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HUBBLE'S LAW AND ITS EXPONENTIAL GENERALIZATION WITH COSMOLOGICAL APPLICATIONS

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Hubble's law reveals how the components of the universe adhere to overarching dynamical rules on a cosmological scale. While it is most renowned for describing the universe's expansion, a general displacement equation derived in alignment with this law, along with a general equation of converging displacement, has been applied to estimate the time remaining before the Milky Way and Andromeda collide. This estimate closely aligns with results from numerical simulations of other studies. Additionally, the implications of this generalized equation provide valuable insights into key cosmological enigmas, including the time variation of the Hubble parameter, the cosmological past incompleteness, and the enduring mystery of the relationship between the subtle value of the cosmological constant and the quantum zero-point energy of the vacuum. It has also been successful in explaining the structure of spiral galaxies.

Keywords: *Hubble's Law, Galaxy Collision, Cosmological Constant,Vacuum's energy, Spiral Galaxy Structure*

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1. INTRODUCTION

Galileo famously demonstrated in a public experiment that, in the uniform gravitational field produced by Earth, all objects fall at the same rate, regardless of their weight. Later, Eötvös provided evidence that gravitational mass and inertial mass are equivalent. Kepler's laws also describe the motion of planets in terms of geometric principles, with the motion being independent of the planets' masses but dependent on the Sun's gravitational influence. These intriguing natural phenomena likely influenced Einstein, who developed the theory of general relativity, positing that gravity is fundamentally connected to the geometry of space and time. This is why it's reasonable to say that gravity is, in essence, geometry [1, 2]. Therefore, if it can be established that an object's motion is influenced solely by a specific gravitational field, we can derive a general displacement relation that depends only on time, regardless of the object's mass or chemistry.

Utilizing a fundamental mathematical approach, a general displacement relation was formulated as a power series [3, 4, 5]. By applying Taylor's expansion around the initial point or origin, specifically, the Maclaurin expansion [3, 4], it was ultimately shown that displacement can be described as an exponential function of time, thereby supporting the theoretical foundation of Hubble's relation. The well-known Hubble's relation, as of now, has been established solely through observational evidence. However, with the advent of the general displacement equation, a theoretical foundation has been established for Hubble's relation, complementing the empirical observations.

Here, the general displacement is derived based on plausible mathematical and physical assumptions, ultimately leading to Hubble's relation, which is consistently and unambiguously supported by numerous empirical observations of large-scale cosmic dynamics [6, 7, 8, 9, 10, 11, 12, 13]. Furthermore, the general displacement relation provides a mathematical explanation to address a cosmic initial incompleteness. This incompleteness is merely an unavoidable mathematical outcome; however, it holds no physical significance. The mathematical inevitability of the cosmic initial incompleteness inherent in the general relativistic approach is re-evaluated using the time exponential expansion of the universe, as inferred from the isotropic and uniform cosmic diverging displacement.

The universe is full of phenomena, many of which remain beyond our current understanding or the scientific models we've developed. However, the general displacement relation has proven quite useful in explaining certain enigmatic cosmic phenomena, particularly those related to large distant gravitational interactions.A similar mathematical and physically plausible approach is employed to derive the general equation for converging dynamical displacement. This approach is further validated by consistently estimating the time required for the upcoming collision between the Milky Way (MW) and Andromeda (M31). It has been effective in estimating the time remaining before the imminent collision between these two galaxies, with results that align with previous numerically simulated analyses where the technique of N-body simulation is employed to estimate the imminent collision. [14, 15].

The general time exponential displacement relation was applied in a quasi-quantum model, serving as the amplitude of classical harmonic oscillation to determine a finite and specific frequency. This frequency was then used to calculate the lowest state energy of the oscillator, modeled as a quantum harmonic oscillator. Since the time-exponential displacement equation is not a periodic function, Fourier transformation is applied to determine the characteristic frequency of the oscillation when the amplitude is modeled as an exponential function of time. This approach proved instrumental in

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demonstrating the consistency between the subtle observational value of the cosmological constant and the zero-point energy of the vacuum. This classical general relationship demonstrates consistency between value of the cosmological constant and the quantum ground-state energy of the vacuum through a quasi-quantum technique. While the quantum field theoretical prediction for the cosmological constant shows a significant degree of inconsistency with observed values.

In addition to the aforementioned applications, one notable application of the time exponential general displacement equation is its use in providing a mathematical explanation for the structure of spiral galaxies.

2. HUBBLE'S LAW AND ITS GENERALIZATION

Edwin Hubble's prolonged observations have unveiled a crucial discovery, indicating that in expansive cosmic distances, celestial units like galaxies are steadily moving away from each other, and this motion is directly and linearly linked to the distances that exist between them [6, 7, 8]. This outcome, subsequently analyzed through numerous further observations, has provided deeper insights into the dynamic relationships between celestial units and their velocities in the vast cosmic distances [9, 10, 11, 12, 13]. Which can be summarized as a simple equation for recession velocity:[16, 17]

$$\frac{dl}{dt} = v = Hl \tag{1}$$

By avoiding the negative and complex solutions, a solution for the above equation can be found by rearranging the terms involved followed by integrating both sides, with respect to the relevant variables as:

$$l = l_0 e^{Ht} \tag{2}$$

The above relation shows a non-degenerate relation, since by differentiating the sides of Hubble's law, we can find the general relation for the time derivatives of displacement variable as: $\ddot{l} \propto \dot{l}$, $\dddot{l} \propto \ddot{l}$, and so on. This implies that the time derivatives of acceleration do not vanish. The third and fourth derivatives of position are known as the jerk and snap, respectively [18]. Furthermore, higher-order time derivatives are a conventional concept in the study of robotics dynamics [19] and aerodynamics. Jerk and snap were also used as higher-order derivatives of the Friedmann–Lemaître–Robertson–Walker (FLRW) scale factor [20]. The term on the right-hand side of equation (2) is differentiable up to infinite order and continuous with respect to time. Importantly, the motion described by equation (2) is general and independent of the individual mass. As a result, developing a Hamiltonian formulation is unnecessary for our mass-independent motion analysis, and therefore any subsequent unbounded Hamiltonian is irrelevant. We will thoroughly examine the mathematical feasibility of equation (2) in principle in the following subsection 2.1. This simple relation is powerful enough to reshape our classical understanding of dynamics, particularly when applied to long-distance motion.

