mass predictions for the 4th radial excitations of S-wave, P-wave, D-wave, and F-wave were developed and compared to previous theoretical studies and existing data.

Figures 1-4 provide the graphical depiction of normalized radial wave functions for B_c mesons. Table 2 displays the anticipated masses (measured in GeV) of both the primary and secondary states of B_c mesons. We compare our findings with previous model predictions and existing experimental evidence. The experimental masses obtained from reference [30], together with the predictions made by Asghar [31], Nosheen [7], and Qi Li [32] have been included for a thorough study. Our model's predictions exhibit strong concordance with both the existing experimental data and other theoretical predictions.

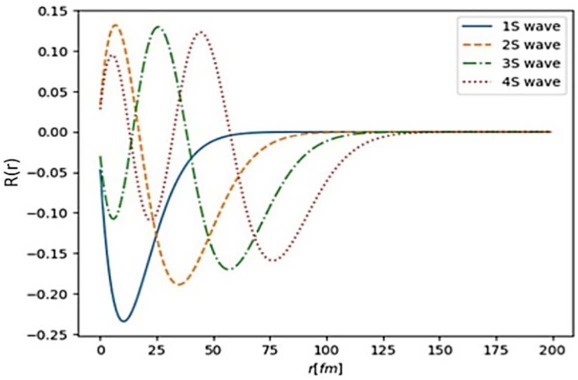

Figure 1. Bottom-Charmed for S-states reduced radial wave functions

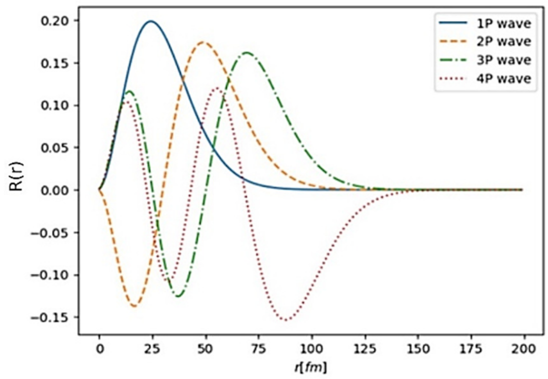

Figure 2. Bottom-Charmed for P-states reduced radial wave functions

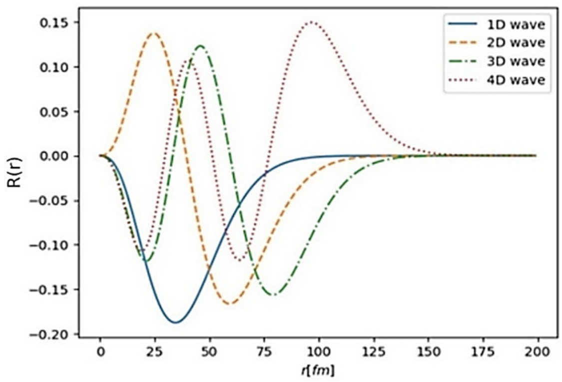

Figure 3. Bottom-Charmed for D-states reduced radial wave functions

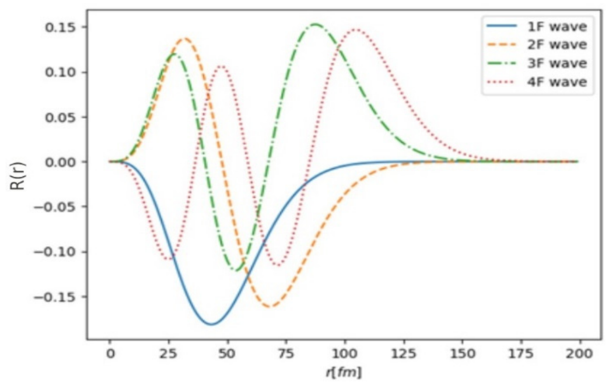

Figure 4. Bottom-Charmed for F-states reduced radial wave functions

The anticipated mass of the well-established 1S ground state B_c ($^1S_0$) is 6.275 GeV, which is in excellent agreement with the observed value of 6.2749 ± 0.008 GeV. The predictions concerning the 2S and 3S excited states are consistent with the evidence obtained from different simulations. When compared to the other computational models, our predictions often fall within a range of 0.01-0.05 GeV, which suggests a high level of accuracy**.** Significantly, our results for the P-wave and D-wave exhibit comparable mass splitting and orderings to those seen in the studies conducted by Asghar et al. and Nosheen et al. This demonstrates assurance that our model accurately replicates the anticipated spectroscopy.

