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KDV AND MKDV ION-ACOUSTIC SOLITARY WAVES IN A POSITRON-BEAM PLASMA WITH KANIADAKIS DISTRIBUTED ELECTRONS

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Theoretical and numerical studies of ion-acoustic solitary waves (IASWs) in an unmagnetized plasma with ions, positron beams under pressure variation, and kaniadakis distributed electrons have been conducted. The potential wave amplitude is calculated by applying the reductive perturbation approach to reduce the controlling set of normalized fluid equations to Korteweg-de Vries (KdV) and modified Korteweg-de Vries (mKdV) equations. In mKdV solutions, only compressive solitons are found, whereas both compressive and rarefactive KdV solitons are found to exist for different values of σ, σ_b, and ν. The parameter k has no effect on the IASWs of the KdV equation, but have contribution in mKdV solitons. It is also shown that the inclusion of nonthermal electrons drastically changes the basic properties of ion-acoustic solitons and creates a new parametric regime.

Keywords: *Ion-acoustic Solitary waves; Positron-beams; KdV equation; mKdV equation; Kaniadakis distribution*

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1. INTRODUCTION

The reductive perturbation approach provides an excellent description of the ion-acoustic solitary wave studies in ions, positron-beams plasma with nonthermal electrons. In order to do this, the development progresses from the typical ion-electron-positron components to ions, positron beams and nonthermal electrons, reaching an intriguing stage of study. Theoretically, several researchers have investigated the existence of ion-acoustic solitary waves in positron beam plasma systems during the past few decades [1–22]. Greaves and Surko [23] have examined the electron-positron beam-plasma experiment for the first time. Coleman [24] has studied positron beams and their applications in various field. Sarma et al. [25] have investigated relativistic positron beams in nonlinear ion-acoustic solitary waves in an electron-positron-ion plasma. Lynn and Schultz [26] have investigated how positron beams interact with surfaces, thin films, and interfaces. The totally nonlinear acoustic waves in a plasma with superthermal electrons and positron beam impact were examined by Shan et al. [27]. Ion acoustic shock waves in the presence of superthermal electrons and the interaction of a classical positron beam have been explored by Shah et al. [28]. Moreover, Shah et al. [29] investigated the interaction of nonlinear waves with a classical positron beam in nonextensive astrophysical plasmas in an astrophysical naturally moderated setting. Hogan [30] has researched plasma acceleration caused by electron and positron beams. Research on nonlinear ion-acoustic solitary waves in a weakly relativistic electron-positron-ion plasma with relativistic electron and positron beams was conducted by Barman and Talukdar [31]. Roy et al. [32] have investigated the effects of relativistic positron beams on ion-acoustic breather, periodic, and solitary waves in the ionospheric area of earth using the mKdV equation and the KdV framework.

A lot of attention has been paid to numerous entropic forms that generalize the conventional Boltzmann-Gibbs Shannon one over the last few decades. Initially, Renyi [33] introduced the generalization of the Boltzmann-Gibbs (BG) statistics. Later it was developed by Tsallis [34]. The Tsallis nonextensive theory was one such theory that showed its effectiveness in managing particular complicated systems and also showed an astonishingly high degree of agreement with experimental evidence [35]. Quantum entanglement, blackbody radiation, and the k-statistics derived from the kaniadakis entropy [36] were all reviewed within the theoretical framework. It has been investigated how the putative k-deformed distributions caused by the kaniadakis entropy relate to cosmic rays [37], quark-gluon plasma production [38], interacting photon and atom kinetics [39], and nonlinear kinetics [40]. It has been investigated [41, 42] for arbitrary amplitude electron-acoustic waves in unmagnetized plasma with a k-deformed kaniadakis electron distribution. Kalita *et al.* [43] have investigated nonlinear dust ion acoustic solitary waves in an unmagnetized plasma involving kaniadakis distributed electrons and temperature ratio. Khalid *et al.* [44] have conducted research using kaniadakis distributed electrons in dust ion acoustic solitary waves in unmagnetized plasma. High relativistic effect on dust-ion-acoustic solitary waves in unmagnetized plasma with kaniadakis distributed electrons has been investigated by Das and Das [45]. Raut *et al.* [46] have investigated dust ion acoustic shock waves, soliton, and bi-soliton in unmagnetized plasma with electrons dispersed according to kaniadakis in both planar and nonplanar geometry. Tribeche and Lourek's [47] have studied on the function of the k-deformed kaniadakis distribution in nonlinear plasma waves. Irshad *et al.* [48] have studied the influence of

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the k-deformed kaniadakis distribution on the modulational instability of electron-acoustic waves in a non-Maxwellian plasma.

In this paper we have established the existence of KdV and mKdV solitons in a three-component plasma consisting of ions, positron beams, and kaniadakis distributed electrons using the reductive perturbation approach. KdV solutions contain both compressive and rarefactive solitons, but only compressive solitons are present in mKdV solitons for the effects of σ, σ_b, ν, and k. The structure of our paper is as follows: in Section [1] , we have presented the usual 'Introduction' ; in Section [2], we have mentioned the 'Basic Equations Governing the Dynamics of Motion'; Section [3] and Section [4] include the 'Derivation of KdV and mKdV Equation and Its Solution'; Section [5] is incorporated the 'Results and Discussions'; and in Section [6] we have presented the 'Conclusion' of the outcome.