2.1. General equation for diverging displacement

Consider an object with an initial position x_0, defined within an inertial reference frame. After a time interval t, measured in the same frame, the object's position becomes x. Hubble's law, which establishes a proportionality between velocity and distance, implies that displacement can be described as a higher-order function of time. Let us express this displacement using a higher-order series expansion as a function of time:

$$x = f(t) = \sum_{n=0}^{\infty} x_n t^n \tag{3}$$

Using the Maclaurin expansion, we can express the displacement $x(t)$ as a higher-order series in terms of time t. The Maclaurin series expands a function about $t = 0$, leading to the following expression:

$$x = f(t) = [x]_{t=0} + \frac{1}{1!}\left[\frac{dx}{dt}\right]_{t=0} t + \frac{1}{2!}\left[\frac{d^2x}{dt^2}\right]_{t=0} t^2 + \frac{1}{3!}\left[\frac{d^3x}{dt^3}\right]_{t=0} t^3 + \cdots + \frac{1}{n!}\left[\frac{d^nx}{dt^n}\right]_{t=0} t^n + \cdots$$

We define the velocity v, acceleration a, the third derivative term a_1, and similarly, the n-th derivative term as $\frac{d^n x}{dt^n} = a_{n-2}$, with any zero subscripts representing the initial values:

$$x = x_0 + \frac{1}{1!}[v]_{t=0}\, t + \frac{1}{2!}[a]_{t=0}\, t^2 + \frac{1}{3!}[a_1]_{t=0}\, t^3 + \cdots + \frac{1}{n!}[a_{n-2}]_{t=0}\, t^n + \cdots$$

$$= x_0 + v_0 t + \frac{1}{2!} a_0 t^2 + \sum_{n=3}^{\infty} \frac{1}{n!}[a_{n-2}]_{t=0}\, t^n$$

Here we assume $x_0 > 0$.

$$x = x_0 \left\{ 1 + \left(\frac{v_0}{x_0} t\right) + \frac{1}{2!}\left(\sqrt[2]{\frac{a_0}{x_0}}\, t\right)^2 + \sum_{n=3}^{\infty} \frac{1}{n!}\left(\sqrt[n]{\frac{(a_{n-2})_{t=0}}{x_0}}\, t\right)^n \right\} \tag{4}$$

In this expression, the coefficients of each time-dependent term are defined using the initial kinematic quantities, such as the initial velocity v_0, acceleration a_0, and higher derivatives $[a_{n-2}]_{t=0}$. By dimensional analysis, each of these coefficients must have the dimension of inverse time, $[T^{-1}]$, which suggests a common physical scaling factor. If we interpret this displacement as occurring within a homogeneous and isotropic system, i.e., one that is directionally invariant and uniformly distributed, then the rate of displacement or expansion must also exhibit isotropic scaling. This concept is well aligned with the cosmological principle employed in Friedmann's models of the universe. In cosmology, under the assumptions of homogeneity and isotropy, the Friedmann–Lemaître–Robertson–Walker (FLRW) metric leads to a scale factor $\alpha(t)$ whose rate of change defines the Hubble parameter: $H = \frac{\dot{\alpha}}{\alpha}$. This formulation implies an exponential-like evolution under certain energy conditions, and hence supports the plausibility of uniform time scaling in the displacement function. Therefore, if the displacement depends only on time, and the system remains homogeneous and isotropic during its evolution, it is physically reasonable to assign the same constant value to all coefficients of the time powers in the expansion. At any given epoch or over a small time interval, we can thus define a single inverse time constant ζ, such that:

$$\zeta = \frac{v_0}{x_0} = \sqrt[2]{\frac{a_0}{x_0}} = \sqrt[n]{\frac{(a_{n-2})_{t=0}}{x_0}}$$

and using equation (4), we can get:

$$x = x_0 \left\{ 1 + \zeta t + \frac{1}{2!}(\zeta t)^2 + \cdots + \sum_{n=3}^{\infty} \frac{1}{n!}(\zeta t)^n \right\}$$

$$x = x_0 e^{\zeta t} \tag{5}$$

From this, Hubble's relation can be easily derived as: $\dot{x} = \zeta x$ or $\dot{x} \propto x$. Indeed, our defined constant ζ is equivalent to the Hubble parameter H in case of displacement due to cosmic expansion, which can be regarded as the constant H_0 for a particular epoch of cosmic time. Equation (5) is identical and equivalent to equation (2) as follows, and it represents the general equation for cosmological diverging displacement:

$$x = x_0 e^{H_0 t} \tag{6}$$

In the Maclaurin expansion, it is important to note that we expand the displacement function from an initial point in time. This approach is essential since time is the only independent variable in the function defined by equation (6). For measurement purposes, we must define a reference point, and the most convenient choice is our current cosmological spacetime point, defined as $(t = 0,\ x = x_0)$. Future time corresponds to positive t, while past time corresponds to negative t, relative to this arbitrarily and conveniently chosen origin of measurement. The Hubble constant is determined by observing the current velocity and distance of cosmic objects with respect to this reference point. As time progresses over a nontrivial interval (e.g., t_f), velocity measurements will no longer be referenced to the initial fixed origin. Instead, they must be referenced to a transformed origin that shifts along the spatial axis as time changes. As a result, the transformed displacement will be:

$$x_{f0} = x_0 e^{H_0 t_f}.$$

Given an initial value for velocity, namely the present-time velocity v_0, the resulting transformed velocity is expressed as:

$$v_{f0} = v_0 + \frac{d^2 x_{f0}}{dt^2} t_f.$$

$$v_{f0} = v_0 + t_f x_0 H_0^2 e^{H_0 t_f}.$$

The Hubble parameter at this future point will then be measured as:

$$H_{f0} = \frac{v_{f0}}{x_{f0}} = \frac{v_0 + x_0 t_f H_0^2 e^{H_0 t_f}}{x_0 e^{H_0 t_f}} = \frac{v_0}{x_0} \frac{1}{e^{H_0 t_f}} + t_f H_0^2,$$

which is evidently $\neq \frac{v_0}{x_0}$. The above argument regarding the inequality of the Hubble parameter across different cosmic epochs, commonly referred to as the Hubble tension, is similarly applicable to the distant past if the Hubble parameter were to be measured. This argument provides a straightforward explanation of the enigmatic concept known as the Hubble tension. The time dependency of the Hubble parameter can be well comprehended through general relativity, in connection with the FLRW scale factor, as a result of employing the FLRW metric in Einstein's field equations, which we will examine in Section 5.

2.2. General equation for converging displacement

Now consider an object with an initial position x_0, defined within an inertial reference frame. After a time interval t, measured in the same frame, the object's position becomes x. In this scenario, as time increases, x decreases. As a result, the time derivatives of displacement, such as $\frac{dx}{dt}$, $\frac{d^2x}{dt^2}$, and so forth, along with their initial values, will have negative signs. Therefore, the Maclaurin series expansion of the general converging displacement will be:

$$x = x_0 - v_0 t - \frac{1}{2!}a_0 t^2 - \sum_{n=3}^{\infty} \frac{1}{n!}\,[a_{\,n-2}]_{t=0}\,t^n.$$

$$x = 2x_0 - \left(x_0 + v_0 t + \frac{1}{2!}a_0 t^2 + \sum_{n=3}^{\infty} \frac{1}{n!}\,[a_{\,n-2}]_{t=0}\,t^n\right).$$

$$x = 2x_0 - \left[x_0\left\{1 + \frac{v_0}{x_0}t + \frac{1}{2!}\left(\sqrt{\frac{2a_0}{x_0}}\,t\right)^2 + \sum_{n=3}^{\infty}\frac{1}{n!}\left(\sqrt{\frac{n[a_{\,n-2}]_{t=0}}{x_0}}\,t\right)^n\right\}\right].$$