For higher excitations, such as the 4S, 4P, and 4D states, where experimental data remain scarce, our model provides predictions that can be tested in future research. The projected 4F states may potentially serve as a starting point for further discoveries. All things considered, our findings are in good agreement with and add to the predictions that have already been made.

Table 2. Predicted masses of ground and excited states of B_c mesons

State	[our work] (GeV)	Qi Li [32] (GeV)	Asghar [31] (GeV)	Nosheen [7] (GeV)	Exp. Masses [30] (GeV)
$B_c\,(1^1S_0)$	6.275	6.271	6.318	6.274	6.2749 ± 0.008
$B_c\,(1^3S_1)$	6.315	6.326	6.336	6.314	
$B_c\,(2^1S_0)$	6.842	6.871	6.741	6.841	6.842±0.004
$B_c\,(2^3S_1)$	6.856	6.890	6.747	6.855	
$B_c\,(3^1S_0)$	7.198	7.239	7.014	7.197	
$B_c\,(3^3S_1)$	7.206	7.252	7.018	7.206	
$B_c\,(4^1S_0)$	7.489	7.540	7.239	7.488	
$B_c\,(4^3S_1)$	7.496	7.550	7.242	7.495	
$B_c\,(1^3P_2)$	6.747	6.787	6.665	6.753	
$B_c\,(1\,P'_1)$	6.769	6.776	6.656	6.744	
$B_c\,(1\,P_1)$	6.774	6.757	6.650	6.725	
$B_c\,(1^3P_0)$	6.746	6.714	6.631	6.701	
$B_c\,(2^3P_2)$	7.111	7.160	6.946	7.111	
$B_c\,(2\,P'_1)$	7.128	7.150	6.939	7.098	
$B_c\,(2\,P_1)$	7.132	7.134	6.930	7.105	
$B_c\,(2^3P_0)$	7.283	7107	6.915	7.086	
$B_c\,(3^3P_2)$	7.408	7.464	7.176	7.406	
$B_c\,(3\,P'_1)$	7.423	7.458	7.168	7.393	
$B_c\,(3\,P_1)$	7.427	7.441	7.162	7.405	
$B_c\,(3^3P_0)$	7.551	7.420	7.147	7.389	
$B_c\,(4^3P_2)$	7. 669	7.732	7.379	-	
$B_c\,(4\,P'_1)$	7.683	7.727	7.373	-	
$B_c\,(4\,P_1)$	7.687	7.710	7.364	-	
$B_c\,(4^3P_0)$	7.794	7.693	7.350	-	
$B_c\,(1^3D_3)$	6.769	7.030	6.847	6.998	
$B_c\,(1\,D'_2)$	6.996	7.032	6.845	6.984	
$B_c\,(1\,D_2)$	6.997	7.024	6.845	6.986	
$B_c\,(1^3D_1)$	6.964	7.020	6.841	6.964	
$B_c\,(2^3D_3)$	7.128	7.348	7.087	7.302	
$B_c\,(2\,D'_2)$	7.304	7.347	7.084	7.293	
$B_c\,(2\,D_2)$	7.304	7.343	7.084	7.294	
$B_c\,(2^3D_1)$	7.271	7.336	7.080	7.280	
$B_c\,(3^3D_3)$	7.423	7.625	7.296	7.570	
$B_c\,(3\,D'_2)$	7.572	7.623	7.293	7.562	
$B_c\,(3\,D_2)$	7.573	7.620	7.293	7.563	
$B_c\,(3^3D_1)$	7.539	7.611	7.289	7.553	
$B_c\,(4^3D_3)$	7.683	-	7.489	-	
$B_c\,(4\,D'_2)$	7.815	-	7.482	-	
$B_c\,(4\,D_2)$	7.816	-	7.482	-	
$B_c\,(4^3D_1)$	7.782	-	7.478	-	
$B_c\,(1^3F_4)$	7.181	7.227	6.9967	-	
$B_c\,(1\,F'_3)$	7.189	7.240	7.001	-	
$B_c\,(1\,F_3)$	7.188	7.224	6.994	-	
$B_c\,(1^3F_2)$	7.179	7.235	6.9972	-	
$B_c\,(2^3F_4)$	7.459	7.514	7.2126	-	
$B_c\,(2\,F'_3)$	7.465	7.525	7.214	-	
$B_c\,(2\,F_3)$	7.465	7.508	7.211	-	
$B_c\,(2^3F_2)$	7.455	7.518	7.2121	-	
$B_c\,(3^3F_4)$	7.71	7.771	-	-	
$B_c\,(3\,F'_3)$	7.715	7.779	-	-	
$B_c\,(3\,F_3)$	7.715	7.768	-	-	
$B_c\,(3^3F_2)$	7.704	7.730	-	-	
$B_c\,(4^3F_4)$	7.941	-	-	-	
$B_c\,(4\,F'_3)$	7.945	-	-	-	
$B_c\,(4\,F_3)$	7.945	-	-	-	
$B_c\,(4^3F_2)$	7.934	-	-	-	

