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PARTIAL EXACT SOLUTIONS OF NONLINEAR DISTRIBUTION ONE-COMPONENT ORDER PARAMETER IN EQUILIBRIUM SYSTEMS

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This paper investigates partial exact solutions of a nonlinear fourth-order differential equation arising from the variational principle for a thermodynamic potential with high derivatives. To describe the spatial distribution of the order parameter, the elliptic cosine function of Jacobi is used, which allows reducing the problem to a system of algebraic equations for amplitude, spatial scale, and modulus. The conditions for the existence of physically admissible solutions were obtained, and it was found that periodic solutions expressed in terms of elliptic cosine are relevant for describing first-order phase transitions. Graphs illustrating the dependence of the main parameters of the solution on the characteristics of the system are presented.
Keywords: *Nonlinear differential equations; Order parameter; Elliptic Jacobi functions; First-order phase transition; Incommensurate phase*
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INTRODUCTION

Phase transitions and spatially inhomogeneous states in condensed matter systems remain one of the key areas of research in modern physics. In many physical models — in particular, in the description of ferroelectrics, magnets, alloys, and during spinodal decomposition — the order parameter is a one-component function of the spatial coordinate [1-4]. Their application goes beyond traditional solid-state physics — similar equations also arise in theories of self-organisation of structures in biological, computational, and non-equilibrium systems [5-7]. In such cases, an important task is to find exact or approximate solutions to the corresponding equations that describe the evolution or equilibrium distribution of this parameter.

In the initial stages of phase transitions, when the amplitude of the order parameter is small, linear analytical approximations are often used. However, with increasing nonlinearity of the system, such approaches lose their accuracy. Particularly difficult to describe are cases where strongly nonlinear, periodic or localised distributions are formed — for example, alternating domains with domain walls. [8]

In classical approaches to modelling such structures, harmonic expansion or expansion in terms of the order parameter is widely used. However, the accurate representation of strongly nonlinear profiles, such as bell solitons or modulation structures, requires the inclusion of a large number of terms, which complicates both analytical analysis and numerical investigation.

An alternative approach is to search for partial exact solutions of the variational equation that allow the order parameter to be represented by special functions, in particular Jacobi elliptic functions [9-11]. In particular, the function

$$\varphi = a * cn(bx, k) \tag{1}$$

the elliptic cosine allows us to describe spatial distributions of order parameters that smoothly interpolate between harmonic and bell-soliton behaviour depending on the modulus k.

The aim of this work is to construct a class of exact partial solutions in the form of elliptic cosine — Jacobi function — for the fourth-order equation derived from the corresponding variational functional Within the framework of Landau theory, analytical solutions in the form of function (1) are constructed, and the dependence of the distribution parameters on the thermodynamic potential parameters has been investigated.

CALCULATION ORDER PARAMETER

Let us write down the thermodynamic potential in the form [9]:

$$F = F_0 + \int dx \left[\frac{1}{2}(\varphi'')^2 - \frac{g}{2}(\varphi\varphi')^2 - \frac{\gamma}{2}(\varphi')^2 + \frac{q}{2}\varphi^2 + \frac{p}{4}\varphi^4 + \frac{1}{6}\varphi^6\right], \tag{2}$$

where g, γ are material parameters and $q = q_0(T - T_c)/T_c$and p depend on other conditions (e.g. pressure).