Defining the constant for converging displacement as

$$\xi = \frac{v_0}{x_0} = \sqrt{\frac{2a_0}{x_0}} = \sqrt{\frac{n[a_{\,n-2}]_{t=0}}{x_0}},$$

we obtain

$$x = 2x_0 - x_0\left\{1 + \xi t + \frac{1}{2!}(\xi t)^2 + \cdots + \sum_{n=3}^{\infty}\frac{1}{n!}(\xi t)^n\right\} = 2x_0 - x_0 e^{\xi t}.$$

$$x = x_0(2 - e^{\xi t}) \qquad (7)$$

The aforementioned equation (7) was derived by considering the convergence of two masses, ignoring any external perturbations, and assuming that the motion is dominated by the center of mass. Therefore, this relation applies to cases of interacting motion between galaxies with well-defined geometric shapes, supported by the presence of significant amounts of non-baryonic dark matter. In these cases, the constituents are strongly bound to the galactic nucleus, which can reasonably be assumed to contain a supermassive black hole.

3. ESTIMATING THE TIME REMAINING FOR MILKY WAY AND ANDROMEDA COLLISION

Contemporary observations confirm that the Andromeda Galaxy is directly approaching the Milky Way, making their collision inevitable [14, 21, 22, 23, 24, 25, 26]. Assuming a direct approach, the time remaining until the collision can be estimated using equation (7). This time represents the duration required for the distance between the two galaxies to decrease from the current initial distance, defined as x_0, to zero, given the current approach velocity of Andromeda, represented as v_0. From equation (7), setting $x = 0$, we calculate the time:

$$0 = x_0\left(2 - e^{\xi t}\right)$$

$$t = \frac{\ln 2}{\xi} \qquad (8)$$

Where $\xi = \frac{v_0}{x_0}$. and considered conversion for 1year = 365.25 days. Using the current data on M31's approaching velocity and distance, the estimated time remaining until the collision is summarized in the Table 1.

Table 1. Assessment of remaining time for collision between the Milky Way and Andromeda using data from different studies.

Current Radial Velocity, v_0 (km/s)	Current Distance, x_0 (Mpc)	ξ (km/s/Mpc)	Remaining Time for Collision, $t = \frac{\ln 2}{\xi}$ (Gyr)
120 [22]	0.78 [21]	153.846	4.405
110 [23]	0.785 [27]	140.127	4.837
115.7 [15]	0.761 [28]	152.037	4.458
109.3 [25]	0.77 [14]	141.948	4.775
110 [23]	0.8 [24]	137.5	4.929

It is important to clarify that the present analysis considers only a single degree of freedom, the radial component of M31's motion. While it is true that M31 also has a tangential velocity component, the radial and tangential motions can be treated as dynamically independent, much like two-dimensional motion under the influence of a potential field.

Furthermore, the effects of dynamical friction and gravitational perturbations from neighboring galaxies tend to moderate the radial motion, making it approximately constant over large timescales. However, the system still evolves dynamically through a spiral infall trajectory, resulting from the continuous interplay between radial decay and angular momentum loss. An analogous coplanar spiral motion, commonly observed within disk galaxies, will be further discussed in section 6.

Since ξ is defined as a constant, the remaining time for collision, $t = \frac{\ln 2}{\xi}$, is likewise a fixed quantity. This is reasonable, as it aligns with our understanding that the unperturbed gravitational interaction time under a given gravitational field is also definite. From Table 1, the estimated time remaining until the upcoming collision between the Milky Way (MW) and Andromeda Galaxy (M31) is approximately 4.4 to 4.93 billion years, aligning closely with contemporary estimates derived from numerical simulation data. For example [14] reported a collision timeline of approximately 4.5 billion years, based on precise measurements of galactic motions. Furthermore, this estimate is also consistent with the timeline of 4 to 5 billion years suggested by [15], which was derived using numerical N-body simulation data.

4. COSMOLOGICAL CONSTANT AND QUANTUM ZERO-POINT ENERGY

The relationship between the cosmological constant and vacuum energy density, interpreted as quantum zero-point energy, was discussed in the seminal work by Zel'dovich [29]. Notable recent studies [30, 31, 32, 33] have further explored this correlation. Notable references can be found in the comprehensive book authored by Hobson [34], and Carroll [35], and the influential article by Steven Weinberg [36] about the cosmological constant value. By employing the concept that vacuum pressure is analogous to perfect fluid pressure [29, 34, 35, 37, 38], the cosmological constant was derived as:

$$\Lambda = \frac{8\pi G\rho_{\text{vac}}}{c^2} \tag{9}$$

This can be achieved through mathematically straightforward and physically plausible assumptions by utilizing Einstein's field equation for empty space with the presence of the cosmological constant. Einstein field equation in covariant form in the presence of matter and employing the cosmological constant [34, 37]:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu} \tag{10}$$

By decomposing the energy-momentum tensor $T_{\mu\nu}$ into components for matter, $[T_{\mu\nu}]_m$ and vacuum, $[T_{\mu\nu}]_{\text{vac}}$, and considering the context of flat spacetime without matter, we set the Einstein tensor, $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0$. Factoring out the $[T_{\mu\nu}]_m$ term for the vacuum, the above equation (10) becomes:

$$\Lambda g_{\mu\nu} = -\kappa[T_{\mu\nu}]_{\text{vac}} = -\kappa[T^{00}]g_{\mu\nu}$$

By utilizing the negative energy property of the vacuum, we obtain from the above equation:

$$\Lambda g_{\mu\nu} = \kappa\rho_{\text{vac}}c^2 g_{\mu\nu}$$

By substituting the value of the constant κ, the above equation becomes:

$$\Lambda = \frac{8\pi G\rho_{\text{vac}}}{c^2}$$

Thus, it can be concluded that equation (9) represents a form of Einstein's field equation under the conditions of empty space, where the energy-momentum tensor does not vanish but is present in the form of vacuum energy, and in the presence of cosmological constant. Applying the Jeans instability condition [22, 39, 40, 41, 42] considering the fluid mechanical analogy to the entire universe reveals that for stability, the universe must be homogeneous and isotropic, without a specific center. Mathematically, it can be demonstrated that for a stable universe, the unperturbed density must be zero for the homogeneous and isotropic static universe with a sufficient extent [22, 40, 41, 42]. While some, like James Binney, have termed this as a "swindle,"[22, 39, 42] several analyses have validated it as a mathematical fact [39, 42, 43]. This concept aligns with the zero-energy density (or zero-mass density) feature of a de Sitter universe, where the cosmological constant dominates and drives an accelerated expansion, resulting in a universe with a constant energy density. It's also noteworthy that the quantum wave function of the entire universe, when in its ground state and under classical conditions, corresponds with the zero-mass density phenomenon of the de Sitter universe [44]. Time exponential equation of cosmic scale factor is in fact a property of de Sitter universe [45].

