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APPLICATIONS OF MULTI-REDUCTION AND MULTI-SOLITON ANALYSIS OF (2+1) ZAKHAROV-KUZNETSOV (ZK) EQUATION

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We study the Zakharov-Kuznetsov (ZK) equation with the triple-power law non-linearity. We determine the invariance properties and construct classes of conservation laws and show how the relationship leads to double reductions of the systems, yielding stable solutions such as travelling waves and solitons. This relationship is determined by recent results involving ‘multipliers’ that lead to ‘total divergent systems’. Multi-solitons analysis is performed using invariance transformation, producing stable multi-soliton structures, alongside vortex soliton solutions that exhibit localized, bell-shaped profiles. A comparison between symmetry and multi-reduction is presented, highlighting the efficacy in achieving integrable outcomes. The physical interpretation of soliton solutions is also discussed in this study, emphasizing their stable propagation and relevance to modeling coherent ion-acoustic and vortex waves in magnetized plasmas.

Keywords: *Double Reduction; Conservation Laws; Multipliers; Zakharov-Kuznetsov Equation; Invariance analysis; Solitons; Multi-solitons; Vortex solitons*

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1. INTRODUCTION AND BACKGROUND

A large class of equations in space science and mathematical physics are special cases of the Zakharov-Kuznetsov (ZK) equation

$$u_t + G(u)u_x + \sigma(u_{xx} + u_{yy})_x = 0. \tag{1}$$

For example, the triple-power law non-linearity [1] is given by $G(u) = au^n + bu^{2n} + cu^{3n}$, which will be the main focus here. Various other versions have been studied, inter Biswas, A and Zerrad [2]. The Zakharov-Kuznetsov (ZK) equation is used to model non-linear ion-acoustic waves [3] through a plasma [4, 5] in the presence of a magnetic field and also construed as an extension of the Korteweg-de Vries (KdV) equation in higher dimension, where the u_{xxx} and u_{xyy} are levels of dispersion. The ZK equation plays a valuable role for understanding multidimensional wave phenomena in plasma physics [6, 7] to study ion-acoustic solitons [8] that maintain shape and weak turbulence. With applications in space physics and fusion research, the ZK equation offers opportunities to explore facts regarding wave interactions, stability, and turbulence in two and three-dimensional plasma systems by building non-linearity and dispersive effects.

The Lie symmetry approach is now an established route for the reduction of differential equations and its advantages in the analysis of nonlinear partial differential equations (PDEs) is vast. The method centres around the algebra of one parameter Lie groups of transformations that are admitted by the PDE; once known, the reduction of the PDE is standard and may lead to exact (symmetry invariant) solutions [9, 10].
There are a number of reasons to find conserved densities of PDEs. Some conservation laws are physical (e.g., conservation of momentum and energy) and others facilitate analysis of the PDE and predicts integrability. Also, some reasons are related to the numerical solution of PDEs. For example, one should check whether the conserved quantities are in fact constant [11]. For instance, if $u = u(t, x, y)$ and $u \to 0$ for $|x| \to \pm\infty$, the conserved form $D_t\Phi^t + D_x\Phi^x + D_y\Phi^y = 0$ implies

$$\int_{-\infty}^{\infty} D_t\Phi^t = constant, \tag{2}$$

for all solutions of the PDE. Lastly, the use of symmetry properties of a given system of partial differential equations to construct or generate new conservation laws from known conservation laws has been investigated extensively [12–14].

