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COSMOLOGICAL DIAGNOSTICS AND STABILITY OF DARK ENERGY MODEL IN NON-METRIC GRAVITY

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In this study, we investigate the dynamics of generalized ghost pilgrim dark energy in the background of $f(Q,C)$ gravity, where Q is the non-metricity scalar and C represents the boundary term. To complete this objective, we take an isotropic and homogeneous universe with an ideal matter distribution. Our analysis includes a scenario with non-interacting fluids, encompassing both dark matter and dark energy. To understand the cosmic dynamics, we reconstruct a $f(Q,C)$ model and examine its influence on the universe evolution. We explore key cosmological factors, i.e., state variable, the behavior of $(\omega_D - \omega'_D)$-plane and the statefinder diagnostic pair, which help to analyze the cosmic expansion. A crucial aspect of our analysis is the stability of generalized ghost pilgrim dark energy model via the squared sound speed method, confirming its viability in supporting the observed accelerated expansion. Our findings are consistent with observational data, demonstrating that $f(Q,C)$ gravity provides a robust theoretical foundation for describing dark energy and the universe large-scale dynamics. This work not only deep our understanding of modified gravity and mysterious energy but also offers new insights into alternative explanation for cosmic acceleration beyond standard paradigms.

Keywords: $f(Q,C)$ *gravity; Dark energy model; Stability analysis*

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1. INTRODUCTION

Einstein theory of gravitation (GR) provides a geometric framework that remains a foundational pillar of modern physics. A significant problem for this theory appears when it addresses the phenomenon of late-time cosmic expansion. The range of astronomical observations such as supernovae type-Ia demonstrate that the cosmos is presently undergoing the expansion phase [1]-[5]. This acceleration is hypothesized to originate from an ambiguous force, named as dark energy (DE). The enigmatic nature and mechanisms of DE constitute one of the most profound unresolved questions in cosmology. To address this issue, the ΛCDM model has been established as a standard model to describe the ambiguous aspects of DE. While ΛCDM aligns well with observational constraints, but it faces cosmic limitations [6]-[11]. These challenges inspired the scientific community to modify GR to gain deep insights for DE and cosmic acceleration [12]. Such modifications redefine the geometric components of the action, thereby proposing alternative gravitational frameworks to resolve the mysteries of the universe. These theories offer new approaches to cosmology and astrophysics, allowing for theoretical and observational advancements.

Modified theories may include exploring alternative geometries beyond Riemannian such as torsion and non-metricity, which offer broader geometric interpretations of gravity. Weyl [13] introduced a geometric framework that extends Riemannian geometry by introducing an additional connection, known as the "length connection," which modifies how vectors behave under parallel transport. Building on this, Weyl-Cartan geometry [14] further incorporates torsion into the structure. Later, Weitzenböck [15] proposed a different approach, defining Weitzenböck spaces characterized by non-zero torsion but zero curvature, leading to the idea of teleparallelism. In this framework, gravity is described through torsion rather than curvature [16]. As a result, GR can be formulated in multiple ways: the traditional curvature-based approach (where torsion and non-metricity vanish) or the teleparallel approach (where curvature and non-metricity are absent). Also, non-metricity describes changes in vector length at the time of parallel transport, where the covariant divergence of metric tensor exists, leading to symmetric teleparallel theory [17]. Different theoretical models and the limitations they face based on observations have been thoroughly investigated [18]-[37].

The symmetric teleparallel gravity is further modified by introducing a boundary term into the functional action, named as $f(Q,C)$ theory [38]. This extension has garnered considerable attention in the scientific community due to its profound implications. This framework offers a compelling alternative with notable theoretical consequences for cosmology and astrophysics. By incorporating non-metricity and boundary effects, this theory enables a rich description

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of gravitational dynamics. This modified proposal offers fresh insights into fundamental questions about mysteries of the universe. By incorporating the boundary term, this approach enhances theoretical flexibility, enabling a wide spectrum of physically viable solutions. This extension addresses GR limitations by probing cosmological singularities and opens new pathways for cosmological models that align with current observational data. It is observed that this alternative proposal elucidates the cosmic evolutionary path and is consistent with current observable data [39]-[42].

Maurya [43] explored the DE model to study the evolutionary picture of the universe in $f(Q,C)$ theory. Further studies delved into DE scenarios incorporating boundary terms to classify DE behavior during cosmic expansion have been discussed in [44]. Myrzakulov et al [45] demonstrated that the $f(Q,C)$ framework presents a viable alternative to the standard cosmological model. The $f(Q,C)$ formalism in FRW spacetime introduces novel perspectives on the standard cosmological paradigms [46]. Maurya [47] analyzed how boundary terms influence cosmological evolution dynamics. Usman et al [48] addressed the observed matter-antimatter asymmetry through gravitational baryogenesis in the same context. The novel insights into the early-universe inflation in this modified framework has been examined in [49]. This approach not only offers a coherent description of the universe dynamics but also exhibits strong agreement with modern observational constraints [50].

Various methods have been developed to understand the influence of DE on the evolution of cosmos. Recently, a novel DE model, known as the pilgrim dark energy (PDE) model was proposed [51]. This model is constructed upon the intriguing properties in quantum field theory while extending their applications to the cosmological framework. This provides a natural framework to connect DE with fundamental physics. The concept of generalized ghost pilgrim dark energy ($GGPDE$) model stems from the necessity to deal with crucial challenges in modern cosmology, particularly the nature of DE which is responsible for the observed rapid growth of the cosmos. This model provides a closer match with observational data, offering an improvement over the PDE model. Thus, by generalizing PDE, researchers aim to deep our comprehension of the universe expansion, bridge the gap between quantum field theory and cosmology. The $GGPDE$ model offers distinct insights and mechanisms that enhance our comprehension of the cosmic expansion. It provides a framework for the accelerated cosmic expansion without depending on a cosmological constant. Thus, $GGPDE$ represents a compelling direction in the quest to decipher the nature of the mysterious energy driving the expansion of our universe.

Modified gravitational theories offer valuable understanding of current phenomenon of accelerated cosmic expansion. Ebrahimi and Sheykhi [52] studied GDE model to examine cosmic acceleration in Brans-Dicke cosmology. Sheykhi and Movahed [53] explored several cosmological parameters to assess cosmic growth using GDE model. Jawad [54] investigated the dynamics and evolutionary trajectories of various cosmological parameters, using PDE in $f(T,T_G)$ theory. Fayaz et al [55] reformed the $f(R,T)$ gravity in relation to the GDE model. Odintsov et al [56] examined several $f(R,G)$ frameworks to clarify the successful emergence of DE and accelerating phase of the universe. The analysis suggested that the EoS parameter reflects a quintessence era. Myrzakulov et al [57] employed the PDE and GDE models to reconstruct the $f(Q)$ gravity. They found that their results align closely with the latest observational data. The study of various DE models to analyze the universe acceleration within the context of $f(R,T^2)$ theory has been explored in [58].

This study aims to rebuild $f(Q,C)$ model by using $GGPDE$ framework to analyze cosmic dynamics. The contents of this article are formatted as follows. Section **2** employs the $GGPDE$ model to rebuild $f(Q,C)$ functional form and explore the effects of non-interacting scenarios between DE and cold DM. Section **3** focuses on the advancement of this theory through the analysis of different cosmographic parameters. Section **4** gives a detailed overview of our findings.