2. BASIC EQUATIONS GOVERNING THE DYNAMICS OF MOTION

We have considered the propagation of three component collisionless ion-acoustic waves in a warm unmagnetized plasma with ion, positron-beam and nonthermal electrons. For such a plasma model, the basic equations governing the plasma dynamics of ion-acoustic waves are described by the following normalized equations

$$\frac{\partial n}{\partial t}+\frac{\partial (nu)}{\partial x}=0 \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\left(\frac{\partial \phi}{\partial x}+\frac{\sigma}{n}\frac{\partial p}{\partial x}\right) \tag{2}$$

$$\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}=-3p\frac{\partial u}{\partial x} \tag{3}$$

$$\frac{\partial n_b}{\partial t}+\frac{\partial (n_b w)}{\partial x}=0 \tag{4}$$

$$\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial x}=-\left(\frac{\partial \phi}{\partial x}+\frac{\sigma_b}{n_b}\frac{\partial p_b}{\partial x}\right) \tag{5}$$

$$\frac{\partial p_b}{\partial t}+w\frac{\partial p_b}{\partial x}=-3p_b\frac{\partial w}{\partial x} \tag{6}$$

$$\frac{\partial^2 \phi}{\partial x^2}=(1-\nu)n_e-n-\nu n_b \tag{7}$$

where $n,\ n_b,\ u,\ w,\ p$, and p_b respectively denote the ion number density, positron-beam number density, the ion fluid velocity, positron-beam fluid velocity, pressure variation of ion and the pressure variation of positron-beam. Also $\nu = n_{b0}/n_0$ is the positron beam to ion number density ratio; $\sigma = T_i/T_e$ is the ion to electron temperature ratio; $\sigma_b = T_b/T_e$ is the positron beam to electron temperature ratio. In the above equations, the particle number densities $n,\ n_b$ and n_e are normalized by unperturbed electron number density n_{e0}; velocity $u,\ w$ by the ion-acoustic speed $c_s = (T_e/m_i)^{1/2}$; pressures p by the characteristic ion pressure eT_i; the time t by the inverse of the characteristic ion plasma frequency $\omega_{pi}^{-1} = \left(m/4\pi n_0 e^2\right)^{1/2}$; the distance x by the Debye length $\lambda_{De} = \left(T_e/4\pi n_0 e^2\right)^{1/2}$.

The electrons follow k-deformed KD [38] which is expressed as:

$$f_e^{(k)}(u_b) = A_k exp_k\left(-\frac{m_e u_b^2/2 - e\varphi}{T_e}\right)$$

with $(u,w) = u_b$

$$exp_k(x) = \left(\sqrt{1+k^2x^2}+kx\right)^{\frac{1}{k}},$$

and A_k is the normalized constant indicated by :

$$A_k = n_{e0}\left(\frac{m_e|k|}{\pi T_e}\right)^{3/2}\frac{\Gamma(\frac{1}{2|k|}+\frac{3}{4})}{\Gamma(\frac{1}{2|k|}-\frac{3}{4})}\left(1+\frac{3}{2}|k|\right).$$

The following standard integration is applied for calculating A_k :

$$\int_0^\infty x^{s-1}exp_k(-x)dx = \frac{[1+(s-2)|k|]|2k|^{-s}\Gamma(\frac{1}{2|k|}-\frac{s}{2})}{[1-(s-1)|k|]^2-k^2\Gamma(\frac{1}{2|k|}+\frac{s}{2})}\Gamma(s)$$

Here, k is a real parameter and Γ stands for the universal gamma function that indicates the degree of deformation. The inequality $|k|<1$ must hold for real value of the parameter.

The quantity k represents the dispersion from the Maxwellian distribution; hence, when $k \longrightarrow 0$, the KD function is transformed into the Maxwell-Boltzmann distribution as follows:

$$\lim_{k\to 0} f_e^{(k)}(u_b) = n_{e0}(\frac{m_e}{2\pi T_e})^{\frac{3}{2}} exp\left(-\frac{(m_e u_b^2/2) - e\varphi}{T_e}\right)$$

with

$$\lim_{k\to 0} exp_k(x) \approx exp(x).$$

Prior to continuing, it is crucial to limit the acceptable range of k. Calculation of the mean square speed $\prec u_b^2 \succ$ requires the following:

$$\prec u_b^2 \succ = \frac{\iiint u_b^2 f_e^{(k)}(u_b) d^3u_b}{\iiint f_e^{(k)}(u_b) d^3u_b} = \frac{2T_e/m_e}{|2k|^{5/2}(1+\frac{5}{2}|k|)} \frac{\Gamma(\frac{1}{2|k|}-\frac{5}{4})}{\Gamma(\frac{1}{2|k|}+\frac{5}{4})}$$

To preserve the physical meaning of $\prec u_b^2 \succ$, which necessitates that $\prec u_b^2 \succ$ be finite and whose value diverges at $k \longrightarrow 0.4$, the appropriate value of k must satisfy the inequality $k < 0.4$. It should be mentioned that this restriction is taken into account while determining A_k and the particles' average kinetic energy, $m \prec u_b^2 \succ /2$, and that the interacting particles are ignored, i.e., $\phi = 0$.

By integrating $f_e^{(k)}(u_b)$ throughout velocity space, the electron density is calculated and is as follows:

$$\begin{aligned} n_e &= \left(\sqrt{1+k^2\phi^2}+k\phi\right)^{\frac{1}{k}}, \\ \implies n_e &= 1+\phi+\frac{1}{2}\phi^2+\frac{(1-k^2)}{6}\phi^3+\ldots \end{aligned} \tag{8}$$