We will discuss the line travelling wave reductions for ZK equation (1) with a 2-dimensional symmetry algebra in (2+1)-dimensions such as line solitons and eikonal waves, via invariance under two commuting translations given by

$$X_1 = \left(\mu^2 + \nu^2\right)\partial_t + \mu\partial_x + \nu\partial_y, \quad X_2 = \nu\partial_x - \mu\partial_y. \tag{3}$$

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Reduction under the above type of commuting translation symmetries is very common in applications and yields interesting types of solutions called line travelling wave solutions, under similarity invariant transformations, $\zeta = \mu x + \nu y - t$ and $u(t, x, y) = U(\zeta)$. The physics of the line travelling wave is presented in [15] and yields the solutions corresponding to the reduction

$$u(t, x, y) \to U(\zeta), \tag{4}$$

and satisfy the reduced ODE obtained from the PDE via double reduction under the condition of symmetry invariance that is associated with a conservation laws and can be expressed by the canonical transformation

$$(\partial_t, \partial_x, \partial_y, \partial_u) \to (\partial_\zeta, \partial_\rho, \partial_\chi, \partial_\nu), \tag{5}$$

under

$$x \to \zeta = \mu x + \nu y - t, \tag{6}$$
$$y \to \rho = c^2 t; \quad c = 1/\sqrt{\mu^2 + \nu^2}, \tag{7}$$
$$t \to \chi = \frac{x}{\nu} \quad (\text{or} = -\frac{y}{\mu}), \tag{8}$$
$$u \to u = U(\zeta), \tag{9}$$

for which the pair of symmetries jointly take the form as

$$\mathrm{X}_1 = \partial_\rho, \quad \mathrm{X}_2 = \partial_\chi. \tag{10}$$

This transformations sends a conservation law $\mathrm{Div}(T, \Phi^x, \Phi^y) = 0$ to an equivalent canonical form $\mathrm{Div}(\bar{T}, \bar{\Phi}^\rho, \bar{\Phi}) = 0$ and can be expressed in conserved vector form

$$(T, \Phi^x, \Phi^y) \to (\bar{T}, \bar{\Phi}^\rho, \bar{\Phi}^\chi) \tag{11}$$

Finally, the reduction of conservation laws can be obtained as every invariant conservation law $D_\zeta \bar{T} + D_\rho \bar{\Phi}^\rho + D_\chi \bar{\Phi}^\chi = 0$ reduces to $D_\zeta \Psi = 0$ which is a first integral $\Psi = k_1$, where k_1 is a constant. The explicit formula for the first integral is given by

$$\Psi(\zeta, U, U', \ldots) = \bar{T} + \int \left(\partial_\rho \bar{\Phi}^\rho + \partial_\chi \bar{\Phi}^\chi\right) d\zeta.$$

This is important to note that the double reduction can only be carried out if the conservation law (T, Φ^x, Φ^y) is associated with X_1 and X_2, as studied in [12]. Since association of symmetries with conservation laws is challenging and the calculations are tedious, the symmetry association with conservation laws is equivalent to and can easily be built using multipliers instead of conservation laws. The association of symmetry with a multiplier is studied in [16].

2. SYMMETRY GENERATORS AND CONSERVATION LAWS

We now present some preliminaries [17, 18] [19, 20]. Consider an rth-order system of partial differential equations (PDEs) of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$

$$G^\mu(x, u, u_{(1)}, \ldots, u_{(r)}) = 0, \quad \mu = 1, \ldots, \tilde{m}, \tag{12}$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(r)}$ denote the collections of all first, second, ..., rth-order partial derivatives, that is, $u^\alpha_i = D_i(u^\alpha), u^\alpha_{ij} = D_j D_i(u^\alpha), \ldots$ respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + \ldots, \quad i = 1, \ldots, n, \tag{13}$$

where the summation convention is used whenever appropriate. A current $\Phi = (\Phi^1, \ldots, \Phi^n)$ is conserved if it satisfies

$$D_i \Phi^i = 0, \tag{14}$$

along the solutions of (12). It can be shown that every admitted conservation law arises from *multipliers* $Q_\mu(x, u, u_{(1)}, \ldots)$ such that

$$Q_\mu G^\mu = D_i \Phi^i, \tag{15}$$

holds identically (i.e., off the solution space) for some current Φ^i.