2. FORMALISM OF $GGPDE$ $f(Q,C)$ MODEL

Here, we aim to develop the $GGPDE$ $f(Q,C)$ model to investigate the enigmatic nature of the universe. To analyze the cosmological characteristics of non-metric gravity, we consider the affine connection as

$$\Gamma^{\upsilon}_{\gamma\lambda} = \check{\Gamma}^{\upsilon}_{\gamma\lambda} + K^{\upsilon}_{\gamma\lambda} + \mathrm{L}^{\upsilon}_{\gamma\lambda}, \tag{1}$$

where Levi-Civita connection ($\check{\Gamma}$), disformation (L^{υ}) and contortion ($K^{\upsilon}_{\gamma\lambda}$) tensors are defined as

$$\check{\Gamma}^{\upsilon}_{\gamma\lambda} \equiv \frac{1}{2}g^{\upsilon\beta}(\partial_{\gamma}g_{\beta\lambda} + \partial_{\lambda}g_{\beta\gamma} - \partial_{\beta}g_{\gamma\lambda}), \tag{2}$$

$$\mathrm{L}^{\upsilon}_{\gamma\lambda} \equiv \frac{1}{2}g^{\upsilon\eta}(-Q_{\gamma\eta\lambda} - Q_{\lambda\eta\gamma} + Q_{\eta\gamma\lambda}), \tag{3}$$

$$K^{\upsilon}_{\gamma\lambda} \equiv \frac{1}{2}g^{\upsilon\eta}(T_{\gamma\eta\lambda} + T_{\lambda\eta\gamma} + T_{\eta\gamma\lambda}). \tag{4}$$

Here, the quantity $T_{\gamma\eta\lambda}$ refers to the torsion tensor, which although not central in the symmetric teleparallel framework (where torsion is usually set to zero), appears here as part of the general affine connection decomposition. Therefore, we use torsion-free condition in this context and thus $T_{\gamma\eta\lambda} = 0$ in our analysis. The non-metricity is given by

$$Q_{\upsilon\gamma\lambda} \equiv \nabla_{\upsilon}g_{\gamma\lambda} = \partial_{\upsilon}g_{\gamma\lambda} - \Gamma^{\eta}_{\upsilon\gamma}g_{\eta\lambda} - \Gamma^{\eta}_{\upsilon\lambda}g_{\gamma\eta}. \tag{5}$$

To construct a boundary term in the action of the metric-affine gravity theories, we need a non-metricity conjugate, known as the superpotential, defined as

$$P^{\upsilon}_{\gamma\lambda} = -\frac{1}{4}Q^{\upsilon}_{\gamma\lambda} + \frac{1}{2}Q_{(\gamma}{}^{\upsilon}{}_{\lambda)} + \frac{1}{4}(Q^{\upsilon} - \tilde{Q}^{\upsilon})g_{\gamma\lambda} - \frac{1}{4}\delta^{\upsilon}_{(\gamma}Q_{\lambda)}.$$

It plays a central role in defining the gravitational field equations. Non-metricity tensor features two different traces as $Q_{\gamma} = Q_{\gamma}{}^{\lambda}{}_{\lambda}$ and $\tilde{Q}^{\gamma} = Q_{\lambda}{}^{\gamma\lambda}$. Thus, we have

$$Q = -\frac{1}{4}Q_{\upsilon\gamma\lambda}Q^{\upsilon\gamma\lambda} + \frac{1}{2}Q_{\gamma\lambda\upsilon}Q^{\lambda\upsilon\gamma} + \frac{1}{4}Q_{\gamma}Q^{\gamma} - \frac{1}{2}Q_{\gamma}\tilde{Q}^{\gamma}. \tag{6}$$

By enforcing the torsion-free and curvature-free conditions, we obtain

$$\check{R}_{\gamma\lambda} + \check{\nabla}_{\upsilon}L^{\upsilon}_{\gamma\lambda} - \check{\nabla}_{\lambda}\tilde{L}_{\gamma} + \tilde{L}_{\upsilon}L^{\upsilon}_{\gamma\lambda} - L_{\upsilon\eta\lambda}L^{\upsilon\eta}{}_{\gamma} = 0, \tag{7}$$

$$\check{R} + \check{\nabla}_{\gamma}(L^{\gamma} - \tilde{L}^{\gamma}) - Q = 0, \tag{8}$$

where $\check{\nabla}_{\upsilon}$ and $\check{R}_{\gamma\lambda}$ represent the covariant derivative and Ricci tensor, respectively, associated with the Levi-Civita connection. These notations are standard in the literature on symmetric teleparallel gravity to distinguish between operations involving the Levi-Civita connection and the general affine connection. Here, $\tilde{L}_{\gamma}$ represents the trace of the disformation tensor, i.e., $\tilde{L}_{\gamma} = L^{\lambda}_{\lambda\gamma}$, where $L^{\lambda}_{\gamma\delta}$ is the disformation tensor given in Eq.(3).

Since $Q^{\gamma} - \tilde{Q}^{\gamma} = L^{\gamma} - \tilde{L}^{\gamma}$, the boundary term is given by

$$C = \check{R} - Q = -\check{\nabla}_{\gamma}(Q^{\gamma} - \tilde{Q}^{\gamma}) = -\frac{1}{\sqrt{-g}}\partial_{\gamma}\big[\sqrt{-g}(Q^{\gamma} - \tilde{Q}^{\gamma})\big]. \tag{9}$$

In the context of $f(Q, C)$ gravity the integral action shown as

$$S = \int \frac{1}{2\kappa}[f(Q, C) + 2\kappa \mathrm{L}_m]\sqrt{-g}d^4x. \tag{10}$$

Here, κ is the coupling constant. The corresponding field equations are

$$\begin{aligned} T_{\gamma\lambda} &= \frac{2}{\sqrt{-g}}\partial_{\upsilon}(\sqrt{-g}f_Q P^{\upsilon}_{\gamma\lambda}) + (P_{\gamma\upsilon\eta}Q^{\upsilon\eta}_{\lambda} - 2P_{\upsilon\eta\lambda}Q^{\upsilon\eta}_{\gamma})f_Q - \frac{f}{2}g_{\gamma\lambda} \\ &+ (\frac{C}{2}g_{\gamma\lambda} - \check{\nabla}_{\gamma}\check{\nabla}_{\lambda} + g_{\gamma\lambda}\check{\nabla}^{\upsilon}\check{\nabla}_{\upsilon} - 2P^{\upsilon}_{\gamma\lambda}\partial_{\upsilon})f_C, \end{aligned} \tag{11}$$

where $f_Q = \frac{\partial f}{\partial Q}$ and $f_C = \frac{\partial f}{\partial C}$. Variation of the action corresponding to affine connection yields

$$(\nabla_{\gamma} - \tilde{L}_{\gamma})(\nabla_{\lambda} - \tilde{L}_{\lambda})\big[4(f_Q - f_C)P^{\gamma\lambda}{}_{\upsilon} + \Delta_{\upsilon}{}^{\gamma\lambda}\big] = 0, \tag{12}$$

with

$$\Delta_{\upsilon}{}^{\gamma\lambda} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta\Gamma^{\upsilon}{}_{\gamma\lambda}}.$$

