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ON SOLUTIONS OF THE KILLINGBECK POTENTIAL AND CLARIFYING COMMENTS ON A RELATED ANALYTICAL APPROACH

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The work presents analytical solutions to the Schrödinger equation for the Killingbeck potential, a hybrid model combining harmonic, linear, and Coulombic terms, as well as an approximate model of Yukawa-type potentials. The radial Schrödinger equation is solved by means of the series expansion method, thus yielding the exact expressions of both bound-state solutions and eigenfunctions for systems such as quarkonium and confined hydrogen-like atoms in plasma environments. Furthermore, we offer a constructive commentary on the work of Obu et al. (East Eur. J. Phys. 3, 146–157, 2023), with the aim of clarifying a mathematical misstatement utilised in their analytical treatment of analogous systems.

Keywords: *Schrödinger equation; Killingbeck potential; Yukawa potential; Series expansion method; Heun equation*

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1. INTRODUCTION

In the domain of quantum mechanics, the accurate modelling of interaction potentials is paramount for the study of particle behaviour in bound states across various scales, ranging from atomic to hadronic systems. The Killingbeck potential is a particularly noteworthy model in this regard, due to its flexibility. The model incorporates a quadratic term, which acts in a manner analogous to a harmonic oscillator, in conjunction with a linear confining term and a Coulomb-like component.

$$V(r) = -\frac{A}{r} + Br + Cr^2. \tag{1}$$

This combination enables the Killingbeck potential to describe both short-range and long-range quantum interactions within a unified analytical framework [1–3]. It has been determined that this subject is of particular value in areas like heavy quarkonium spectroscopy, meson physics, quantum dots and Hydrogen-like systems embedded in a plasma environment, where both confinement and screening effects are present [4–6]. The notable attribute of the Killingbeck potential is twofold: firstly, its inherent solvability, and secondly, its capacity to establish a linkage between disparate potential models. The linear term captures the long-range confining force seen in quark confinement, while the Coulomb term accounts for the dominant one-gluon exchange interaction, which is an essential component of effective QCD potentials [7, 8]. The harmonic term is a useful regulator in hadronic systems, despite its unphysical behaviour at large distances. It facilitates analytic solutions in non-relativistic quantum mechanics, a feature that is frequently advantageous for theoretical research [1, 9].

The Killingbeck potential exhibits noteworthy mathematical and physical affinities with exponential-type potentials, including the Yukawa and its various variants of screened Coulomb forms. These potentials describe interactions involving massive bosons, and which result in short-range forces characterised by exponential decay [10–14]. They also describe the confinement potential of hydrogen-like atoms in plasma [15–17]. When the screening effects are weak, these exponential potentials simplify to the Killingbeck form by employing appropriate series expansions [18–20]. The analogies employed in this context transcend the confines of formalism. They capture a deeper physical intent, with an equilibrium of attraction and screening achieved in quantum confinement models. For systems like quarkonium, where the interplay between asymptotic freedom and confinement is particularly pronounced, these potentials facilitate the calculation of more precise spectral predictions [4, 8, 20]. Furthermore, the employment of various analytical approaches has been demonstrated to be beneficial in this context. The Nikiforov-Uvarov method, the perturbation theory, and the series expansions have all been applied effectively, thereby reinforcing the underlying structural coherence of these potentials [2, 5, 19].

In this work, we present the complete analytical solutions of the Schrödinger equation for the Killingbeck potential using the series expansion method. In addition, we take this opportunity to address a related methodological point in the recent literature (Obu et al. East Eur. J. Phys. 3, 146–157, 2023) [20], where an analytical misstatement affects the interpretation of a series solution in a similar spectral problem.

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2. CLARIFYING COMMENT ON THE WORK BY OBU ET AL.

In their recent article, Obu et al. present a "Comparative Study of the Mass Spectra of Heavy Quarkonium System with an Interacting Potential Model" (East Eur. J. Phys. 3, 146–157, 2023) [20]. This study makes a significant contribution to the field of hadronic physics through analytical comparisons by employing the Nikiforov-Uvarov and the Series Expansion Methods (SEM). The authors' approach to solving the Schrödinger equation with Yukawa-type potentials is methodologically rigorous and relevant for charmonium and bottomonium systems.

