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BARROW HOLOGRAPHIC DARK ENERGY WITHIN SAEZ-BALLESTER SCALAR FIELD AND LYRA GEOMETRY

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This paper investigates the dynamical behavior of hypersurface homogeneous spacetime cosmological models within the framework of the scalar-tensor theory of gravitation formulated by Saez and Ballester (Phys. Lett. A, 113, 467 1986) in Lyra geometry. We present two cosmological models derived from this theory by solving the field equations using: (i) Special law of variation for Hubble's parameter and (ii) the proportional relationship between the shear scalar σ^2 and scalar expansion θ as described by Collins et al. (Gen. Rel. Grav. 12, 805 1980). For each model, we evaluate key dynamical parameters, including the equation of state (EoS) parameter, the deceleration parameter, the statefinder parameter, and the total energy density parameter of dark energy. Additionally, we determine the scalar field in both models. Our findings indicate that these models describe an accelerated expansion of the universe, with theoretical results showing reasonable agreement with observational data.

Keywords: *Hypersurface Homogeneous space-time; Holographic dark energy; Scalar-Tensor Theory of Gravitation*

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1. INTRODUCTION

The recent observational studies have given evidence for the accelerated expansion of the universe [1, 2] and fluctuation of Cosmic Microwave Background Radiation (CMBR) [3], Sloan Digital Sky Survey (SDSS) [4], Wilkinson Microwave Anisotropy Probe (WMAP) [5], Large Scale Structure (LSS) [6], Baryonic Acoustic Oscillations [7], Gravitational Lensing [8] etc. It has also been suggested that the main reason for this is an exotic negative pressure named as 'dark energy'. It is surmised that the universe embedded of 68.3% dark energy as well as 26.8% dark matter. There have been numerous other dark energy models proposed, including quintessence [9], phantom [10], quintom [11], tachyon [12], ghost [13], K-essence [14], phantom [15], Chaplygin gas [16], polytropic gas [17] and holographic dark energy (HDE) [18] and many more to explain the accelerated expansion of the universe. To explain this accelerated expansion of the universe two different approaches have been advocated: to construct different dark energy candidates and to modify Einstein's theory of gravitation.

Very recently, a holographic dark energy model has been conjectured to explain the dark energy (Thomas [19], Horaya and Minic [20]). Li [18] has constructed a viable holographic dark energy model based on the holographic principle of the quantum gravity theory. Brans-Dicke (BD) [21] and Saez-Ballester (SB) [22] scalar tensor theories of gravitations. Therefore, there have been several investigations of DE cosmological model in the above alternative theories of gravitation to explain DE models by studying their dynamical aspects. In most of the above cases, the researchers concentrated on the anisotropic Bianchi type DE models. Recently, Naidu et al., [24] discussed Kaluza Klein FRW dark energy models in Saez-Ballester theory of gravitation. Oliveros et al., [25] investigated Barrow holographic dark energy [26, 27] with Granda-Oliveros cutoff [30]. In this study, we focus on the scalar-tensor theory of gravity proposed by Saez- Ballester. In the Saez-Ballester theory, a scalar field ϕ is introduced alongside the metric tensor field, modifying gravitational interactions. This modification can lead to attractive or repulsive forces, depending on the form of the scalar field and its coupling.. This inclusion aims to more fully incorporate Mach's principle. Saez and Ballester later developed a new scalar-tensor theory where the metric of spacetime is simply coupled with a dimensionless scalar field. However, this theory includes an antigravity regime. The gravitational action was first introduced by Saez [23] and is given by:

$$I = \int_{\Sigma} (L + GL_m)\sqrt{-g}d^4x \tag{1}$$

where L_m is the matter Lagrangian. Varying this gravitational action, δI, leads to the field equations of the Saez-Ballester scalar-tensor theory and Lyra geometry:

$$G_{ij} + \frac{3}{2}\psi_i\psi_j - \frac{3}{4}g_{ij}\psi_k\psi^k = -\left(T_{ij} + \bar{T}_{ij}\right) + \omega\phi^n\left(\phi_{.i}\phi_{.j} - \frac{1}{2}g_{ij}\phi_{.k}\phi^{.\hat{k}}\right) \tag{2}$$

The scalar field ϕ satisfies the following equation:

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$$2\phi^n \phi_{,i}^{;i} + n\phi^{n-1}\phi_{,k}\phi^{,k} = 0 \tag{3}$$

Here, $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$, R_{ij} is the Ricci tensor, R is the Ricci scalar, ψ^i is displacement vector field of Lyra's geometry w is a dimensionless constant, and T_{ij} and $\bar{T}_{ij}$ are the energy-momentum tensor of matter and Barrow holographic dark energy, respectively. Relativistic units $8\pi G = c = 1$ are used in these equations.

