East European Journal of Physics. 4. 141–156 (2025)
DOI: 10.26565/2312-4334-2025-4-12 ISSN 2312-4334

QUIESCENT SOLITONS IN MAGNETO-OPTIC WAVEGUIDES WITH NONLINEAR CHROMATIC DISPERSION AND KUDRYASHOV'S FORM OF SELF-PHASE MODULATION HAVING GENERALIZED TEMPORAL EVOLUTION

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Received July 1, 2025; revised September 18, 2025; accepted September 23, 2025

The article discusses how Kudryashov's proposed self-phase modulation scheme and nonlinear chromatic dispersion cause the evolution of quiescent optical solitons in magneto-optic waveguides. Provide a comprehensive understanding of the governing model; generalised temporal evolution is considered. The modified sub-ODE approach is employed to facilitate the recovery of such solitons. This leads to a complete range of optical solitons and the necessary conditions that must be met for these solitons to exist, which are also provided.

Keywords: *Solitons; Self-Phase Modulation; Integrability; Chromatic Dispersion*

PACS: 02.30.Jr; 05.45.Yv; 02.30.Ik; 47.40.Nm

1. INTRODUCTION

The three main features of how solitons move in optical waveguides and crystals are chromatic dispersion (CD), self-phase modulation (SPM), and how pulses change over time after they are introduced at the beginning of the waveguides [1–40]. Subsequently, these pulses evolve with temporally and spatially and thus serve as bit carriers for information transmission through such fibers across transcontinental and transoceanic distances. There are many hiccups one faces during the transmission of such pulses. One issue is the soliton clutter. Therefore in order to declutter these solitons, one considers magneto–optic waveguides. This leads to the interest in magneto–optic waveguides.

This work examines soliton propagation in magneto-optic waveguides characterised by nonlinear cross-dispersion and a particular kind of self-phase modulation, as postulated by Kudryashov. We preserve the temporal evolution of the pulses using a general framework known as generalised temporal evolution. The inconsistent characteristics of waveguides would cause the solitons to get impeded during their propagation via subterranean or underwater cables. This property will be examined in depth in the present research, facilitating a clear comprehension and retrieval of the quiescent optical solitons relevant to the model discussed in this work. This understanding will not only enhance the theoretical framework surrounding self-phase modulation but also offer practical suggestions for optimising soliton transmission in complex waveguide environments. By addressing these challenges, we aim to improve the reliability and efficiency of optical communication systems that use these solitonic structures.

The mathematical algorithm that would be selected is the modified version of the sub–ODE (ordinary differential equation) approach. This integration scheme would lead to the emergence of a wide spectrum of soliton solutions along with their respective classifications. The results are reconstructed and presented in the subsequent sections of the paper following a brief review of the model, its technical characteristics, and an explanation of the integration algorithm.

Cite as: E.M.E. Zayed, M. El–Shater, A.H. Arnous, O.M.K. Al-Dulaimi, F.M. Mohammed, I.Z. Chaloob, O. González–Gaxiola, A. Biswas, C.M.B. Dragomir, East Eur. J. Phys. 4, 141 (2025), https://doi.org/10.26565/2312-4334-2025-4-12

It must be noted that the model with the quintuple form of SPM has been addressed in the previous round for optical fibers [9]. However, the current work retrieves quiescent optical solitons for a different kind of optical waveguide, namely magneto-optic waveguides, by the implementation of the improved sub-ODE approach.

Unlike prior analyses of Kudryashov-type nonlinearities that focused on single-field optical fibers (including quintuple SPM in fibers) or on sixth/eighth-order single-field NLS equations, we study magneto–optic waveguides described by a coupled magneto–optic (MO) waveguides system with generalized nonlinear chromatic dispersion and generalized temporal evolution. This formulation introduces cross-SPM/CD terms and magneto-optic couplings (Q_1, Q_2) absent from fiber-only models, and enables the construction of new stationary families (bright, dark, straddled hyperbolic–rational, Jacobi elliptic, and Weierstrass) with explicit existence constraints in the MO setting. In short, the novelty lies in the medium and model (coupled MO system), the analytic framework (addendum to the improved sub-ODE within the MO model), and the solution classification with parameter regimes tailored to magneto–optic waveguides [10, 11].

1.1. Governing model

Zayed et al. [32] analysed the nonlinear Schrödinger equation (NLSE) with non-local nonlinearity, Kudryashov's general quintuple power law, generalised nonlinear chromatic dispersion (CD), and a generalised temporal evolution, expressed as follows:

$$i\left(\Psi^l\right)_t + a\left(|\Psi|^r\,\Psi^l\right)_{xx} + \left[b_1\,|\Psi|^{2m} + b_2\,|\Psi|^{2m+n} + b_3\,|\Psi|^{2m+n+p} + b_4\,|\Psi|^{2m+2n} + b_5\,|\Psi|^{2m+2n+p} + b_6\left(|\Psi|^p\right)_{xx}\right]\Psi^l = 0, \tag{1}$$

In this context, $\Psi^l(x,t)$ denotes a complex-valued function that characterises the generalised wave shape, with the condition that $l \geq 1$ and $i = \sqrt{-1}$. The parameters r, l, m, n, and p are positive real constants. The initial term relates to the generalised temporal evolution dictated by the evolution parameter l. The constant a represents the coefficient of the generalised nonlinear CD, defined by the nonlinear parameter r. The constants b_j $(j = 1, 2, 3, 4, 5)$ denote the coefficients of SPM arising from the nonlinear intensity-dependent refractive index framework, as examined by Kudryashov [1–3].

In magneto-optic waveguides, Eq. (1) separates into two components, as follows:

$$\begin{aligned} &i\left(\eta^l\right)_t + a_1\left(|\eta|^k\,\eta^l\right)_{xx} + [b_1\,|\eta|^{2m} + c_1\,|\eta|^{2m+n} + d_1\,|\eta|^{2m+n+p} + e_1\,|\eta|^{2m+2n} + f_1\,|\eta|^{2m+2n+p} + g_1\left(|\eta|^p\right)_{xx} \\ &+h_1\,|\rho|^{2m} + l_1\,|\rho|^{2m+n} + s_1\,|\rho|^{2m+n+p} + n_1\,|\rho|^{2m+2n} + p_1\,|\rho|^{2m+2n+p} + q_1\left(|\rho|^p\right)_{xx}]\eta^l = Q_1\rho^l, \end{aligned} \tag{2}$$

and

$$\begin{aligned} &i\left(\rho^l\right)_t + a_2\left(|\rho|^k\,\rho^l\right)_{xx} + [b_2\,|\rho|^{2m} + c_2\,|\rho|^{2m+n} + d_2\,|\rho|^{2m+n+p} + e_2\,|\rho|^{2m+2n} + f_2\,|\rho|^{2m+2n+p} + g_2\left(|\rho|^p\right)_{xx} \\ &h_2\,|\eta|^{2m} + l_2\,|\eta|^{2m+n} + s_2\,|\eta|^{2m+n+p} + n_2\,|\eta|^{2m+2n} + p_2\,|\eta|^{2m+2n+p} + q_2\left(|\eta|^p\right)_{xx}]\rho^l = Q_2\eta^l, \end{aligned} \tag{3}$$

Here, $\eta(x,t)$ and $\rho(x,t)$ are complex-valued functions representing the wave profiles, with $i = \sqrt{-1}$. The first terms in Eqs. (2) and (3) describe the generalized temporal evolution, characterized by the parameter $l \geq 1$. The constants a_j $(j = 1, 2)$ are the coefficients of nonlinear CD, governed by the parameter $k \geq 0$. The constants b_j, c_j, d_j, e_j, f_j, g_j, h_j, l_j, p_j,n_j, and q_j $(j = 1, 2)$ correspond SPM terms, structured according to the formulation introduced by Kudryashov [33,34]. Finally, Q_j $(j = 1, 2)$ denote the coefficients associated with the magneto–optic waveguides. The parameters m and n are rational numbers, not necessarily integers.

This work aims to resolve Eqs. (2) and (3) by an enhancement of the modified Sub-ODE approach. This methodology is used to derive several categories of soliton solutions, including dark soliton solutions, solitary soliton solutions, Jacobi's elliptic functions, Weierstrass elliptic functions, bright soliton solutions, and straddled soliton solutions. Consequently, a broad array of soliton solutions would arise.

This article is organised as follows: Section 2 presents the mathematical study. In Section 3, a modification of the modified sub-ODE method is implemented. Section 4 presents more findings. Conclusions are presented in Section 6.