The estimated root mean square (r_{ms}) radii of ground and excited B_c meson states are presented in Table 3, which compares the results of our model to those of earlier calculations carried out by Nosheen et al [7]. There are some radial excitations of S-wave, P-wave, D-wave, and F-wave states that are taken into consideration.

Table 3. Root mean square radii of ground and excited states of B_c mesons

State	[our work] in *fm*	Nosheen in *fm* [7]
$B_c\,(1^1S_0)$	0.319	0.318
$B_c\,(1^3S_1)$	0.335	0.334
$B_c\,(2^1S_0)$	0.724	0.723
$B_c\,(2^3S_1)$	0.733	0.732
$B_c\,(3^1S_0)$	1.053	1.052
$B_c\,(3^3S_1)$	1.06	1.059
$B_c\,(4^1S_0)$	1.338	1.337
$B_c\,(4^3S_1)$	1.343	1.342
$B_c\,(1^3P_2)$	0.595	0.594
$B_c\,(1\,P'_1)$	0.612	--
$B_c\,(1\,P_1)$	0.618	--
$B_c\,(1^3P_0)$	0.757	0.562
$B_c\,(2^3P_2)$	0.944	0.940
$B_c\,(2\,P'_1)$	0.96	--
$B_c\,(2\,P_1)$	0.965	--
$B_c\,(2^3P_0)$	1.076	0.920
$B_c\,(3^3P_2)$	1.24	1.235
$B_c\,(3\,P'_1)$	1.254	--
$B_c\,(3\,P1)$	1.259	--
$B_c\,(3^3P_0)$	1.354	1.220
$B_c\,(4^3P_2)$	1.504	--
$B_c\,(4\,P'_1)$	1.518	--
$B_c\,(4\,P1)$	1.522	--
$B_c\,(4^3P_0)$	1.606	--
$B_c\,(1^3D_3)$	0.612	0.793
$B_c\,(1\,D'_2)$	0.788	--
$B_c\,(1\,D_2)$	0.791	--
$B_c\,(1^3D_1)$	0.743	0.752
$B_c\,(2^3D_3)$	0.96	--
$B_c\,(2\,D'_2)$	1.107	1.107
$B_c\,(2\,D_2)$	1.109	--
$B_c\,(2^3D_1)$	1.062	1.083
$B_c\,(3^3D_3)$	1.254	1.382
$B_c\,(3\,D'_2)$	1.385	1.382
$B_c\,(3\,D_2)$	1.387	--
$B_c\,(3^3D_1)$	1.339	1.364
$B_c\,(4^3D_3)$	1.518	–
$B_c\,(4\,D'_2)$	1.637	–
$B_c\,(4\,D_2)$	1.639	–
$B_c\,(4^3D_1)$	1.59	–
$B_c\,(1^3F_4)$	0.95	–
$B_c\,(1\,F'_3)$	0.951	–
$B_c\,(1\,F_3)$	0.952	–
$B_c\,(1^3F_2)$	0.933	–
$B_c\,(2^3F_4)$	1.248	–
$B_c\,(2\,F'_3)$	1.249	–
$B_c\,(2\,F_3)$	1.25	–
$B_c\,(2^3F_2)$	1.231	–
$B_c\,(3^3F_4)$	1.512	–
$B_c\,(3\,F'_3)$	1.513	–
$B_c\,(3\,F_3)$	1.514	–
$B_c\,(3^3F_2)$	1.496	–
$B_c\,(4^3F_4)$	1.755	–
$B_c\,(4\,F'_3)$	1.756	–
$B_c\,(4\,F_3)$	1.757	–
$B_c\,(4^3F_2)$	1.738	–