We determine the conserved flows by first constructing the multipliers Q_μ which are obtained by noting that **the Euler(-Lagrange) operator, $\frac{\delta}{\delta u^\alpha}$, annihilates total divergences**, i.e., a defining equation, for Q_μ, would be

$$\frac{\delta}{\delta u^\alpha}[Q_\mu G^\mu] = 0. \tag{16}$$

The conserved flow Φ may be obtained in a number of ways, for instance, by a well known 'homotopy' formula. We consider second multiplier in the space of $(x, y, t, u, u_x, u_t, u_y, u_{xx}, u_{tx}, u_{xy}, u_{ty}, u_{yy}, u_{tt})$ given by

$$Q := f\left(x, y, t, u, u_x, u_t, u_y, u_{xx}, u_{tx}, u_{xy}, u_{ty}, u_{yy}, u_{tt}\right)$$

The symbolic computation provide us the set of multipliers for ZK equation (1) given by

$$Q = c_1 u_{tt} + c_1 u_{xx} + \frac{c_1 u^{n+1}}{\sigma\ (n+1)} + \frac{b c_1 u^{2\,n+1}}{\sigma\ (2\,n+1)} + \frac{c c_1 u^{3\,n+1}}{\sigma\ (3\,n+1)} + c_2 u + F(y)$$

The multiplier Q_1 is arbitrary function of y given by

$$Q_1 = F(y) = y.$$

Multiplier Q_2 and Q_3 are

$$Q_2 = u, \quad Q_3 = u_{tt} + u_{xx} + \frac{u^{n+1}}{\sigma\ (n+1)} + \frac{b u^{2\,n+1}}{\sigma\ (2\,n+1)} + \frac{c u^{3\,n+1}}{\sigma\ (3\,n+1)}.$$

The Conservation laws corresponding to multiplier $Q_1 = 1$ and $Q_2 = u$ respectively are given by

$$\begin{aligned}(T_1, \Phi_1^y, \Phi_1^x) = (u, \tfrac{2}{3}\sigma u_{xy}, \tfrac{1}{6\,n^3+11\,n^2+6\,n+1}[&6\,n^3\sigma\, u_{xx} + 2\,n^3\sigma\, u_{yy} + 6\,u^{n+1} a n^2 + 3\,u^{2\,n+1} b n^2\\ &+2\,u^{3\,n+1} c n^2 + 11\,n^2\sigma\, u_{xx} + \tfrac{11}{3}\,n^2\sigma\, u_{yy} + 5\,u^{n+1} a n + 4\,u^{2\,n+1} b n + 3\,u^{3\,n+1} c n\\ &+6\,n\sigma\, u_{xx} + 2\,n\sigma\, u_{yy} + u^{n+1} a + u^{2\,n+1} b + u^{3\,n+1} c + \sigma\, u_{xx} + \tfrac{1}{3}\,\sigma\, u_{yy}]).\end{aligned}$$

and

$$\begin{aligned}(T_1, \Phi_1^y, \Phi_1^x) = (\tfrac{1}{2}u^2, \tfrac{1}{3}\sigma\left(2u u_{xy} - u_x u_y\right), \tfrac{1}{18\,n^3+66\,n^2+72\,n+24}[&-33\,n^2\sigma\, {u_x}^2 - 36\,n\sigma\, {u_x}^2\\ &-9\,n^3\sigma\, {u_x}^2 + 30\,u^{2+n} a n + 18\,u^{2+3\,n} c n - 3\,n^3\sigma\, {u_y}^2 - 11\,n^2\sigma\, {u_y}^2 - 12\,n\sigma\, {u_y}^2\\ &+18\,u^{2+n} a n^2 + 24\,u^{2+2\,n} b n + 6\,u^{2+3\,n} c n^2 + 9\,u^{2+2\,n} b n^2 + 24\,u\sigma\, u_{xx} + 8\,u\sigma\, u_{yy}\\ &+12\,u^{2+n} a + 12\,u^{2+2\,n} b + 12\,u^{2+3\,n} c - 12\,\sigma\, {u_x}^2 - 4\,\sigma\, {u_y}^2 + 18\,n^3\sigma\, u u_{xx} + 66\,n^2\sigma\, u u_{xx}\\ &+72\,n\sigma\, u u_{xx} + 6\,n^3\sigma\, u u_{yy} + 22\,n^2\sigma\, u u_{yy} + 24\,n\sigma\, u u_{yy}]).\end{aligned}$$

In the next section, we will check the association of multipliers with symmetries so that we can use conservation law theorem to perform double reduction.