3. DERIVATION OF KDV EQUATION AND ITS SOLUTIONS

To derive the KdV equation from (1)-(8), we apply the reductive perturbation method. we consider a new set of stretched variables as

$$\xi = \epsilon^{1/2}(x - Ut), \quad \tau = \epsilon^{3/2}t \tag{9}$$

where U is the waves' phase velocity and ϵ is a small, non-dimensional parameter that quantifies the degree of nonlinearity. The flow variables are then expanded asymptotically about the stable equilibrium state in terms of ϵ as follows:

$$\left.\begin{aligned} n &= n_0+\epsilon n_1+\epsilon^2 n_2+\epsilon^3 n_3+\ldots \\ u &= \epsilon u_1+\epsilon^2 u_2+\epsilon^3 u_3+\ldots\ldots\ldots \\ p &= p_0+\epsilon p_1+\epsilon^2 p_2+\epsilon^3 p_3+\ldots \\ n &= n_{b0}+\epsilon n_{b1}+\epsilon^2 n_{b2}+\epsilon^3 n_{b3}+\ldots \\ w &= \epsilon w_1+\epsilon^2 w_2+\epsilon^3 w_3+\ldots\ldots\ldots \\ p &= p_{b0}+\epsilon p_{b1}+\epsilon^2 p_{b2}+\epsilon^3 p_{b3}+\ldots \\ \phi &= \epsilon\phi_1+\epsilon^2\phi_2+\epsilon^3\phi_3+\ldots\ldots\ldots \end{aligned}\right\} \tag{10}$$

Using the transformation (9) and the expansions (10) in the equations (1)-(8) and equating the coefficients of the first lowest-order of ϵ with the use of the boundary conditions $n_1 = n_{b1} = 0$, $u_1 = w_1 = 0$, $\phi_1 = 0$ at $|\xi| \longrightarrow \infty$, we get

$$\left.\begin{aligned} n_1 &= \frac{n_0\phi_1}{U^2-3\sigma}; \quad u_1 = \frac{U\phi_1}{U^2-3\sigma}; \quad p_1 = \frac{3p_0\phi_1}{U^2-3\sigma}; \\ n_{b1} &= \frac{n_{b0}\phi_1}{U^2-3\sigma_b}; \quad w_{b1} = \frac{U\phi_1}{U^2-3\sigma_b}; \quad p_{b1} = \frac{3p_{b0}\phi_1}{U^2-3\sigma_b} \end{aligned}\right\} \tag{11}$$

From equation (4) we get,

$$(1-\nu)\phi_1 - n_1 - \nu n_{b1} = 0 \tag{12}$$

Using the expressions of n_1 and n_{b1} from (11), the expression for phase speed U is found as

$$U^2 = \frac{-y \pm \sqrt{y^2 - 4xz}}{2x} \tag{13}$$

where $x = 1 - \nu$

$y = -3(1-\nu)(\sigma+\sigma_b) - n_0 - \nu n_{b0}$

$$z = 9\sigma\sigma_b(1-\nu) + 3(n_0\sigma_b + \nu\sigma n_{b0})$$

Again, equating the coefficients of second higher-order terms of ϵ from (1)-(6), we get the followings,

$$\frac{\partial n_1}{\partial \tau} - U\frac{\partial n_2}{\partial \xi} + n_0\frac{\partial u_2}{\partial \xi} + \frac{\partial (n_1 u_1)}{\partial \xi} = 0 \tag{14}$$

$$n_0\frac{\partial u_1}{\partial \tau} - Un_0\frac{\partial u_2}{\partial \xi} - Un_1\frac{\partial u_1}{\partial \xi} + n_0 u_1\frac{\partial u_1}{\partial \xi} + \sigma\frac{\partial p_2}{\partial \xi} + n_0\frac{\partial \phi_2}{\partial \xi} + n_1\frac{\partial \phi_1}{\partial \xi} = 0 \tag{15}$$

$$\frac{\partial p_1}{\partial \tau} - U\frac{\partial p_2}{\partial \xi} + u_1\frac{\partial p_1}{\partial \xi} + 3p_0\frac{\partial u_2}{\partial \xi} + 3p_1\frac{\partial u_1}{\partial \xi} = 0 \tag{16}$$

$$\frac{\partial n_{b1}}{\partial \tau} - U\frac{\partial n_{b2}}{\partial \xi} + n_{b0}\frac{\partial w_2}{\partial \xi} + \frac{\partial (n_{b1} w_1)}{\partial \xi} = 0 \tag{17}$$

$$n_{b0}\frac{\partial w_1}{\partial \tau} - Un_{b0}\frac{\partial w_2}{\partial \xi} - Un_{b1}\frac{\partial w_1}{\partial \xi} + n_{b0} w_1\frac{\partial w_1}{\partial \xi} + \sigma_b\frac{\partial p_{b2}}{\partial \xi} + n_{b0}\frac{\partial \phi_2}{\partial \xi} + n_{b1}\frac{\partial \phi_1}{\partial \xi} = 0 \tag{18}$$

$$\frac{\partial p_{b1}}{\partial \tau} - U\frac{\partial p_{b2}}{\partial \xi} + w_1\frac{\partial p_{b1}}{\partial \xi} + 3p_{b0}\frac{\partial w_2}{\partial \xi} + 3p_{b1}\frac{\partial w_1}{\partial \xi} = 0 \tag{19}$$

Similarly, from equation(7), we obtain

$$\frac{\partial^2 \phi_1}{\partial \xi^2} = (1-\nu)(\phi_2 + \frac{1}{2}\phi_1^2) - n_2 - \nu n_{b2}$$

$$\Longrightarrow \frac{\partial^3 \phi_1}{\partial \xi^3} = (1-\nu)\frac{\partial \phi_2}{\partial \xi} + (1-\nu)\phi_1\frac{\partial \phi_1}{\partial \xi} - \frac{\partial n_2}{\partial \xi} - \nu\frac{\partial n_{b2}}{\partial \xi} \tag{20}$$