However, in section 4 of their paper, specifically at equation (62), there is a conceptual misstatement regarding the nature of linear independence in a power series expansion. The authors state the following:

"Equation (62) is linearly independent implying that each of the terms is separately equal to zero, noting that r is a non-zero function; therefore, it is the coefficient of r that is zero. The coefficients a_n are independent."

This statement confuses the independence of the functions r^n with the independence of the coefficients a_n. The correct interpretation is that the *functions r^n* form a linearly independent set in the polynomial on any open interval around $r = 0$. Therefore, for a power series to vanish identically on such an interval, it is necessary that each *coefficient* of these *functions r^n* vanishes separately. This structural phenomenon gives rise to recurrence relations between the coefficients, rather than ensuring their independence from each other.

We can see from equation (63) in [20] that it yields the result $L = -\frac{1}{2}(2N + 1)$. This is in direct contradiction to the established definitions of the principal quantum number N and the orbital quantum number L (in the limit where $\alpha_3 = 0$).

Equation (62) in [20] must therefore be expressed as a polynomial given that it is written in [20] as a sum of polynomials. Consequently, a more accurate formulation would be as follows:

"Since the powers of r are linearly independent, the coefficient of each power must vanish separately. This, in turn, leads to recurrence relations among these coefficients."

This clarification is important to maintain the mathematical rigour of the derivation and to ensure the educational value of the work for future researchers.

3. SCHRÖDINGER ENERGIES FOR THE KILLINGBECK POTENTIAL USING SEM

We will follow the steps outlined in [20] with some adjustments. In [20], the Potential is defined as follows:

$$V(r) = -\frac{a}{r} + \frac{b}{r}e^{-\alpha_I r} - \frac{c}{r^2}e^{-2\alpha_I r}, \tag{2}$$

where a,b and c are potential strengths and where the screening parameter is represented by the symbol α_I.

By expanding eq.(2) with Taylor series up to order three of α_I, the form of the Killingbeck potential is obtained:

$$V(r) = \frac{-\alpha_0}{r} + \alpha_1 r + \alpha_2 r^2 + \frac{\alpha_3}{r^2} + \alpha_4, \tag{3}$$

with:

$$-\alpha_0 = -a + b + 2c\alpha_I; \alpha_1 = \frac{1}{2}b\alpha_I^2 + \frac{4}{3}c\alpha_I^3; \alpha_2 = -\frac{1}{6}b\alpha_I^3; \alpha_3 = -c; \alpha_4 = -b\alpha_I - 2c\alpha_I^2. \tag{4}$$

Here we mention that the parameter c is omitted in the vicinity of the parameter α_I^3 in the α_1 term in [20].

Due to the spherical symmetry of the interaction, the radial Schrödinger equation is the primary focus:

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} + \left(\frac{2\mu}{\hbar^2}(E - V(r)) - \frac{l(l+1)}{r^2}\right)R(r) = 0, \tag{5}$$

here, l denotes the angular quantum number, while μ represents the reduced mass for the quarkonium. The variable r is the internuclear separation.

Substituting eq.(3) into eq.(5) gives:

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} + \left(\varepsilon + \frac{A}{r} - Br - Cr^2 - \frac{L(L+1)}{r^2}\right)R(r) = 0, \tag{6}$$

where:

$$\varepsilon = \frac{2\mu}{\hbar^2}(E - \alpha_4); A = \frac{2\mu}{\hbar^2}\alpha_0; B = \frac{2\mu}{\hbar^2}\alpha_1; C = \frac{2\mu}{\hbar^2}\alpha_2, \tag{7}$$

$$L(L+1) = \left(l(l+1) + \frac{2\mu}{\hbar^2}\alpha_3\right). \tag{8}$$

From eq.(8), we get the new "orbital" quantum number L

$$L = -\frac{1}{2} + \frac{1}{2}\sqrt{(2l+1)^2 + \frac{8\mu}{\hbar^2}\alpha_3}. \tag{9}$$

Following [20], we write the solution in the form:

$$R(r) = e^{-\left(\alpha r^2+\beta r\right)} F\left(r\right). \tag{10}$$

Substituting eq.(10) into eq.(6) and dividing by $e^{-\left(\alpha r^2+\beta r\right)}$, we obtain:

$$\begin{aligned} &F''\left(r\right) + \left[-4\alpha r - 2\beta + \frac{2}{r}\right] F'\left(r\right) \\ &\left[\left(\varepsilon + \beta^2 - 6\alpha\right) + \frac{\left(A - 2\beta\right)}{r} + \left(4\alpha\beta - B\right) r + \left(4\alpha^2 - C\right) r^2 - \frac{L\left(L+1\right)}{r^2}\right] F\left(r\right) = 0. \end{aligned} \tag{11}$$