The variational equation for functional (1) is the following nonlinear fourth-order differential equation:

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$$\varphi^{(IV)} + g\left[\varphi^2\varphi'' + \varphi\varphi'^2\right] + \gamma\varphi'' + q\varphi + p\varphi^3 + \varphi^5 = 0. \tag{3}$$

Let us represent the order parameter as (1). Substituting (1) into (3) we obtain the following system of equations

$$\begin{cases} a^4 - 3ga^2b^2k^2 + 24b^4k^4 = 0 \\ b^4(20k^2 - 40k^4) + 2ga^2b^2(2k^2 - 1) - 2\gamma b^2k^2 + pa^2 = 0 \\ b^4(16k^4 - 16k^2 + 1) + ga^2b^2(1 - k^2) + \gamma b^2(2k^2 - 1) + q = 0 \end{cases} \tag{4}$$

From the first equation in (4) we can find the parameter a:

$$a^2 = 3\frac{g \pm \sqrt{g^2 - \frac{32}{3}}}{2} b^2k^2 = \acute{a}_{\pm}{}^2 b^2k^2 \tag{5}$$

We see that parameter a is valid only when

$$g \geq \sqrt{\frac{32}{3}} \tag{6}$$

Let's look at the limiting case when $k = 0$. From the third equation (4) we obtain

$$b^4 - \gamma b^2 + q = 0 \tag{7}$$

or

$$b^2 = \frac{\gamma \pm \sqrt{\gamma^2 - 4q}}{2} \tag{8}$$

From this, we can see that a second-order phase transition occurs from a highly symmetric phase to a modulated phase at the point $q = \gamma^2/4$.

Now let's look at the limiting case when $k^2 = 1$. We obtain the following system of equations

$$\begin{cases} -20b^4 + 2g\acute{a}^2b^2 - 2\gamma b^2 + p\acute{a}^2b^2 = 0 \\ b^4 + \gamma b^2 + q = 0 \end{cases} \tag{9}$$

From here, we can find an expression for b^2:

$$b^2 = \frac{2\gamma - p\acute{a}^2}{2(g\acute{a}^2 - 10)} \tag{10}$$

Parameter b is valid only when either $g\acute{a}^2 > 10$ and $\gamma \geq p\acute{a}^2/2$, or $g\acute{a}^2 < 10$ and $\gamma \leq p\acute{a}^2/2$.

Substitute (5) into the second equation (4):

$$b^2 = \frac{2\gamma - p\acute{a}^2}{2(2k^2 - 1)(g\acute{a}^2 - 10)} = \frac{\xi\sigma}{2(2k^2 - 1)}, \tag{11}$$

where

$$\xi = 2\gamma - p\acute{a}^2, \tag{12}$$

$$\frac{1}{\sigma} = g\acute{a}^2 - 10. \tag{13}$$

Substituting (11) into the third equation (4) we obtain the following expression

$$k^4 - k^2 + \frac{\sigma^2\xi^2 + 2\gamma\sigma\xi + 4q}{(6\sigma^2 - \sigma)\xi^2 + 8\gamma\sigma\xi + 16q} = 0 \tag{14}$$

Let's calculate the discriminant of this equation for k^2:

$$D = \frac{(2\sigma^2 - \sigma)\xi^2}{(6\sigma^2 - \sigma)\xi^2 + 8\gamma\sigma\xi + 16q} \tag{15}$$

The solution to equation (14) will be

$$k^2 = \frac{1 \pm \sqrt{D}}{2}. \tag{16}$$

It can be seen that these solutions (16) take values either [0, 1/2] or [1/2, 1]. For k^2 to take all values, the discriminant (15) must be zero at some point. From (15) it can be seen that at any temperature $D \neq 0$, but at $\xi = 0$ (or $p = 2\gamma/\acute{a}^2$) – D=0, i.e. we are dealing with first-order transitions.

At $\sigma = 0$ or $\sigma = 1/2$, we obtain a degenerate solution $k^2 = 1/2$, which will not be considered further.

A first-order transition occurs at a constant temperature from $0 \leq k^2 \leq 1$. From (14) we find this temperature:

$$\frac{\sigma^2{\xi_0}^2 + 2\gamma\sigma\xi_0 + 4q_0}{(6\sigma^2 - \sigma){\xi_0}^2 + 8\gamma\sigma\xi_0 + 16q_0} = 0 \tag{17}$$

$$q_0 = -\frac{\sigma^2{\xi_0}^2 + 2\gamma\sigma\xi_0}{4} = \frac{\gamma^2}{4} - \frac{(\gamma + \sigma\xi_0)^2}{4} \tag{18}$$

where ξ_0 is the initial value of ξ from p_0.