Using the coincident gauge, we obtain

$$\partial_{\gamma}\partial_{\lambda}\big[\sqrt{-g}\big(4(f_Q - f_C)P^{\gamma\lambda}{}_{\upsilon} + \Delta_{\upsilon}{}^{\gamma\lambda}\big)\big] = 0. \tag{13}$$

Here, $\partial_{\lambda}\sqrt{-g} = -\sqrt{-g}\tilde{L}_{\lambda}$. Using the previous relations, we have

$$\begin{aligned} \big(\frac{1}{2}Qg_{\gamma\lambda} + \check{G}_{\gamma\lambda} + 2P^{\upsilon}{}_{\gamma\lambda}\partial_{\upsilon}\big)f_Q &= \frac{2}{\sqrt{-g}}\partial_{\upsilon}(\sqrt{-g}f_Q P^{\upsilon}{}_{\gamma\lambda}) + (P_{\gamma\upsilon\eta}Q_{\lambda}{}^{\upsilon\eta} \\ &- 2P_{\upsilon\eta\lambda}Q^{\upsilon\eta}{}_{\gamma})f_Q, \end{aligned} \tag{14}$$

The corresponding field equations are

$$\begin{aligned} T_{\gamma\lambda} &= -\frac{f}{2}g_{\gamma\lambda} + 2P^{\upsilon}{}_{\gamma\lambda}\nabla_{\upsilon}(f_Q - f_C) + \big(\frac{Q}{2}g_{\gamma\lambda} + \check{G}_{\gamma\lambda}\big)f_Q \\ &+ \big(\frac{C}{2}g_{\gamma\lambda} - \check{\nabla}_{\gamma}\check{\nabla}_{\lambda} + g_{\gamma\lambda}\check{\nabla}^{\upsilon}\check{\nabla}_{\upsilon}\big)f_C. \end{aligned} \tag{15}$$

By re-arranging this equation, we get

$$G_{\gamma\lambda} = \frac{1}{f_Q}\big(T^{(m)}_{\gamma\lambda} + T^{(D)}_{\gamma\lambda}\big), \tag{16}$$

where

$$\begin{aligned} T^{(D)}_{\gamma\lambda} &= \frac{f}{2}g_{\gamma\lambda} - 2P^{\upsilon}{}_{\gamma\lambda}\nabla_{\upsilon}(f_Q - f_C) - \big(\frac{1}{2}Cg_{\gamma\lambda} - \check{\nabla}_{\gamma}\check{\nabla}_{\lambda} + g_{\gamma\lambda}\check{\nabla}^{\upsilon}\check{\nabla}_{\upsilon}\big)f_C \\ &- \frac{Q}{2}f_Q g_{\gamma\lambda}. \end{aligned} \tag{17}$$

To explore the mysterious characteristics of the universe, we assume an isotropic and homogenous spacetime as

$$ds^2 = -dt^2 + (dx^2 + dy^2 + dz^2)\mathrm{a}^2(\mathrm{t}). \tag{18}$$

The ideal fluid encompasses the matter density (ρ_m) and pressure (p_m) is expressed as

$$T_{\gamma\lambda} = (\rho_m + p_m)U_{\gamma}U_{\lambda} + p_m g_{\gamma\lambda}. \tag{19}$$

Using Eqs.(17)-(19), we obtain

$$\begin{aligned} \rho_D + \rho_m &= 3H^2, \\ p_D + p_m &= -(2\dot{H} + 3H^2), \end{aligned} \tag{20}$$

where

$$\rho_D = 3H^2(1 - 2f_Q) - \frac{f}{2} + (3H^2 + 3\dot{H})f_C - 3H\dot{f}_C, \tag{21}$$

$$\begin{aligned} p_D &= -2\dot{H}(1 - f_Q) - 3H^2(1 - 2f_Q) + \frac{f}{2} + 2H\dot{f}_Q \\ &- (3H^2 + 3\dot{H})f_C + \ddot{f}_C. \end{aligned} \tag{22}$$

These modified equations represent a significant framework to comprehend the mysteries of the cosmos. To simplify these equations, we take our analysis to a specific class of functional in this theory with an ordinary parameter (α) as

$$f(Q, C) = f(Q) + \alpha C^2. \tag{23}$$

This functional form allows for a systematic investigation of deviations from standard gravitational dynamics, offering insights into the behavior of the universe under certain conditions. The field equations corresponding to this model turn out to be

$$\rho_D = 3H^2(1 - 2f_Q) - \frac{1}{2}(f(Q) + \alpha C^2) + 2\alpha C(3H^2 + 3\dot{H}) - 6\alpha H\dot{C} \tag{24}$$

$$\begin{aligned} p_D &= -2\dot{H}(1 - f_Q) - 3H^2(1 - 2f_Q) + \frac{1}{2}(f(Q) + \alpha C^2) + 2H\dot{f}_Q \\ &- 2\alpha C(3H^2 + 3\dot{H}) + 2\alpha\ddot{C}. \end{aligned} \tag{25}$$

The expressions for fractional energy densities are given by

$$\Omega_D = \frac{\rho_D}{3H^2}, \quad \Omega_m = \frac{\rho_m}{3H^2}. \tag{26}$$

The continuity equations of DM and DE for non-interacting case are

$$0 = \dot{\rho}_m + 3H(\rho_m + p_m), \tag{27}$$

$$0 = \dot{\rho}_D + 3H(\rho_D + p_D), \tag{28}$$

To quantifies how the universe expands over time, we study the behavior of scale factor which is considered as the key parameter in cosmology. It is a positive function that progresses over cosmic time, governing the expansion and the development of the universe. The decreasing behavior of scale factor suggests a cosmic contraction phase and its increasing behavior implies cosmic expansion era. It is directly related to the redshift of light from distant galaxies, allowing scientists to track the universe expansion history. Its study provides key insights into cosmic evolution from the big bang to the possible long-term fate of the universe, whether through continuous expansion or ultimate collapse.

We assume the scale parameter as

$$\mathrm{a}(\mathrm{t}) = \mathrm{a}_0\mathrm{t}^{\mu}. \tag{29}$$

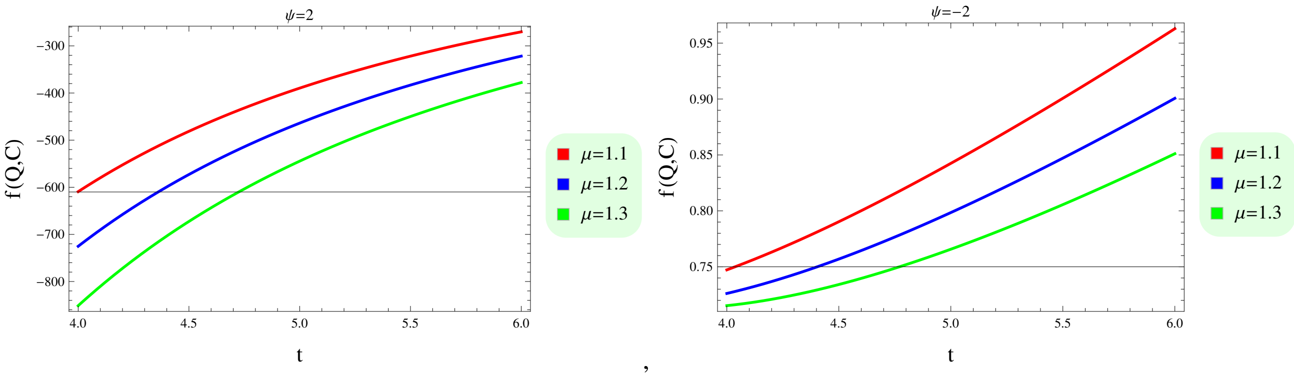

Figure 1. Plot of $f(Q,C)$ versus cosmic time for ($\psi = \pm 2$).