2. BARROW HOLOGRAPHIC DARK ENERGY

A new proposal that has recently caught the attention of the community is the so-called Barrow holographic dark energy (BHDE) [26], which has its roots in the idea introduced by Barrow [27], inspired by illustrations of the COVID-19 virus. Barrow proposed that quantum gravitational effects modify the black hole Bekenstein-Hawking entropy [28, 29] by introducing a fractal structure for the geometry of the horizon. In this section, we delve into the theoretical framework and cosmological implications of an interacting Barrow holographic dark energy (BHDE) model. The concept of BHDE was introduced by Barrow, building upon the modification of black hole horizon entropy represented by:

$$S_B = \left(\frac{A}{A_0}\right)^{1+\frac{\Delta}{2}}, 0 \leq \Delta \leq 1 \tag{4}$$

where A denotes the standard horizon area and A_0 is the Planck area. The parameter Δ quantifies the effect of quantum deformation on the structure of the horizon. In particular, $\Delta = 1$ represents maximal deformation, whereas $\Delta = 0$ corresponds to the simplest horizon structure, recovering the conventional Bekenstein-Hawking entropy. In the area of cosmology, this modified entropy leads to a holographic dark-energy (HDE) model described by:

$$\rho_B = CL^{\Delta-2}, \tag{5}$$

where C is an unknown parameter with dimensions $[L]^{-2-\Delta}$. This formulation extends beyond the standard HDE scenario ($\Delta = 0$), where $\rho_D \propto L^{-2}$. The BHDE model is thus a more comprehensive framework, particularly focusing on cases where $\Delta > 0$ and quantum deformation effects are significant. The energy density of BHDE, employing the Hubble horizon $\left(H^{-1}\right)$ as the IR cutoff (L), is given by:

$$\rho_B = CH^{2-\Delta} \tag{6}$$

The choice of the Hubble horizon as the IR cut-off is noteworthy because of its inherent relevance in cosmology. Various models of HDE have explored different IR cut-off, each influencing cosmological dynamics differently. For instance, Li demonstrated that selecting the future event horizon as the IR cut-off yields an accelerating universe in the absence of interaction between dark matter (DM) and dark energy (DE), whereas the particle horizon leads to a decelerating universe. Recent attention has been drawn to the BHDE model within the context of the Granda-Oliveros (G-O) cutoff [30], which incorporates both the square of the Hubble parameter and its time derivative:

$$L_{\mathbb{R}} = \left(\alpha H^2 + \gamma \dot{H}\right)^{-1/2} \tag{7}$$

where α and γ are arbitrary dimensionless parameters. Recent studies have further explored BHDE with the G-O length as the IR cutoff, considering BHDE as a dynamical vacuum and accounting for interactions between matter and dark energy sectors. We are implementing the BHDE density with the G-O IR cutoff, where the holographic DE density ρ_B is given by:

$$\rho_B = 3M_p^2\left(\alpha H^2 + \gamma \dot{H}\right)^{1-\frac{\Delta}{2}} \tag{8}$$

Here, α and β are parameters with dimensions $[L]^{\frac{2\Delta}{\Delta-2}}$, ensuring the correct dimensionality of ρ_B.

Unlike the original HDE model where $\Delta = 0$, the parameter C in equation (6) is replaced by $3M_p^2$. The barotropic equation of state $p_B = \omega_B \rho_B$, the equation of state HDE parameter is obtained as

$$\omega_B = -1 - \frac{2\alpha H\dot{H} + \gamma \ddot{H}}{3H\left(\alpha H^2 + \gamma \dot{H}\right)} \tag{9}$$

The above discussion and investigations, we consider in this paper the minimally interacting holographic dark energy model in hypersurface homogeneous spacetime within the framework of the SB scalar-tensor theory of gravitation. This work is organized as follows: In Sect. 2, we derive the SB field equations with the help of a hypersurface homogeneous spacetime metric in the presence of two minimally interacting fields: dark matter and holographic dark energy. Sect. 3 is devoted to the solution of SB field equations with the help of a special law of variation for Hubble's parameter proposed by Berman [31] and using physically relevant conditions. In Sect. 4, physical and kinematical parameters of the model are also computed and discussed. The last section contains some concluding remarks.