We will now model the vacuum as a classical harmonic oscillator instead of modeling the oscillation of the entire universe. Since the equations governing harmonic oscillation require oscillation around a center, this could lead to instability arising from the assumption of a specific center for the universe, as previously discussed.
We recall the fundamental equation of classical harmonic oscillator [46]:

$$m\ddot{x} = -kx \tag{11}$$

To model the vacuum as an oscillator, we will consider the vacuum's constituent point mass m_0 and the stiffness of its oscillation k_0. Using equation (6) along with the equation for the classical harmonic oscillator mentioned above equation (11):

$$m_0 x_0 H_0^2 e^{H_0 t} = -k_0 x_0 e^{H_0 t}$$

$$m_0 H_0^2 = -k_0$$

$$H_0 = i\sqrt{\frac{k_0}{m_0}} = i\omega \tag{12}$$

Where ω represents the angular frequency of the oscillator. The general solution for equation (11) in exponential form, with C_1 and C_2 as constant coefficients, is:

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$

Using Equation (12), the general solution becomes:

$$x(t) = C_1 e^{H_0 t} + C_2 e^{-H_0 t} \tag{13}$$

The above equation (13) is evidently not a periodic function. Therefore, to determine the characteristic angular frequency in relation to Hubble constant, we can analyze the Fourier transformation of the function represented by equation (13). Now the Fourier transformation [47] of x(t) will be as below:

$$\hat{f}(x) = \int_{-\infty}^{\infty} \left(C_1 e^{H_0 t} + C_2 e^{-H_0 t}\right) e^{-i\omega t}\, dt$$

$$\hat{f}(x) = C_1 \int_{-\infty}^{\infty} e^{t(H_0 - i\omega)}\, dt + C_2 \int_{-\infty}^{\infty} e^{-t(H_0 + i\omega)}\, dt \tag{14}$$

Since $e^{H_0 t}$ grows exponentially as t approaches ∞, its transformation converges only when $t \to -\infty$. Similarly, $e^{-H_0 t}$ grows exponentially as t approaches $-\infty$, so its transformation converges only when $t \to \infty$. Therefore, for $H_0 > 0$, equation (14) takes the following form:

$$\hat{f}(x) = C_1 \int_{-\infty}^{0} e^{t(H_0 - i\omega)}\, dt + C_2 \int_{0}^{\infty} e^{-t(H_0 + i\omega)}\, dt$$

$$\hat{f}(x) = \frac{C_1}{H_0 - i\omega} + \frac{C_2}{H_0 + i\omega}$$

As Hubble constant has a real value, therefore in accordance to equation (12), we cannot determine a conventional real-valued frequency like that of an ordinary oscillating system; instead, we can evaluate the characteristic frequency. The characteristic frequency corresponds to the frequency scale where the real and imaginary parts of the denominator are comparable. The real part is H_0 and the imaginary part is ω, therefore the characteristic angular frequency is :

$$\omega_c = |H_0|.$$

By applying Euler's formula to Equation (13), we obtain:

$$x(t) = C_1\left[\cos(\omega t) + i\sin(\omega t)\right] + C_2\left[\cos(\omega t) - i\sin(\omega t)\right],$$

$$x(t) = (C_1 + C_2)\cos(\omega t) + i(C_1 - C_2)\sin(\omega t),$$

Which is a periodic function having the characteristic frequency

$$\omega_c = |H_0|.$$

It has been demonstrated that when modeling the vacuum as an ordinary harmonic oscillator, an imaginary frequency emerges instead of a real one. However, we can define a characteristic frequency with a value equal to H_0. The vacuum is not ordinary, and the challenge of obtaining a real frequency can be resolved through the well-established property of the vacuum's extraordinary negative energy. We can formulate the Lagrangian for the vacuum as:

$$L_{\text{vac}} = T_{\text{vac}} - V_{\text{vac}},$$

where T_{vac} represents the negative kinetic energy of the vacuum, defined as

$$T_{\text{vac}} = -\frac{1}{2} m_0 \dot{x}^2,$$

and the displacement-dependent potential is given by

$$V_{\text{vac}} = -\frac{1}{2}k_0x^2.$$

Using these relations, the Euler–Lagrange equation for the vacuum becomes:

$$\frac{\partial L_{\text{vac}}}{\partial x} - \frac{d}{dt}\left(\frac{\partial L_{\text{vac}}}{\partial \dot{x}}\right) = 0,$$

which yields

$$-k_0x + m_0\ddot{x} = 0. \tag{15}$$

By applying the exponential time-dependent amplitude solution, we have:

$$m_0x_0H_0^2e^{H_0t} = k_0x_0e^{H_0t}, \qquad H_0 = \sqrt{\frac{k_0}{m_0}} = \omega.$$

Thus, by leveraging the vacuum's negative energy property, we obtain a real-valued frequency, which matches the characteristic frequency determined earlier.

If we align with the de Sitter universe, assuming the entire universe is a vast vacuum with a constant energy density ε_{vac} and mass density ρ_{vac}, the first Friedmann equation with the cosmological constant term Λ results in the following equation for the cosmic scale factor α: [34]

$$\ddot{\alpha} = -\frac{4\pi G}{3}\left(\rho_{\text{vac}} + \frac{3p}{c^2}\right)\alpha + \frac{1}{3}\Lambda c^2\alpha. \tag{16}$$

By substituting the perfect fluid analog of vacuum, where the energy exerts negative pressure, i.e., $p = -\varepsilon_{\text{vac}} = -\rho_{\text{vac}}c^2$, into the above equation (16), we obtain:

$$\ddot{\alpha} = -\frac{4\pi G}{3}\left(\rho_{\text{vac}} - 3\rho_{\text{vac}}\right)\alpha + \frac{1}{3}\Lambda c^2\alpha,$$

$$\ddot{\alpha} - \frac{8\pi G\rho_{\text{vac}}}{3}\alpha - \frac{1}{3}\Lambda c^2\alpha = 0. \tag{17}$$

By defining the quantity

$$H_\Lambda = \sqrt{\frac{1}{3}\Lambda c^2 + \frac{8\pi G\rho_{\text{vac}}}{3}},$$

the equation (17) becomes:

$$\ddot{\alpha} - H_\Lambda^2\alpha = 0.$$

Thus, a general solution for equation (17) can be determined, which is analogous to equation (13), as follows with scalar constants C_3 and C_4:

$$\alpha(t) = C_3e^{H_\Lambda t} + C_4e^{-H_\Lambda t} \tag{18}$$

Another form of the solution can be expressed in terms of hyperbolic trigonometric functions:

$$\alpha(t) = \frac{C_3 + C_4}{2}\cosh(H_\Lambda t) + \frac{C_3 - C_4}{2}\sinh(H_\Lambda t) \tag{19}$$