Substituting (18) the discriminant (15) can be rewritten as:

$$D = \frac{(2\sigma - 1)\xi^2}{(6\sigma - 1)\xi^2 + 8\gamma(\xi - \xi_0) - 4\sigma{\xi_0}^2} \tag{19}$$

Expression (11) can be rewritten by substituting (16) and (19):

$$b^2 = \frac{|\sigma|}{2}\sqrt{\frac{(6\sigma - 1)\xi^2 + 8\gamma(\xi - \xi_0) - 4\sigma{\xi_0}^2}{2\sigma - 1}} \tag{20}$$

ξ will take such values that the discriminant (19) takes values from 0 to 1. There is a condition

$$D(\xi_0) = D(\xi_{max}) = 1 \tag{21}$$

From condition (21) it follows that

$$\xi \in [\xi_0, -\xi_0 - 2\gamma/\sigma] \tag{22}$$

Expression (11) must be true so we have two cases:

$$\begin{aligned} &\xi_0 > 0\left(p_0 < \frac{2\gamma}{\acute{a}^2}\right),\ \sigma < 0 \\ &\xi_0 < 0\left(p_0 > \frac{2\gamma}{\acute{a}^2}\right),\ \sigma > 0 \end{aligned}. \tag{23}$$

If $\sigma > 0$, there are two cases:

$$\begin{aligned} -\xi_0 - 2\gamma/\sigma > 0 &\rightarrow p_0 > \frac{2\gamma}{\acute{a}^2}\left(1 + \frac{1}{\sigma}\right) \\ -\xi_0 - 2\gamma/\sigma < 0 &\rightarrow \frac{2\gamma}{\acute{a}^2} < p_0 < \frac{2\gamma}{\acute{a}^2}\left(1 + \frac{1}{\sigma}\right) \end{aligned}. \tag{24}$$

In the first case (24) – $0 \leq k^2 \leq 1$, in the second – $0 \leq k^2 \leq k_{max}^2 < \frac{1}{2}$ where k_{max}^2 is found from the extremum condition (19). The extremum itself is equal to