Here, $a_0 = 1$ and μ is an arbitrary constant. This form solves the field equations that govern the evolution of the universe, which helps to understand the broader dynamics of cosmic expansion and the behavior of DE. By exploring how the parameter μ influences evolution and correlating this with empirical observations, we gain a deep understanding of the characteristics of DE and the core principles of gravity. By comparing the power-law scale factor with cosmic observations, researchers can evaluate how $GGPDE$ model matches the cosmic observations. This process aids in validating or refining these models, providing valuable information about their accuracy and relevance.

To provide crucial insights into the cosmic acceleration, we analyze the behavior of Hubble parameter. The expansion of the cosmos is associated with a positive Hubble parameter, whereas contraction occurs with a negative value. The Hubble parameter changes over time depending on the universe energy composition, playing a crucial role in cosmic dynamics. It is fundamental in measuring cosmic distances, estimating the universe age and analyzing the influence of DE on expansion. The Hubble parameter is expressed as

$$H = \frac{\dot{\mathrm{a}}}{\mathrm{a}} = \frac{\mu}{t}. \tag{30}$$

Using this relation, we have

$$Q = 6\frac{\mu^2}{t^2}, \quad C = 6\mu(\frac{\mu-1}{t^2}). \tag{31}$$

The cosmic accelerated expansion can also be explored using DE models. A novel dynamical model of DE known as PDE helps to understand the ongoing cosmic expansion. This model maintains consistency with observational constraints and can be characterized through multiple energy density parameterizations. Crucially, its inherent energy dissipation mechanism naturally resolves the coincidence problem triggered by DE dominance. Furthermore, the introduction of the PDE concept leads to the definition of a $GGPDE$ model. The $GGPDE$ model for energy density can be characterized by the following expression

$$\rho_D = (\xi H + \zeta H^2)^{\psi}. \tag{32}$$

Using Eqs.(24) and (32), we obtain

$$f(Q,C) = \frac{1}{3}(-3\sqrt{6}\alpha\dot{C}\sqrt{Q} + 2\alpha CQ + \frac{3c_1}{\sqrt{Q}} - \frac{6^{1-\frac{\psi}{2}}(\sqrt{Q}(2\zeta+\xi))^{\psi}}{\psi+1} + Q), \tag{33}$$

where c_1 is an integration constant that arises during the reconstruction of the $f(Q,C)$ model using the $GGPDE$ ansatz. Inserting Eqs.(30) and (31) into (33), we get

$$\begin{aligned} f(Q,C) \quad &= \quad \frac{1}{3}(\frac{\sqrt{\frac{3}{2}}c_1}{\sqrt{\frac{\mu^2}{t^2}}} + \frac{72\alpha(\mu-1)\mu^3}{t^4} - \frac{6((2\zeta+\xi)\sqrt{\frac{\mu^2}{t^2}})^{\psi}}{\psi+1} \\ &+ \quad \frac{6\mu^2}{t^2} + \frac{216\alpha(\mu-1)\mu\sqrt{\frac{\mu^2}{t^2}}}{t^3}). \end{aligned} \tag{34}$$

Specifically, we consider $\psi = \pm 2$, $\alpha = 9 \times 10^{-6}$, $c_1 = 0.04$, $\zeta = 10$ and $\xi = 90$. This consistent specification ensures the viable numerical results. As for the physical reasoning behind the choice of these parameter values, we emphasize that the selected values fall within the phenomenologically viable ranges used in prior studies involving ghost dark energy models and modified gravity frameworks. All these parameter values are chosen not arbitrarily. The parameter α, which

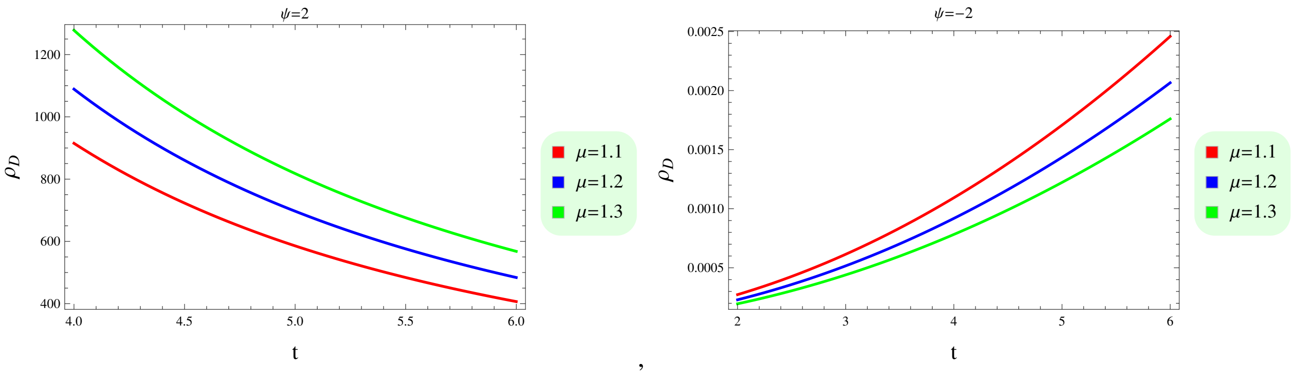

Figure 2. Graphical behavior of energy density against cosmic time.

controls the strength of the boundary term in the $f(Q,C)$ function is chosen to be small to ensure that the modification remains a perturbative correction to GR, consistent with Solar System constraints and cosmological observations. The constants ξ and ζ appear in the GGPDE density, and their values are selected to reflect a balance between the linear and quadratic Hubble terms, capturing the interaction of DE with the expansion rate while avoiding divergences or nonphysical behavior at early or late times. Figure **1** demonstrates the dynamics of reconstructed $GGPDE\ f(Q,C)$ model for different values of $\mu = 1.1, 1.2, 1.3$. As t increases, the function $f(Q,C)$ shows a decreasing trend for $\psi = 2$. In contrast, the function $f(Q,C)$ increases as t grows for $\psi = -2$. Substituting Eq.(33) in (21) and (22), we get

$$\rho_D = \sqrt{6}\alpha\dot{C}\sqrt{Q} + \frac{1}{3}\alpha Q(Q-3C) + 6^{-\frac{\psi}{2}}\left(\sqrt{Q}(2\zeta+\xi)\right)^{\psi}, \tag{35}$$

$$p_D = \frac{1}{9}\left(-7\sqrt{6}\alpha\dot{C}\sqrt{Q} - 3\alpha Q(Q-3C)\right) - 6^{-\frac{\psi}{2}}\left(\sqrt{Q}(2\zeta+\xi)\right)^{\psi}. \tag{36}$$