3. METRIC AND FIELD EQUATIONS

We consider the hypersurface homogeneous space time as follows

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 \left[dy^2 + \Sigma^2(y,k)dz^2\right] \tag{10}$$

where A and B are functions of time (t) and $\Sigma^2(\mathrm{y,k}) = \sin y, y, \sinh y$ for $k = 1, 0, -1$, respectively. T_{ij} and $\bar{T}_{ij}$ are energy momentum tensors for matter and holographic dark energy, respectively. which are defined as

$$\begin{aligned} T_{ij} &= \rho_m u_i u_j \\ \bar{T}_{ij} &= (\rho_B + p_B) u_i u_j + g_{ij} p_B \end{aligned} \tag{11}$$

Here ρ_m and ρ_h are the energy densities of matter and barrow holographic dark energy and p_B is the pressure of holographic dark energy.

In a co-moving coordinate system, the field equation (2) for the metric (10), using equation (11) can be written as

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{k}{B^2} + \frac{3}{4}\beta^2 + \frac{\omega}{2}\phi^n\dot{\phi}^2 = -p_B \tag{12}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{3}{4}\beta^2 + \frac{\omega}{2}\phi^n\dot{\phi}^2 = -p_B \tag{13}$$

$$2\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} + \frac{k}{B^2} - \frac{3}{4}\beta^2 - \frac{\omega}{2}\phi^n\dot{\phi}^2 = (\rho_m + \rho_B) \tag{14}$$

$$\ddot{\phi} + \dot{\phi}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right) + \frac{n}{2}\frac{\dot{\phi}^2}{\phi^n} = 0 \tag{15}$$

where an overhead dot denotes differentiation with respect to t. Now the average scale factor and the volume of the universe are defined as

$$V = AB^2 \tag{16}$$

Subtracting (12) from (13), we get

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{\dot{B}^2}{B^2} - \frac{k}{B^2} = 0 \tag{17}$$

We obtain

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{c_1}{V} + \frac{1}{V}\int \left[\frac{k}{B^2}\right] V dt \tag{18}$$

where λ represents a constant of integration.

Taking $k = 0$, the equation (18) leads to

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{c_1}{V} \tag{19}$$

From equation (19), we obtain

$$A^3 = \frac{V}{c}\exp\left(\int \frac{c_1}{V} dt\right) \tag{20}$$

The directional Hubble parameter in the direction of x, y, and z axes respectively are as follows

$$H_x = \frac{\dot{A}}{A}, H_y = H_z = \frac{\dot{B}}{B} \tag{21}$$

The average Hubble parameter is

$$H = \frac{1}{3}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right) \tag{22}$$

The expressions for the scalar expansion θ and the shear scalar σ^2 are

$$\theta = 3H \tag{23}$$

$$\sigma^2 = \frac{1}{2}\left(\sum_{i=1}^{3} H_i^2 - 3H^2\right) \tag{24}$$

The average anisotropy parameter is

$$A_B = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{\Delta H_i}{H}\right)^2 \tag{25}$$

where $\Delta H_i = H_i - H, i = 1, 2, 3$ obtained as

$$\dot{\rho}_m + \dot{\rho}_B + 3H\left(\rho_m + \rho_B + p_B\right) = 0 \tag{26}$$

The continuity equation of the matter is

$$\dot{\rho}_m + 3H\rho_m = 0 \tag{27}$$

The continuity equation of the HDE is

$$\dot{\rho}_B + 3H\left(\rho_B + p_B\right) = 0 \tag{28}$$

4. SOLUTIONS AND THE MODEL

We have three equations (12)-(14) in four unknowns A, B, p_B, ρ_B. To solve the system completely, we need one extra condition. We solve the field equations for following physical conditions.

1. Special law of variation for Hubble's parameter;
2. Shear scalar proportional to expansion scalar.

4.1. Model Special law of variation for Hubble's parameter

We consider the relation between H and a, which was proposed by Berman (1983)

$$H = na^{\frac{-1}{n}} \tag{29}$$

where $n \geq 0$ are constants.