2. MATHEMATICAL STRUCTURE

To resolve Eqs. (2) and (3), we suggest that the wave profiles exhibit the following forms:

$$\begin{aligned} \eta\left(x,t\right) &= \phi_1(x)e^{i\lambda t}, \\ \rho\left(x,t\right) &= \phi_2(x)e^{i\lambda t}, \end{aligned} \tag{4}$$

where $\phi_j(x)$ $(j = 1, 2)$ are real functions and λ is a constant representing the wave number. Substituting (4) into Equations (2) and (3), we obtain:

$$\begin{aligned}
&-\lambda l\phi_1^l(x) + a_1(k+l)(k+l-1)\phi_1^{k+l-2}(x)\phi_1'^2(x) + a_1(k+l)\phi_1^{k+l-1}(x)\phi_1''(x)\\
&+b_1\phi_1^{2m+l}(x) + c_1\phi_1^{2m+n+l}(x) + d_1\phi_1^{2m+n+p+l}(x) + e_1\phi_1^{2m+2n+l}(x) + f_1\phi_1^{2m+2n+p+l}(x)\\
&+g_1p(p-1)\phi_1^{p-2+l}(x)\phi_1'^2(x) + g_1p\phi_1^{p-1+l}(x)\phi_1''(x) + h_1\phi_2^{2m}(x)\phi_1^l(x) + l_1\phi_2^{2m+n}(x)\phi_1^l(x)\\
&+s_1\phi_2^{2m+n+p}(x)\phi_1^l(x) + n_1\phi_2^{2m+2n}(x)\phi_1^l(x) + p_1\phi_2^{2m+2n+p}(x)\phi_1^l(x) + q_1p(p-1)\phi_2^{p-2+l}(x)\phi_2'^2(x)\\
&+q_1p\phi_1^{p-1+l}(x)\phi_1''(x) = Q_1\phi_2^l(x),
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
&-\lambda l\phi_2^l(x) + a_2(k+l)(k+l-1)\phi_2^{k+l-2}(x)\phi_2'^2(x) + a_2(k+l)\phi_2^{k+l-1}(x)\phi_2''(x)\\
&+b_2\phi_2^{2m+l}(x) + c_2\phi_2^{2m+n+l}(x) + d_2\phi_2^{2m+n+p+l}(x) + e_2\phi_2^{2m+2n+l}(x) + f_2\phi_2^{2m+2n+p+l}(x)\\
&+g_2p(p-1)\phi_2^{p-2+l}(x)\phi_2'^2(x) + g_2p\phi_2^{p-1+l}(x)\phi_2''(x) + h_2\phi_1^{2m}(x)\phi_2^l(x)\\
&+l_2\phi_1^{2m+n}(x)\phi_2^l(x) + s_2\phi_1^{2m+n+p}(x)\phi_2^l(x) + n_2\phi_1^{2m+2n}(x)\phi_2^l(x) + p_2\phi_1^{2m+2n+p}(x)\phi_2^l(x)\\
&+q_2p(p-1)\phi_1^{p-2}(x)\phi_1'^2(x)\phi_2^l(x) + q_2p\phi_1^{p-1}(x)\phi_1''(x)\phi_2^l(x) = Q_2\phi_1^l(x),
\end{aligned} \tag{6}$$

Now, for the sake of simplicity, let us put

$$\phi_2(x) = \chi\phi_1(x), \tag{7}$$

where χ is a nonzero constant and $\chi \neq 1$. Eqs. (5) and (6) became:

$$\begin{aligned}
&-(\lambda l + Q_1\chi^l)\phi_1^l(x) + a_1(k+l)(k+l-1)\phi_1^{k+l-2}(x)\phi_1'^2(x)\\
&+a_1(k+l)\phi_1^{k+l-1}(x)\phi_1''(x) + \left(b_1 + h_1\chi^{2m}\right)\phi_1^{2m+l}(x)\\
&+\left(c_1 + l_1\chi^{2m+n}\right)\phi_1^{2m+n+l}(x) + \left(d_1 + s_1\chi^{2m+n+p}\right)\phi_1^{2m+n+p+l}(x)\\
&+\left(e_1 + n_1\chi^{2m+2n}\right)\phi_1^{2m+2n+l}(x) + \left(f_1 + p_1\chi^{2m+2n+p}\right)\phi_1^{2m+2n+p+l}(x)\\
&+\left(g_1 + q_1\chi^p\right)p(p-1)\phi_1^{p+l-2}(x)\phi_1'^2(x) + \left(g_1 + q_1\chi^p\right)p\phi_1^{p+l-1}(x)\phi_1''(x) = 0,
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
&-(\lambda l\chi^l + Q_2)\phi_1^l(x) + a_2(k+l)(k+l-1)\chi^{k+l}\phi_1^{k+l-2}(x)\phi_1'^2(x)\\
&+a_2(k+l)\chi^{k+l}\phi_1^{k+l-1}(x)\phi_1''(x) + \left(b_2\chi^{2m+l} + h_2\chi^l\right)\phi_1^{2m+l}(x)\\
&+\left(c_2\chi^{2m+2n+l} + l_2\chi^l\right)\phi_1^{2m+n+l}(x) + \left(d_2\chi^{2m+n+p+l} + s_2\chi^l\right)\phi_1^{2m+n+p+l}(x)\\
&+\left(e_2\chi^{2m+2n+l} + n_2\chi^l\right)\phi_1^{2m+2n+l}(x) + \left(f_2\chi^{2m+2n+p+l} + p_2\chi^l\right)\phi_1^{2m+2n+p+l}(x)\\
&+\left(g_2\chi^{p+l} + q_2\chi^l\right)p(p-1)\phi_1^{p+l-2}(x)\phi_1'^2(x) + \left(g_2\chi^{p+l} + q_2\chi^l\right)p\phi_1^{p+l-1}(x)\phi_1''(x) = 0,
\end{aligned} \tag{9}$$

Eqs. (8) and (9) are equivalent along with constraints conditions:

$$\begin{aligned}
\lambda l + Q_1\chi^l &= \lambda l\chi^l + Q_2,\\
a_1 &= a_2\chi^{k+l}\\
b_1 + h_1\chi^{2m} &= b_2\chi^{2m+l} + h_2\chi^l,\\
c_1 + l_1\chi^{2m+n} &= c_2\chi^{2m+2n+l} + l_2\chi^l,\\
d_1 + s_1\chi^{2m+n+p} &= d_2\chi^{2m+n+p+l} + s_2\chi^l,\\
e_1 + n_1\chi^{2m+n} &= e_2\chi^{2m+2n+l} + n_2\chi^l,\\
f_1 + p_1\chi^{2m+2n+p} &= f_2\chi^{2m+2n+p+l} + p_2\chi^l,\\
g_1 + q_1\chi^p &= g_2\chi^{p+l} + q_2\chi^l.
\end{aligned} \tag{10}$$

Solving Eq. (8), let $k = 6m, p = 4m, n = 2m$ then Eq.(8) is now

$$\begin{aligned}
&-(\lambda l + Q_1\chi^l) + a_1(6m+l)(6m+l-1)\phi_1^{6m-2}(x)\phi_1'^2(x)\\
&+a_1(6m+l)\phi_1^{6m-1}(x)\phi_1''(x) + \left(b_1 + h_1\chi^{2m}\right)\phi_1^{2m}(x)\\
&+\left(c_1 + l_1\chi^{4m}\right)\phi_1^{4m}(x) + \left(d_1 + s_1\chi^{8m}\right)\phi_1^{8m}(x)\\
&+\left(e_1 + n_1\chi^{6m}\right)\phi_1^{6m}(x) + \left(f_1 + p_1\chi^{10m}\right)\phi_1^{10m}(x)\\
&+\left(g_1 + q_1\chi^{4m}\right)(4m)(4m-1)\phi_1^{4m-2}(x)\phi_1'^2(x) + \left(g_1 + q_1\chi^{4m}\right)(4m)\phi_1^{4m-1}(x)\phi_1''(x) = 0,
\end{aligned} \tag{11}$$

Balancing $\phi_1^{10m}(x)$ with $\phi_1^{6m-1}(x)\phi_1''(x)$ in Eq.(11) gives $N = \frac{1}{2m}$, $m \neq 0$. In addition, considering the transformation

$$\phi_1(x) = V^{\frac{1}{2m}}(x), \tag{12}$$

where $V(x)$ denotes a novel function. By putting (12) into Eq. (11), we have

$$\begin{aligned}
&\Phi_1 + \Phi_2V(x)V'^2(x) + \Phi_3V^2(x)V''(x) + \Phi_4V(x) + \Phi_5V^2(x) + \Phi_6V^3(x)\\
&+\Phi_7V^4(x) + \Phi_8V^5(x) + \Phi_9V'^2(x) + \Phi_{10}V(x)V''(x) = 0,
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
\Phi_1 &= -(\lambda l + Q_1 \chi^l), \\
\Phi_2 &= \tfrac{a_1}{2m}(\tfrac{4m+l}{2m})(6m+l), \\
\Phi_3 &= \tfrac{a_1}{2m}(6m+l), \\
\Phi_4 &= b_1 + h_1 \chi^{2m}, \\
\Phi_5 &= c_1 + l_1 \chi^{4m}, \\
\Phi_6 &= e_1 + n_1 \chi^{6m}, \\
\Phi_7 &= d_1 + s_1 \chi^{8m}, \\
\Phi_8 &= f_1 + p_1 \chi^{10m}, \\
\Phi_9 &= 2\left(g_1 + q_1 \chi^{4m}\right), \\
\Phi_{10} &= 2\left(g_1 + q_1 \chi^{4m}\right).
\end{aligned} \tag{14}$$

Next, we will employ the integration method explained below to generate the solitons of Eqs. (2) and (3).