Using Eqs.(30) and (31), we have

$$\rho_D = [-24\alpha\mu^4 + 36\alpha\mu^3 + 72\alpha\mu t\sqrt{\frac{\mu^2}{t^2}} + t^4((2\zeta+\xi)\sqrt{\frac{\mu^2}{t^2}})^{\psi} - 72\alpha t^3(\frac{\mu^2}{t^2})^{3/2}][t^4]^{-1}, \tag{37}$$

$$p_D = [12\alpha\mu^3(2\mu-3) - 56\alpha\mu t\sqrt{\frac{\mu^2}{t^2}} - t^4((2\zeta+\xi)\sqrt{\frac{\mu^2}{t^2}})^{\psi} + 56\alpha t^3(\frac{\mu^2}{t^2})^{3/2}][t^4]^{-1}. \tag{38}$$

Figures **2** and **3** show the behavior aligns with the characteristics of DE as energy density is positive and pressure is negative for all values of μ and ψ.

3. ANALYSIS OF COSMOGRAPHIC FACTORS

Here, we study the characteristics of various cosmological parameters, which serve as essential tools for describing the universe expansion history. They allow researchers to analyze cosmic evolution without relying on specific gravitational theories, making them useful for testing various cosmological frameworks.

3.1. Study of State Variable

The EoS variable ($\omega_D = \frac{p_D}{\rho_D}$) determines the nature of DE and other cosmological components by relating pressure to energy density. It plays a pivotal role in distinguishing various phases of cosmic evolution. Various values of ω_D determines the distinct cosmic eras, i.e., matter-dominated regions such as dust, radiative fluid and stiff matter are characterized by the values $\omega_D = 0$, $\omega_D = \frac{1}{3}$ and $\omega_D = 1$, respectively. The vacuum energy phase is represented by $\omega_D = -1$, the phantom energy phase corresponds to $\omega_D < -1$, whereas the quintessence phase is defined in the range $-1 < \omega_D < -\frac{1}{3}$. These different values of EoS parameter help to classify the evolution of the universe and its various expansion phases. Recent cosmological observations, such as data from the Planck satellite, indicate that the EoS parameter for DE may lie below -1. This supports the existence of phantom DE, which is an exotic form with negative kinetic energy. Such behavior explains the accelerated expansion of the universe. The corresponding expression for EoS parameter is given by

$$\omega_D = -\frac{\frac{1}{9}\left(7\sqrt{6}\alpha\dot{C}\sqrt{Q} + 3\alpha Q(Q-3C)\right) + 6^{-\frac{\psi}{2}}\left(\sqrt{Q}(2\zeta+\xi)\right)^{\psi}}{\sqrt{6}\alpha\dot{C}\sqrt{Q} + \frac{1}{3}\alpha Q(Q-3C) + 6^{-\frac{\psi}{2}}\left(\sqrt{Q}(2\zeta+\xi)\right)^{\psi}}. \tag{39}$$

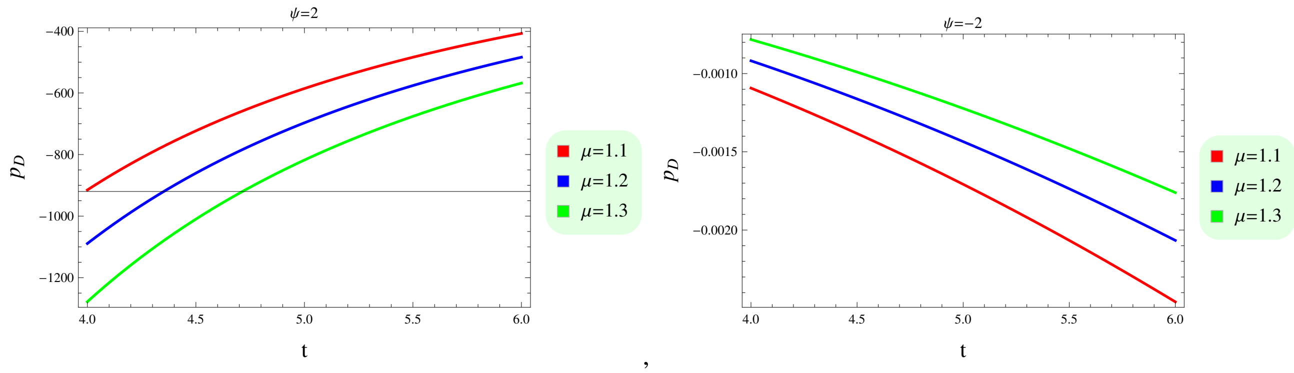

Figure 3. Evolution of pressure against cosmic time.

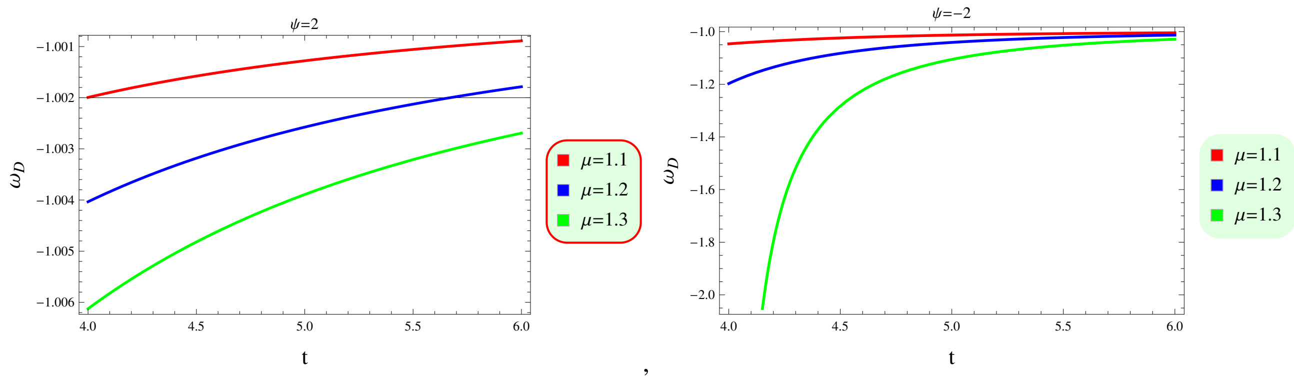

Figure 4. Plot of EoS parameter against cosmic time.

Using Eqs.(30) and (31), we have

$$\begin{aligned}\omega_D &= [-24\alpha\mu^4 + 36\alpha\mu^3 + 56\alpha\mu t\sqrt{\frac{\mu^2}{t^2}} + t^4((2\zeta+\xi)\sqrt{\frac{\mu^2}{t^2}})^{\psi} \\ &- 56\alpha t^3(\frac{\mu^2}{t^2})^{3/2}][12\alpha\mu^3(2\mu-3) - 72\alpha\mu t\sqrt{\frac{\mu^2}{t^2}} - t^4((2\zeta \\ &+ \xi)\sqrt{\frac{\mu^2}{t^2}})^{\psi} + 72\alpha t^3(\frac{\mu^2}{t^2})^{3/2}]^{-1}. \end{aligned} \tag{40}$$

Figure **4** determines the behavior of EoS parameter for three different values of μ. This indicates that the model shows phantom era of DE for $\psi = 2$ and $\psi = -2$, satisfying the conditions for GGPDE phenomenon.