Now, putting the values of $\partial n_0/\partial\xi$ and $\partial n_{b0}/\partial\xi$ from (14)-(19) and using the relations (20), the KdV equation is obtained as

$$\frac{\partial \phi_1}{\partial \tau} + p\phi_1\frac{\partial \phi_1}{\partial \xi} + q\frac{\partial^3 \phi_1}{\partial \xi^3} = 0 \tag{21}$$

where the nonlinear coefficients p and dispersion coefficient q are given by

$$p = \frac{3n_0(U^2-3\sigma_b)^3(U^2+\sigma) + 3\nu n_{b0}(U^2-3\sigma)^3(U^2+\sigma_b) - (1-\nu)(U^2-3\sigma)^3(U^2-3\sigma_b)^3}{2U(U^2-3\sigma)(U^2-3\sigma_b)\left[(U^2-3\sigma_b)^2 n_0 + (U^2-3\sigma)^2\nu n_{b0}\right]};$$

$$q = \frac{(U^2-3\sigma)^2(U^2-3\sigma_b)^2}{2U\left[(U^2-3\sigma_b)^2 n_0 + (U^2-3\sigma)^2\nu n_{b0})\right]}$$

A new transformation $\eta = \xi - V\tau$ is introduced to discover the stationary solitary wave solutions to the Korteweg-de Vries(KdV) equation(21). In this transformation, V represents the soliton speed in the linear $\eta-$space. The solution for the solitary wave can be obtained by integrating this transformation into the partial differential equation (21) as

$$\phi_1 = \phi_0 \operatorname{sech}^2\left(\frac{\eta}{\Delta}\right) \tag{22}$$

where $\phi_0 = 3V/p$ is the amplitude of the ion-acoustic soliton, and $\Delta = \sqrt{4q/V}$ is the width of the ion-acoustic soliton.

4. DERIVATION OF THE MKDV AND ITS SOLUTION

Equation (21) shows that the nonlinear coefficient p is determined by the plasma parameters ν, σ, σ_b. Higher order nonlinearity requires $p = 0$ in the KdV Eqn. (21). Consequently, the critical density ν_c is obtained, which may be computed as

$$\nu_c = \frac{2(U^2-3\sigma)^3(U^2-3\sigma_b)^3 - 3n_0(U^2-3\sigma)^3(U^2+\sigma)}{2(U^2-3\sigma)^3(U^2-3\sigma_b)^3 + 3n_{b0}(U^2-3\sigma)^3(U^2+\sigma_b)} \tag{23}$$

In order to characterize the system at or near the critical point ν_c given in (23). Different stretched variables are provided for higher-order non-linearity as:

$$\eta = \epsilon(x - Ut)\ , \quad \tau = \epsilon^3 t \tag{24}$$

Utilizing the new stretched variables of (24) and expansions (10), the equations (1)-(6) provide the same (13) for the phase velocity U. The second higher order perturbation of ϵ can be obtained as

$$\left.\begin{aligned} n_2 &= \frac{3n_0(U^2+\sigma)\phi_1^2}{2(U^2-3\sigma)^3} + \frac{n_0\phi_2}{U^2-3\sigma}, \quad n_{b2} = \frac{3n_{b0}(U^2+\sigma_b)\phi_1^2}{2(U^2-3\sigma_b)^3} + \frac{n_{b0}\phi_2}{U^2-3\sigma_b}, \\ u_2 &= \frac{(U^3+9U\sigma)\phi_1^2}{2(U^2-3\sigma)^3} + \frac{U\phi_2}{U^2-3\sigma}, \quad w_2 = \frac{(U^3+9U\sigma_b)\phi_1^2}{2(U^2-3\sigma_b)^3} + \frac{U\phi_2}{U^2-3\sigma_b}, \\ p_2 &= \frac{3n_0(5U^2-3\sigma)\phi_1^2}{2(U^2-3\sigma)^3} + \frac{3n_0\phi_2}{U^2-3\sigma}, \quad p_{b2} = \frac{3n_{b0}(5U^2-3\sigma_b)\phi_1^2}{2(U^2-3\sigma_b)^3} + \frac{3n_{b0}\phi_2}{U^2-3\sigma_b} \end{aligned}\right\} \tag{25}$$

Next, we enter the values of n_2 in the associated second-order partial differential equation to the relation

$$(1-\nu_c)(\phi_2 + \frac{1}{2}\phi_1^2) - n_2 - \nu_c n_{b2} = 0 \tag{26}$$

We obtain the following equations by taking into account the third higher order terms of ϵ from (1)-(6).

$$\frac{\partial n_1}{\partial \tau} - U\frac{\partial n_3}{\partial \xi} + n_0\frac{\partial u_3}{\partial \xi} + \frac{\partial (n_1u_2)}{\partial \xi} + \frac{\partial (n_2u_1)}{\partial \xi} = 0 \tag{27}$$

$$\begin{aligned} n_0\frac{\partial u_1}{\partial \tau} - Un_0\frac{\partial u_3}{\partial \xi} - Un_1\frac{\partial u_2}{\partial \xi} - Un_2\frac{\partial u_1}{\partial \xi} + n_0u_1\frac{\partial u_2}{\partial \xi} + n_0u_2\frac{\partial u_1}{\partial \xi} + n_1u_1\frac{\partial u_1}{\partial \xi} + \sigma\frac{\partial p_3}{\partial \xi} \\ + n_0\frac{\partial \phi_3}{\partial \xi} + n_1\frac{\partial \phi_2}{\partial \xi} + n_2\frac{\partial \phi_1}{\partial \xi} = 0 \end{aligned} \tag{28}$$

$$\frac{\partial p_1}{\partial \tau} - U\frac{\partial p_3}{\partial \xi} + u_1\frac{\partial p_2}{\partial \xi} + u_2\frac{\partial p_1}{\partial \xi} + 3p_0\frac{\partial u_3}{\partial \xi} + 3p_1\frac{\partial u_2}{\partial \xi} + 3p_2\frac{\partial u_1}{\partial \xi} = 0 \tag{29}$$

$$\frac{\partial n_{b1}}{\partial \tau} - U\frac{\partial n_{b3}}{\partial \xi} + n_{b0}\frac{\partial w_3}{\partial \xi} + \frac{\partial (n_{b1}w_2)}{\partial \xi} + \frac{\partial (n_{b2}w_1)}{\partial \xi} = 0 \tag{30}$$