3. DOUBLE REDUCTION AND INVARIANT SOLUTIONS

Applications to double reductions has been studied in various ways [21, 22] via conservation Law theorem under double reduction theory. In this study, we discussed the double reduction via multiplier association. Furthermore, for the $(1+2)$ case, if $X = \xi\partial_x + \theta\partial_y + \tau\partial_t + \eta\partial_u$ is a Lie point symmetry that leaves a scalar PDE in (12), say,

$$G(x, t, u, u_x, u_y, u_t, u_{xx}\ldots) = 0, \tag{17}$$

invariant with

$$XG = RG, \tag{18}$$

and

$$\lambda = D_t\tau + D_x\xi + D_y\theta, \tag{19}$$

such that

$$XQ + (R+\lambda)Q = 0, \tag{20}$$

then X is associated with the corresponding conserved flow $\Phi = (T, \phi^x, \phi^y)$ and, via X and Φ, double reduction may be obtained [15, 16]. Here, $D_t T + D_x\phi^x + D_y\phi^y = 0|_{(17)}$. Moreover, in transformed coordinates (ζ, ρ, χ, U), $\bar{X}_1 = \partial_\rho$ and $\bar{X}_2 = \partial_\chi$ so that

$$X_1(\rho) = 1, \qquad X_2(\chi) = 1,$$

and $\bar{\Phi} = (\bar{T}, \phi^{\rho}, \phi^{\chi})$ leads to $D_{\zeta}\bar{T} + D_{\rho}\phi^{\rho} + D_{\chi}\phi^{\chi} = 0$. Thus,

$$D_t T + D_x \phi^x + D_y \phi^y = -\frac{dT}{d\zeta} + c^2\frac{dT}{d\rho} + \mu\frac{d\Phi^x}{d\zeta} + \frac{1}{\nu}\frac{d\Phi^x}{d\chi} + \nu\frac{d\Phi^y}{d\zeta}, \tag{21}$$

so that

$$\begin{aligned} \bar{T} &= -T + \mu\Phi^x + \nu\Phi^y, \\ \phi^{\rho} &= c^2 T, \\ \phi^{\chi} &= \tfrac{1}{\nu}\Phi^x. \end{aligned} \tag{22}$$

Finally, we obtain the double reduction of the original system by following

$$\Psi = \bar{T} + \int_{-\infty}^{\infty}(\frac{d\Phi^{\rho}}{d\rho} + \frac{d\Phi^{\chi}}{d\chi})d\zeta = 0. \tag{23}$$

given by

$$k = \bar{T}|_{\zeta,U,U',\ldots}, \tag{24}$$

where k is a constant - for details, see [15]. Finally the equation (24) provide us reduced ODEs, which are second order ODEs given by

$$\begin{aligned} &\mu\,(1+n)b(1+3\,n)U(\zeta)^{1+2\,n} - 6\,(-\tfrac{1}{3}\,c\mu\,(1+n)U(\zeta)^{1+3\,n} \\ &+(-\mu\,\sigma\,(\mu^2+\nu^2)(1+n)\tfrac{d^2}{d\zeta^2}U(\zeta) - U(\zeta)^{1+n}a\mu \\ &+U(\zeta)(1+n))(\tfrac{1}{3}+n))(\tfrac{1}{2}+n) = k_1(6\,n^3+11\,n^2+6\,n+1), \end{aligned} \tag{25}$$

ODE (25) is second order reducible ODE with missing independent variable ζ.