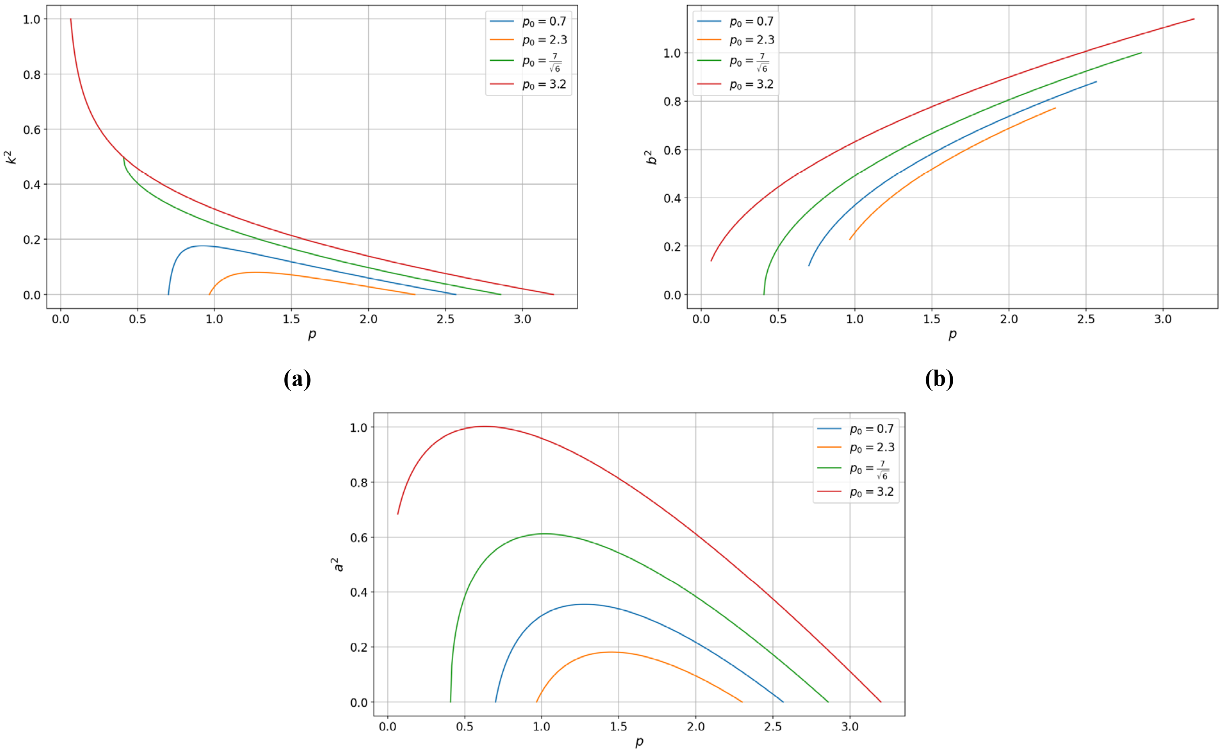
$$\xi_e = 2\xi_0 + \frac{\sigma\xi_0^2}{\gamma} \tag{25}$$

When $-\xi_0 - 2\gamma/\sigma = \xi_0$ (or $\xi_0 = -\gamma/\sigma$), $k_{max}^2 = 0$, i.e. the modulated phase is generally not realised.

RESULTS

Let's move on to the graphs. Let $\gamma = 1$ for all graphs.

Let's take $g^2 = 32/3$ or $\sigma = 1/6$. Let's look at the graphs for small values of p_0 and we get the following Figure 1.



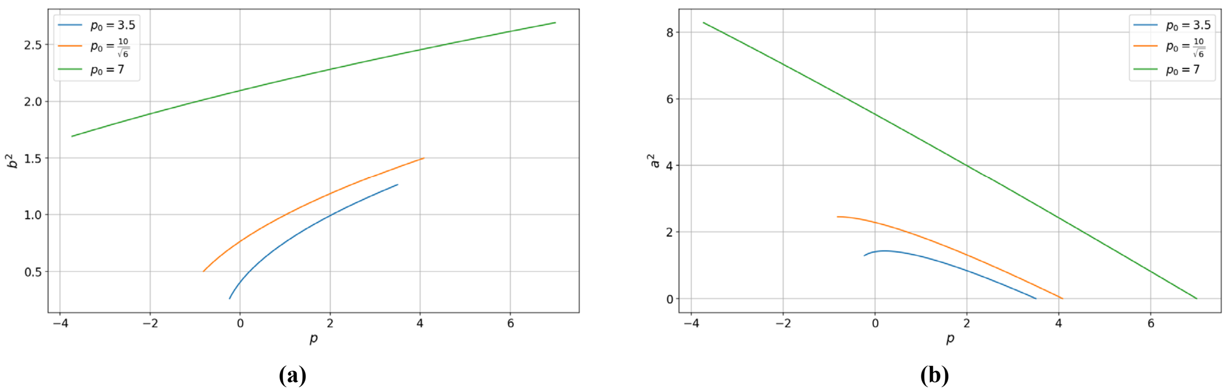
(a) **(b)** **(c)**

Figure 1. Graphs of dependence on p at $g^2 = 32/3$ for different initial values of p_0 for **(a)** k^2 **(b)** b^2 **(c)** a^2

As can be seen from Fig. 1, there are three types of solutions:

- $\frac{1}{\sqrt{6}} < p_0 < \frac{4}{\sqrt{6}}$ – the parameter p increases and the final phase remains modulated (lock-in transition).
- $\frac{4}{\sqrt{6}} < p_0 \leq \frac{7}{\sqrt{6}}$ – the parameter p decreases and the final phase remains modulated (lock-in transition).
- $p_0 > \frac{7}{\sqrt{6}}$– the parameter p decreases and the final phase transitions to a commensurate phase.