$$\begin{aligned} n_{b0}\frac{\partial w_1}{\partial \tau} - n_{b0}U\frac{\partial w_3}{\partial \xi} - Un_{b1}\frac{\partial w_2}{\partial \xi} - Un_{b2}\frac{\partial w_1}{\partial \xi} + n_{b0}w_1\frac{\partial w_2}{\partial \xi} + n_{b0}w_2\frac{\partial w_1}{\partial \xi} + n_{b1}w_1\frac{\partial w_1}{\partial \xi} + \sigma_b\frac{\partial p_{b3}}{\partial \xi} \\ + n_{b0}\frac{\partial \phi_3}{\partial \xi} + n_{b1}\frac{\partial \phi_2}{\partial \xi} + n_{b2}\frac{\partial \phi_1}{\partial \xi} = 0 \end{aligned} \tag{31}$$

$$\frac{\partial p_{b1}}{\partial \tau} - U\frac{\partial p_{b3}}{\partial \xi} + w_1\frac{\partial p_{b2}}{\partial \xi} + w_2\frac{\partial p_{b1}}{\partial \xi} + 3p_{b0}\frac{\partial w_3}{\partial \xi} + 3p_{b1}\frac{\partial w_2}{\partial \xi} + 3p_{b2}\frac{\partial w_1}{\partial \xi} = 0 \tag{32}$$

Eliminating n_3, n_{b3}, u_3 and w_3 from equations (27)–(32) we get,

$$\frac{\partial \phi}{\partial \tau} + p'\phi^2\frac{\partial \phi}{\partial \xi} + q'\frac{\partial^3 \phi}{\partial \xi^3} = 0 \tag{33}$$

where the higher order nonlinear coefficient p' and dispersion coefficient q' are given by

$$p' = \frac{15U^2\left[U^2+6\sigma+\nu_c(U^2+6\sigma_b)\right] + 27(\sigma^2+\nu_c\sigma_b^2) - (1-k^2)(1-\nu_c)(U^2-3\sigma)^5(U^2-3\sigma_b)^5}{4U(U^2-3\sigma)^3(U^2-3\sigma_b)^3\left[(U^2-3\sigma_b)^2n_0 + (U^2-3\sigma)^2\nu_c n_{b0})\right]},$$

$$q' = \frac{(U^2-3\sigma)^2(U^2-3\sigma_b)^2}{2U\left[(U^2-3\sigma_b)^2n_0 + (U^2-3\sigma)^2\nu_c n_{b0}\right]}$$

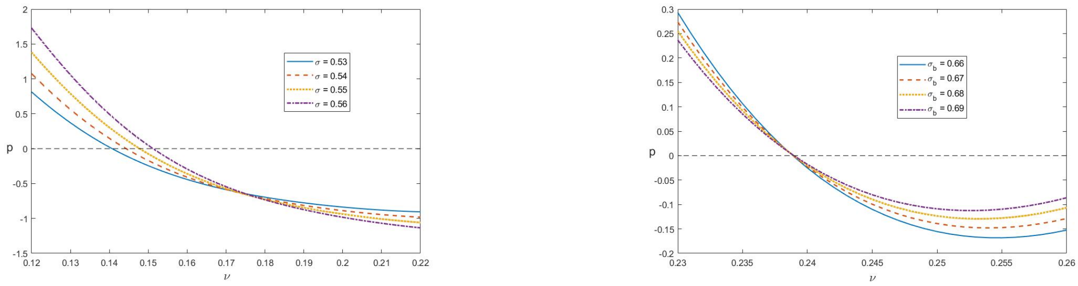
Using the same transformation and following the steps outlined in section[3], the solutions of mKdV equation (33) is obtained as

$$\phi_1' = \sqrt{\frac{6V}{p'}}\,\text{sech}\left(\sqrt{\frac{V}{q'}}\chi\right) \tag{34}$$

where V represents the soliton speed; $\phi_0' = \sqrt{6V/p'}$ and $\Delta' = \sqrt{q'/V}$ indicate, respectively, the amplitude and width of the solitary waves represented by the mKdV equation(33).

5. RESULTS AND DISCUSSIONS

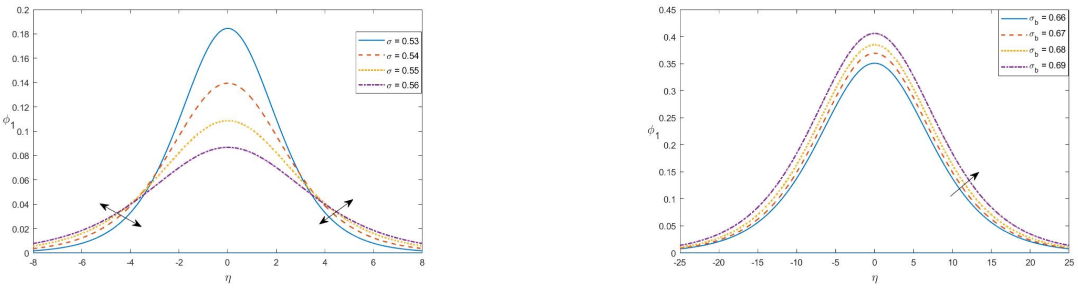
The present work has examined the characteristics of ion-acoustic solitary waves (IASWs) in the context of kaniadakis-deformed electrons k. A numerical analysis is done on the effects of different plasma parameters on the variations of nonlinear term p and dispersion term q given in (21). These parameters include ion-to-electron temperature ratio $\sigma(= T_i/T_e)$, positron beam-to-electron temperature ratio $\sigma_b(= T_b/T_e)$, positron-beam to ion number density ν and the deformation parameter k.



(a) **(b)**

Figure 1. Variation of p versus ν for different values of (a) σ and (b) σ_b.