3.2. Analysis of $\omega_D - \omega'_D$ Plane

Here, we use $(\omega_D - \omega'_D)$ analysis to study the dynamics of DE, where prime denotes the derivative corresponding to non-metricity. This analysis helps in understanding how the modified terms influence the deceleration parameter and the transition between different cosmic phases. This phase plane is a useful tool in cosmology to comprehend the behavior and stability of DE models. The behavior of DE models incorporating a scalar field has been investigated in [59]. They also classified DE models in two main categories, i.e., the thawing region (characterized by a short-duration cosmic acceleration) which is shown when ω_D has a negative value and ω'_D is positive. The other one is freezing region (where acceleration continues over a long period) which is shown when both ω_D and ω'_D have negative value. The expression for ω'_D is given by

$$\omega'_D = \frac{\alpha\dot{C}6^{\frac{\psi+1}{2}}\left(\alpha Q6^{\psi/2}(C-Q) - (\psi-1)\left(\sqrt{Q}(2\zeta+\xi)\right)^{\psi}\right)}{\sqrt{Q}\left(\alpha\sqrt{Q}6^{\psi/2}\left(3\sqrt{6}\dot{C} + \sqrt{Q}(Q-3C)\right) + 3\left(\sqrt{Q}(2\zeta+\xi)\right)^{\psi}\right)^2}. \tag{41}$$

Using Eqs.(30) and (31), we have

$$\omega'_D = \left[4\alpha(\mu-1)\mu t(36\alpha\mu^3 + t^4(\psi-1)((2\zeta+\xi)\sqrt{\frac{\mu^2}{t^2}})^{\psi})\right]$$

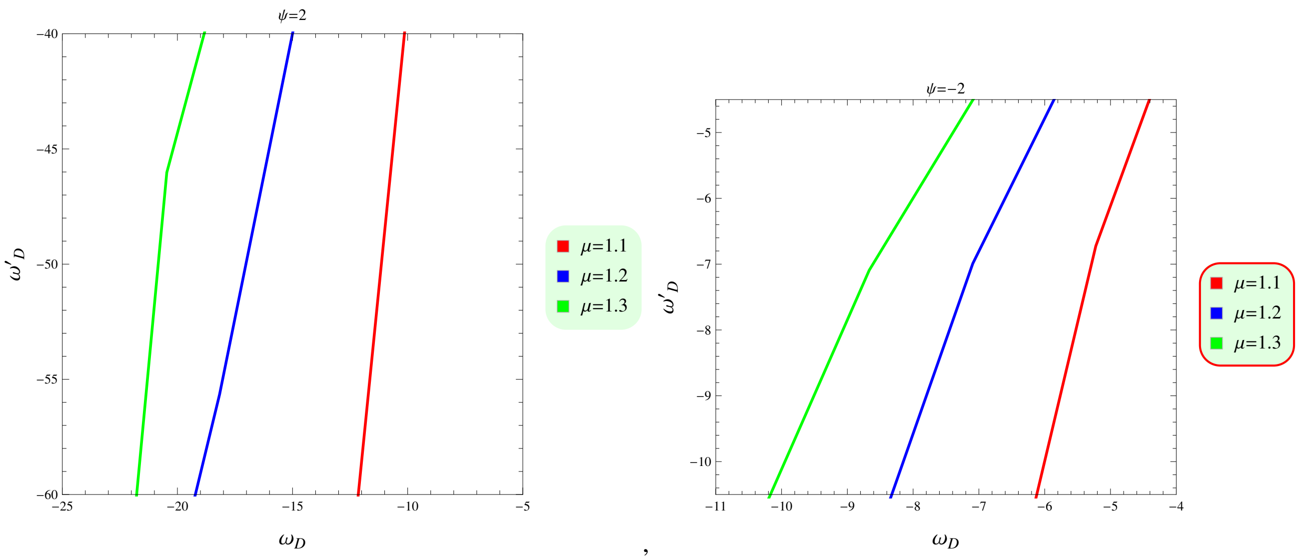

Figure 5. Graphical behavior of ω_D against ω'_D.

$$\times \quad \left[3\sqrt{\frac{\mu^2}{t^2}}(12\alpha\mu^3(3-2\mu)+72\alpha\mu t\sqrt{\frac{\mu^2}{t^2}}+t^4((2\zeta+\xi)\right.$$
$$\times \quad \left.\sqrt{\frac{m^2}{t^2}})^{\psi}-72\alpha t^3(\frac{\mu^2}{t^2})^{3/2})^2\right]^{-1}. \tag{42}$$

Figure **5** shows that for every value of μ and $\psi=\pm2$, $\omega_D<0$ and $\omega'_D<0$, showing the freezing region.

3.3. Study of (r,s) Parameters

These parameters offer a deep insight into the dynamic behavior and evolutionary phases of DE models [60]. They help to differentiate between various cosmological models by mapping their distinct trajectories in (r,s) plane. Specifically, the cold DM model is obtained for $(r,s)=(1,0)$ while the standard model is found at $(r,s)=(1,1)$. Additionally, the conditions $r<1$ and $s>0$ correspond to phantom and quintessence phases of DE. The Chaplygin gas model is represented by $s<0$ and $r>1$. These parameters are represented as

$$r = \frac{\ddot{a}}{aH^3} = 1+\frac{9\omega_D}{2}\Omega_D(1+\omega_D)-\frac{3\omega'_D}{2H}\Omega_D, \tag{43}$$
$$s = \frac{r-1}{3(q-\frac{1}{2})} = 1+\omega_D-\frac{\omega'_D}{3\omega_D H}. \tag{44}$$

Inserting ω_D and ω'_D values, we obtain

$$\begin{aligned} r &= [\alpha Q6^{\psi/2}(-28\alpha\dot{C}^2Q+\dot{C}(-6\alpha C+3\sqrt{6}Q^{3/2}(2\alpha C+1)-2\sqrt{6}\alpha Q^{5/2}\\ &+ 6\alpha Q)+Q^2(Q-3C))+3(2\alpha\dot{C}(-\sqrt{6}Q^{3/2}+\psi-1)+Q^2)(\sqrt{Q}(2\zeta\\ &+ \xi))^{\psi}][\alpha Q^{5/2}6^{\psi/2}(3\sqrt{6}\dot{C}+\sqrt{Q}(Q-3C))+3Q^2(\sqrt{Q}(2\zeta+\xi))^{\psi}]^{-1}, \end{aligned} \tag{45}$$
$$\begin{aligned} s &= [\alpha^2\dot{C}Q2^{\psi+1}3^{\psi}(Q(14\dot{C}+\sqrt{6}Q^{3/2}-3)+C(3-3\sqrt{6}Q^{3/2}))-\alpha\dot{C}6^{\frac{\psi}{2}+1}\\ &\times (-\sqrt{6}Q^{3/2}+\psi-1)(\sqrt{Q}(2\zeta+\xi))^{\psi}][Q(7\alpha^2\dot{C}^2Q2^{\psi+1}3^{\psi+2}+\alpha\dot{C}\\ &\times \sqrt{Q}2^{\frac{\psi+9}{2}}3^{\frac{\psi+1}{2}}(\alpha Q6^{\psi/2}(Q-3C)+3(\sqrt{Q}(2\zeta+\xi))^{\psi})+3(\alpha Q6^{\psi/2}\\ &\times (Q-3C)+3(\sqrt{Q}(2\zeta+\xi))^{\psi})^2)]^{-1}. \end{aligned} \tag{46}$$