Now let's look at the graphs for larger p_0 (Figure 2). As can be seen from Figure 2(b), when $p_0 < \frac{10}{\sqrt{6}}$, the maximum amplitude a is reached at $\xi < \xi_{max}$, in other cases the maximum amplitude a is reached at $\xi = \xi_{max}$.

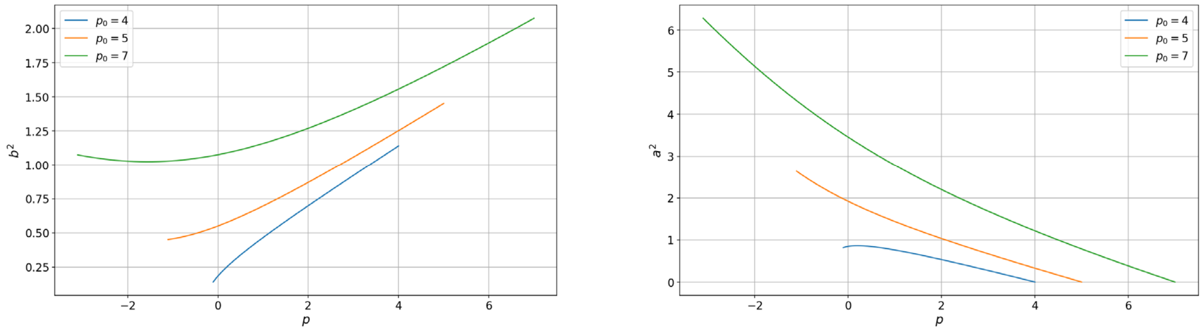


(a) **(b)**

Figure 2. Graphs of dependence on p at $g^2 = 32/3$ for different initial values of p_0 for **(a)** b^2 **(b)** a^2

Since in (5) there are two expressions for the amplitude let us consider them separately:

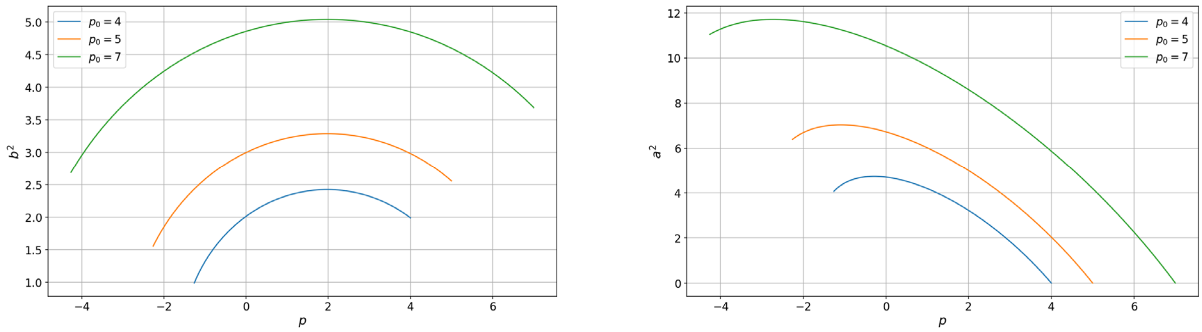
1. When $\acute{a}_+$ σ takes the value $(0,1/6)$. The graphs for small p_0 will be similar to Figure 1, but for larger p_0 ${b_+}^2$ will have a minimum at $\xi < \xi_{max}$, for example, for $g^2 = 11$ we obtain Figure 3.