From equations (28) and (30) we obtain

$$q = -1 + \frac{1}{n} \tag{30}$$

Now, using Eq. (30) and Eq. (31), the solution of Eq. (28) gives the law of variation of the average scale factor of the form

$$a(t) = (t+b)^n, n \neq 0. \tag{31}$$

Using equation (31), we get

$$V = a^3(t) = (t+b)^{3n} \tag{32}$$

Now from equations (17), (21) and (32) we obtain

$$\begin{aligned} A(t) &= c_2^{\frac{2}{3}}(t+b)^n \exp\left(\frac{2c_1}{3(1-3n)}(t+b)^{1-3n}\right), \\ B(t) &= c_2^{\frac{-1}{3}}(t+b)^n \exp\left(-\frac{c_1}{3(1-3n)}(t+b)^{1-3n}\right) \end{aligned} \tag{33}$$

Now from equation (33) in equation (15) we obtain the scalar field as

$$\phi^{\frac{n+2}{2}} = \left(\frac{n+2}{2}\right)\left[\frac{\phi_0}{3(1-3n)}(t+b)^{1-3n}\right] \tag{34}$$

where ϕ_0 are constants of integration.

The directional and average Hubble parameter is

$$\begin{aligned} H_x &= \frac{n}{(t+b)} + \frac{2c_1}{3}(t+b)^{-3n} \\ H_y = H_z &= \frac{n}{(t+b)} - \frac{c_1}{3}(t+b)^{-3n} \end{aligned} \tag{35}$$

And

$$H = \frac{n}{(t+b)} \tag{36}$$

The values of the directional parameters are infinite at $t = 0$ and tend to zero as $t \to \infty$. The mean Hubble parameter $H \to 0$ as $t \to \infty$ i.e. the rate of expansion of the universe is decreasing.

The scalar expansion θ is

$$\theta = 3\frac{n}{(t+b)} \tag{37}$$

The average anisotropy parameter is

$$A_B = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 = \frac{2c_1^2}{9}(t+b)^{2-6n} \tag{38}$$

The shear scalar σ^2 is

$$\sigma^2 = \frac{c_1^2}{2}(t+b)^{-6n} \tag{39}$$

It is found that $\frac{\sigma^2}{\theta^2} \neq 0$ and the anisotropy parameter do not vanish except at n = 1.

Applying the conservation condition for the left-hand side of equation (2), we get

$$\beta\left(\dot{\beta} + \beta\left[\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right]\right) = 0.$$

From Eq. (43) by integrating, we have

$$\beta = \beta_0(t+b)^{-3n}$$

Using equation (32) in equation (28) we get the energy density of dark matter as

$$\rho_m = \rho_0(t+b)^{-3n} \tag{40}$$

where $\rho_0 > 0$ is a real constant of integration.

Using equation (36) in equation (28) the BHDE density is written as,

$$\rho_B = 3M_p^2\left(\frac{n^2(\alpha - n\gamma)}{(t+b)^2}\right)^{1-\frac{\Delta}{2}} \tag{41}$$

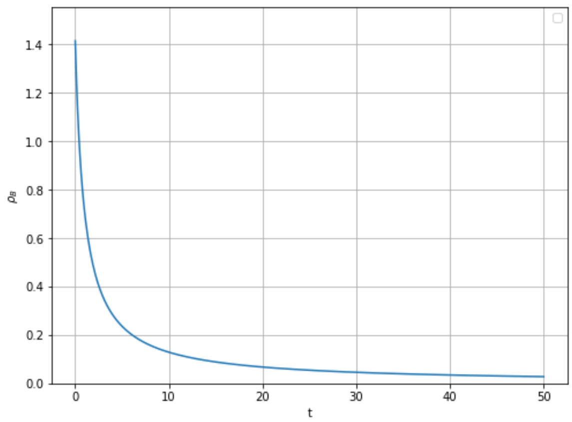

Figure 1. Plot of Density of Barrow HDE vs time, the energy density of Hypersurface homogeneous Barrow HDE model in Granda-Oliveros cut-off decreases.