Form of the equation (18) is similar to the displacement solution expressed in equation (13). This similarity further demonstrates the consistency between the solution for the Hubble parameter in equation (2) and that in equation (25), expressed in terms of distance and the cosmic scale factor, respectively. We have already shown that in the displacement-based solution with referencing equation (13), H_0 cannot be equal to an ordinary real frequency of a harmonic oscillation; a similar argument applies to the scale factor–based solution, that is, the equations (18) and (19) do not allow H_Λ to be interpreted as a direct harmonic oscillation frequency. After applying a Fourier transformation, H_Λ can be regarded as a characteristic frequency, which in this case is real. Therefore, corresponding to the equation (17), if we model the universe as an undamped free harmonic oscillator, the characteristic angular frequency can be given by:

$$H_\Lambda = \sqrt{\frac{1}{3}\Lambda c^2 + \frac{8\pi G\rho_{\text{vac}}}{3}}.$$

Using equation (9) for Λ, we can derive:

$$H_\Lambda = \sqrt{\frac{16\pi G\rho_{\text{vac}}}{3}} \tag{20}$$

This is equivalent to the expression for the Hubble constant, H_0, in a flat, exponentially expanding universe and can be verified numerically using observational data [48, 49]. Equations (18) and (19) present fascinating results, as they indicate that the cosmic scale factor remains non-zero for any finite time variable domain. This implies a cosmic solution for an eternal universe without spacetime incompleteness. Unlike a comoving solution, this result is derived relative to a fixed spacetime framework. In this type of oscillating cosmic model, each cycle of expansion, from a non-zero minimum to a maximum, and subsequent contraction back to the minimum point will have a defined period:

$$T_\Lambda = \frac{2\pi}{H_\Lambda} = 2\pi\sqrt{\frac{3}{16\pi G\rho_{\text{vac}}}},$$

$$T_\Lambda = \frac{1}{2}\sqrt{\frac{3\pi}{G\rho_{\text{vac}}}} \tag{21}$$

Equation (21) is valid only if the harmonically oscillating model of an exponentially expanding universe is assumed through utilizing the Fourier transformation, rather than the standard exponentially ever-expanding de Sitter universe. Any model that introduces a fixed cosmic spacetime center directly contradicts the standard cosmological principle. Additionally, this type of oscillation conflicts with the second law of thermodynamics, which states that the entropy of the universe must increase over time. Thus, it is more reasonable to associate this undamped constant oscillation frequency, $\omega = H_0$, as derived from the equation (15) by using the general displacement equation, with the vacuum rather than with the entire universe.

It is worth noting that the cyclic model of the universe has been regarded as a fascinating theory in science, philosophy, and even in common-sense interpretations, as it suggests cosmic perpetuity. Various versions of the cyclic model were discussed by Kragh, including a constant cyclic period proposed by Einstein and Friedmann, as well as a varying period of oscillation proposed by Tolman, Zanstra, and many others between 1922 and 1960 [50]. In a more recent work, Ijjas & Steinhardt proposes a novel cyclic model in which the scale factor grows exponentially with each cycle, addressing several unresolved cosmological problems, notably flatness, the monopole problem, initial conditions, and singularity [51].

Up until now, we have been analyzing using classical methods. Now, we will shift to a quantum approach to directly achieve consistency between the value of the cosmological constant derived from analogy with the negative vacuum pressure and the lowest, or zero-point, energy of the quantum harmonic oscillator. Writing the time-independent Schrödinger equation for a harmonic oscillator with frequency H_0 and wavefunction ψ_0:

$$-\frac{\hbar^2}{2m_0}\frac{d^2\psi_0}{dx^2} + \frac{1}{2}m_0 H_0^2 x^2\psi_0 = E_0\psi_0 \tag{22}$$

Using standard techniques, we obtain the solution for the vacuum eigen wavefunction:

$$\psi_0(x) = \left(\frac{m_0 H_0}{\pi\hbar}\right)^{1/4} e^{-\frac{m_0 H_0 x^2}{2\hbar}}. \tag{23}$$

And the lowest or zero-point energy of the vacuum:

$$E_0 = \frac{1}{2}\hbar H_0. \tag{24}$$

While the general equation for the quantum harmonic oscillator's eigen wave function is [52, 53]:

$$\psi_n(x) = \sqrt{\frac{1}{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right),$$

where H_n is the nth Hermite polynomial. The parameters in equations (9) and (24) were observationally determined to be constants, allowing for the numerical evaluation of Λ using the value $\rho_{\text{vac}} = 60.3 \times 10^{-29}\,\text{kg}\,\text{m}^{-3}$ as shown below [48]:

$$\Lambda = \frac{8\pi G\rho_{\text{vac}}}{c^2} = \frac{8 \times 3.1416 \times 6.67 \times 10^{-11}\,\text{Nm}^2\,\text{kg}^{-1} \times 60.3 \times 10^{-28}\,\text{kg}\,\text{m}^{-3}}{(2.9979 \times 10^8\,\text{m}\,\text{s}^{-1})^2} = 1.1247 \times 10^{-52}\,\text{m}^{-2}.$$

And the numerical value of E_0 using $H_0 = 69.8\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1} = 2.2615 \times 10^{-18}\,\text{s}^{-1}$ [49]:

$$E_0 = \frac{1}{2}\hbar H_0 = \frac{1}{2} \times 1.05 \times 10^{-34}\,\text{Js} \times 2.2615 \times 10^{-18}\,\text{s}^{-1} = 1.1873 \times 10^{-52}\,\text{J}.$$

Since this analysis is based on the vacuum, flat, homogeneous, and isotropic Einstein field equations consistent with a de Sitter universe, it does not constitute a conventional model of quantum field theory in curved spacetime. Rather, it can be described as a quasi-quantum model aimed at deriving the vacuum's zero-point energy. Despite its straightforward nature, this model demonstrates remarkable consistency with observational data, as illustrated in the earlier numerical comparison between the cosmological constant Λ and the vacuum energy density E_0.

To determine a specific angular frequency of oscillation, we began with the classical harmonic oscillator equation rather than considering the infinite modes of oscillation associated with the continual creation and annihilation of vacuum particles. This approach was chosen to avoid the so-called "worst theoretical prediction." Subsequently, we utilized the equation for the lowest energy eigenvalue of the quantum harmonic oscillator, leading to a well-aligned theoretical prediction. The obtained numerical values $1.1247 \times 10^{-52}\,\mathrm{m}^{-2}$ for the cosmological constant Λ, and $1.1873 \times 10^{-52}\,\mathrm{J}$ for E_0, representing the modeled energy of a quasi-quantum harmonically oscillating vacuum, are remarkably close. There exists a coupling constant with the unit J·m^2 and a value very close to unity:

$$\frac{1.1873 \times 10^{-52}\,\mathrm{J}}{1.1247 \times 10^{-52}\,\mathrm{m}^{-2}} = 1.05566\,\mathrm{J\,m}^2.$$

An expression for this coupling constant can be derived using equations (24) and (20), considering equivalence between H_Λ and H_0, as follows:

$$\frac{E_0}{\Lambda} = \frac{1}{2}\frac{\hbar c^2}{\sqrt{12\pi G \rho_{\mathrm{vac}}}}.$$

Since the properties of a vacuum can be effectively explained through its fluid-mechanical similarities, it is not surprising to find a relationship between its intrinsic curvature and its energy. Just as any external force acting on a continuum mechanical body results in the development of internal energy and bending, the curvature in a vacuum arises from the energy associated with its own mass,that is energy of the lowest possible or ground state undamped free vibration. In this scenario, the cosmological constant, represented by the vacuum's curvature, corresponds to its geometric curvature per unit length.