(a) **(b)**

Figure 3. Graphs of dependence on p at $g^2 = 11$ for different initial values of p_0 for **(a)** ${b_+}^2$ **(b)** ${a_+}^2$

2. When $\acute{a}_-$ σ takes the values $(1/6,1/2)$, $(1/2,+\infty)$ and $(-\infty,-1/2)$.
For $g^2 = 11$ (see Figure 4) ${b_-}^2$ i ${a_-}^2$will have a maximum at $\xi < \xi_{max}$.

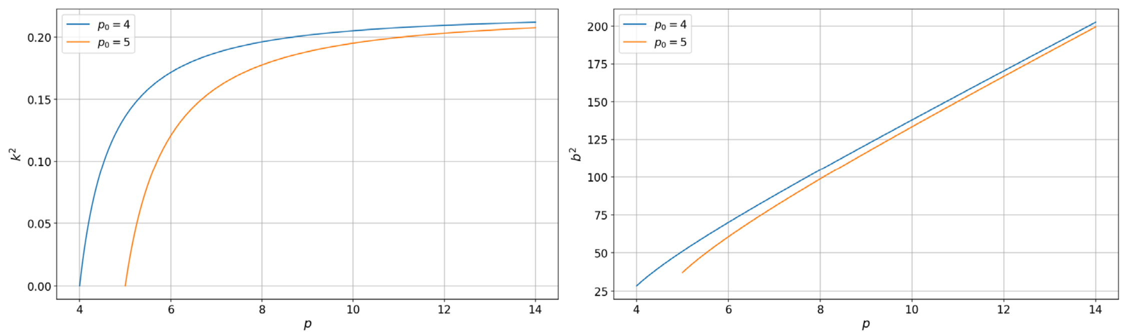


(a) **(b)**

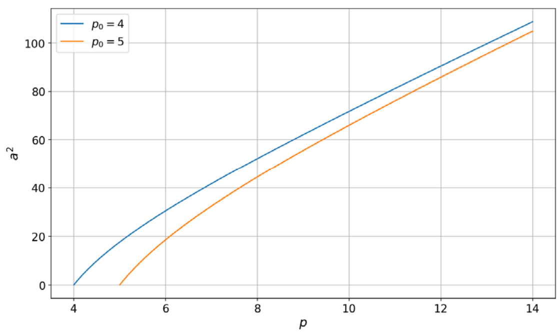
Figure 4. Graphs of dependence on p at $g^2 = 11$ for different initial values of p_0 for **(a)** ${b_-}^2$ **(b)** ${a_-}^2$

For $\acute{a}_-$ ($g^2 > 12$), there are no such values of p_0 where k^2 took all values from 0 to 1. For $g^2 > 12$, with an increase in p (see Figures 5-6)

$$k^2 \to \left(1 + \sqrt{\frac{(2\sigma - 1)}{(6\sigma - 1)}}\right) \Big/ 2 \tag{26}$$