Using equations (32), (33) in equation (13) we get, the pressure of Barrow HDE as

$$p_B = \frac{n^2}{(t+b)^2} + \left(\frac{2nc_1}{3} + \frac{3}{4}\beta_0\right)(t+b)^{-3n} + \left(\frac{\omega\phi_0^2}{2} + \frac{c_1^2}{9}\right)(t+b)^{-6n} \tag{42}$$

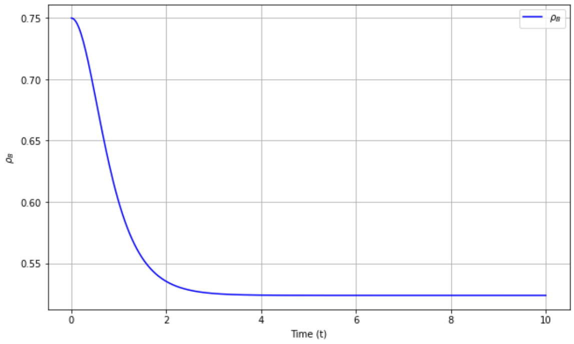

Figure 2. Plot of pressure of BHDE vs time, the pressure p_B of BHDE decreases with time and approaches zero as time goes to infinity.

Using equations (41), (42) and the barotropic equation of state $p_B = \omega_B \rho_B$, the equation of state BHDE parameter is obtained as

$$\omega_B = -1 - \frac{(2-\Delta)(2\alpha H\dot{H} + \beta\ddot{H})}{6H\left(\alpha H^2 + \beta\dot{H}\right)} \tag{43}$$

EoS parameter is obtained as,

$$\omega_B = -1 + \frac{(2-\Delta)}{3n}. \tag{44}$$

From equation (44) shows (i) For small Δ and n : The dark energy behaves less like a cosmological constant and more like matter or radiation, with ω_B greater than -1. (ii) For large Δ and n : The system behaves like a cosmological constant with ω_B approaching -1. (iii) For intermediate values of Δ and n : ω_B transitions smoothly between matter-like and dark energy-like behavior, offering a flexible model for the evolution of the universe.

This form of ω_B in the Barrow holographic dark energy model provides a way to model the evolution of dark energy and its impact on the universe's expansion, with flexibility to explain both early-time and late-time acceleration.

Matter density parameter Ω_m and the holographic dark energy parameter Ω_B are given by

$$\Omega_m = \frac{\rho_m}{3H^2} \text{ and } \Omega_B = \frac{\rho_B}{3H^2} \tag{45}$$

Using equations (36), (40), (41) and (45) we get the overall density parameter as

$$\begin{aligned}\Omega &= \Omega_m + \Omega_B \\ &= \frac{3n^2}{(t+b)^2} + \frac{1}{3n^2}\left(\frac{\omega\phi_0^2}{2} + \frac{5c_1^2}{9} + \frac{3}{4}\beta_0\right)(t+b)^{2-6n} + \frac{kc_2^{\frac{2}{3}}}{3n^2}(t+b)^{2(1-n)}\exp\left(\frac{2c_1}{3(1-3n)}(t+b)^{1-3n}\right)\end{aligned} \tag{46}$$

From equation (46), the total energy density parameter of Hypersurface homogeneous Barrow HDE model in Granda-Oliveros cut-off decreases below 1, indicating an open universe.

4.2. Model for Shear scalar proportional to expansion scalar

The shear scalar σ^2 is proportional to scalar expansion so that we can take (Collins et al., [32])

$$A = B^n \tag{47}$$

where $n \neq 1$ is a positive constant and preserves the isotropic character of the spacetime.

Using the equation (47) in the equation (17), we get

$$\frac{\ddot{B}}{B} + (n+1)\frac{\dot{B}^2}{B^2} = \frac{k}{(n-1)B^2} \tag{48}$$

We get

$$2\ddot{B} + 2(n+1)\frac{\dot{B}^2}{B} = \frac{2k}{(n-1)B} \tag{49}$$

equation (49) further reduces to

$$\dot{B}^2 = \frac{k}{n^2 - 1} + CB^{-2n-2} \tag{50}$$

where C is a constant of integration and for $n = -2$ we get

$$\begin{aligned} B &= \cosh\left(\ell_1 t + \ell_2\right)^{1/3} \\ A &= \cosh\left(\ell_1 t + \ell_2\right)^{n/3} \end{aligned} \tag{51}$$

The spatial volume is given by

$$V = \left[\cosh\left(\ell_1 t + \ell_2\right)\right]^{\frac{n+2}{3}} \tag{52}$$

The average Hubble parameter is

$$H = \frac{1}{3}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right) = \frac{(n+2)}{9} \cdot \tanh\left(\ell_1 t + \ell_2\right) \tag{53}$$

The values of the directional parameters are infinite at $t = 0$ and tend to zero as $t \longrightarrow \infty$. The mean Hubble parameter $H \longrightarrow 0$ as $t \longrightarrow \infty$ i.e. the rate of expansion of the universe is decreasing.