$$\frac{d^2}{d\zeta^2}U(\zeta) = \frac{1}{6}\frac{C}{(1+n)\mu\sigma(\mu^2+\nu^2)(\frac{1}{3}+n)(\frac{1}{2}+n)}, \tag{26}$$

where

$$\begin{aligned} C = &[k_1(6n^3+11n^2+6n+1) - 3\mu(1+n)b(\tfrac{1}{3}+n)(U(\zeta))^{1+2n} \\ &+6(-\tfrac{1}{3}c\mu(1+n)(U(\zeta))^{1+3n} + (-(U(\zeta))^{1+n}a\mu \\ &+U(\zeta)(1+n))(\tfrac{1}{3}+n))(\tfrac{1}{2}+n)]. \end{aligned}$$

For $n = 1$, (27) has solution in integral representation takes the form given by

$$\zeta - \left(\int^{U(\zeta)} \frac{30\mu\sigma\left(\mu^2+\nu^2\right)}{\sqrt{-30\mu\sigma\left(\mu^2+\nu^2\right)\left(3\Omega^5 c\mu + 5\Omega^4 b\mu + 10\Omega^3 a\mu + 60c_1\mu^3\sigma + 60c_1\mu\nu^2\sigma - 30\Omega^2 - 60\Omega_1\right)}} d\Omega\right) - c_2 = 0$$

On the other hand, ODE (27) is second order non-linear ODE with missing independent variable ζ presented by

$$\begin{aligned} &3\,(\tfrac{2}{3}+n)\mu\,(2+n)b(U(\zeta))^{2+2\,n} + 6\,(1+n)(\tfrac{1}{3}\,c\mu\,(2+n)(U(\zeta))^{2+3\,n} \\ &+((U(\zeta))^{2+n}a\mu - \tfrac{1}{2}\,(-2\,\mu\,\sigma\,U(\zeta)(\mu^2+\nu^2)\tfrac{d^2}{d\zeta^2}U(\zeta) + \mu\,\sigma\,(\mu^2+\nu^2)(\tfrac{d}{d\zeta}U(\zeta))^2 \\ &+(U(\zeta))^2)(2+n))(\tfrac{2}{3}+n)) = k_2(6\,n^3+22\,n^2+24\,n+8). \end{aligned} \tag{27}$$

For $n = 1$, (27) has solution in its integral representation given by

$$\int^{U(\zeta)}\left(-\frac{\sqrt{30}\mu\sigma\left(\mu^2+\nu^2\right)}{\sqrt{\mu\sigma\left(\mu^2+\nu^2\right)\left(-3\Omega^5 c\mu - 5\Omega^4 b\mu + 30\Omega\mu^3\sigma c_1 + 30\Omega\mu\nu^2\sigma c_1 - 10\Omega^3 a\mu + 30\Omega^2 - 60k_2\right)}}\right)d\Omega - \zeta - c_2 = 0$$

3.1. Analysis of ODE (25) and Travelling Wave Solution

An analysis of (25) is presented in this section. ODE (25) can further reduce using invariance analysis. The ODE (25) reduces to first order ODE by single reduction using the symmetry $X_1 = \partial\zeta$. The reduced ODE is given by

$$6\,w\,(\alpha)\left(\tfrac{d}{d\alpha}w\,(\alpha)\right)\mu\,n^3\nu^2\sigma + 11\,w\,(\alpha)\left(\tfrac{d}{d\alpha}w\,(\alpha)\right)\mu\,n^2\nu^2\sigma + 6\,w\,(\alpha)\left(\tfrac{d}{d\alpha}w\,(\alpha)\right)\mu\,n\nu^2\sigma$$