We use the parameters α and β to simplify the given equation, thereby eliminating the terms in r and r^2 in the equation.

$$\begin{cases} 4\alpha^2 - C = 0 \implies \alpha = \frac{\sqrt{C}}{2}, \\ 4\alpha\beta - B = 0 \implies \beta = \frac{B}{2\sqrt{C}}. \end{cases} \tag{12}$$

and we get the new radial equation

$$F''\left(r\right) + \left(-4\alpha r - 2\beta + \frac{2}{r}\right) F'\left(r\right) + \left(\varepsilon + \beta^2 - 6\alpha + \frac{\left(A - 2\beta\right)}{r} - \frac{L\left(L+1\right)}{r^2}\right) F\left(r\right) = 0. \tag{13}$$

We present the solutions of this equation in a polynomial form:

$$F\left(r\right) = \sum_{k=0}^{\infty} a_k r^{k+s}. \tag{14}$$

In this study, the approach taken differs from that of [20] in terms of the chosen polynomial form. Specifically, the latter authors select $F\left(r\right) = \sum_{n=0}^{\infty} a_n r^{2n+L}$ in [20], yet no rationale is provided for the selection of this particular polynomial (it starts at power r^L) nor for the choice of an even power of r in the series. We also use the letter k in place of n in the summation, thus ensuring clarity and avoiding any potential confusion with the principal quantum number n, which is generally employed in standard textbooks. Putting the solution (14) and its derivatives into the radial equation (13) results in the following equation:

$$\begin{aligned} &\sum_{k=0}^{\infty} \left[\left(k+s\right)\left(k+s+1\right) - L\left(L+1\right)\right] a_k r^{k+s-2} + \sum_{k=0}^{\infty} \left[A - 2\beta\left(k+s+1\right)\right] a_k r^{k+s-1} \\ &+ \sum_{k=0}^{\infty} \left[\varepsilon + \beta^2 - 2\alpha\left(2k+2s+3\right)\right] a_k r^{k+s} = 0. \end{aligned} \tag{15}$$

We rearrange the summation terms to write:

$$\begin{aligned} &\sum_{k=-2}^{\infty} \left[\left(k+s+2\right)\left(k+s+3\right) - L\left(L+1\right)\right] a_{k+2} r^{k+s} + \sum_{k=-1}^{\infty} \left[A - 2\beta\left(k+s+2\right)\right] a_{k+1} r^{k+s} \\ &+ \sum_{k=0}^{\infty} \left[\varepsilon + \beta^2 - 2\alpha\left(2k+2s+3\right)\right] a_k r^{k+s} = 0, \end{aligned} \tag{16}$$

and we get the following form of a single polynomial series:

$$\begin{aligned} &\sum_{k=0}^{\infty} \left[\left[\left(k+s+2\right)\left(k+s+3\right) - L\left(L+1\right)\right] a_{k+2} + \left[A - 2\beta\left(k+s+2\right)\right] a_{k+1} + \left[\varepsilon + \beta^2 - 2\alpha\left(2k+2s+3\right)\right] a_k\right] r^{k+s} \\ &+ \left[\left[\left(s+1\right)\left(s+2\right) - L\left(L+1\right)\right] a_1 + \left[A - 2\beta\left(s+1\right)\right] a_0\right] r^{s-1} + \left[\left[s\left(s+1\right) - L\left(L+1\right)\right] a_0\right] r^{s-2}\} = 0. \end{aligned} \tag{17}$$