Now using equations (51) in equation (15) we obtain the scalar field as

$$\phi^{\frac{n+2}{2}} = \left(\frac{n+2}{2}\right) \operatorname{sech}^{\frac{2(n+2)}{3}}\left(\ell_1 t + \ell_2\right) + \phi_0 \tag{54}$$

where ϕ_0 are constants of integration. Taking $\phi_0 = 0$, we have

$$\phi^{\frac{n+2}{2}} = \left(\frac{n+2}{2}\right) \operatorname{sech}^{\frac{2(n+2)}{3}}\left(\ell_1 t + \ell_2\right) \tag{55}$$

The scalar expansion θ is

$$\theta = 3H = (n+2)\tanh\left(\ell_1 t + \ell_2\right) \tag{56}$$

The shear scalar σ^2 is

$$\sigma^2 = \frac{3}{2}\Delta H^2 = \left[\frac{(n-1)}{3} \cdot \tanh\left(\ell_1 t + \ell_2\right)\right]^2 \tag{57}$$

The average anisotropy parameter is

$$A_B = \frac{2(n-1)^2}{(n+2)^2} \tag{58}$$

It is found that $\frac{\sigma^2}{0^2} \neq 0$ and the anisotropy parameter do not vanish except at n = 1. We observe that at $t = 0$ the mean anisotropy parameter is not zero i.e. in the early stage the universe found to be anisotropic. The shear scalar $\sigma = 0$ as $t \longrightarrow \infty$ i.e. at late time matter has no shear.

Using equation (52) in equation (27) we get the energy density of dark matter as

$$\rho_m = \rho_0 \left[\operatorname{sech}\left(\ell_1 t + \ell_2\right)\right]^{\frac{n+2}{3}} \tag{59}$$

where $\rho_0 > 0$ is a real constant of integration.

Using equation (54) in equation (8) we get, the density of Barrow HDE as

$$\rho_B = 3M_p^2\left[\frac{(n+2)}{9}\left(\frac{\alpha(n+2)}{81} - \frac{\gamma\ell_1}{9}\right)\tanh^2\left(\ell_1 t - \ell_2\right) + \frac{\gamma(n+2)\ell_1}{9}\right]^{1-\frac{\Delta}{2}} \tag{60}$$

Using equations (51) and (52) in equation (13) we get

$$\begin{aligned} p_B = -M_p^2 &\frac{k_1\tanh^2\left(\ell_1 t + \ell_2\right) + k_2\Big]^{1-\frac{\Delta}{2}} \cdot \left(1 - \frac{\Delta}{2}\right) \cdot 2k_1\tanh\left(\ell_1 t + \ell_2\right)k_3}{\frac{n+2}{3}\tanh\left(\ell_1 t + \ell_2\right)} \\ &- \left[3k_1\tanh^2\left(\ell_1 t + \ell_2\right) + 3k_2\right]^{1-\frac{\Delta}{2}} \end{aligned} \tag{61}$$

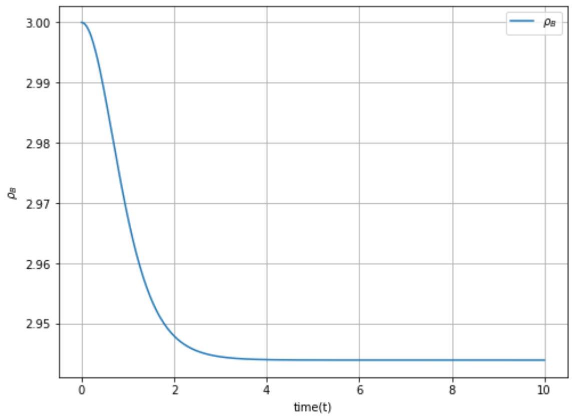

Figure 3. Plot of Density of Barrow HDE vs time, the energy density of Hypersurface homogeneous Barrow HDE model in Granda-Oliveros cut-off decreases.