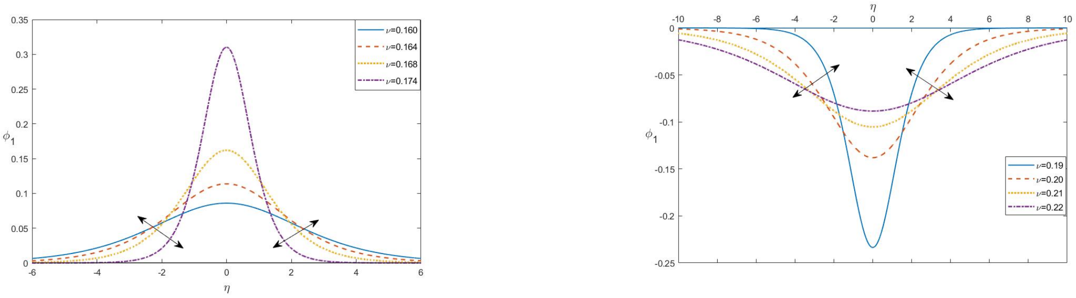
Figure(1a-1b) describes the variation of nonlinearity of the plasma system for different values of (a) $\sigma = 0.53, 0.54, 0.55, 0.56$ with $\sigma_b = 0.11$ and (b) $\sigma_b = 0.66, 0.67, 0.68, 0.69$ with $\sigma = 0.65$ and other parameters $n_0 = 0.1$, $n_{b0} = 0.1$, $\nu = 0.12$, and $V = 0.05$. The nonlinear coefficient p of the KdV Eq.(21) can have positive, zero and negative values. So, p represents compressive or rarefactive according as $p > 0$ or $p < 0$ and p vanishes or $p \approx 0$ at specific sets of critical values. This represents the singularity shown in Figures (1a) and (1b).



(a) **(b)**

Figure 2. Variation of ϕ_1 versus η for different values of (a) σ and (b) σ_b.

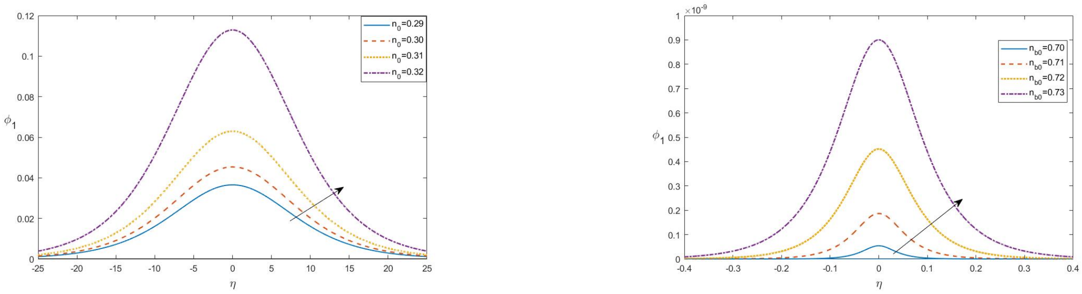
The ion acoustic solitary wave is compressive, as can be seen from Figures (2a-2b). In Figure (2a), the amplitude decreases and the width slightly increases for increasing values of σ with the same fixed values as mentioned in Figure (1), while in Figure (2b), both the amplitude and width rises as increase values of σ_b.



(a) **(b)**

Figure 3. Variation of ϕ_1 versus η for different values of ν.

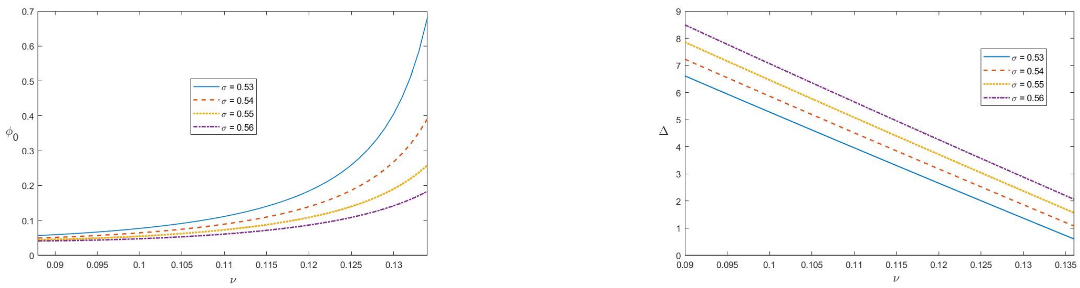
For fixed parameters as $\sigma_b = 0.11$, $\sigma = 0.65$, $n_0 = 0.1$, $n_{b0} = 0.1$, and $V = 0.05$, how the number density ν effects on the propagating ion-acoustic solitary waves are shown in Figure [3]. The soliton type is observed to have changed from compressive to rarefactive at ν depending on σ and σ_b. With rising values of $\nu < \nu_c$, the soliton is compressive (Fig.-[3a]) and amplitude of positive potential ion-acoustic solitary waves increases notably while width decreases slightly. Conversely, for increasing values of $\nu > \nu_c$, the soliton is rarefactive (Fig.-[3b]) and decreases the pulse of negative potential ion-acoustic solitary waves in both amplitude and width. It is noticed that ν plays a crucial role for the existence of both compressive and rarefactive solitons.