Since this relation holds for all values of the variable r, each coefficient of the r^k-functions must vanish. The following equation therefore holds:

$$\left[\left(k+s-L+2\right)\left(k+s+L+3\right)\right] a_{k+2} + \left[A - 2\beta\left(k+s+2\right)\right] a_{k+1} + \left[\varepsilon + \beta^2 - 2\alpha\left(2k+2s+3\right)\right] a_k = 0, \tag{18}$$

$$\left[\left(s+1\right)\left(s+2\right) - L\left(L+1\right)\right] a_1 + \left[A - 2\beta\left(s+1\right)\right] a_0 = 0, \tag{19}$$

$$\left[s\left(s+1\right) - L\left(L+1\right)\right] a_0 = 0. \tag{20}$$

We impose the condition $a_0 \neq 0$ to ensure that $F(r) \neq 0$; otherwise, it follows from eq.(19) that $a_1 = 0$. Consequently, all $a_k = 0$ from eq.(18). Therefore, from eq.(20), we derive the following result:

$$[s(s+1) - L(L+1)]\, a_0 = 0 \quad \text{and} \quad a_0 \neq 0 \Longrightarrow s = L \quad \text{or} \quad s = -L-1. \tag{21}$$

We reject the solution $s = -L-1$ on the basis of the expressions of $R(r)$ in eq.(10) and $F(r)$ in eq.(14). These expressions imply that $R(r)$ is divergent at the origin of r. Therefore, it can be concluded that $s = L$. It is evident here that the minimal power of the polynomial $F(r)$ is r^L; this is in contrast to the approach taken in [20], where the rationale for this choice is not provided. Replacing this value in the recurrence relations (18) and (19), we write:

$$a_{k+2} = \frac{2\beta(k+L+2) - A}{(k+2)(k+2L+3)} a_{k+1} + \frac{\varepsilon + \beta^2 - 2\alpha(2k+2L+3)}{(k+2)(k+2L+3)} a_k, \tag{22}$$

$$a_1 = \frac{2\beta(L+1) - A}{2(L+1)} a_0. \tag{23}$$

In the context of the probabilistic interpretation of the wave function, it is imperative to impose the condition that $R(r)$ must be convergent when $r \longrightarrow \infty$ and, consequently, the function $F(r)$ must terminate as a finite polynomial. To accomplish this objective, it is necessary to truncate the series (22).

We can follow the method used in [21] and impose that for some value $k = n_r$, the coefficients of both a_{n_r} and a_{n_r+1} vanish while we have $a_{n_r} \neq 0$ and $a_{n_r+1} \neq 0$:

$$a_{n_r+2} = 0 \quad \text{if} \quad 2\beta(n_r + L + 2) - A = 0 \quad \text{and} \quad \varepsilon + \beta^2 - 2\alpha(2n_r + 2L + 3) = 0. \tag{24}$$

This will give us the energies and a relation between the coefficients β and A:

$$\varepsilon_{n_r,l} = 2\alpha(2n_r + 2L + 3) - \beta^2, \tag{25}$$

$$\beta = \frac{A}{2(n_r + L + 2)}. \tag{26}$$

These two relations are equivalent to eq.(65) and eq.(68) in [20] when we replace n_r by $2n$, because we have employed a more general expression for $R(r)$.

In order to show that we have a combination of the energies of both a harmonic oscillator and a Coulomb potential, we write the energies as follows:

$$\varepsilon_{n_r,l} = 2\sqrt{C}\left(n_r + L + \frac{3}{2}\right) - \frac{A^2}{4(n_r + L + 2)^2}. \tag{27}$$

Upon substituting the expressions of A, C and the α terms from relations (4), (6) and (8), we obtain the same energies as in eq.(70) in [20]. It is noteworthy that $2n \longrightarrow n_r$ in the expressions of [20].

$$E_{n_r,l} = \sqrt{-\frac{\hbar^2 b}{12\mu}\alpha_I^3}\left(2n_r + 2 + \sqrt{(2l+1)^2 - \frac{8\mu}{\hbar^2}c}\right) - \frac{2\mu}{\hbar^2}\frac{[a - b - 2c\alpha_I]^2}{\left(2n_r + 1 + \sqrt{(2l+1)^2 - \frac{8\mu}{\hbar^2}c}\right)^2} - b\alpha_I - 2c\alpha_I^2. \tag{28}$$

At this point, it has been demonstrated that the energy spectrum of the Killingbeck potential is obtained by applying rigorously the SEM method and correcting the inaccuracies in the application of this SEM method made by Obu et al. in [20].