Using Eqs.(30) and (31), we have

$$\begin{aligned} r &= [1728\alpha^2\mu^7t\sqrt{\frac{\mu^2}{t^2}}+\frac{216\alpha\mu^7t}{\sqrt{\frac{\mu^2}{t^2}}}-4320\alpha^2\mu^6t\sqrt{\frac{\mu^2}{t^2}}-\frac{216\alpha\mu^6t}{\sqrt{\frac{\mu^2}{t^2}}}\\ &+ 2592\alpha^2\mu^5t\sqrt{\frac{\mu^2}{t^2}}-72\alpha\mu^5t^3(2\zeta+\xi)((2\zeta+\xi)\sqrt{\frac{\mu^2}{t^2}})^{\psi-1} \end{aligned}$$

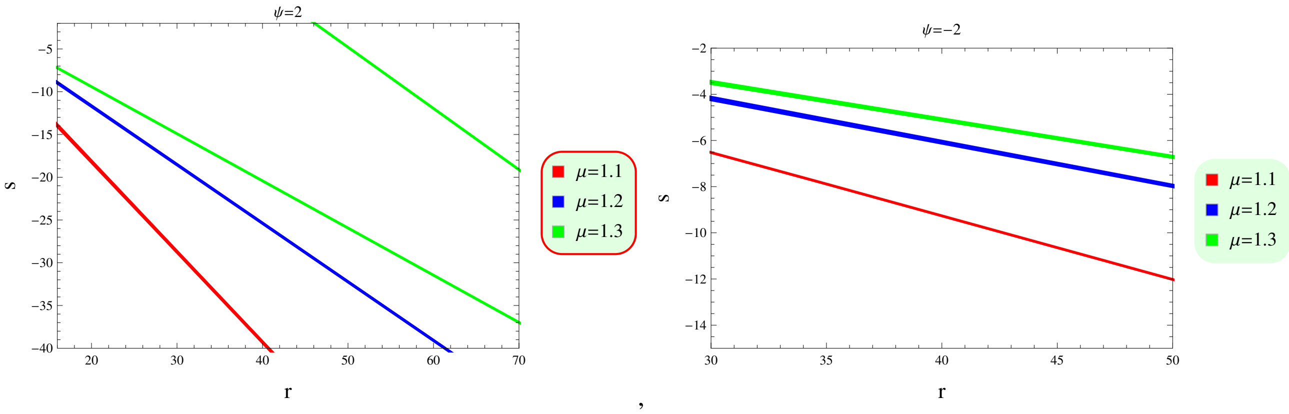

Figure 6. Graphical representation of diagnostic pair.

$$
\begin{aligned}
&+ \quad 72\alpha\mu^4t^3(2\zeta+\xi)((2\zeta+\xi)\sqrt{\frac{\mu^2}{t^2}})^{\psi-1}+t^6(-3\mu^3+2\alpha\mu t(\psi-1)\\
&- \quad 2\alpha t(\psi-1))((2\zeta+\xi)\sqrt{\frac{\mu^2}{t^2}})^{\psi}+36\alpha\mu^3((2\mu-3)\mu^3t^2+2\alpha(\mu\\
&- \quad 1)(56(\mu-1)\mu^2+t^3))][3\mu^3t^2(12\alpha\mu^3(2\mu-3)-72\alpha\mu t\sqrt{\frac{\mu^2}{t^2}}\\
&- \quad t^4((2\zeta+\xi)\sqrt{\frac{\mu^2}{t^2}})^{\psi}+72\alpha t^3(\frac{\mu^2}{t^2})^{3/2})]^{-1},
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
s \quad &= \quad [4\alpha(\mu-1)(36\alpha\mu^3t^3+t^7(\psi-1)((2\zeta+\xi)\sqrt{\frac{\mu^2}{t^2}})^{\psi}+288\alpha\mu^6\\
&\times \quad (3t\sqrt{\frac{\mu^2}{t^2}}+7)-144\alpha\mu^5(9t\sqrt{\frac{\mu^2}{t^2}}+14)-36\mu^4t^3(2\zeta+\xi)((2\zeta\\
&+ \quad \xi)\sqrt{\frac{\mu^2}{t^2}})^{\psi-1})][9\mu(144\alpha^2\mu^4(\mu(\mu(4(\mu-3)\mu+37)-56)+28)\\
&+ \quad t^8((2\zeta+\xi)\sqrt{\frac{\mu^2}{t^2}})^{2\psi}+3072\alpha^2\mu^6t\sqrt{\frac{\mu^2}{t^2}}-7680\alpha^2\mu^5t\sqrt{\frac{\mu^2}{t^2}}\\
&+ \quad 4608\alpha^2\mu^4t\sqrt{\frac{\mu^2}{t^2}}-128\alpha\mu^4t^3(2\zeta+\xi)((2\zeta+\xi)\sqrt{\frac{\mu^2}{t^2}})^{\psi-1}\\
&- \quad 24\alpha\mu^3(2\mu-3)t^4((2\zeta+\xi)\sqrt{\frac{\mu^2}{t^2}})^{\psi}+128\alpha\mu^3t^3(2\zeta+\xi)((2\zeta\\
&+ \quad \xi)\sqrt{\frac{\mu^2}{t^2}})^{\psi-1})]^{-1}.
\end{aligned}
\tag{48}
$$

Figure **6** demonstrates Chaplygin gas model ($r > 1$ and $s < 0$) for $\psi = \pm 2$ and different values of μ.

3.4. Stability Criteria

The ability of an object to regain its equilibrium state after being subjected to external forces is termed as stability. The squared sound speed (v_s^2) parameter plays a key role in cosmological models, as it dictates the stability of cosmic models. When v_s^2 is positive, it indicates stability within the model, whereas its negative value indicates instability. This relationship can be expressed as follows

$$
v_s^2=\frac{\dot{p}_D}{\dot{\rho}_D}=\frac{\rho_D}{\dot{\rho}_D}\omega_D'+\omega_D. \tag{49}
$$

Manipulating this equation, we have

$$
v_s^2=\frac{-7\alpha\dot{C}\sqrt{Q}6^{\frac{\psi+1}{2}}+\alpha Q6^{\frac{\psi}{2}+1}(3C-2Q)-9\psi\left(\sqrt{Q}(2\zeta+\xi)\right)^{\psi}}{3\left(\alpha\sqrt{Q}6^{\psi/2}\left(3\sqrt{6}\dot{C}+2\sqrt{Q}(2Q-3C)\right)+3\psi\left(\sqrt{Q}(2\zeta+\xi)\right)^{\psi}\right)}. \tag{50}
$$