5. USING GENERAL EQUATION OF COSMIC EXPANSION TO REVIEW THE REAL COSMIC PAST INCOMPLETENESS

Hubble's law, based on observations, fundamentally describes the dynamics of cosmic structures. This relationship, where distances between cosmic structures grow over time, reflects the expansion of the universe [54]. Since the universe encompasses everything, its expansion can only be understood and measured internally, through its own inherent dynamics. However, this expansion preserves the universe's large-scale homogeneity and isotropy while maintaining short-distance inhomogeneity and anisotropy. Using equation (2) and an equivalent to the equation (3), we can clearly express the equation for the cosmic scale factor as:

$$\alpha = \alpha_0 e^{Ht} \tag{25}$$

The above equation (25) can also be derived by exploring the relationship between the Hubble constant H and the scale factor α [16, 17, 55]:

$$H = \frac{\dot{\alpha}}{\alpha}$$

For the mathematical validity of the above relationship in equation (25), it is crucial that the Hubble parameter H remains constant. In the derivation of the general equation for diverging displacement in subsection 2.1 , we demonstrated that H is indeed constant when measured relative to a fixed cosmic space-time point of origin. This constancy holds true for a specific cosmic epoch, such as the current epoch of the universe. This relation (equation 25) is consistent with the exponential cosmic expansion predicted in the de Sitter universe model [17, 56, 57, 58, 59] and aligns with the inflationary solutions for the evolution of the universe [45, 60] and expanding steady state theory of Bondi & Gold [61].

While addressing the metric of the maximally symmetric universe, known as the Friedmann-Lemaître-Robertson-Walker (FLRW) metric [59, 62] in a temporal (t) and comoving spatial spherical polar coordinate system (r, θ, ϕ), with Gaussian curvature constant, k :

$$ds^2 = (c\,dt)^2 - \alpha^2 \left\{ \frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right\}$$

Expressing the time-exponential cosmic scale factor α, allows us to rewrite the FLRW metric:

$$ds^2 = (c\,dt)^2 - \left(\alpha_0 e^{Ht}\right)^2 \left\{ \frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right\} \tag{26}$$

The corresponding covariant metric tensor is:

$$g_{\mu\nu} = \begin{bmatrix} c^2 & 0 & 0 & 0 \\ 0 & -\frac{\left(\alpha_0 e^{Ht}\right)^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & -\left(\alpha_0 e^{Ht}\right)^2 r^2 & 0 \\ 0 & 0 & 0 & -\left(\alpha_0 e^{Ht}\right)^2 r^2 \sin^2\theta \end{bmatrix}$$

And the resulting contravariant metric tensor is:

$$g^{\mu\nu} = \begin{bmatrix} \frac{1}{c^2} & 0 & 0 & 0 \\ 0 & -\frac{(1-kr^2)}{(\alpha_0 e^{Ht})^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{(\alpha_0 e^{Ht})^2 r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{(\alpha_0 e^{Ht})^2 r^2 \sin^2\theta} \end{bmatrix}$$

Here $\mu\nu$ represents permutations of coordinates labeled as $[0,1,2,3]$, with the coordinates conventionally defined as $(x^0 = t, x^1 = r, x^2 = \theta, x^3 = \phi)$. Now, it is straightforward, though tedious, to find the non-zero Christoffel symbol terms $\Gamma^0_{11}, \Gamma^0_{22}, \Gamma^0_{33}, \Gamma^1_{01}, \Gamma^1_{01}, \Gamma^1_{11}, \Gamma^1_{22}, \Gamma^1_{33}, \Gamma^2_{02}, \Gamma^2_{12}, \Gamma^2_{33}, \Gamma^3_{03}, \Gamma^3_{13}, \Gamma^3_{23}$ using the equation below [36]:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\frac{\partial}{\partial x^\nu} g_{\rho\mu} + \frac{\partial}{\partial x^\mu} g_{\rho\nu} - \frac{\partial}{\partial x^\rho} g_{\mu\nu}\right)$$

We can evaluate the non-zero Christoffel symbol terms using the above equation:

$$\Gamma^0_{11} = \frac{\alpha_0^2 e^{2Ht}\,(H + \dot{tH})}{c^2(1-kr^2)}, \quad \Gamma^0_{22} = \frac{\alpha_0^2 e^{2Ht}(H + t\dot{H})r^2}{c^2}, \quad \Gamma^0_{33} = \frac{\alpha_0^2 (H + t\dot{H}) e^{2Ht} r^2 \sin^2\theta}{c^2}.$$

$$\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = H + t\dot{H}, \quad \Gamma^1_{11} = \frac{kr}{1-kr^2}, \quad \Gamma^1_{22} = -r(1-kr^2), \quad \Gamma^1_{33} = -r(1-kr^2)\sin^2\theta,$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^2_{33} = -\sin\theta\cos\theta, \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta.$$

Utilizing the Christoffel symbols as determined above, one can readily obtain the non-zero Ricci curvature tensor components (R_{00}, R_{11}, R_{22}, R_{33}) from the equation by contracting the Riemann tensor by raising the index σ as $R_{\mu\nu} = R^\sigma_{\mu\sigma\nu}$[63]:

$$R_{\mu\nu} = \frac{\partial}{\partial x^\sigma}\Gamma^\sigma_{\mu\nu} - \frac{\partial}{\partial x^\nu}\Gamma^\sigma_{\mu\sigma} + \Gamma^\rho_{\mu\nu}\Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu} \tag{27}$$

Evaluating R_{00} using equation (27) yields:

$$R_{00} = \frac{\partial}{\partial x^\sigma}\Gamma^\sigma_{00} - \frac{\partial}{\partial x^0}\Gamma^\sigma_{0\sigma} + \Gamma^\rho_{00}\Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{0\sigma}\Gamma^\sigma_{\rho 0} \tag{28}$$

From equation (28) we obtain for the relevant non-zero terms for the σ and ρ indices:

$$R_{00} = -3\left\{(2\dot{H} + t\ddot{H}) + (H + t\dot{H})^2\right\}. \tag{29}$$