Our radii predictions correlate with those of Nosheen et al. in a very tight manner, with deviations of less than 0.01 fm across the board for all states. Our model provides a realistic description of the spatial sizes of B_c mesons, as this explains how it works. The fact that our model is able to accurately predict spatial features is validated by the fact that it is in close agreement with earlier computations.

The β values of the ground and excited states of B_c mesons are presented in Table 4. This table also includes a comparison between our findings and the anticipated results made by Asghar [31]. It is essential to take into consideration the fact that Asghar's predictions are not accessible for certain stages in the F-wave, and our findings are given in comparison to those predictions. In general, the comparison of β values between our calculations and Asghar's prediction indicates the existence of reliable and favorable compatibility throughout a wide range of states of B_c mesons.

Table 4. β values of ground and excited states of B_c mesons

State	[our work] in GeV	Asghar in GeV [31]
$B_c\ (1^1S_0)$	0.741	0.653
$B_c\ (1^3S_1)$	0.689	0.634
$B_c\ (2^1S_0)$	0.513	0.515
$B_c(2^3S_1)$	0.503	0.508
$B_c(3^1S_0)$	0.446	0.442
$B_c(3^3S_1)$	0.437	0.439
$B_c\ (4^1S_0)$	0.412	0.402
$B_c\ (4^3S_1)$	0.402	0.401
$B_c\ (1^3P_2)$	0.525	0.468
$B_c\ (1P'_1)$	0.51	0.471
$B_c\ (1P_1)$	0.505	0.468
$B_c\ (1^3P_0)$	0.612	0.468
$B_c\ (2^3P_2)$	0.443	0.428
$B_c\ (2P'_1)$	0.436	0.430
$B_c\ (2P_1)$	0.434	0.428
$B_c\ (2^3P_0)$	0.502	0.428
$B_c\ (3^3P_2)$	0.406	0.395
$B_c\ (3P'_1)$	0.401	0.397
B_c (3P1)	0.4	0.395
$B_c\ (3^3P_0)$	0.449	0.395
$B_c\ (4^3P_2)$	0.382	0.373
$B_c\ (4P'_1)$	0.379	0.374
$B_c\ (4P_1)$	0.378	0.373
$B_c\ (4^3P_0)$	0.379	0.373
$B_c\ (1^3D_3)$	0.471	0.417
$B_c\ (1D'_2)$	0.468	0.417
$B_c\ (1D_2)$	0.467	0.417
$B_c\ (1^3D_1)$	0.623	0.417
$B_c\ (2^3D_3)$	0.419	0.395
$B_c\ (2D'_2)$	0.418	0.395
$B_c\ (2D_2)$	0.417	0.395
$B_c\ (2^3D_1)$	0.509	0.395
$B_c\ (3^3D_3)$	0.391	0.374
$B_c\ (3D'_2)$	0.39	0.374
$B_c\ (3D_2)$	0.39	0.374
$B_c\ (3^3D_1)$	0.454	0.374
$B_c\ (4^3D_3)$	0.372	0.357
$B_c\ (4D'_2)$	0.372	0.358
$B_c\ (4D_2)$	0.371	0.357
$B_c\ (4^3D_1)$	0.383	0.357
$B_c\ (1^3F_4)$	0.441	0.390
$B_c\ (1F'_3)$	0.44	0.390
$B_c\ (1F_3)$	0.439	0.390
$B_c\ (1^3F_2)$	0.539	0.390
$B_c\ (2^3F_4)$	0.403	0.375
$B_c\ (2F'_3)$	0.403	0.375
$B_c\ (2F_3)$	0.402	0.375