3. AN ADDENDUM TO THE IMPROVED SUB-ODE METODOLOGY

It is presumed that Eq. (13) has a formal solution [35–38]:

$$V(x) = \sum_{s=0}^{N} A_s \, [H(x)]^s \; . \tag{15}$$

The constants A_s, for $s = 0, 1, 2, \ldots, N$ are specified, with $A_N \neq 0$, and the condition $H(x)$ is satisfied:

$$H'^2(x) = A\, H^{2-2\alpha}(x) + B\, H^{2-\alpha}(x) + C\, H^2(x) + D\, H^{2+\alpha}(x) + E\, H^{2+2\alpha}(x). \tag{16}$$

In this particular case, $E, D, C, B,$ and A are constants, whereas α is a positive integer. Given $D[\phi(x)] = N$, then $D[\phi'(x)] = N + \alpha$, $D[\phi''(x)] = N + 2\alpha$, and so $D[\phi^{(r)}(x)] = N + r\alpha$. Therefore, $D[\phi^{(r)}(x)\phi^s(x)] = (s+1)N + \alpha r$. When the highest derivative $V''(x)V^2(x)$ is balanced with the nonlinear term $V^5(x)$ in Eq. (13), the result is:

$$3N + 2\alpha = 5N \implies N = \alpha. \tag{17}$$

With $\alpha = 1$ and therefore $N = 1$, Eq. (15) becomes:

$$V(x) = A_0 + A_1 H(x), \tag{18}$$

given that A_0 and A_1 are constants with $A_1 \neq 0$, $H(\xi)$ admits:

$$H'^2(x) = A\; + B\, H(x) + C\, H^2(x) + D\, H^3(x) + E\, H^4(x), \quad E \neq 0. \tag{19}$$

Substituting Eqs. (18) and (19) into Eq. (13) and grouping all terms of $[H(x)]^l [H]'^f$, where $l = 0, 1, 2, \ldots, 5$ and $f = 0, 1$ to zero yields:

$$\begin{cases}
H^5(x) : \Phi_8 A_1^5 + (\Phi_2 + 2\Phi_3) E A_1^3 = 0, \\
H^4(x) : (\Phi_2 + 4\Phi_3) E A_0 A_1^2 + \Phi_7 A_1^4 + \Phi_9 A_1^2 E + \Phi_2 A_1^3 D + 5\Phi_8 A_0 A_1^4 + \frac{3}{2} D \Phi_3 A_1^3 + 2\Phi_{10} A_1^2 E = 0, \\
H^3(x) : \Phi_6 A_1^3 + 10\Phi_8 A_0^2 A_1^3 + (\Phi_2 + \Phi_3) C A_1^3 + 2\Phi_3 E A_0^2 A_1 + D\Phi_2 A_0 A_1^2 + 3D\Phi_3 A_0 A_1^2 \\
\qquad + 4\Phi_7 A_0 A_1^3 + \frac{3}{2} D \Phi_{10} A_1^2 + D\Phi_9 A_1^2 + 2E\Phi_{10} A_0 A_1 = 0, \\
H^2(x) : \Phi_5 A_1^2 + 2\Phi_3 C A_0 A_1^2 + \frac{3}{2}\Phi_3 D A_0^2 A_1 + \frac{3}{2} D \Phi_{10} A_0 A_1 + \Phi_2 C A_0 A_1^2 + \Phi_2 A_1^3 B + \Phi_9 A_1^2 C \\
\qquad + \Phi_{10} C A_1^2 + \frac{1}{2}\Phi_3 B A_1^3 + 3\Phi_6 A_0 A_1^2 + 6\Phi_7 A_0^2 A_1^2 + 10\Phi_8 A_0^3 A_1^2 = 0, \\
H(x) : \Phi_4 A_1 + B A_0 A_1^2 (\Phi_2 + \Phi_3) + C\Phi_3 A_0^2 A_1 + \Phi_{10} A_0 A_1 C + \Phi_2 A_1^3 A + B\Phi_9 A_1^2 \\
\qquad + 3\Phi_6 A_0^2 A_1 + 2\Phi_5 A_0 A_1 + 4\Phi_7 A_0^3 A_1 + 5\Phi_8 A_0^4 A_1 + \frac{1}{2}\Phi_{10} A_1^2 B = 0, \\
H^0(x) : \Phi_9 A A_1^2 + \Phi_7 A_0^4 + \Phi_8 A_0^5 + \Phi_4 A_0 + \Phi_5 A_0^2 + \Phi_6 A_0^3 + \Phi_1 + \Phi_2 A_0 A_1^2 A + \frac{1}{2}\Phi_3 B A_0^2 A_1 \\
\qquad + \frac{1}{2}\Phi_{10} B A_0 A_1 = 0.
\end{cases} \tag{20}$$

We will now examine each of the following sets:

Set–1: Considering $D = B = A = 0$ in Eq. (20), the results achieved are:

$$A_0 = A_0, \quad A_1 = \sqrt{-\frac{E\,(\Phi_2 + 2\Phi_3)}{\Phi_8}}, \tag{21}$$

and

$$\begin{aligned}\Phi_1 &= -\frac{A_0^2\Phi_9\left(CA_1^2+EA_0^2\right)}{A_1^2},\\ \Phi_4 &= -\frac{A_0\left(C\Phi_2A_0A_1^2+E\Phi_2A_0^3-CA_1^2(2\Phi_9+\Phi_{10})-2EA_0^2(2\Phi_9+\Phi_{10})\right)}{A_1^2},\\ \Phi_5 &= \frac{CA_0A_1^2(2\Phi_2+\Phi_3)+2EA_0^3(2\Phi_2+\Phi_3)-CA_1^2(\Phi_9+\Phi_{10})-6EA_0^2(\Phi_9+2\Phi_{10})}{A_1^2},\\ \Phi_6 &= -\frac{CA_1^2(\Phi_2+\Phi_3)+6EA_0^2(\Phi_2+\Phi_3)-2EA_0^2(2\Phi_9+3\Phi_{10})}{A_1^2},\\ \Phi_7 &= \frac{2EA_0^2(2\Phi_2+3\Phi_3)-E(\Phi_9+2\Phi_{10})}{A_1^2},\end{aligned} \tag{22}$$

given $C > 0$ and $E < 0$, the solutions represent the bright solitons, which are:

$$\eta(x,t) = \varepsilon\left[A_0 + \sqrt{\frac{C\,(\Phi_2+2\Phi_3)}{\Phi_8}}\operatorname{sech}\sqrt{C}\,x\right]^{\frac{1}{2m}} e^{i\lambda t}, \tag{23}$$

and

$$\rho(x,t) = \chi\varepsilon\left[A_0 + \sqrt{\frac{C\,(\Phi_2+2\Phi_3)}{\Phi_8}}\operatorname{sech}\sqrt{C}\,x\right]^{\frac{1}{2m}} e^{i\lambda t}, \tag{24}$$

where $\Phi_8(\Phi_2 + 2\Phi_3) > 0$.

Set–2: Considering $B = D = 0$ and $A = \frac{C^2}{4E}$ are replaced into the algebraic equations (20), the results achieved are:

$$A_0 = 0,\quad A_1 = \sqrt{-\frac{E\,(\Phi_9+2\Phi_{10})}{\Phi_7}}, \tag{25}$$

and

$$\begin{aligned}\Phi_1 &= -\frac{A_1^2\Phi_9C^2}{4E},\\ \Phi_2 &= -\frac{\Phi_8A_1^2+2E\Phi_3}{E},\\ \Phi_4 &= \frac{A_1^2C^2\left(2E\Phi_3+\Phi_8A_1^2\right)}{4E^2},\\ \Phi_5 &= -C\,(\Phi_9+\Phi_{10}),\\ \Phi_6 &= \frac{C\left(\Phi_8A_1^2+E\Phi_3\right)}{E},\end{aligned} \tag{26}$$

where $E\Phi_7(\Phi_9 + 2\Phi_{10}) < 0$. Hence, the dark solitons take the form of:

$$\eta(x,t) = \varepsilon\left[\sqrt{\frac{C\,(\Phi_9+2\Phi_{10})}{2\Phi_7}}\tanh\left(\sqrt{-\frac{C}{2}}\,x\right)\right]^{\frac{1}{2m}} e^{i\lambda t}, \tag{27}$$

and

$$\rho(x,t) = \chi\varepsilon\left[\sqrt{\frac{C\,(\Phi_9+2\Phi_{10})}{2\Phi_7}}\tanh\left(\sqrt{-\frac{C}{2}}\,x\right)\right]^{\frac{1}{2m}} e^{i\lambda t}, \tag{28}$$

provided $C < 0$, $\Phi_7(\Phi_9 + 2\Phi_{10}) < 0$, $\varepsilon = \pm 1$.

Set–3: Considering $B = D = 0$ and $A = \frac{e_1C^2}{E}$ are replaced into the algebraic equations (20), the results achieved are:

$$A_0 = 0,\quad A_1 = A_1, \tag{29}$$

and

$$\begin{aligned}\Phi_1 &= -\frac{C^2e_1\Phi_9A_1^2}{E},\\ \Phi_3 &= -\frac{\Phi_8A_1^2+E\Phi_2}{2E},\\ \Phi_4 &= -\frac{C^2e_1\Phi_2A_1^2}{E},\\ \Phi_5 &= -C\,(\Phi_9+\Phi_{10}),\\ \Phi_6 &= -\frac{CE\Phi_2-C\Phi_8A_1^2}{2E},\\ \Phi_7 &= -\frac{E(\Phi_9+2\Phi_{10})}{A_1^2},\end{aligned} \tag{30}$$

assuming that e_1 remains constant, the following scenarios are derived:

(I) Considering $e_1 = \frac{m_1^2(m_1^2-1)}{(2m_1^2-1)^2}$, A is given by $\frac{C^2m_1^2(m_1^2-1)}{E(2m_1^2-1)^2}$, and $0 < m_1 < 1$, the Jacobi elliptic solutions are derived:

$$\eta(x,t) = \varepsilon\left[A_1\left(-\frac{Cm_1^2}{E(2m_1^2-1)}\right)^{\frac{1}{2}} \operatorname{cn}\left(\sqrt{\frac{C}{2m_1^2-1}}\, x,\ m_1\right)\right]^{\frac{1}{2m}} e^{i\lambda t}, \tag{31}$$

and

$$\rho(x,t) = \chi\varepsilon\left[A_1\left(-\frac{Cm_1^2}{E(2m_1^2-1)}\right)^{\frac{1}{2}} \operatorname{cn}\left(\sqrt{\frac{C}{2m_1^2-1}}\, x,\ m_1\right)\right]^{\frac{1}{2m}} e^{i\lambda t}, \tag{32}$$

provided $C\ (2m_1^2-1) > 0$, $E < 0$.