In a similar manner, utilizing equation (27) yields further Ricci curvature tensor components:

$$R_{11} = \frac{\alpha_0^2 e^{2Ht}}{c^2(1-kr^2)}\left\{3(H + t\dot{H})^2 + 2\dot{H} + t\ddot{H} + \frac{2kc^2}{\alpha_0^2 e^{2Ht}}\right\} \tag{30}$$

$$R_{22} = \frac{\alpha_0^2 e^{2Ht} r^2}{c^2}\left\{3(H + t\dot{H})^2 + 2\dot{H} + t\ddot{H} + \frac{2kc^2}{\alpha_0^2 e^{2Ht}}\right\} \tag{31}$$

$$R_{33} = \frac{\alpha_0^2 e^{2Ht} r^2 \sin^2\theta}{c^2}\left\{3(H + t\dot{H})^2 + 2\dot{H} + t\ddot{H} + \frac{2kc^2}{\alpha_0^2 e^{2Ht}}\right\} \tag{32}$$

The Ricci curvature scalar, R, is obtained by summing the non-zero covariant Riemann tensor and the contravariant metric tensor over the repeated indices:

$$R = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33}.$$

In this case, after simplifying:

$$R = -\frac{3}{c^2}\left\{(2\dot{H} + t\ddot{H}) + (H + t\dot{H})^2\right\} - \frac{3}{c^2}\left\{3(H + t\dot{H})^2 + 2\dot{H} + t\ddot{H} + \frac{2kc^2}{\alpha_0^2 e^{2Ht}}\right\} \tag{33}$$

The above equation (33) represents the Ricci curvature scalar in terms of the time-varying Hubble parameter. As previously explained, the time dependency of the Hubble parameter arises from its measurement based on the recession velocities and distances of cosmic objects at different cosmic comoving epochs. However, if the Hubble parameter is measured at a fixed point in spacetime or within a specific cosmic epoch, the equation (33) can be rewritten in terms of the Hubble constant, with its time derivatives vanishing, as follows:

$$R = -\frac{3H_0^2}{c^2} - 3\left\{\frac{3H_0^2}{c^2} + \frac{2k}{\left(\alpha_0 e^{H_0 t}\right)^2}\right\},$$

$$R = -\frac{12H_0^2}{c^2} - \frac{6k}{\left(\alpha_0 e^{H_0 t}\right)^2} \tag{34}$$

The negative sign of the Ricci curvature scalar, as determined above, arises from the contraction of the Riemann tensor using $R_{\mu\nu} = R^{\sigma}_{\mu\sigma\nu}$, as referenced in [63]. However, if the contraction $R_{\mu\nu} = R_{\mu\nu\sigma}^{\ \ \ \sigma} = -R^{\sigma}_{\mu\sigma\nu}$, as referenced in [3], were used instead, the value of R would be the same but with a positive sign. This is not a contradiction, as both approaches yield the same magnitude; the difference in sign simply reflects the choice of convention or the system from which the curvature is measured.

Importantly, whether considering the time-dependent case (33) or the time-independent case (34) of the Hubble parameter, the Ricci curvature scalar R remains finite for any finite value of the time variable. The only exception occurs at $t = -\infty$, which represents the sole instance of past incompleteness. However, this incompleteness is not related to any real domain of the time coordinate, which serves as the sole independent variable in the continuous function of the Ricci curvature scalar. Therefore, this past incompleteness is not a physical reality for this type of exponentially expanding cosmological solution but rather an unavoidable mathematical artifact, as highlighted by the Borde–Guth–Vilenkin (BGV) theorem [64]. Optionally, for a spatially flat universe ($k = 0$), equation (34) indicates a constant cosmic Ricci curvature scalar.

6. GENERAL EQUATION OF DISPLACEMENT AND LOGARITHMIC SPIRAL MOTION IN SPIRAL GALAXY

It is fascinating to realize that the equation for a logarithmic, or equiangular, spiral is an alternative expression of our time-based exponential general displacement equation. It is worth noting that, despite having a uniform linear velocity, an object can still experience continuously varying acceleration, as well as higher-order time derivatives of displacement, by continuously changing its orbital distance from its center. This is the key insight when drawing a correlation between the time-exponential displacement equation and the logarithmic spiral equation.

In a system following a spiral trajectory defined by an angle θ measured relative to its center, a growth or decay rate K, and a varying radius r, the equation of the logarithmic spiral is [65]:

$$r(\theta) = r_0 e^{K\theta} \tag{35}$$

In terms of t with uniform angular velocity $\dot{\theta}$, the above equation becomes:

$$r(t) = r_0 e^{K\dot{\theta}t} \tag{36}$$

In the equation above, $K = \tan\alpha$, where α is referred to as the pitch angle. This angle is defined as the constant angle between the tangent to the spiral at any given radial distance from the spiral's center and the tangent to the circle at the point of intersection with the same radius. For a logarithmic spiral, this angle remains constant, which is why logarithmic spirals are also called equiangular spirals. Consequently, the growth rate K is also constant. If we represent the time-exponential displacement equation (5) with the displacement being radial, it can be written as:

$$r = r_0 e^{\zeta t} \tag{37}$$

In the equation of the logarithmic spiral, the growth rate K and angular velocity $\dot{\theta}$ are both treated as constants, similar to the constant ζ in the case of radial displacement. By comparing equations (36) and (37), we can establish the equivalent relationship:

$$\zeta = K\dot{\theta} \tag{38}$$

Therefore, the logarithmic spiral displacement equation can be interpreted as a transformed two-dimensional displacement equation in a coplanar polar coordinate system. This transformation originates from the one-degree-of-freedom time-exponential displacement equation, which describes the motion of an object orbiting a fixed center.

In the Cartesian coordinate system, the equation $r = r_0 e^{\zeta t}$ can be re-expressed in the polar coordinate system as two distinct components using equation (35), as a matrix representation shown below:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\dot{\theta}t) & 0 \\ \sin(\dot{\theta}t) & 0 \end{bmatrix} \begin{bmatrix} r_0 e^{K\dot{\theta}t} \\ 0 \end{bmatrix} = \begin{bmatrix} r_0 e^{K\dot{\theta}t}\cos(\dot{\theta}t) \\ r_0 e^{K\dot{\theta}t}\sin(\dot{\theta}t) \end{bmatrix}.$$

Using the above relations and the fact that the locus of the coordinate (x, y) forms a circle , one can easily derive the relation presented in equation (36) as below:

$$r^2 = x^2 + y^2 = r_0^2 e^{2K\dot{\theta}t} \left\{\cos^2(\dot{\theta}t) + \sin^2(\dot{\theta}t)\right\},$$
$$r = r_0 e^{K\dot{\theta}t}.$$

For a non-periodic angular motion in which the angle continuously increases with time and is not limited to 4π, an object will follow a logarithmic spiral trajectory instead of a circular one .In terms of the measured angle considering the center as the origin of the polar coordinate system, the above equation becomes:

$$r(\theta) = r_0 e^{K\theta}, \qquad r(\theta) = r_0 e^{\tan\alpha\,\theta} \tag{39}$$

The above equation (39) is a general expression for a logarithmic spiral or equiangular spiral. The radial distance r progressively changes over time, following spiral trajectories.