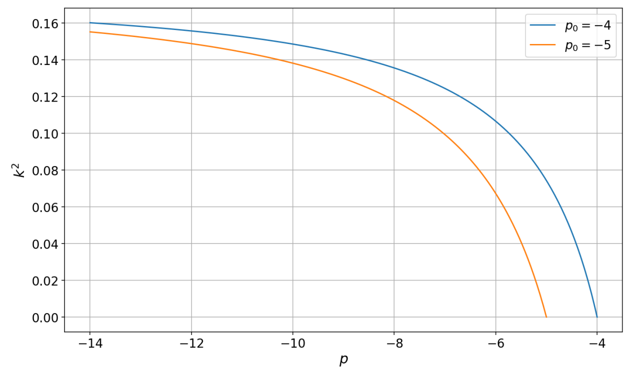


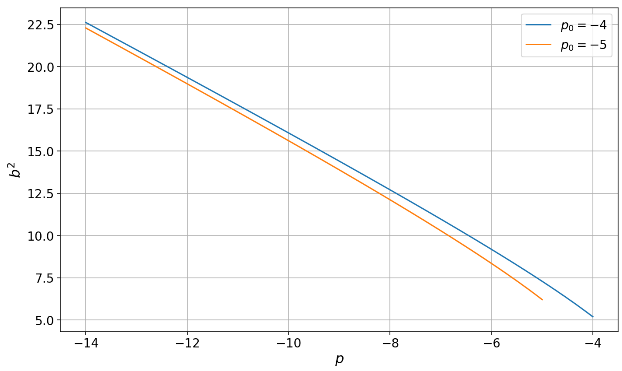
(a) **(b)**



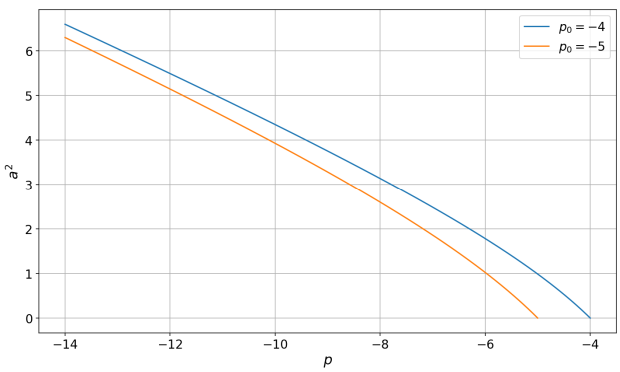
(c)

Figure 5. Graphs of dependence on p at $g^2 = 16$ for different initial values of p_0 for **(a)** ${k_-}^2$ **(b)** ${b_-}^2$ **(b)** ${a_-}^2$





(a) **(b)**



(c)

Figure 6. Graphs of dependence on p at $g^2 = 25$ for different initial values of p_0 for **(a)** $k\_^2$ **(b)** $b\_^2$ **(b)** $a\_^2$

CONCLUSIONS

This paper investigates a nonlinear fourth-order differential equation arising from the variational principle for a thermodynamic potential with high derivatives. For the order parameter, we used a distribution in the form of an elliptic cosine of Jacobi, which allowed us to reduce the problem to a system of algebraic equations for the amplitude, spatial scale, and modulus.

We established conditions for the existence of a physically realisable solution, in particular, a restriction on the parameter g, which guarantees the validity of the amplitude. It was found that initially a second-order phase transition occurs from a highly symmetric phase to an incommensurate phase at the point $q = \gamma^2/4$. Further, it was found that solutions based on the elliptic cosine are relevant for describing first-order phase transitions.

Graphs of the dependence of parameters a, b, k on the initial values p_0 and g were constructed.

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ЧАСТКОВІ ТОЧНІ РІШЕННЯ НЕЛІНІЙНОГО РОЗПОДІЛУ ОДНОКОМПОНЕНТНОГО ПАРАМЕТРА ПОРЯДКУ В РІВНОВАЖНИХ СИСТЕМАХ

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У цій роботі досліджено частинні точні розв'язки нелінійного диференційного рівняння четвертого порядку, що виникає з варіаційного принципу для термодинамічного потенціалу з високими похідними. Для опису просторового розподілу параметра порядку використано функцію еліптичного косинуса Якобі, що дало змогу звести задачу до системи алгебраїчних рівнянь для амплітуди, просторового масштабу та модуля. Отримано умови існування фізично допустимих розв'язків та виявлено, що періодичні розв'язки, виражені через еліптичний косинус, релевантні для опису фазових переходів першого роду. Представлено графіки, що ілюструють залежність основних параметрів рішення від характеристик системи.

Ключові слова: *нелінійні диференційні рівняння; параметр порядку; еліптичні функції Якобі; фазовий перехід першого роду; несумірна фаза*