$$+3\,\mu\,b\alpha^{1+2\,n}n^2+2\,c\mu\,\alpha^{1+3\,n}n^2+6\,n^2\alpha^{1+n}a\mu+4\,\mu\,b\alpha^{1+2\,n}n+3\,c\mu\,\alpha^{1+3\,n}n$$
$$+5\,n\alpha^{1+n}a\mu+w\,(\alpha)\left(\tfrac{d}{d\alpha}w\,(\alpha)\right)\mu^3\sigma+\alpha^{1+n}a\mu+\mu\,b\alpha^{1+2\,n}+c\mu\,\alpha^{1+3\,n}-6\,k_1\,n^3-11\,k_1\,n^2$$
$$-6\,k_1\,n-\alpha-k_1-6\,\alpha\,n-11\,\alpha\,n^2-6\,\alpha\,n^3+6\,w\,(\alpha)\left(\tfrac{d}{d\alpha}w\,(\alpha)\right)\mu^3n\sigma+w\,(\alpha)\left(\tfrac{d}{d\alpha}w\,(\alpha)\right)\mu\,\nu^2\sigma$$
$$+6\,w\,(\alpha)\left(\tfrac{d}{d\alpha}w\,(\alpha)\right)\mu^3n^3\sigma+11\,w\,(\alpha)\left(\tfrac{d}{d\alpha}w\,(\alpha)\right)\mu^3n^2\sigma=0, \tag{28}$$

under the similarity variables $w\,(\alpha)=\frac{d}{dx}U\,(\zeta)$ and $\alpha=U\,(\zeta)$. Using Lie symmetry, we get a single reduction as symmetry reduces second order ODE (25) to first order ODE (28), solving with respect to α get us to the invariant solution given by

$$w\,(\alpha)=-\frac{\sqrt{A}}{\sigma\,\mu\,\,(3\,n+2)\,(1+2\,n)\,(1+3\,n)\,(n+2)\,(1+n)\left(\mu^2+\nu^2\right)}, \tag{29}$$

where

$$A=-648(1+n)(\tfrac{1}{3}+n)(\tfrac{1}{9}(1+n)\mu(\tfrac{1}{2}+n)c(n+2)\alpha^{3n+2}$$
$$+(\tfrac{1}{3}+n)(\tfrac{1}{4}b\mu(n+2)\alpha^{2n+2}+(\tfrac{1}{2}+n)(a\mu\alpha^{n+2}-1/2(n+2)(1+n)(c_1\mu^3\sigma+$$
$$c_1\mu\nu^2\sigma+\alpha^2+2k_1\alpha)))(n+\tfrac{2}{3}))(\mu^2+\nu^2)\mu(\tfrac{1}{2}+n)\sigma(n+2/3)(n+2).$$

The invariant solution can be written as

$$U(\zeta)=w(\alpha)\zeta+c_1.$$

The invariant solution in integral representation is given by

$$\zeta+\int^{U(\zeta)}\frac{\sigma\,\mu\,\,(3\,n+2)\,(1+2\,n)\,(1+3\,n)\,(n+2)\,(1+n)\left(\mu^2+\nu^2\right)}{\sqrt{A}}d\Omega+c_1=0, \tag{30}$$

where

$$A=-648(1+n)(\mu^2+\nu^2)(\frac{1}{2}+n)(\frac{1}{3}+n)\sigma(n+\frac{2}{3})\mu(n+2)(\frac{1}{9}(1$$
$$+n)(\frac{1}{2}+n)c\mu(n+2)\Omega^{3n+2}+(\frac{1}{3}+n)(n+\frac{2}{3})(\frac{1}{4}b\mu(n+2)\Omega^{2n+2}+$$
$$(\frac{1}{2}+n)(a\mu\Omega^{n+2}-\frac{1}{2}(1+n)(c_1\mu^3\sigma+c_1\mu\nu^2\sigma+(\Omega)(\Omega+2k_1))(n+2)))).$$