Logarithmic spiral geometries naturally occur in numerous living and inanimate phenomena, including in spiral galaxies, seashells, hurricanes, and the growth patterns of plants, such as sunflower seeds and pinecones [66, 67, 68]. Notably, in the asymptotic limit, the ratio of successive radii in a logarithmic spiral converges to the same value as the ratio of consecutive Fibonacci numbers, namely the golden ratio or golden mean. These spirals are often observed in nature due to their efficient growth patterns, optimizing space and resources. The same geometric principles expressed mathematically by equations (36), (37), and (39) can be applied to describe the structure of spiral galaxies. The logarithmic spiral provides a reasonable approximation for the structure of galactic spiral arms [69, 70]. In these galaxies, the arms follow a logarithmic spiral pattern, giving rise to the term "Galactic spiral." Approximately 60% of galaxies in the local universe are spiral galaxies [71].

In spiral galaxies such as our MW and its closest neighbor, M31, the spiral arms form in regions where stars and their systems are sufficiently far from the galactic center. In these regions, the gravitational pull toward the center becomes negligible compared to the frictional drag experienced by the stars. This frictional drag causes the stars to orbit at nearly uniform velocities, regardless of their distance from the center. In this scenario, acceleration is achieved through the continual change in distance from the galactic center, which aids in the formation of spiral trajectories.

Moreover, the relatively homogeneous nature of the galaxy, where gravitational forces dominate locally but diminish at larger distances, results in the galaxy behaving continuum-mechanically, like a rotating disk. These combined phenomena lead to a fluid-mechanical analogy, where the spiral arms eject along an equiangular spiral trajectory, as described by equations (36), (37), and (39). It is important to note that dark matter plays a crucial role in forming gravitationally bound, stiff, and geometrically well-defined galaxies. This stands in contrast to the expected pattern of decreasing rotational velocity with distance from the galactic center, as predicted by the Newton-Keplerian model, which is inconsistent with observational data.

The logarithmic spiral trajectory is also consistent with explaining the radial migration of constituents, such as stars or star systems, which move outward from the more unstable central region to more stable locations, following a spiral path within their host galaxy. For example, the radial migration concept suggests that the Solar System, including Earth, may have originally formed comparatively closer to the Milky Way's central region, around 5 kpc from MW's center, and later migrated to its present location, approximately $8.5 \sim 9$ kpc from the galactic center, thereby avoiding any major destructive collisions [72].

7. RESULTS AND DISCUSSION

We began with a power series approximation of the displacement function, using time as the independent variable. From this, we derived the time exponential displacement relation, which aligns with large-distance observational cosmic motion. In contrast, the short-distance motions we experience in everyday life are simply lower-order approximations of this exponential relation, which is particularly significant for large-scale motion. This general time exponential displacement relation was directly applied to derive Hubble's relation, an equation describing cosmic diverging displacement and providing a plausible explanation for cosmic expansion. The recession of cosmic units due to cosmic expansion is significant for large-scale cosmic motion. In contrast, at shorter distances, the attractive gravitational force dominates.

We then derived a general equation for time exponential converging displacement in a manner similar to how the general equation for time exponential diverging displacement was formulated. This converging relation was then utilized to estimate in principle the time required for the upcoming collision between the Milky Way and Andromeda. The result obtained, ranging from 4.4 to 4.9 billion years, was highly consistent with the findings of 4 to 5 billion years from numerical simulations based on observational data.

A noteworthy application of the general diverging equation is to derive the subtle value of energy for a vacuum unit, considering it as quantum harmonically oscillated in a quasi-quantum approach. This gives a value of 1.1873×10^{-52} J, which is very close to the value determined from the general relativistic approach, $1.1247 \times 10^{-52}\,\mathrm{m}^{-2}$. Therefore, the coupling constant relating the vacuum unit's energy to the curvature of the vacuum has a value very close to unity. By relating the large-scale cosmic receding distances to the cosmic expansion scale factor, a general relativistic mathematical approach is reviewed to avoid a physical past incompleteness while addressing an unphysical mathematical singularity.

Lastly, the general time exponential displacement was applied to explain the spiraling motion of galactic units in spiral galaxies. This spiral motion can also account for the radial migration of a galaxy's constituents, providing support for the radial migration theory of the solar system.

Despite several successful applications of the general time exponential motion, it does have some limitations. During its derivation, the Maclaurin series was employed, which requires a fixed space-time coordinate origin around which the series expansion is performed. This necessitates assuming the Hubble parameter is constant to derive the time exponential equation. However, in any comoving coordinate system, the Hubble parameter must be time-dependent. This time dependency was addressed during the derivation and was considered in the context of reviewing the cosmic past incompleteness. Another limitation arises from the uncertainty during the final stages of galaxy merging. At this stage, tidal effects, frictional interactions between galaxy constituents, and the influence of nearby cosmic neighbors may impact the time estimation. However, the consistency of the estimated time for the Milky Way (MW) and Andromeda (M31) collision with results from numerical simulations suggests that our model, which neglects perturbations, is reasonable. This is because galactic motion is primarily dominated by their centers, and during galaxy merging, the likelihood of individual components colliding is very low.

8. CONCLUSION

The cosmos is infinite, so it's natural that no finite set of principles can fully explain its phenomena. It's also sensible to expect the universe to have infinite spatial and temporal dimensions. We reasonably assumed the infinite order differentiability of displacement with respect to time and subsequently formulated a time-exponential relation for displacement. This formulation is shown to be consistent with Hubble's law and provides a plausible explanation for the time dependency of the Hubble parameter. This relation proved to be valuable in understanding the infinite cosmic past and future, as well as in explaining the structure of spiral galaxies. Additionally, it provided insights into the enigmatic relationship between the quantum harmonically oscillating vacuum's zero-point energy and the cosmological constant. Furthermore, it was applied to estimate the timing of a significant cosmic event: the collision between the Milky Way and Andromeda.

Data Availability

The data that support the findings of this study are available within the article and the cited references.

Disclosure of Interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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ЗАКОН ГАББЛА ТА ЙОГО ЕКСПОНЕНЦІАЛЬНЕ УЗАГАЛЬНЕННЯ З КОСМОЛОГІЧНИМИ ЗАСТОСУВАННЯМИ

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Закон Габбла показує, як компоненти Всесвіту дотримуються загальних динамічних правил у космологічному масштабі. Хоча він найбільш відомий для опису розширення Всесвіту, загальне рівняння зміщення, отримане відповідно до цього закону, разом із загальним рівнянням збіжного зміщення, було застосовано для оцінки часу, що залишився до зіткнення Чумацького Шляху та Андромеди. Ця оцінка тісно узгоджується з результатами числового моделювання інших досліджень. Крім того, наслідки цього узагальненого рівняння дають цінне розуміння ключових космологічних загадок, включаючи зміну параметра Габбла з часом, неповноту космологічного минулого та незмінну таємницю зв'язку між тонким значенням космологічної константи та квантовою енергією нульової точки вакууму. Він також успішно пояснює структуру спіральних галактик.

Ключові слова: *закон Габбла; зіткнення галактик; космологічна стала; енергія вакууму; структура спіральної галактики*