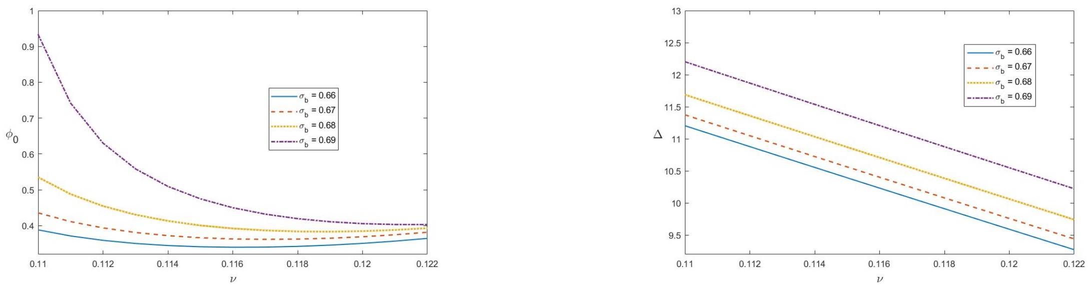
(a) **(b)**

Figure 4. Variation of ϕ_1 versus η for different values of (a) n_b and (b) n_{b0}.

The variation of the amplitude and width of the fast compressive KdV soliton profiles for different values of ion number density n_0 effect is shown in Figure (4a) and positron beam number density n_{b0} effect is shown in Figure (4b). It is seen that both the amplitude and the width of the fast compressive KdV soliton increases with the increase of n_0 and n_{b0}.



(a) **(b)**



(c) **(d)**

Figure 5. Variation of ϕ_0 versus ν for different values of (a) σ and (c) σ_b and variation of Δ versus ν for different values of (b) σ and (d) σ_b

In Figure (5a), the fast compressive KdV soliton's amplitude gradually increases in the lower range of $\nu \leq 0.14$ for different values of σ, and for fixed values of $\sigma_b = 0.11$, $n_0 = 0.1$, $n_{b0} = 0.1$, and $V = 0.05$, while in Figure (5c), the amplitude reduces in the lower range of $\nu \leq 0.122$ for different values of σ_b. However, the width decreases uniformly for both the cases. In Figure (6a), the fast compressive KdV soliton amplitude increases as the increasing values of $\nu = 0.160, 0.164, 0.168, 0.174$ for fixed other parameters $\sigma = 0.65$, $n_0 = 0.1$, $n_{b0} = 0.1$, $k = 0.3$ and $V = 0.05$, and Figure (6b) shows that for the fast compressive mKdV soliton, the amplitude monotonically increases for the same various values of ν and also same fixed values. In Figure (6c), the fast rarefactive KdV soliton amplitude increases as the increasing values of $\nu = 0.19, 0.20, 021, 0.22$ for fixed same values in Figure (6a), and Figure (6d) shows that for the fast compressive mKdV soliton, amplitude decreases for the same various values of ν. However, calculation reveals that the width exhibits a consistent character, with the same set of parametric values for every graph in Figure (6).

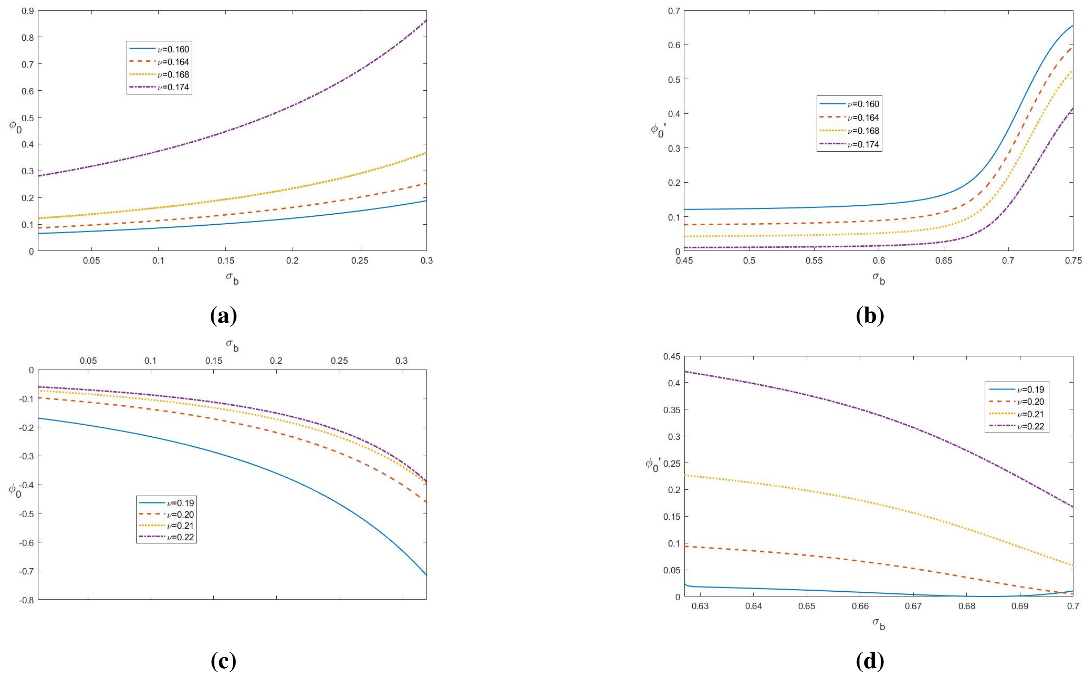


Figure 6. Variation of ϕ_0 versus σ_b for different values of ν in (a) and (c), and variation of ϕ_0' versus σ_b in different values of ν for (b) and (d).

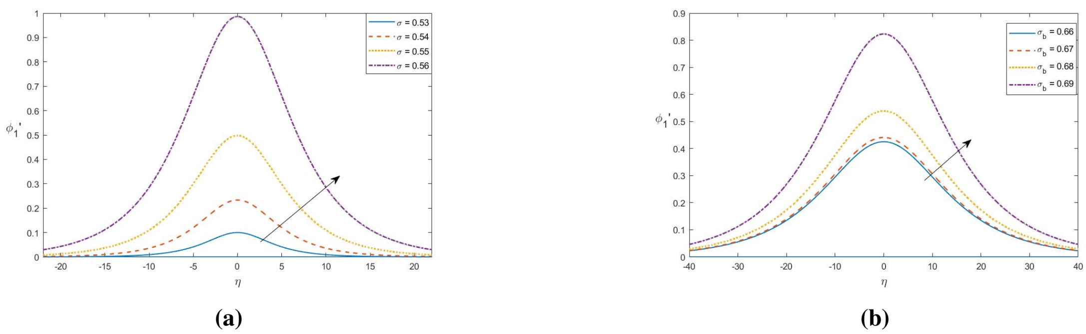


Figure 7. Variation of ϕ_1' versus η for different values of (a) σ and (b) σ_b.