(II) When $e_1 = \frac{(1-m_1^2)}{(2-m_1^2)^2}$, A is given by $\frac{C^2(1-m_1^2)}{E(2-m_1^2)^2}$, and $0 < m_1 < 1$, the Jacobi elliptic solutions are obtained:

$$\eta(x,t) = \varepsilon\left[A_1\left(-\frac{C}{E(2-m_1^2)}\right)^{\frac{1}{2}} \operatorname{dn}\left(\sqrt{\frac{C}{2-m_1^2}}\, x,\ m_1\right)\right]^{\frac{1}{2m}} e^{i\lambda t}, \tag{33}$$

and

$$\rho(x,t) = \chi\varepsilon\left[A_1\left(-\frac{C}{E(2-m_1^2)}\right)^{\frac{1}{2}} \operatorname{dn}\left(\sqrt{\frac{C}{2-m_1^2}}\, x,\ m_1\right)\right]^{\frac{1}{2m}} e^{i\lambda t}, \tag{34}$$

assuming $C > 0$, $E < 0$. The bright solitons are given when m_1 approaches 1^- in the Eqs. (31) - (34):

$$\eta(x,t) = \varepsilon\left[A_1\sqrt{-\frac{C}{E}}\operatorname{sech}\left(\sqrt{C}\, x\right)\right]^{\frac{1}{2m}} e^{i\lambda t}, \tag{35}$$

and

$$\rho(x,t) = \chi\varepsilon\left[A_1\sqrt{-\frac{C}{E}}\operatorname{sech}\left(\sqrt{C}\, x\right)\right]^{\frac{1}{2m}} e^{i\lambda t}. \tag{36}$$

(III) When $e_1 = \frac{m_1^2}{(m_1^2+1)^2}$ and $A = \frac{C^2m_1^2}{E(m_1^2+1)^2}$ with $0 < m_1 < 1$, the resulting Jacobi elliptic solutions are:

$$\eta(x,t) = \varepsilon\left[A_1\left(-\frac{Cm_1^2}{E(m_1^2+1)}\right) \operatorname{sn}\left(\sqrt{-\frac{C}{m_1^2+1}}\, x, m_1\right)\right]^{\frac{1}{2m}} e^{i\lambda t}, \tag{37}$$

and

$$\rho(x,t) = \chi\varepsilon\left[A_1\left(-\frac{Cm_1^2}{E(m_1^2+1)}\right) \operatorname{sn}\left(\sqrt{-\frac{C}{m_1^2+1}}\, x, m_1\right)\right]^{\frac{1}{2m}} e^{i\lambda t}, \tag{38}$$

provided $E > 0$ and $C < 0$. The dark solitons, when m_1 approaches 1^- in Eqs. (37) and (38), are:

$$\eta(x,t) = \varepsilon\left[A_1\sqrt{-\frac{C}{2E}}\tanh\left(\sqrt{-\frac{C}{2}}\, x\right)\right]^{\frac{1}{2m}} e^{i\lambda t}, \tag{39}$$

and

$$\rho(x,t) = \chi\varepsilon\left[A_1\sqrt{-\frac{C}{2E}}\tanh\left(\sqrt{-\frac{C}{2}}\, x\right)\right]^{\frac{1}{2m}} e^{i\lambda t}. \tag{40}$$

Set–4: Considering $A = B = 0$, $C > 0$, and $D^2 = 4CE$ are replaced into the algebraic equations (20), the results achieved are:

$$A_0 = 0, \quad A_1 = A_1, \tag{41}$$

and

$$\begin{aligned}
\Phi_2 &= -\frac{\Phi_8 A_1^2+2E\Phi_3}{E},\\
\Phi_5 &= \frac{C\left(2\sqrt{CE}\Phi_8 A_1^3+\sqrt{CE}\Phi_3 A_1 E+\Phi_7 E A_1^2+E^2\Phi_{10}\right)}{E^2},\\
\Phi_6 &= -\frac{\sqrt{CE}\left(3\sqrt{CE}\Phi_8 A_1^3+\sqrt{CE}\Phi_3 A_1 E+2\Phi_7 E A_1^2+E^2\Phi_{10}\right)}{E^2 A_1},\\
\Phi_9 &= -\frac{2\sqrt{CE}\Phi_8 A_1^3+\sqrt{CE}\Phi_3 A_1 E+\Phi_7 E A_1^2+2E^2\Phi_{10}}{E^2 A_1},\\
\Phi_1 &= \Phi_4 = 0.
\end{aligned} \tag{42}$$

Therefore, the dark solitons become:

$$\eta(x,t) = \varepsilon\left[\frac{A_1}{2}\sqrt{\frac{C}{E}}\left(1+\tanh\frac{\sqrt{C}}{2}\,x\right)\right]^{\frac{1}{2m}} e^{i\lambda t}, \tag{43}$$

and

$$\rho(x,t) = \chi\varepsilon\left[\frac{A_1}{2}\sqrt{\frac{C}{E}}\left(1+\tanh\frac{\sqrt{C}}{2}\,x\right)\right]^{\frac{1}{2m}} e^{i\lambda t}. \tag{44}$$

The singular solitons emerge as:

$$\eta(x,t) = \varepsilon\left[-\frac{A_1}{2}\sqrt{\frac{C}{E}}\left(1+\coth\frac{\sqrt{C}}{2}\,x\right)\right]^{\frac{1}{2m}} e^{i\lambda t}, \tag{45}$$

and

$$\rho(x,t) = \chi\varepsilon\left[-\frac{A_1}{2}\sqrt{\frac{C}{E}}\left(1+\coth\frac{\sqrt{C}}{2}\,x\right)\right]^{\frac{1}{2m}} e^{i\lambda t}. \tag{46}$$

The bright soliton solutions are:

$$\eta(x,t) = \left[\frac{A_1}{2}\sqrt{\frac{C}{E}}\frac{2\operatorname{sech}^2\frac{\sqrt{C}}{2}\,x}{[1+\tanh^2\frac{\sqrt{C}}{2}\,x]}\right]^{\frac{1}{2m}} e^{i\lambda t}, \tag{47}$$

and

$$\rho(x,t) = \chi\left[\frac{A_1}{2}\sqrt{\frac{C}{E}}\frac{2\operatorname{sech}^2\frac{\sqrt{C}}{2}\,x}{[1+\tanh^2\frac{\sqrt{C}}{2}\,x]}\right]^{\frac{1}{2m}} e^{i\lambda t}. \tag{48}$$

Then the singular solitons are:

$$\eta(x,t) = \left[\frac{A_1}{2}\sqrt{\frac{C}{E}}\frac{2\operatorname{csch}^2\frac{\sqrt{C}}{2}\,x}{[1+\coth^2\frac{\sqrt{C}}{2}\,x]}\right]^{\frac{1}{2m}} e^{i\lambda t}, \tag{49}$$

and

$$\rho(x,t) = \chi\left[\frac{A_1}{2}\sqrt{\frac{C}{E}}\frac{2\operatorname{csch}^2\frac{\sqrt{C}}{2}\,x}{[1+\coth^2\frac{\sqrt{C}}{2}\,x]}\right]^{\frac{1}{2m}} e^{i\lambda t}. \tag{50}$$

Set–5: Considering $A = 0$, $B = \frac{8C^2}{27D}$, $E = \frac{D^2}{4C}$, and $C < 0$ are replaced into the algebraic equations (20), the results achieved are:

$$A_0 = 0,\ \ A_1 = A_1, \tag{51}$$

and

$$\begin{aligned}
\Phi_1 &= 0,\\
\Phi_2 &= -\frac{6CD\Phi_3 A_1+4C\Phi_7 A_1^2+D^2\Phi_9+2D^2\Phi_{10}}{4CDA_1},\\
\Phi_4 &= -\frac{4C^2 A_1(2\Phi_9+\Phi_{10})}{27D},\\
\Phi_5 &= \frac{C(8CD\Phi_3 A_1+8C\Phi_7 A_1^2-25D^2\Phi_9-23D^2\Phi_{10})}{27D^2},\\
\Phi_6 &= \frac{2CD\Phi_3 A_1+4C\Phi_7 A_1^2-3D^2\Phi_9-2D^2\Phi_{10}}{4CA_1}.
\end{aligned} \tag{52}$$

Consequently, the corresponding solutions employing hyperbolic functions are:

$$\eta(x,t) = \left[A_1\left(-\frac{8C\tanh^2\frac{\sqrt{-3C}}{2}\,x}{3D\left(3+\tanh\frac{\sqrt{-3C}}{2}\,x\right)}\right)\right]^{\frac{1}{2m}} e^{i\lambda t}, \tag{53}$$

and

$$\rho(x,t)=\chi\left[A_1\left(-\frac{8C\tanh^2\frac{\sqrt{-3C}}{2}x}{3D\left(3+\tanh\frac{\sqrt{-3C}}{2}x\right)}\right)\right]^{\frac{1}{2m}}e^{i\lambda t}. \tag{54}$$

Also,

$$\eta(x,t)=\left[A_1\left(-\frac{8C\coth^2\frac{\sqrt{-3C}}{2}x}{3D\left(3+\coth\frac{\sqrt{-3C}}{2}x\right)}\right)\right]^{\frac{1}{2m}}e^{i\lambda t}, \tag{55}$$

and

$$\rho(x,t)=\chi\left[A_1\left(-\frac{8C\coth^2\frac{\sqrt{-3C}}{2}x}{3D\left(3+\coth\frac{\sqrt{-3C}}{2}x\right)}\right)\right]^{\frac{1}{2m}}e^{i\lambda t}, \tag{56}$$

provided $DCA_1<0$.