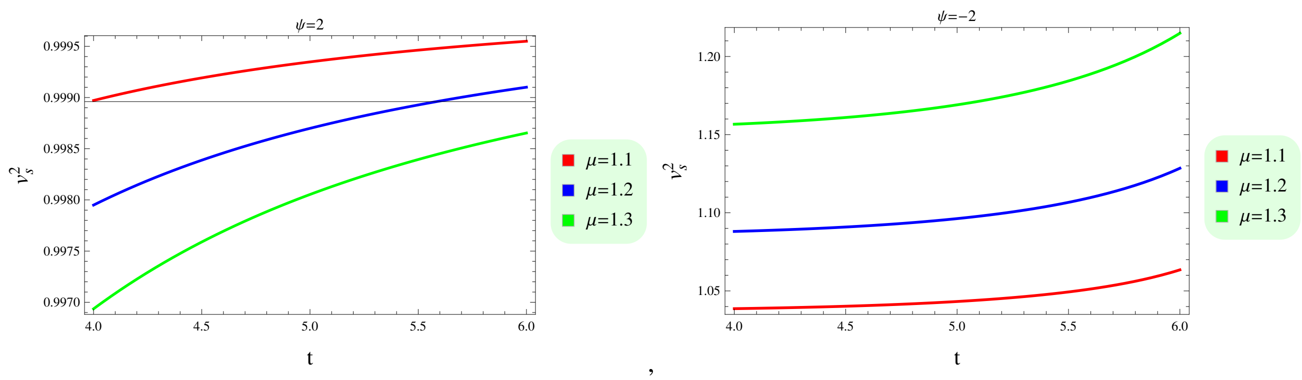

Figure 7. Plot of sound speed parameter against cosmic time.

Using Eqs.(30) and (31), it follows that

$$\begin{aligned} v_s^2 &= [-24\alpha(\mu-3)\mu^3 + 56\alpha\mu t\sqrt{\frac{\mu^2}{t^2}} + t^4\psi((2\zeta+\xi)\sqrt{\frac{\mu^2}{t^2}})^{\psi} \\ &- 56\alpha t^3(\frac{\mu^2}{t^2})^{3/2}][24\alpha(\mu-3)\mu^3 - 72\alpha\mu t\sqrt{\frac{\mu^2}{t^2}} - t^4\psi((2\zeta \\ &+ \xi)\sqrt{\frac{\mu^2}{t^2}})^{\psi} + 72\alpha t^3(\frac{\mu^2}{t^2})^{3/2}]^{-1}. \end{aligned} \tag{51}$$

Figure **7** demonstrates that the reconstructed $GGPDE\ f(Q,C)$ model is stable across different cosmic epochs.

4. FINAL RESULTS

The reconstruction method in modified theories serves as a useful approach to develop a viable DE model and understand the cosmic evolution. The inspiration for exploring a reconstructed $f(Q,C)$ framework comes form its crucial theoretical and empirical factors in gravitational physics and cosmology. The boundary term in the functional action is vital to the dynamical equations, providing novel clarity on cosmic progression. This proposed theory offers an explanation for the accelerating universe without depending on a cosmological constant, resolving DE problem. This modified proposal has garnered significant attention for their capacity to expand GR by aligning with various observational and experimental constraints. The connection between boundary term and non-metricity enhance the viability and enables a broad spectrum of feasible solutions. Consequently, this theoretical framework opens up the new avenues for developing cosmic frameworks that align with current observations. The main motivation for exploring the GGPDE model in $f(Q,C)$ theory is to better understand the accelerated cosmic expansion.

This analysis has investigated GGPDE model in the framework of non-metric gravity. We have used standard diagnostic instruments and the state finder pair to analyze different cosmic eras. We have assessed stability of the model using squared sound speed method. The main results are summarized as follows.

- The reconstructed $f(Q,C)$ model indicates a decreasing pattern for $\psi = 2$ and increasing behavior for $\psi = -2$, which demonstrates the realistic model behavior (Figure **1**).
- The decreasing behavior of the matter contents is consistent with the expected behavior of DE for all values of μ and ψ (Figures **2** and **3**).
- Our analysis reveals that the EoS parameter lies within the phantom region for all values of μ (Figure **4**).
- The upward trend in the $(\omega_D - \omega'_D)$ reveals the freezing region, which suggests that non-interacting case leads to a rapid cosmic evolution (Figure **5**).
- The $(r-s)$-plane determines Chaplygin gas model associated with non-interacting GGPDE $f(Q,C)$ model (Figure **6**).
- Our results show that v_s^2 is positive for each value of μ, indicating that GGPDE $f(Q,C)$ model is stable (Figure **7**).

The GGPDE $f(Q,C)$ model exhibits stable features and consistently corresponds with the present cosmic expansion. It is important to note that our findings are consistent with the existing observational data [61]

$$\omega_D = -1.023^{+0.091}_{-0.096}(PlanckTT + LowP + ext),$$

$$\omega_D = -1.006^{+0.085}_{-0.091}(PlanckTT + LowP + lensing + ext),$$
$$\omega_D = -1.019^{+0.075}_{-0.080}(PlanckTT, TE, EE + LowP + ext).$$

With a 95% confidence level, the data was gathered using a variety of observational techniques. Notably, our results align well with the predictions of the quantum chromodynamics ghost model within modified gravitational theories [62, 63]. Moreover, they are in agreement with recent theoretical and observational studies [57].

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КОСМОЛОГІЧНА ДІАГНОСТИКА ТА СТАБІЛЬНІСТЬ МОДЕЛІ ТЕМНОЇ ЕНЕРГІЇ В НЕМЕТРИЧНІЙ ГРАВІТАЦІЇ

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У цій роботі ми досліджуємо динаміку узагальненої темної енергії привида-піломника на тлі гравітації $f(Q,C)$, де Q – це неметричний скаляр, а C – граничний член. Для досягнення цієї мети ми беремо ізотропний та однорідний Всесвіт з ідеальним розподілом матерії. Наш аналіз включає сценарій з невзаємодіючими рідинами, що охоплює як темну матерію, так і темну енергію. Щоб зрозуміти космічну динаміку, ми реконструюємо модель $f(Q,C)$ та досліджуємо її вплив на еволюцію Всесвіту. Ми досліджуємо ключові космологічні фактори, тобто змінну стану, поведінку $(\omega_D-\omega'_D)$-площини та діагностичну пару statefinder, які допомагають аналізувати розширення космосу. Ключовим аспектом нашого аналізу є стабільність узагальненої моделі темної енергії пілігрима-привида, отриманої за допомогою методу квадрата швидкості звуку, що підтверджує її життєздатність у підтримці спостережуваного прискореного розширення. Наші результати узгоджуються з даними спостережень, демонструючи, що гравітація $f(Q,C)$ забезпечує надійну теоретичну основу для опису темної енергії та динаміки Всесвіту у великих масштабах. Ця робота не лише поглиблює наше розуміння модифікованої гравітації та таємничої енергії, але й пропонує нові погляди на альтернативне пояснення космічного прискорення поза стандартними парадигмами.

Ключові слова: $f(Q,C)$ *гравітація; модель темної енергії; аналіз стабільності*