Next, we plot the higher-order solitary wave potential $\phi_1'(\eta)$ versus η for fixed values of $n_0 = 0.1$, $n_{b0} = 0.1$, $k = 0.3$ and $V = 0.05$ in Figure (7a-7b) with varying values of (a) σ with $\nu = 0.12$ and (b) σ_b with $\nu = 0.191$. For both the cases, only compressive mKdV ion-acoustic soliton is found to exist.

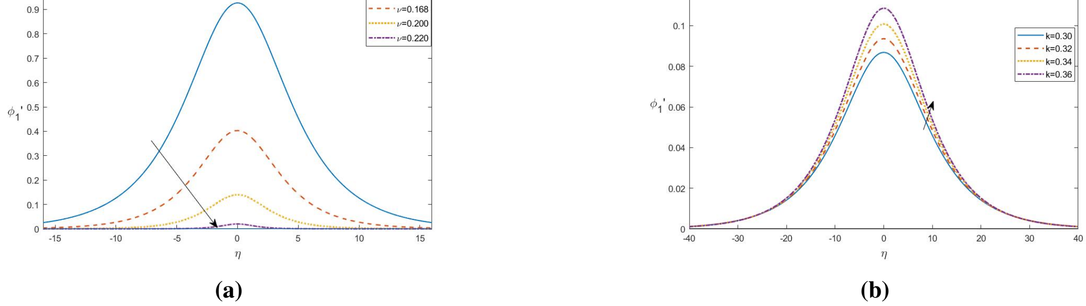


Figure 8. Variation of ϕ_1' versus η for different values of (a) ν with $k = 0.3$ and (b) k.

The variations of $\phi_1'(\eta)$ versus η for various values of (a) $\nu = 0.160, 0.168, 0.200, 0.220$ and (b) $k = 0.30, 0.32, 0.34, 0.36$ respectively, with other parameters $n_0 = 0.1$, $n_{b0} = 0.1$, $\sigma = 0.65$, $\sigma_b = 0.11$, $\nu = 0.12$ and $V = 0.05$ are shown in Figures (8a-8b). From both the figures we observed that ϕ_1' is compressive. It can be observed that as ν increases, the ion-acoustic solitary waves amplitude and width decrease monotonically in Figure (8a) and in Figure (8b) the amplitude of compressive ion-acoustic solitary waves is shown to increase as k increases, but the width does not change significantly.

6. CONCLUSION

We have investigated the propagation of ion-acoustic solitary waves in an unmagnetized plasma model with kaniadakis-distributed electrons, ions, and positron beams. The reductive perturbation method is used to derive the KdV and mKdV equations and to obtain their solitary wave solutions. Physical parameters such as ν, the positron beam to ion number density ratio; σ, the ion to electron temperature ratio; σ_b, the positron to electron temperature ratio and the parameter k play a crucial role in giving the soliton character. The outcomes that have been observed in this study can be contracted as follows:

1. It is found that there are two different types of wave modes in the current plasma model: slow acoustic modes and fast ion-acoustic modes. However, we only take into account fast ion-acoustic modes for extracting KdV and mKdV equations, because slow modes do not give any possibility for the existence of soliton.
2. The first order non-linear coefficient p in the KdV equation can be a positive and a negative quantity, while the second order non-linear coefficient p' of the mKdV equation is a positive quantity, depending on the plasma parametric values. Therefore, there exists both compressive and rarefactive KdV solitons in the present plasma system.
3. The change in the soliton types from compressive to rarefactive is predicting mainly through the variation of positron beam to ion density ratio parameter ν, depending on σ and σ_b. It is seen that compressive and rarefactive solitons are to exist for the range of $\nu \leq 0.1793$ and $\nu > 0.1793$ respectively.
4. At the critical ν_c, we consider a second order nonlinearity and determine mKdV equation. Only compressive ion acoustic solitary wave structures are feasible in present plasma system.

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KdV ТА mKdV ІОННО-АКУСТИЧНІ ПООДИНОКІ ХВИЛІ В ПОЗИТРОННО-ПУЧКОВІЙ ПЛАЗМІ З РОЗПОДІЛЕНИМИ ЕЛЕКТРОНАМИ КАНІАДАКІСА

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Проведено теоретичні та чисельні дослідження іонно-акустичних поодиноких хвиль (IAX) у ненамагніченній плазмі з іонами, пучками позитронів при зміні тиску та розподіленими електронами Каніадакіса. Потенційна амплітуда хвилі розраховується шляхом застосування підходу редуктивного збурення для зведення контрольного набору нормалізованих рівнянь рідини до рівнянь Кортевега-де Фріза (KdV) і модифікованих рівнянь Кортевега-де Фріза (mKdV). У mKdV рішеннях зустрічаються лише солітони стиснення, тоді як Встановлено, що для різних значень σ, σ_b і ν існують як стискаючі, так і розріджені солітони KdV. Параметр k також не впливає на IASW рівняння KdV, але має внесок у солітонах mKdV. Також показано, що включення нетеплових електронів різко змінює основні властивості іонно-акустичних солітонів і створює новий параметричний режим.

Ключові слова: *іонно-акустичні одиночні хвилі; позитронні пучки; рівняння KdV; рівняння mKdV; розподіл Каніадакіса*