Set–6: When $A=B=0$ and $E=\frac{D^2}{4C}-C$ are replaced into the algebraic equations (20), the results achieved are derived in the following cases:

$$A_0=0,\quad A_1=A_1, \tag{57}$$

and

$$\begin{aligned}
&\Phi_5=-C(\Phi_9+\Phi_{10}),\\
&\Phi_6=-\frac{2CA_1(\Phi_2+\Phi_3)+D(2\Phi_9+3\Phi_{10})}{2A_1},\\
&\Phi_7=\frac{C(-4D\Phi_2A_1-6D\Phi_3A_1+4C\Phi_9+8C\Phi_{10}-D^2(\Phi_9+2\Phi_{10})}{4CA_1^2},\\
&\Phi_8=\frac{(2C^2-D^2)(\Phi_2+2\Phi_3)}{4CA_1^2},\\
&\Phi_1=\Phi_4=0.
\end{aligned} \tag{58}$$

As a result, the bright solitons become:

$$\eta(x,t)=\varepsilon A_1\left[\frac{1}{\cosh\sqrt{C}\,x-\frac{D}{2C}}\right]^{\frac{1}{2m}}e^{i\lambda t}, \tag{59}$$

and

$$\rho(x,t)=\chi A_1\left[\frac{1}{\cosh\sqrt{C}\,x-\frac{D}{2C}}\right]^{\frac{1}{2m}}e^{i\lambda t}, \tag{60}$$

where $C>0$.

Set–7: Considering $B=D=0$ are replaced into the algebraic equations (20), the results achieved are:

$$A_0=0,\quad A_1=\sqrt{-\frac{(\Phi_2+2\Phi_3)E}{\Phi_8}}, \tag{61}$$

and

$$\begin{aligned}
&\Phi_1=-A\Phi_9A_1^2,\\
&\Phi_4=-A\Phi_2A_1^2,\\
&\Phi_5=-C(\Phi_9+\Phi_{10})\\
&\Phi_6=-C(\Phi 2+\Phi_3),\\
&\Phi_7=-\frac{E(\Phi_9+2\Phi_{10})}{A_1^2},
\end{aligned} \tag{62}$$

where $E\Phi_8(\Phi_2+2\Phi_3)<0$. In view of this, the following are the four solutions that take the form of Weierstrass elliptic functions:

(I)

$$\eta(x,t)=\varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)}{E\Phi_8}}\left(\wp\left[(x),g_2,g_3\right]-\frac{C}{3}\right)\right]^{\frac{1}{4m}}e^{i\lambda t}, \tag{63}$$

and

$$\rho(x,t)=\chi\varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)}{E\Phi_8}}\left(\wp\left[(x),g_2,g_3\right]-\frac{C}{3}\right)\right]^{\frac{1}{4m}}e^{i\lambda t}. \tag{64}$$

(II)

$$\eta(x,t)=\varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)E}{\Phi_8}}\left(\frac{A}{\wp[(x),g_2,g_3]-\frac{C}{3}}\right)\right]^{\frac{1}{4m}}e^{i\lambda t}, \tag{65}$$

and

$$\rho(x,t) = \chi\varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)E}{\Phi_8}}\left(\frac{A}{\wp[(x),g_2,g_3]-\frac{C}{3}}\right)\right]^{\frac{1}{4m}} e^{i\lambda t}, \tag{66}$$

where the solutions of the Weierstrass elliptic function (63)-(66) are characterised by the invariants g_2 and g_3, which are determined by the following equation:

$$g_2 = \frac{4C^2-12AE}{3} \text{ and } g_3 = \frac{4C(-2C^2+9AE)}{27}. \tag{67}$$

(III)

$$\eta(x,t) = \varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)E}{9\Phi_8}}\left(\frac{6\sqrt{A}\wp[(x),g_2,g_3]+C\sqrt{A}}{3\wp'[(x),g_2,g_3]}\right)\right]^{\frac{1}{4m}} e^{i\lambda t},\ A>0, \tag{68}$$

and

$$\rho(x,t) = \chi\varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)E}{9\Phi_8}}\left(\frac{6\sqrt{A}\wp[(x),g_2,g_3]+C\sqrt{A}}{3\wp'[(x),g_2,g_3]}\right)\right]^{\frac{1}{4m}} e^{i\lambda t},\ A>0. \tag{69}$$

Set–8: Considering $B = D = 0$ and $A = \frac{5C^2}{36E}$ are replaced into the algebraic equations (20), the results achieved are:

$$A_0 = 0, \quad A_1 = \sqrt{-\frac{(\Phi_9+2\Phi_{10})E}{\Phi_7}}, \tag{70}$$

and

$$\begin{aligned} \Phi_1 &= -\frac{5C^2\Phi_9A_1^2}{36E}, \\ \Phi_2 &= -\frac{\Phi_8A_1^2+2E\Phi_3}{E}, \\ \Phi_4 &= -\frac{(5\Phi_8A_1^2+2E\Phi_3)C^2A_1^2}{36E^2}, \\ \Phi_5 &= -C(\Phi_9+\Phi_{10}) \\ \Phi_6 &= \frac{C(\Phi_8A_1^2+E\Phi_3)}{E}, \end{aligned} \tag{71}$$

provided $E > 0, \Phi_7(\Phi_9 + 2\Phi_{10}) < 0$, $\varepsilon = \pm 1$. In view of this, the following are the two solutions that take the form of Weierstrass elliptic functions:

$$\eta(x,t) = \varepsilon\left[\sqrt{-\frac{5C^2(\Phi_9+2\Phi_{10})}{36\Phi_7}}\frac{6\wp[(x),g_2,g_3]+C}{3\wp'[(x),g_2,g_3]}\right]^{\frac{1}{2m}} e^{i\lambda t}, \tag{72}$$

and

$$\rho(x,t) = \chi\varepsilon\left[\sqrt{-\frac{5C^2(\Phi_9+2\Phi_{10})}{36\Phi_7}}\frac{6\wp[(x),g_2,g_3]+C}{3\wp'[(x),g_2,g_3]}\right]^{\frac{1}{2m}} e^{i\lambda t}, \tag{73}$$

where the solutions of the Weierstrass elliptic function (72) and (73) are characterised by the invariants g_2 and g_3, which are determined by the following equities:

$$g_2 = \frac{2C^2}{9} \text{ and } g_3 = \frac{C^3}{54}. \tag{74}$$

4. ADDITIONAL RESULTS

The Weierstrass elliptic function $\wp(x; g_2, g_3)$ is typically represented as [39,40]:

$$\left.\begin{aligned} \wp(x;g_2,g_3) &= l_2-(l_2-l_3)\text{cn}^2\left(\sqrt{l_1-l_3}x;m_1\right), \\ \wp(x;g_2,g_3) &= l_3+(l_1-l_3)\text{ns}^2\left(\sqrt{l_1-l_3}x;m_1\right). \end{aligned}\right\} \tag{75}$$

The modulus is derived from $m_1 = \sqrt{\frac{l_2-l_3}{l_1-l_3}}$, as described by the Jacobian elliptic functions, where l_j with $j = 1, 2, 3$, and $l_1 \geq l_2 \geq l_3$, are the three roots of the equation $4y^3 - g_2y - g_3 = 0$.
Upon substituting (75) into (63) and (64), the following are the four solutions that take the form of Jacobi elliptic functions:

$$\eta(x,t) = \varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)}{E\Phi_8}}\left(\left[l_2-(l_2-l_3)\text{cn}^2\left(\sqrt{l_1-l_3}\,x;m_1\right)\right]-\frac{C}{3}\right)\right]^{\frac{1}{4m}} e^{i\lambda t}, \tag{76}$$

and

$$\rho(x,t) = \chi\varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)}{E\Phi 8}}\left(\left[l_2-(l_2-l_3)\text{cn}^2\left(\sqrt{l_1-l_3}\,x;m_1\right)\right]-\frac{C}{3}\right)\right]^{\frac{1}{4m}} e^{i\lambda t}, \tag{77}$$

also,

$$\eta(x,t) = \varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)}{E\Phi_8}}\ \left(\left[l_3 + (l_1 - l_3)\mathrm{ns}^2\left(\sqrt{l_1 - l_3}\, x; m_1\right)\right] - \frac{C}{3}\right)\right]^{\frac{1}{4m}} e^{i\lambda t}, \tag{78}$$

and

$$\rho(x,t) = \chi\varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)}{E\Phi_8}}\ \left(\left[l_3 + (l_1 - l_3)\mathrm{ns}^2\left(\sqrt{l_1 - l_3}\, x; m_1\right)\right] - \frac{C}{3}\right)\right]^{\frac{1}{4m}} e^{i\lambda t}, \tag{79}$$