In this section, we followed the condition (24) as done in [21] to truncate the series (22). Notwithstanding the utilisation of this condition by the authors of [21] in numerous recent works [22–25], it is imperative to acknowledge that this condition does not guarantee the truncation. An examination of the parameter a_{n_r+3} as depicted from eq.(22) and eq.(24) substantiates this assertion:

$$a_{n_r+3} = \frac{\varepsilon_{n_r,l} + \beta^2 - 2\alpha(2n_r + 2L + 5)}{(n_r+3)(n_r+2L+4)} a_{n_r+1} = -\frac{4\alpha}{(n_r+3)(n_r+2L+4)} a_{n_r+1}. \tag{29}$$

It is evident that $a_{n_r+3} \neq 0$ and so is all the parameters beside it. The error when employing this method is attributable to the confusion arising from the erroneous identification of the index k of the polynomial coefficients a_k (which is denoted n in [20]), and the index n_r of the energies, which is determined by the level under consideration. Consequently, n_r possesses a fixed value for all the values of k in eq.(22) (A parallel observation concerning this error is documented in [26]). This leads us to consider alternative conditions that could be utilised to truncate the series. A comprehensive discussion of these alternatives will be presented in the subsequent section.

4. SCHRÖDINGER ENERGIES FOR THE KILLINGBECK POTENTIAL USING HEUN FUNCTIONS

Now we use the Heun formulation of the Schrödinger equation (6), deriving from the same form of the solutions $R(r)$ in eq.(10) with two additional transformations $F(r) = r^{L+1}g(r)$ and $\rho = \sqrt{2\mu/\hbar^2}r$. This results in the Biconfluent Heun equation (BHE):

$$\rho g''(\rho) + \left(1 + \alpha' - \beta'\rho + 2\rho^2\right) g'(\rho) + \left((\gamma' - \alpha' - 2)\rho - \frac{1}{2}(\delta' + \beta'(1 + \alpha'))\right) g(\rho) = 0. \tag{30}$$

The parameters of this equation are defined as follows:

$$\alpha' = 2L + 1; \beta' = \frac{B}{\sqrt{C}} C^{-\frac{1}{4}}; \gamma' = \frac{1}{\sqrt{C}}\left(\varepsilon + \frac{B^2}{4C}\right); \delta' = \frac{-2A}{\sqrt{C}} C^{\frac{1}{4}}, \tag{31}$$

where A, B, C, L, ε, α and β are defined in the relations (7), (8) and (12).

The solution of eq.(30) is given by the biconfluent Heun functions [27]:

$$g(\rho) = H_b(\alpha', \beta', \gamma', \delta', \rho) = \sum_{n\geq 0} a_n \frac{\Gamma(1+\alpha')}{\Gamma(1+\alpha'+n)} \frac{\rho^n}{n!}. \tag{32}$$

Thus, we have obtained the radial part $R(r) \propto e^{-(\alpha r^2 + \beta r)} r^{L+1} g(r)$ of the eigenfunctions of the Schrödinger equation for the Killingbeck potential. The angular part consists of the usual spherical harmonic functions $Y_{L,M}(\theta, \phi)$.

As a consequence of the recurrence relation (22), there exists a value $k = n_r$, for which we have [27]:

$$a_{n_r+2} = 0 \quad \text{if} \quad a_{n_r+1} = 0 \quad \text{and} \quad \left[\text{the coefficient of } a_{n_r}\right] = 0. \tag{33}$$

Equivalently:

$$a_{n_r+2} = 0 \quad \text{if} \quad a_{n_r+1} = 0 \quad \text{and} \quad 4\alpha(n_r + L) - \left(\varepsilon + \beta^2 - 6\alpha\right) = 0. \tag{34}$$

Using the second condition and the relation (12), we obtain the energies as follows:

$$\varepsilon_{n_r,l} = 2\sqrt{C}\left(n_r + L + \frac{3}{2}\right) - \beta^2 = 2\sqrt{C}\left(n_r + L + \frac{3}{2}\right) - \frac{B^2}{4C}. \tag{35}$$

It should be noted that this is analogous to eq.(25), with the exception that eq.(26) is not applicable in this instance.

To determine the value of the $B^2/4C$ term, we use the first condition $a_{n+1} = 0$, which establishes a relationship for each value of the radial quantum number n_r.