State	[our work] in GeV	Asghar in GeV [31]
$B_c\,(2^3F_2)$	0.467	0.375
$B_c\,(3^3F_4)$	0.38	-
$B_c\,(3F'_3)$	0.38	-
$B_c\,(3F_3)$	0.38	-
$B_c\,(3^3F_2)$	0.428	-
$B_c\,(4^3F_4)$	0.364	-
$B_c\,(4F'_3)$	0.364	-
$B_c\,(4F_3)$	0.364	-
$B_c\,(4^3F_2)$	0.368	-

4. CONCLUSIONS

In this work, we have used the Tridiagonal Matrix Numerov (TMN) approach to estimate the mass spectra of the B_c meson system's newly anticipated excited states. The TMN technique produced precise and stable eigenvalues corresponding to different radial and orbital quantum numbers by numerically solving the radial Schrödinger equation under a suitable potential framework.

Our findings show that the TMN approach is a reliable and effective computational technique for simulating heavy-heavy quark systems, such as the B_c meson, especially when it comes to Predicting higher excited states that are now unattainable through experimentation.

The calculated masses provide accurate predictions for undiscovered excited states that might direct future experimental efforts at facilities like LHCb and Belle II. They are also compatible with established theoretical models and, when accessible, current experimental evidence.

The TMN approach's efficacy in hadron spectroscopy is demonstrated by its ability to capture the fine structure of the B_c spectrum. In addition to improving our theoretical knowledge of double-heavy mesons, this study provides a framework for Predicting further findings in heavy quarkonium physics.

The most noteworthy outcomes of this work are that the decay widths and differential cross sections for B_c states can be computed by utilizing the values of r_{ms} and the β coefficients that have been figured out. Furthermore, by obtaining additional meson characteristics, the effectiveness of Numerov's matrix technique can be thoroughly analyzed. The results of this evaluation are found to correspond well with the findings reported in the literature, which means a high level of accuracy.

5. RECOMMENDATIONS

Using the Tridiagonal Matrix-Numerov (TMN) approach to predict the excited states of B_c mesons, the following suggestions are made considering the results:

- Adoption in Other Quarkonium Systems: Due to its accuracy and efficiency, the TMN technique is advised for use in other heavy meson systems, including Y\Upsilon (bottomonium), ψ\psi (charmonium), and mixed-flavor mesons like B_s or D_s.
- Enhanced Potential Models: To better depict both short-range and long-range interactions, it is recommended to include more realistic potentials, such as the Cornell, logarithmic, or QCD-inspired screened potentials.
- To facilitate the search for and possible confirmation of novel B_c meson states, experimental collaborations (such as LHCb, CMS, and Belle II) should be provided with the expected excited states.
- Cross-Validation with Other Methods: To achieve consistency and dependability, results from TMN should be cross-checked with those from other numerical approaches, such as lattice QCD, variational techniques, or the shot method.

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Declarations Conflict of interest

The authors declare that no conflicts of interest or personal relationships have influenced this work.

Data availability statement

My manuscript and associated personal data will be shared with Research Square for the delivery of the author dashboard.