where $E\Phi_8(\Phi_2 + 2\Phi_3) < 0$. In detail, as $m_1 \to 1$, $l_1 \to l_2$, resulting in $\mathrm{ns}(x,1) \to \coth(x)$ and $\mathrm{cn}(x,1) \to \mathrm{sech}(x)$. Bright and singular solitons appear prominently in the solutions:

$$\eta(x,t) = \varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)}{E\Phi_8}}\ \left(\left[l_2 - (l_2 - l_3)\mathrm{sech}^2\left(\sqrt{l_1 - l_3}\, x\right)\right] - \frac{C}{3}\right)\right]^{\frac{1}{4m}} e^{i\lambda t}, \tag{80}$$

and

$$\rho(x,t) = \chi\varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)}{E\Phi_8}}\ \left(\left[l_2 - (l_2 - l_3)\mathrm{sech}^2\left(\sqrt{l_1 - l_3}\, x\right)\right] - \frac{C}{3}\right)\right]^{\frac{1}{4m}} e^{i\lambda t}, \tag{81}$$

also,

$$\eta(x,t) = \varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)}{E\Phi_8}}\ \left(\left[l_2 - (l_2 - l_3)\coth^2\left(\sqrt{l_1 - l_3}\, x\right)\right] - \frac{C}{3}\right)\right]^{\frac{1}{4m}} e^{i\lambda t}, \tag{82}$$

and

$$\rho(x,t) = \chi\varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)}{E\Phi_8}}\ \left(\left[l_2 - (l_2 - l_3)\coth^2\left(\sqrt{l_1 - l_3}\, x\right)\right] - \frac{C}{3}\right)\right]^{\frac{1}{4m}} e^{i\lambda t}, \tag{83}$$

where $E\Phi_8(\Phi_2 + 2\Phi_3) < 0$.

Upon substituting (75) into (65) and (66), Jacobi elliptic solutions are achieved:

$$\eta(x,t) = \varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)E}{\Phi_8}}\left(\frac{A}{l_2-(l_2-l_3)\mathrm{cn}^2\left(\sqrt{l_1-l_3}\, x;m_1\right)-\frac{C}{3}}\right)\right]^{\frac{1}{4m}} e^{i\lambda t}, \tag{84}$$

and

$$\rho(x,t) = \chi\varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)E}{\Phi_8}}\left(\frac{A}{l_2-(l_2-l_3)\mathrm{cn}^2\left(\sqrt{l_1-l_3}\, x;m_1\right)-\frac{C}{3}}\right)\right]^{\frac{1}{4m}} e^{i\lambda t}, \tag{85}$$

also,

$$\eta(x,t) = \varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)E}{\Phi_8}}\left(\frac{A}{l_3+(l_1-l_3)\mathrm{ns}^2\left(\sqrt{l_1-l_3}\, x;m_1\right)-\frac{C}{3}}\right)\right]^{\frac{1}{4m}} e^{i\lambda t}, \tag{86}$$

and

$$\rho(x,t) = \chi\varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)E}{\Phi_8}}\left(\frac{A}{l_3+(l_1-l_3)\mathrm{ns}^2\left(\sqrt{l_1-l_3}\, x;m_1\right)-\frac{C}{3}}\right)\right]^{\frac{1}{4m}} e^{i\lambda t}, \tag{87}$$

where $E\Phi_8(\Phi_2 + 2\Phi_3) < 0$. Singular and dark solitons appear prominently in the solutions:

$$\eta(x,t) = \varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)E}{\Phi 8}}\left(\frac{A}{l_2-(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_1-l_3}\, x\right)-\frac{C}{3}}\right)\right]^{\frac{1}{4m}} e^{i\lambda t}, \tag{88}$$

and

$$\rho(x,t) = \chi\varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)E}{\Phi 8}}\left(\frac{A}{l_2-(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_1-l_3}\, x\right)-\frac{C}{3}}\right)\right]^{\frac{1}{4m}} e^{i\lambda t}, \tag{89}$$

also,

$$\eta(x,t) = \varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)E}{\Phi_8}}\left(\frac{A}{l_3+(l_1-l_3)\coth^2\left(\sqrt{l_1-l_3}\, x\right)-\frac{C}{3}}\right)\right]^{\frac{1}{4m}} e^{i\lambda t}, \tag{90}$$

and

$$\rho(x,t) = \chi\varepsilon\left[\sqrt{-\frac{(\Phi_2+2\Phi_3)E}{\Phi_8}}\left(\frac{A}{l_3+(l_1-l_3)\coth^2\left(\sqrt{l_1-l_3}\, x\right)-\frac{C}{3}}\right)\right]^{\frac{1}{4m}} e^{i\lambda t}, \tag{91}$$

where $E\Phi_8(\Phi_2 + 2\Phi_3) < 0$. By inserting (75) into (68) and (69), Jacobi elliptic solutions are generated:

$$\eta(x,t) = \varepsilon\left[\frac{1}{18}\sqrt{-\frac{(\Phi_2+2\Phi_3)E}{\Phi_8}}\left(\frac{6\sqrt{A}\left[l_2-(l_2-l_3)\mathrm{cn}^2\left(\sqrt{l_1-l_3}\, x;m_1\right)\right]+C\sqrt{A}}{\sqrt{l_1-l_3}(l_2-l_3)\mathrm{cn}\left(\sqrt{l_1-l_3}\, x;m_1\right)\mathrm{sn}\left(\sqrt{l_1-l_3}\, x;m_1\right)\mathrm{dn}\left(\sqrt{l_1-l_3}\, x;m_1\right)}\right)\right]^{\frac{1}{4m}} e^{i\lambda t}, \tag{92}$$

and

$$\rho(x,t)=\chi\varepsilon\left[\frac{1}{18}\sqrt{-\frac{(\Phi_2+2\Phi_3)E}{\Phi_8}}\left(\frac{6\sqrt{A}\left[l_2-(l_2-l_3)\mathrm{cn}^2\left(\sqrt{l_1-l_3}\;x;m_1\right)\right]+C\sqrt{A}}{\sqrt{l_1-l_3}(l_2-l_3)\mathrm{cn}\left(\sqrt{l_1-l_3}\;x;m_1\right)\mathrm{sn}\left(\sqrt{l_1-l_3}\;x;m_1\right)\mathrm{dn}\left(\sqrt{l_1-l_3}\;x;m_1\right)}\right)\right]^{\frac{1}{4m}}e^{i\lambda t}, \tag{93}$$

also,

$$\eta(x,t)=\varepsilon\left[\frac{1}{18}\sqrt{-\frac{(\Phi_2+2\Phi_3)E}{\Phi_8}}\left(-\frac{6\sqrt{A}\left[l_3+(l_1-l_3)\mathrm{ns}^2\left(\sqrt{l_1-l_3}\;x;m_1\right)\right]+C\sqrt{A}}{\sqrt{l_1-l_3}\;(l_1-l_3)\mathrm{cn}\left(\sqrt{l_1-l_3}\;x;m_1\right)\mathrm{dn}\left(\sqrt{l_1-l_3}\;x;m_1\right)\mathrm{ns}^3\left(\sqrt{l_1-l_3}\;x;m_1\right)}\right)\right]^{\frac{1}{4m}}e^{i\lambda t}, \tag{94}$$

and

$$\rho(x,t)=\chi\varepsilon\left[\frac{1}{18}\sqrt{-\frac{(\Phi_2+2\Phi_3)E}{\Phi_8}}\left(-\frac{6\sqrt{A}\left[l_3+(l_1-l_3)\mathrm{ns}^2\left(\sqrt{l_1-l_3}\;x;m_1\right)\right]+C\sqrt{A}}{\sqrt{l_1-l_3}\;(l_1-l_3)\mathrm{cn}\left(\sqrt{l_1-l_3}\;x;m_1\right)\mathrm{dn}\left(\sqrt{l_1-l_3}\;x;m_1\right)\mathrm{ns}^3\left(\sqrt{l_1-l_3}\;x;m_1\right)}\right)\right]^{\frac{1}{4m}}e^{i\lambda t}, \tag{95}$$

where $E\Phi_8(\Phi_2+2\Phi_3)<0$. Notably, the solutions includes additional forms of straddled quiescent optical solitons:

$$\eta(x,t)=\varepsilon\left[\frac{1}{18}\sqrt{-\frac{(\Phi_2+2\Phi_3)E}{\Phi_8}}\left(\frac{6\sqrt{A}\left[l_2-(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_2-l_3}\;x;m_1\right)\right]+C\sqrt{A}}{\sqrt{l_2-l_3}(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_2-l_3}\;x;m_1\right)\tanh\left(\sqrt{l_2-l_3}\;x;m_1\right)}\right)\right]^{\frac{1}{4m}}e^{i\lambda t}, \tag{96}$$

and

$$\rho(x,t)=\chi\varepsilon\left[\frac{1}{18}\sqrt{-\frac{(\Phi_2+2\Phi_3)E}{\Phi_8}}\left(\frac{6\sqrt{A}\left[l_2-(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_2-l_3}\;x;m_1\right)\right]+C\sqrt{A}}{\sqrt{l_2-l_3}(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_2-l_3}\;x;m_1\right)\tanh\left(\sqrt{l_2-l_3}\;x;m_1\right)}\right)\right]^{\frac{1}{4m}}e^{i\lambda t}, \tag{97}$$