For instance, when $n_r = 0$, the result obtained from eq.(23) is:

$$a_1 = 0 \Longrightarrow \beta = \frac{A}{2(L+1)} \Longrightarrow \beta^2 = \frac{B^2}{4C} = \frac{A^2}{4(L+1)^2}. \tag{36}$$

And we derive the following expression for the corresponding energy levels:

$$\begin{aligned} \varepsilon_{0,l} &= 2\sqrt{C}\left(L + \frac{3}{2}\right) - \frac{B^2}{4C} \\ \Longrightarrow E_{0,l} &= \sqrt{-\frac{\hbar^2 b}{12\mu}\alpha_I^3}\left(2 + \sqrt{(2l+1)^2 - \frac{8\mu}{\hbar^2}c}\right) - \frac{2\mu}{\hbar^2}\frac{[a - b - 2c\alpha_I]^2}{\left(1 + \sqrt{(2l+1)^2 - \frac{8\mu}{\hbar^2}c}\right)^2} - b\alpha_I - 2c\alpha_I^2. \end{aligned} \tag{37}$$

The result obtained here is the same result derived in the previous section in eq.(28) ($n_r = 0$).

In the case of $n_r = 1$, it is necessary to express the value of a_2. This is obtained from eq.(22) and eq.(23):

$$a_2 = \frac{2\beta(L+2) - A}{(2)(2L+3)} a_1 + \frac{\varepsilon_{1,l} + \beta^2 - 2\alpha(2L+3)}{2(2L+3)} a_0. \tag{38}$$

We recall here that:

$$\varepsilon_{1,l} = 2\sqrt{C}\left(1 + L + \frac{3}{2}\right) - \beta^2 \text{ and } \alpha = \frac{\sqrt{C}}{2}.$$

We have the condition $a_2 = 0$, so we write:

$$\Longrightarrow [4(L+2)(L+1)]\beta^2 - [2A(2L+3)]\beta + \left[A^2 + 4(L+1)\sqrt{C}\right] = 0$$

$$\Longrightarrow \beta_{1,2} = \frac{A(2L+3) \pm \sqrt{A^2 - 16(L+2)(L+1)^2\sqrt{C}}}{2(L+2)(L+1)}$$

$$\Longrightarrow \beta^2 = \frac{B^2}{4C} = \left(\frac{A}{2(L+2)(L+1)}\left((2L+3) \pm \sqrt{1 - 16(L+2)(L+1)^2\frac{\sqrt{C}}{A^2}}\right)\right)^2. \tag{39}$$

We have two expressions for the energies, corresponding to the two possible solutions of β:

$$E^{+}_{1,l} = \sqrt{-\frac{\hbar^2 b}{12\mu}\alpha_I^3}\left(2L+5+\frac{4}{L+2}\right) - b\alpha_I - 2c\alpha_I^2$$
$$- \frac{2\hbar^2\left(a-b-2c\alpha_I\right)^2}{\mu(L+1)^2}\left(1 - \frac{(2L+3)}{2(L+2)^2}\left(1 + \sqrt{1 - 16(L+2)(L+1)^2\frac{\sqrt{-\frac{\hbar^2}{12\mu}b\alpha_I^3}}{(a-b-2c\alpha_I)^2}}\right)\right), \tag{40}$$

$$E^{-}_{1,l} = \sqrt{-\frac{\hbar^2 b}{12\mu}\alpha_I^3}\left(2L+5+\frac{4}{L+2}\right) - b\alpha_I - 2c\alpha_I^2$$
$$- \frac{2\hbar^2\left(a-b-2c\alpha_I\right)^2}{\mu(L+1)^2}\left(1 - \frac{(2L+3)}{2(L+2)^2}\left(1 - \sqrt{1 - 16(L+2)(L+1)^2\frac{\sqrt{-\frac{\hbar^2}{12\mu}b\alpha_I^3}}{(a-b-2c\alpha_I)^2}}\right)\right). \tag{41}$$

It is evident, from the general form eq.(35), that both expressions yield the result obtained in the previous section when $n_r = 1$ in eq.(28), which is similar to the one found by [20], with some corrections in the Coulomb parts of the relations. It is indeed the case that, upon substituting the value of the parameters $\alpha_0 = 0$ and $\alpha_1 = 0$ in the potential from eq. (3), we obtain the standard energies of the harmonic oscillator, and they represent the first contributions observed in both eq.(27) and eq.(28).