In the similar way, solution ODE (25) in integral representation corresponding to the values of $n=1$, $n=2$ and $n=3$ are presented, respectively

$$\int^{U(\zeta)}\frac{\sqrt{30}\mu\,\sigma\,\,\left(\mu^2+\nu^2\right)}{\sqrt{-10\,\left(-3\,c_1\,\mu^3\sigma+\left(3/10\,\Omega^5c+\frac{1}{2}\,\Omega^4b+\Omega^3a-3\,\nu^2c_1\,\sigma\right)\mu-3\,\,(\Omega)\,\,(\Omega+2\,k_1)\right)\sigma\,\,\left(\mu^2+\nu^2\right)\mu}}d\Omega-\zeta-c_2=0,$$

$$\int^{U(\zeta)}\frac{2\sqrt{105}\mu\,\sigma\,\,\left(\mu^2+\nu^2\right)}{\sqrt{-70\,\sigma\,\,\left(\mu^2+\nu^2\right)\mu\,\,\left(-6\,c_1\,\mu^3\sigma+\left(3/14\,\Omega^8c+2/5\,\Omega^6b+\Omega^4a-6\,\nu^2c_1\,\sigma\right)\mu-6\,\,(\Omega)\,\,(\Omega+2\,k_1)\right)}}d\Omega-\zeta-c_2=0,$$

$$\int^{U(\zeta)}\frac{2\sqrt{385}\mu\,\sigma\,\,\left(\mu^2+\nu^2\right)}{\sqrt{-154\,\sigma\,\,\left(\mu^2+\nu^2\right)\left(-10\,c_1\,\mu^3\sigma+\left(2/11\,c\Omega^{11}+\frac{5\,b\Omega^8}{14}+a\Omega^5-10\,\nu^2c_1\,\sigma\right)\mu-10\,\,(\Omega)\,\,(\Omega+2\,k_1)\right)\mu}}d\Omega-\zeta-c_2=0.$$

The ODE (25) is cumbersome and its solution is presented above in integral form as the general value of n and k_1. Suppose a solution of the form given by

$$U(\zeta)=G(\zeta)e^{\lambda\zeta}.$$

Where $G(\zeta)$ is to be determined, and λ is a constant. This assumption reduces the ODE (25) to a manageable equation

$$-6(-\mu\sigma(\mu^2+\nu^2)(1+n)e^{\lambda\zeta}\lambda^2+e^{\lambda\zeta}(1+n))(\tfrac{1}{3}+n)(\tfrac{1}{2}+n)G(\zeta)$$
$$+3\mu(1+n)b(\tfrac{1}{3}+n)(G(\zeta)e^{\lambda\zeta})^{1+2n}-6(-\tfrac{1}{3}c\mu(1+n)(G(\zeta)e^{\lambda\zeta})^{1+3n}$$
$$+(-\mu\sigma(\mu^2+\nu^2)(1+n)(2(\tfrac{d}{d\zeta}G(\zeta))\lambda e^{\lambda\zeta}+(\tfrac{d^2}{d\zeta^2}G(\zeta))e^{\lambda\zeta})$$

$$-(G(\zeta)e^{\lambda\zeta})^{1+n}a\mu)(\tfrac{1}{3}+n))(\tfrac{1}{2}+n) = k_1(6n^3+11n^2+6n+1). \tag{31}$$

By factoring out $e^{\lambda\zeta}$ from the above equation, comparing nonlinear terms of $G(\zeta)$ and by simplifying, we get the following condition

$$6\left(\frac{1}{2}+n\right)\left(-1+\mu\sigma\left(\mu^2+\nu^2\right)\lambda^2\right)(1+n)\left(\frac{1}{3}+n\right) = 0, \tag{32}$$

that yields the values of λ given by

$$\lambda_1 = \frac{1}{\sqrt{\mu^3\sigma+\mu\nu^2\sigma}}, \quad \lambda_2 = -\frac{1}{\sqrt{\mu^3\sigma+\mu\nu^2\sigma}}. \tag{33}$$

Finally, it provide us the solutions of the form $U(\zeta) = Ge^{\lambda\zeta}$, where G is a constant, as we received $G'(\zeta) = 0$. The solutions are given by

$$U(\zeta) = c_1 e^{\frac{1}{\sqrt{\mu^3\sigma+\mu\nu^2\sigma}}\zeta} + c_2 e^{-\frac{1}{\sqrt{\mu^3\sigma+\mu\nu^2\sigma}}\zeta}. \tag{34}$$