also,

$$\eta(x,t)=\varepsilon\left[\frac{1}{18}\sqrt{-\frac{(\Phi_2+2\Phi_3)E}{\Phi_8}}\left(-\frac{6\sqrt{A}\left[l_3+(l_2-l_3)\coth^2\left(\sqrt{l_2-l_3}\;x;m_1\right)\right]+C\sqrt{A}}{\sqrt{l_2-l_3}(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_2-l_3}\;x;m_1\right)\coth^3\left(\sqrt{l_2-l_3}\;x;m_1\right)}\right)\right]^{\frac{1}{4m}}e^{i\lambda t}, \tag{98}$$

and

$$\rho(x,t)=\chi\varepsilon\left[\frac{1}{18}\sqrt{-\frac{(\Phi_2+2\Phi_3)E}{\Phi_8}}\left(-\frac{6\sqrt{A}\left[l_3+(l_2-l_3)\coth^2\left(\sqrt{l_2-l_3}\;x;m_1\right)\right]+C\sqrt{A}}{\sqrt{l_2-l_3}(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_2-l_3}\;x;m_1\right)\coth^3\left(\sqrt{l_2-l_3}\;x;m_1\right)}\right)\right]^{\frac{1}{4m}}e^{i\lambda t}, \tag{99}$$

where $E\Phi_8(\Phi_2+2\Phi_3)<0$, $A>0$.
By inserting (75) into (72) and (73), Jacobi elliptic solutions are formed:

$$\eta(x,t)=\varepsilon\left[\sqrt{-\frac{5C^2(\Phi_9+2\Phi_{10})}{36\Phi_7}}\left(\frac{6\left[l_2-(l_2-l_3)\mathrm{cn}^2\left(\sqrt{l_1-l_3}\;x;m_1\right)\right]+C}{3\sqrt{l_1-l_3}(l_2-l_3)\mathrm{cn}\left(\sqrt{l_1-l_3}\;x;m_1\right)\mathrm{sn}\left(\sqrt{l_1-l_3}\;x;m_1\right)\mathrm{dn}\left(\sqrt{l_1-l_3}\;x;m_1\right)}\right)\right]^{\frac{1}{2m}}e^{i\lambda t}, \tag{100}$$

and

$$\rho(x,t)=\chi\varepsilon\left[\sqrt{-\frac{5C^2(\Phi_9+2\Phi_{10})}{36\Phi_7}}\left(\frac{6\left[l_2-(l_2-l_3)\mathrm{cn}^2\left(\sqrt{l_1-l_3}\;x;m_1\right)\right]+C}{3\sqrt{l_1-l_3}(l_2-l_3)\mathrm{cn}\left(\sqrt{l_1-l_3}\;x;m_1\right)\mathrm{sn}\left(\sqrt{l_1-l_3}\;x;m_1\right)\mathrm{dn}\left(\sqrt{l_1-l_3}\;x;m_1\right)}\right)\right]^{\frac{1}{2m}}e^{i\lambda t}, \tag{101}$$

also,

$$\eta(x,t)=\varepsilon\left[\sqrt{-\frac{5C^2(\Phi_9+2\Phi_{10})}{36\Phi_7}}\left(-\frac{6\left[l_3+(l_1-l_3)\mathrm{ns}^2\left(\sqrt{l_1-l_3}\;x;m_1\right)\right]+C}{3\sqrt{l_1-l_3}\;(l_1-l_3)\mathrm{cn}\left(\sqrt{l_1-l_3}\;x;m_1\right)\mathrm{dn}\left(\sqrt{l_1-l_3}\;x;m_1\right)\mathrm{ns}^3\left(\sqrt{l_1-l_3}\;x;m_1\right)}\right)\right]^{\frac{1}{2m}}e^{i\lambda t}, \tag{102}$$

and

$$\rho(x,t)=\chi\varepsilon\left[\sqrt{-\frac{5C^2(\Phi_9+2\Phi_{10})}{36\Phi_7}}\left(-\frac{6\left[l_3+(l_1-l_3)\mathrm{ns}^2\left(\sqrt{l_1-l_3}\;x;m_1\right)\right]+C}{3\sqrt{l_1-l_3}\;(l_1-l_3)\mathrm{cn}\left(\sqrt{l_1-l_3}\;x;m_1\right)\mathrm{dn}\left(\sqrt{l_1-l_3}\;x;m_1\right)\mathrm{ns}^3\left(\sqrt{l_1-l_3}\;x;m_1\right)}\right)\right]^{\frac{1}{2m}}e^{i\lambda t}, \tag{103}$$

where $\Phi_7(\Phi_9+2\Phi_{10})<0$. A further set of additional form of straddled optical solitons is enlisted here:

$$\eta(x,t)=\varepsilon\left[\sqrt{-\frac{5C^2(\Phi_9+2\Phi_{10})}{36\Phi_7}}\left(\frac{6\left[l_2-(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_2-l_3}\;x\right)\right]+C}{3\sqrt{l_2-l_3}(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_2-l_3}\;x\right)\tanh\left(\sqrt{l_2-l_3}\;x\right)}\right)\right]^{\frac{1}{2m}}e^{i\lambda t}, \tag{104}$$

and

$$\rho(x,t)=\chi\varepsilon\left[\sqrt{-\frac{5C^2(\Phi_9+2\Phi_{10})}{36\Phi_7}}\left(\frac{6\left[l_2-(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_2-l_3}\;x\right)\right]+C}{3\sqrt{l_2-l_3}(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_2-l_3}\;x\right)\tanh\left(\sqrt{l_2-l_3}\;x\right)}\right)\right]^{\frac{1}{2m}}e^{i\lambda t}, \tag{105}$$

also,

$$\eta(x,t)=\varepsilon\left[\sqrt{-\frac{5C^2(\Phi_9+2\Phi_{10})}{36\Phi_7}}\left(-\frac{6\left[l_3+(l_2-l_3)\coth^2\left(\sqrt{l_2-l_3}\;x\right)\right]+C}{3\sqrt{l_2-l_3}(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_2-l_3}\;x\right)\coth^3\left(\sqrt{l_2-l_3}\;x\right)}\right)\right]^{\frac{1}{2m}}e^{i\lambda t}, \tag{106}$$

and

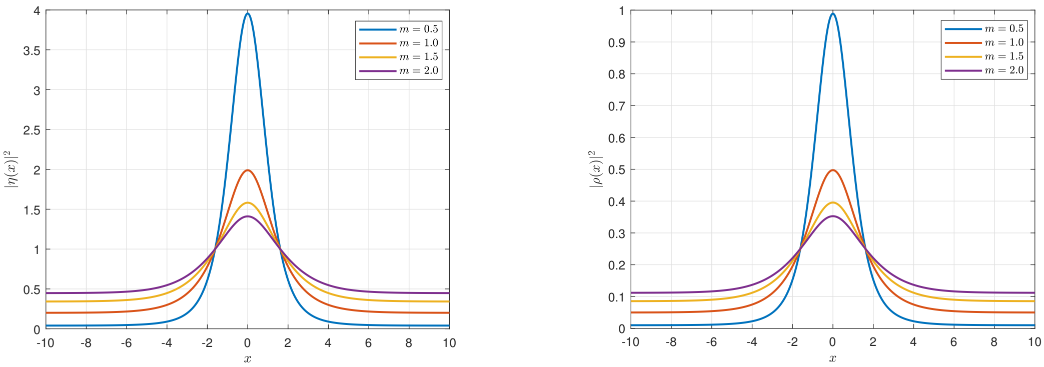
$$\rho(x,t)=\chi\varepsilon\left[\sqrt{-\frac{5C^2(\Phi_9+2\Phi_{10})}{36\Phi_7}}\left(-\frac{6\left[l_3+(l_2-l_3)\coth^2\left(\sqrt{l_2-l_3}\;x\right)\right]+C}{3\sqrt{l_2-l_3}(l_2-l_3)\mathrm{sech}^2\left(\sqrt{l_2-l_3}\;x\right)\coth^3\left(\sqrt{l_2-l_3}\;x\right)}\right)\right]^{\frac{1}{2m}}e^{i\lambda t}, \tag{107}$$

where $\Phi_7(\Phi_9+2\Phi_{10})<0$.

5. RESULTS AND DISCUSSION

The application of the modified sub-ODE methodology to the coupled nonlinear Schrödinger–type system yields a wide spectrum of quiescent soliton solutions in magneto–optic waveguides subject to nonlinear chromatic dispersion and Kudryashov's generalized self-phase modulation. In this section, we focus on three representative families of stationary localized states: bright, dark, and straddled (hyperbolic–rational) quiescent solitons. Their spatial structures, as reconstructed from the analytic formulas, are depicted in Figures 1–3.

Figure 1 illustrates the family of bright quiescent solitons described by Eqs. (23)–(24). Both field components, $\eta(x)$ and $\rho(x)$, exhibit a single localized hump centered at $x = 0$, decaying exponentially to the background. The parameter m plays a dual role, modulating both amplitude and spatial width of the soliton. For smaller values of m, the profiles become broader with lower peak intensity, whereas increasing m sharpens the localization and elevates the peak amplitude. Importantly, the bright states maintain their stability across the tested range of m, reflecting the robustness of the balance between nonlinear self-phase modulation and dispersive spreading.