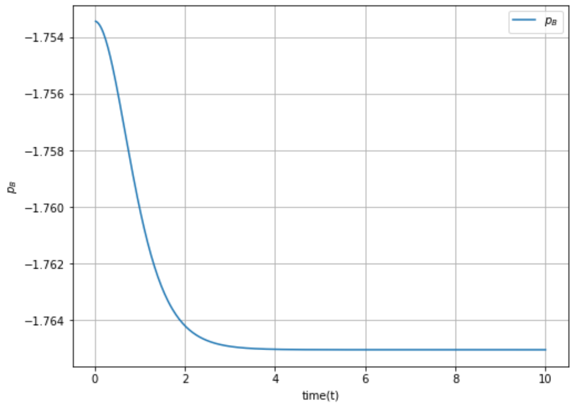

Figure 4. Plot of pressure of BHDE vs time, the pressure p_B of BHDE decreases with time and approaches zero as time goes to infinity.

where

$$k_1 = \frac{n+2}{9}\left(\frac{\alpha(n+2)}{81} - \frac{\gamma\ell_1}{9}\right), k_2 = \frac{\gamma(n+2)\ell_1}{9}, k_3 = \left|\left(1-\tanh^2(\ell_1 t + \ell_2)\right)\cdot \ell_1\right|$$

Using equation (53) in equation (43) the equation of state BHDE parameter is obtained as

$$\omega_B = -1 - \frac{(2-\Delta)\ell_1\left(2\alpha\frac{n+2}{9} - 2\gamma\right)\operatorname{sech}^2(\ell_1 t + \ell_2)}{6\left[\alpha\left(\frac{(n+2)^2}{81}\right)\tanh^2(\ell_1 t + \ell_2) + \gamma\left(\frac{n+2}{9}\right)\operatorname{sech}^2(\ell_1 t + \ell_2)\right]} \tag{62}$$

Matter density parameter Ω_m and the holographic dark energy parameter Ω_h are given by

$$\Omega_m = \frac{\rho_m}{3H^2} \text{ and } \Omega_B = \frac{\rho_B}{3H^2} \tag{63}$$

Using equations (53), (59), (60) and (63) we get the overall density parameter as

$$\Omega = \Omega_m + \Omega_B$$
$$= \frac{27M_p^2\rho_0\left[\operatorname{sech}(\ell_1 t + \ell_2)\right]^{\frac{n+2}{3}}\left[\frac{(n+2)}{9}\left(\frac{\alpha(n+2)}{81} - \frac{\gamma\ell_1}{9}\right)\tanh^2(\ell_1 t - \ell_2) + \frac{\gamma(n+2)\ell_1}{9}\right]^{1-\frac{\Delta}{2}}}{(n+2)^2\tanh^2(\ell_1 t + \ell_2)} \tag{64}$$

From equation (64) show how the density parameter evolves as a function of time. Here are the key features we expect: 1. Early Evolution: When t is small, Ω may be larger than 1 , indicating a matter-dominated or radiation-dominated

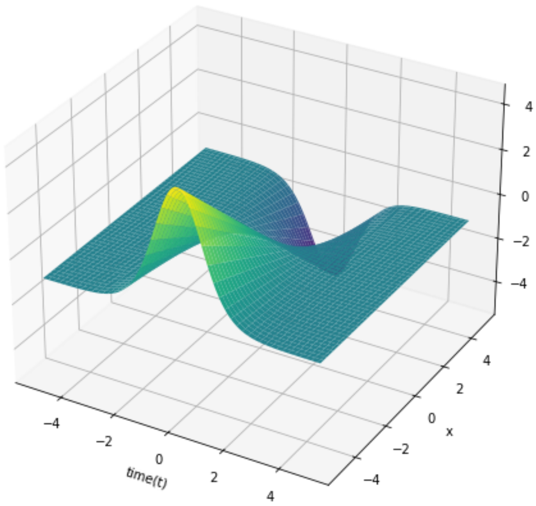

Figure 5. Plot of EoS parameter versus cosmic time t

universe. 2. Transition Period: As time progresses, the function will likely show a smooth transition where the contribution from dark energy becomes dominant. 3. Late Evolution: For large values of t, Ω might approach 1 , signifying that the universe has reached a state dominated by dark energy or a cosmological constant, and the expansion is accelerating. From figure (5) observed that the regime of the EOS parameter ω_B changes with Δ, transitioning from a phantom energy regime $\omega_B < -1$) for $\Delta < 2$ to a vacuum energy regime ($\omega_B = -1$) for $\Delta = 2$ and then to a quintessence regime ($\omega_B > -1$) for $\Delta > 2$. The specific plots of ω_B against t for different Δ values will illustrate these transitions and provide a visual understanding of the behavior of the parameter in different cosmological regimes.