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REFERENCES

[1] M.S. Ali, *Spectra of quarkonium*, (LAP LAMBERT Academic Publishing, Germany, 2015). http://dx.doi.org/10.13140/2.1.4598.8166
[2] D. Griffiths, *Introduction to elementary particles*, (John Wiley & Sons, 2020).
[3] M. Thomson, *Modern particle physics*, (Cambridge University Press, 2013).
[4] G. Kane, *Modern elementary particle physics: explaining and extending the standard model*, (Cambridge University Press, 2017).
[5] L. Apolinário, Y.-J. Lee, and M. Winn, "Heavy quarks and jets as probes of the QGP," Progress in Particle and Nuclear Physics, **127**, 103990 (2022). https://doi.org/10.1016/j.ppnp.2022.103990
[6] M. Mai, Ulf-G. Meißner, and C. Urbach, "Towards a theory of hadron resonances," Physics Reports, **1001**, 1-66 (2023). https://doi.org/10.1016/j.physrep.2022.11.005
[7] N. Akbar, F. Akram, B. Masud, and M.A. Sultan, "Conventional and hybrid B_c mesons in an extended potential model," The European Physical Journal A, **55**(5), 82 (2019). https://doi.org/10.1140/epja/i2019-12735-1
[8] K.K. Pathak, S. Bhattacharya, and T. Das, "Weak decays of B_c meson in a QCD potential model," Indian Journal of Physics, **97**, 3685–3689 (2023). https://doi.org/10.1007/s12648-023-02716-w
[9] P. Gambino, S. Hashimoto, S. Mächler, M. Panero, F. Sanfilippo, S. Simula, A. Smecca, and N. Tantalo, "On the study of inclusive semileptonic decays of B_s-meson from lattice QCD," arXiv preprint arXiv:2311.09892 (2023). https://doi.org/10.48550/arXiv.2311.09892
[10] A.M. Yasser, G.S. Hassan, and T.A. Nahool, "A study of some properties of bottomonium," Journal of Modern Physics, **5**(17), 1938 (2014). http://dx.doi.org/10.4236/jmp.2014.517188
[11] M.A. Nobes, and R.M. Woloshyn, "Decays of the B_c meson in a relativistic quark-meson model," Journal of Physics G: Nuclear and Particle Physics, **26**(7), 1079 (2000). https://doi.org/10.1088/0954-3899/26/7/308
[12] M.S. Ali, G.S. Hassan, A.M. Abdelmonem, S.K. Elshamndy, F. Elmasry, and A.M. Yasser, "The spectrum of charmed quarkonium in non-relativistic quark model using matrix Numerov's method," Journal of Radiation Research and Applied Sciences, **13**(1), 226-233 (2020). https://doi.org/10.1080/16878507.2020.1723949
[13] A.M. Yasser, and T.A. Nahool, "A new gate to Numerov's method," Open Access Journal of Physics, **2**(3), (2018). https://doi.org/10.22259/2637-5826.0203001
[14] S.M. Ryan, D.J. Wilson, and Hadron Spectrum Collaboration, "Excited and exotic bottomonium spectroscopy from lattice QCD," Journal of High Energy Physics, **2021**(2), 1-23 (2021). https://doi.org/10.1007/JHEP02(2021)214
[15] J. Killingbeck, "Shooting methods for the Schrodinger equation," Journal of Physics A: Mathematical and General, **20**(6), 1411 (1987). https://doi.org/10.1088/0305-4470/20/6/024
[16] V. Kumar, R.M. Singh, S.B. Bhardwaj, R. Rani, and F. Chand, "Analytical solutions to the Schrodinger equation for generalized Cornell potential and its applications to diatomic molecules and heavy mesons," Modern Physics Letters A, **37**(02), 2250010 (2022). https://doi.org/10.1142/S0217732322500109
[17] F.J. Ynduráin, *The theory of quark and gluon interactions*, (Springer Science & Business Media, 2007).
[18] M.A. Ivanov, Yu.L. Kalinovsky, and C.D. Roberts, "Survey of heavy-meson observables." Physical Review D, **60**(3), 034018 (1999). https://doi.org/10.1103/PhysRevD.60.034018