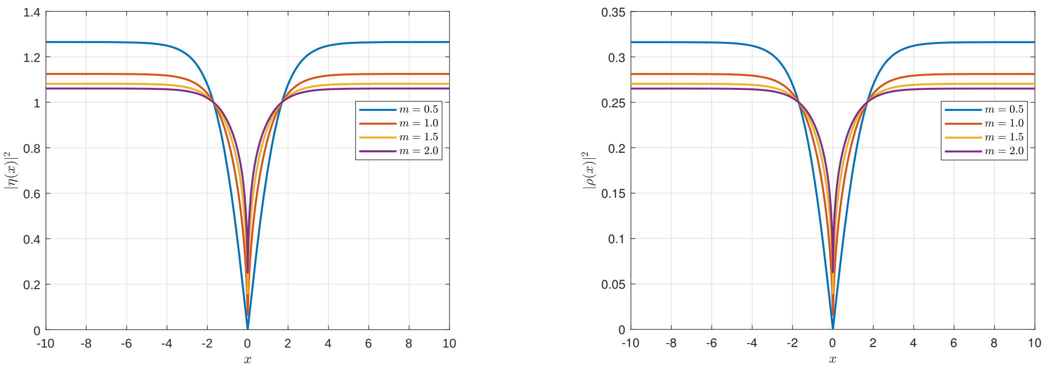
(a) Bright quiescent soliton for $\eta(x)$ given by Eq. (23), shown for different values of m.

(b) Bright quiescent soliton for $\rho(x)$ given by Eq. (24), shown for different values of m.

Figure 1. Bright quiescent solitons: localized hump solutions described by Eqs. (23)–(24) ($C = 0.8,\ \ \Phi_2 = 2,\ \ \Phi_3 = 1,\ \ \Phi_8 = 1,\ \ A_0 = 0.2,\ \ \varepsilon = 1,\ \ \chi = 0.5$).

In contrast, Figure 2 presents the dark quiescent solitons obtained from Eqs. (27)–(28). Here, the fields $\eta(x)$ and $\rho(x)$ retain a finite background, featuring a localized notch at the origin. The depth of this dip depends sensitively on m: shallow depressions for small m evolve into deeper and narrower notches as m increases. This behavior mirrors the expected dynamics of dark solitons under normal dispersion conditions. The presence of a finite pedestal ensures that the solutions remain bounded, which is a crucial feature for stable pulse transmission. The consistency of these results with the parameter restrictions in Set-2 highlights the versatility of the sub-ODE method in capturing both bright and dark stationary structures within a unified framework.

A more exotic class of solutions is displayed in Figure 3, where the hybrid hyperbolic–rational structures emerge from Eqs. (53)–(54). These "straddled" solitons exhibit a shallow hyperbolic-type core near $x = 0$, accompanied by algebraically

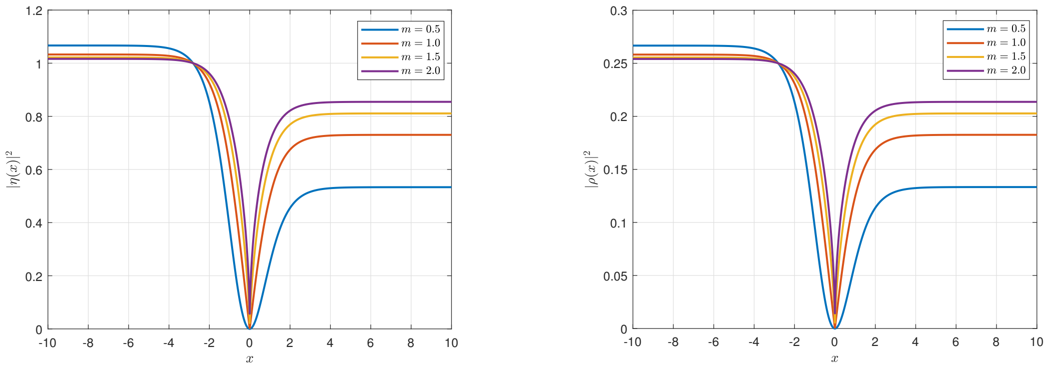


(a) Dark quiescent soliton for $\eta(x)$ given by Eq. (27), shown for different values of m.

(b) Dark quiescent soliton for $\rho(x)$ given by Eq. (28), shown for different values of m.

Figure 2. Dark quiescent solitons: localized notch solutions described by Eqs. (27)–(28) ($\varepsilon = 1,\ \ \chi = 0.5,\ \ C = -0.8,\ \ \Phi_7 = -1,\ \ \Phi_9 = 2,\ \ \Phi_{10} = 1.$).

decaying shoulders extending into the far field. Such composite profiles differ from the exponentially localized bright and dark solitons: while they maintain localization, their slower tail decay introduces long-range interactions that may influence multi-pulse dynamics in realistic magneto–optic media. The dependence on m again tunes the relative prominence of the core versus the shoulders, suggesting that the soliton morphology can be engineered through parameter control. The analytical derivation of these solutions (Set-5) demonstrates the flexibility of Kudryashov's SPM structure in generating hybrid states that combine features of both hyperbolic and rational forms.



(a) Straddled (hyperbolic–rational) quiescent soliton for $\eta(x)$ given by Eq. (53), shown for different values of m.

(b) Straddled (hyperbolic–rational) quiescent soliton for $\rho(x)$ given by Eq. (54), shown for different values of m.

Figure 3. Straddled (hyperbolic–rational) quiescent solitons: hybrid profiles described by Eqs. (53)–(54) ($C = -0.8, \quad D = 1, \quad A_1 = 1, \quad \chi = 0.5.$)

Taken together, the three families highlight the diversity of stationary quiescent solitons admitted by the model. Bright and dark solutions represent classical solitary waveforms associated with anomalous and normal dispersion regimes, respectively, while straddled solitons introduce hybrid decay patterns with potential for novel interaction dynamics. The modulation parameter m consistently acts as a control knob for amplitude–width trade-offs, underscoring its physical role in tailoring pulse shapes. From an applications perspective, the ability to sustain such varied soliton classes in magneto–optic waveguides suggests opportunities for optimizing pulse transmission, mitigating clutter, and exploring nonstandard optical switching regimes.

6. CONCLUSIONS

This work examines quiescent optical solitons in magneto-optic waveguides, including nonlinear CD and generalised temporal evolution, using Kudryashov's generalised self-phase modulation structure. The generalised version of sub-ODE is the mathematical framework that has facilitated the successful recovery of quiescent optical solitons. We establish a diverse array of classified optical solitons and the parametric conditions that guarantee their existence. These solitons exhibit unique stability properties and can be manipulated by varying the external magnetic field and waveguide parameters. Our results deepen the understanding of soliton dynamics in complex media and open avenues for potential applications in optical communications and information processing. The results of the work serve as a stark reminder to the telecommunication engineers that rough handling of optical fibers would stall the progress of such solitons under the sea as well as underground. This would catastrophically dismantle the internet traffic flow across the planet.

While magneto–optic waveguides is one of the optoelectronic devices where the dynamics of quiescent optical solitons are established in the current paper, the study would be later extended to additional such devices that has been left untouched in the literature. Examples of such devices include Bragg gratings, optical couplers, optical metamaterials, fibres exhibiting polarization-mode dispersion, and dispersion-flattened fibres, among others. The mathematical algorithms that would be adopted while addressing such devices would be plentiful including the most powerful of them namely Lie symmetry analysis. The results would be disseminated all across the board sequentially over time.

Acknowledgements

This work of one of the authors (AB) was funded by the budget of Grambling State University for the Endowed Chair of Mathematics. The author thankfully acknowledges this support.

Author contributions

All authors contributed to the study conception and design. Material preparation, data collection, review & editing and analysis were performed by [Mona El-Shater], [Ahmed H. Arnous], [Omer Mohammed Khodayer Al-Dulaimi], [Farag Mahel Mohammed], [Ibrahim Zeghaiton Chaloob], and [Carmelia Mariana Balanica Dragomir]. The first draft of the manuscript was written by [Elsayed M. E. Zayed]. Investigation, writing—review and editing was made by [Anjan Biswas] and [O. González-Gaxiola]. All authors provided comments on previous drafts of the manuscript and then reviewed and approved the final version.

Funding

The authors confirm that they did not receive any financial, grant, or other forms of help during the preparation of this manuscript.

Data availability

The article contains all the information required to comprehend the results of this investigation.

Conflict of interest

We are ignorant of any financial conflicts of interest or personal relationships that could have potentially impacted the research described in this article.

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СПОКІЙНІ СОЛІТОНИ В МАГНІТООПТИЧНИХ ХВИЛЕВОДАХ З НЕЛІНІЙНОЮ ХРОМАТИЧНОЮ ДИСПЕРСІЄЮ ТА ФОРМОЮ САМОФАЗОВОЇ МОДУЛЯЦІЇ ЗА КУДРЯШОВИМ, ЩО МАЮТЬ УЗАГАЛЬНЕНУ ЧАСОВУ ЕВОЛЮЦІЮ

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У статті обговорюється, як запропонована Кудряшовим схема самофазової модуляції та нелінійна хроматична дисперсія спричиняють еволюцію спокійних оптичних солітонів у магнітооптичних хвилеводах. Забезпечується повне розуміння моделі управління; розглядається узагальнена часова еволюція. Для полегшення відновлення таких солітонів використовується модифікований підхід суб-ODE. Це призводить до повного спектру оптичних солітонів та необхідних умов, які повинні бути виконані для існування цих солітонів, які також наводяться.

Ключові слова: *солітони; самофазова модуляція; інтегрованість; хроматична дисперсія*