From equation (57), it is clear that the expansion scalar is decreasing function of time which approaches constant at large time. The universe is expanding with constant rate at the present. Scalar expansion versus time. The scalar expansion (θ) and the shear scalar (σ^2) diverge with $t \to 0$. The parameters σ^2 and θ are a decreasing function of time t and vanish as $t \to \infty$. The deceleration parameter tends to be negative at $t \to \infty$, i.e. the universe is accelerating at present, which is in accordance with the observational result. One can see that the mean anisotropic parameter is not zero i.e. the model is anisotropic. In this regime, we also note that the anisotropy of the fluid does not act so as to increase the anisotropy of the expansion. The density and the EoS parameter are dynamical quantities $\rho_B \to 0, \omega \to$ -1 as $t \to \infty$. i.e. the model represents a vacuum universe and is mathematically equivalent to cosmological constant.

5. CONCLUSIONS

In this paper we have investigated a hypersurface homogeneous space-time filled with two minimally interacting fluids, dark matter and Barrow holographic dark energy in the Saez-Ballester scalar-tensor theory of gravitation in Lyra gemetry. To obtain exact solutions of the Saez-Ballester field equations we use (i) models of the constant deceleration parameter of the universe and (ii) the relation between the scalar field ϕ and the average scale factor. We have also computed some of the physical and kinematical parameters of the model, and their cosmological significance is described. It is found that $\frac{\sigma^2}{\sigma^2} \neq 0$ and the anisotropy parameter do not vanish except at $n = 1$. The coincidence parameter is a decreasing function of time. The spatial volume (V) of the universe increases with cosmic time so that there is a spatial expansion of the universe with time t. The parameters H, θ, σ^2 all diverge at the initial epoch, i.e.at $t = 0$ and they all tend to zero for infinite time. Also, the physical parameters $\rho_m, \rho_h, p_B, \phi$ diverge for $t = 0$ while they all vanish for infinite time. Therefore, the model has a big bang singularity at $t = 0$. We can observe that $A_B \neq 0$ and this indicates that this present model is anisotropic throughout the evaluation of the universe. The average anisotropy parameter and shear scalar vanish for $n = 1$. This shows that the universe will become shear free and isotropic in finite time. It may be observed that the EoS parameter turns out to be vacuum universe and mathematically equivalent to cosmological constant for $\Delta = 2$. Also, the average density decreases with time.

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ГОЛОГРАФІЧНА ТЕМНА ЕНЕРГІЯ БАРРОУ В СКАЛЯРНОМУ ПОЛІ САЄЗА-БАЛЛЕСТЕРА ТА ГЕОМЕТРІЇ ЛІРИ

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У цій статті досліджується динамічна поведінка гіперповерхневих однорідних просторово-часових космологічних моделей у рамках скалярно-тензорної теорії гравітації, сформульованої Саезом і Баллестером (Phys. Lett. A, 113, 467 1986) у геометрії Ліри. Ми представляємо дві космологічні моделі, отримані з цієї теорії шляхом вирішення рівнянь поля з використанням: (i) спеціального закону зміни для параметра Хаббла та (ii) пропорційного співвідношення між скалярним зсувом σ^2 і скалярним розширенням θ, як описано Коллінзом та ін. (Gen. Rel. Grav. 12, 805 1980). Для кожної моделі ми оцінюємо ключові динамічні параметри, включаючи параметр рівняння стану (EoS), параметр уповільнення, параметр вимірювача стану та параметр загальної густини енергії темної енергії. Додатково визначаємо скалярне поле в обох моделях. Наші висновки вказують на те, що ці моделі описують прискорене розширення Всесвіту, причому теоретичні результати демонструють розумну згоду з даними спостережень.

Ключові слова: *гіперповерхневий однорідний простір-час; голографічна темна енергія; Скалярно-тензорна теорія тяжіння*