We can use the Coulomb limit of these energies to test the validity of the two expressions.

$$\alpha_I \to \infty \implies E^{+}_{1,l} \to -\frac{2\hbar^2 a^2}{\mu(L+1)^2}, \tag{42}$$

$$\alpha_I \to \infty \implies E^{-}_{1,l} \to -\frac{2\hbar^2 a^2}{\mu(L+2)^2}. \tag{43}$$

As we can see from eq.(42), that $E^{+}_{1,l}$ yields a result analogous to that obtained from eq.(37) which is the Coulomb energy of the $n_r = 0$ level. However, it should be noted that this is not the level under consideration in this particular context. Conversely, eq.(43) shows that the limit of $E^{-}_{1,l}$ corresponds to the Coulomb energy of the $n_r = 1$ level which is the case considered here. This is congruent with the finding of the precedent section, where the energies (28) were employed. Consequently, we conclude that $E^{-}_{1,l}$ represents the appropriate generalisation of the result previously found in [20].

For $n_r = 2$, we have the condition $a_3 = 0$, which gives us the following algebraic equation for β:

$$(L+1)(L+2)(L+3)\beta^3 - \left(3L^2+12L+11\right)\frac{A}{\sqrt{2}}\beta^2 - \left[(L+1)(4L+9)\frac{C}{\sqrt{2}} - (L+2)\frac{3A^2}{2}\right]\beta - \frac{A^3}{2\sqrt{2}} = 0. \tag{44}$$

It has been established that the solutions of this equation are real [27–29]. The same procedure as for $n_r = 1$ is employed to write the energies and to check their Coulombian limits, in order to compare with the solutions written in [20].

5. CONCLUSION

In this study, we have provided exact analytical solutions to the radial Schrödinger equation for the Killingbeck potential using both the general series expansion method and the biconfluent Heun formalism. The Killingbeck potential, a composite of harmonic, linear, and Coulomb terms, emerges naturally as a limiting case of screened Coulomb potentials, particularly under weak screening conditions relevant for quarkonium and plasma-embedded systems. Through systematic expansion and appropriate transformations, we derived the explicit expressions of both energy eigenvalues and wavefunctions, thus confirming the applicability of the model across various quantum regimes.

A salient feature of the derived solutions is their capacity to interpolate seamlessly between two classical regimes of quantum mechanics. In certain limiting cases, specifically, the vanishing linear and repulsive terms, or the dominant

Coulomb coupling, the spectrum reduces, correspondingly, to that of the harmonic oscillator and the hydrogen-like systems. However, the general expressions go further, capturing a hybrid structure that reflects both screening effects and long-range confinement. This specificity renders the Killingbeck potential a valuable tool for modelling systems where purely Coulombic or harmonic oscillator models fail to capture essential physical features.

We have also revised and clarified a conceptual misinterpretation found in a recent work by Obu et al. [20], related to the treatment of linear independence in power series expansions. Furthermore, a critical re-examination of the analytical approach employed by Guvendi and Mustafa [21] was undertaken, leading to the rectification of a significant mathematical flaw in the truncation conditions for the series. These aforementioned corrections serve a dual purpose; firstly, they ensure the maintenance of the methodology's integrity, and secondly, they serve to enhance the pedagogical and physical insights into spectral problem-solving techniques.

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ПРО РОЗВ'ЯЗАННЯ ПОТЕНЦІАЛУ КІЛЛІНГБЕКА ТА УТОЧНЮЮЧІ КОМЕНТАРІ ЩОДО ПОВ'ЯЗАНОГО АНАЛІТИЧНОГО ПІДХОДУ

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У роботі представлені аналітичні розв'язки рівняння Шредінгера для потенціалу Кіллінгбека, гібридної моделі, що поєднує гармонічні, лінійні та кулонівські члени, а також наближену модель потенціалів типу Юкави. Радіальне рівняння Шредінгера розв'язується методом розкладання в ряд, що дає точні вирази як для розв'язків у зв'язаних станах, так і для власних функцій для таких систем, як кварконій та обмежені воднеподібні атоми в плазмових середовищах. Крім того, ми пропонуємо конструктивний коментар до роботи Обу та ін. (East Eur. J. Phys. 3, 146–157, 2023) з метою уточнення математичної помилки, використаної в їх аналітичному обробці аналогічних систем.
Ключові слова: *рівняння Шредінгера; потенціал Кіллінгбека; потенціал Юкави; метод розкладання в ряд; рівняння Гойна*