[19] J. Ahmed, R. Manzoor, L. Chang, A. Raya, and K. Raya, "Heavy Quark Mesons: Mass Spectrum and Mass Relations," Few-Body Systems, **62**(3), 39 (2021). https://doi.org/10.1007/s00601-021-01624-1
[20] A.M. Yasser, G.S. Hassan, S.K. Elshamndy, and M.S. Ali, "Comparison Between Two Numerical Schemes to Study the Spectra of Charmed Quarkonium," arXiv preprint arXiv:2010.07436 (2020). https://doi.org/10.48550/arXiv.2010.07436
[21] M. Allosh, M.S. Abdelaal, F. Alshowaikh, and A. Ismail, "Numerical Solutions of a Three-Dimensional Schrodinger Equation for a Non–Relativistic Quark Model," Applied Mathematics & Information Sciences, **17**(3), 447-452 (2023). http://dx.doi.org/10.18576/amis/170306
[22] M.S. Ali, and A.M. Yasser, "Spectra of quark-antiquark bound states via two derived QCD potential," arXiv preprint arXiv:1502.06569 (2015). https://arxiv.org/abs/1502.06569
[23] T.A. Nahool, M. Yasser, M. Anwar, and G.A. Yahya, "Charmonium properties," East European Journal of Physics, (3) 31-38 (2020). https://doi.org/10.26565/2312-4334-2020-3-04
[24] T. Barnes, S. Godfrey, and E.S. Swanson, "Higher charmonia," Physical Review D, **72**(5), 054026 (2005). https://doi.org/10.1103/PhysRevD.72.054026
[25] Semay, Claude, and B. Silvestre-Brac. "Potential models and meson spectra." Nuclear Physics A 618, no. 4 (1997): 455-482.
[26] M. Allosh, Y. Mustafa, N. Khalifa Ahmed, A. Sayed Mustafa, Few-Body Syst (2021) 62:26 https://doi.org/10.1007/s00601-021-01608-1
[27] M.I. Hapareer, M. Allosh, G.S. Hassan, and A.M. Yasser, "Bottomonia Under Effect Three Inspired QCD Potentials in the Framework of Non-Relativistic Quark Model," East Eur. J. Phys. (2), 348 (2023). https://doi.org/10.26565/2312-4334-2023-2-41
[28] T. Das, and D.K. Choudhury, "Root mean square radii of heavy flavoured mesons in a quantum chromodynamics potential model," Pramana, **87**, 1-7 (2016). https://doi.org/10.1007/s12043-016-1242-6
[29] C.-Y. Wong, "Molecular states of heavy quark mesons," Physical Review C, **69**(5), 055202 (2004). https://doi.org/10.1103/PhysRevC.69.055202
[30] P.A. Zyla, and Particle Data Group, "Review of Particle Physics," PTEP, **2020**, 083C01 (2020). https://doi.org/10.1093/ptep/ptaa104
[31] I. Asghar, F. Akram, B. Masud, and M.A. Sultan, "Properties of excited charmed-bottom mesons," Physical Review D, **100**(9), 096002 (2019). https://doi.org/10.1103/PhysRevD.100.096002
[32] Q. Li, M.-S. Liu, L.-S. Lu, Q.-F. Lü, L.-C. Gui, and X.-H. Zhong, "Excited bottom-charmed mesons in a nonrelativistic quark model," Physical Review D, **99**(9), 096020 (2019). https://doi.org/10.1103/PhysRevD.99.096020
[33] A.A. Aly, "Heavy meson spectra in the non-relativistic quark model," M. Sc. Thesis, South Valley University, Egypt, 2012.

ПРОГНОЗУВАННЯ ЗБУДЖЕНИХ СТАНІВ НОВИХ *Bc*-МЕЗОНІВ: ТРИДІАГОНАЛЬНИЙ МАТРИЧНО-ЧИСЛОВИЙ ПІДХІД

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Властивості станів нижньо-чарованих мезонів були ретельно досліджені з використанням тридіагонального матричного підходу Нумерова для прогнозування радіальних хвильових функцій. На основі отриманих значень ми передбачили значення ангармонізму β та середньоквадратичні радіуси для різних збуджених станів Bc-мезонів. Було проведено комплексне порівняння між оціненими результатами та нещодавно опублікованими теоретичними та експериментальними даними, яке показало порівнянний добуток з високим ступенем точності.

Ключові слова: *нижньо-чарований мезон; тридіагональна матриця Нумерова; коефіцієнт β; середньоквадратичні радіуси*