Thus, the solution of the ZK equation (1) can be obtained in original variables by substituting the value of $\zeta = \mu x + \nu y - t$ and $u(t,x,y) = U(\zeta)$ presented by

$$u(t,x,y) = c_1 e^{\frac{1}{\sqrt{\mu^3\sigma+\mu\nu^2\sigma}}(\mu x+\nu y-t)} + c_2 e^{-\frac{1}{\sqrt{\mu^3\sigma+\mu\nu^2\sigma}}(\mu x+\nu y-t)}. \tag{35}$$

This solution represent travelling wave solution and its behaviour over time for different values of parameters is presented in Figure-1.

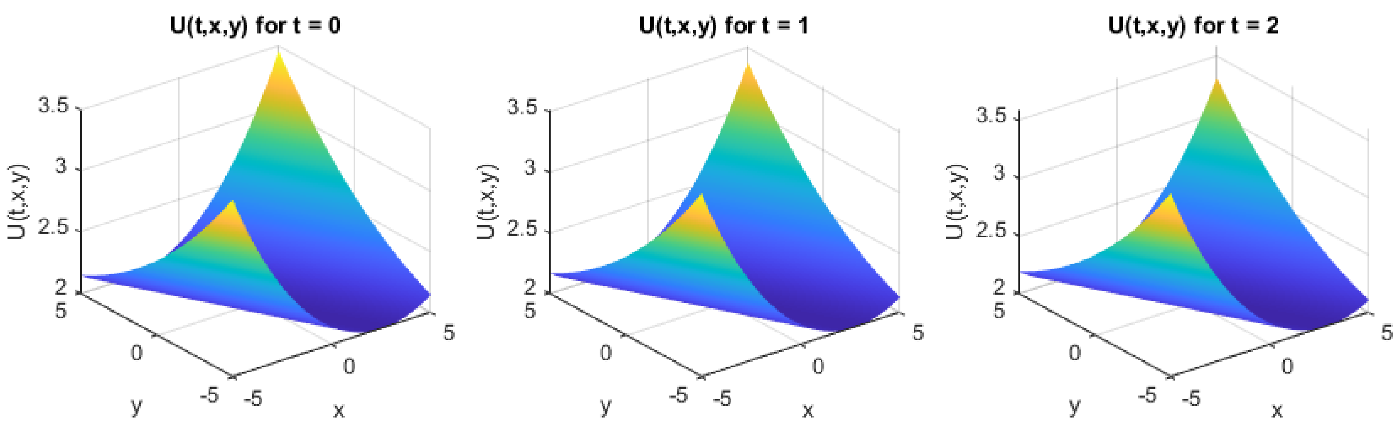

Figure 1. Travelling wave $u(t,x,y)$ over time

For another set of parameters, the solution $u(t,x,y)$ is presented in Figure-2, which is depicting the wave behaviour over time t.

3.2. Analysis of ODE (27) and Travelling Wave Solution

An analysis of (27) is presented in this section. ODE (27) can further reduce using invariance analysis. The ODE (27) reduces to first order ODE by single reduction using the symmetry $X_1 = \partial\zeta$. The reduced ODE is given by

$$\begin{gathered}
-6k_2n^3 - 22k_2n^2 - 24k_2n - 8k_2 + 4c\mu\alpha^{2+3n} + 6w(\alpha)(\tfrac{d}{d\alpha}w(\alpha))\alpha\mu^3n^3\sigma \\
+22w(\alpha)(\tfrac{d}{d\alpha}w(\alpha))\alpha\mu^3n^2\sigma + 24w(\alpha)(\tfrac{d}{d\alpha}w(\alpha))\alpha\mu^3n\sigma + 8w(\alpha)(\tfrac{d}{d\alpha}w(\alpha))\alpha\mu\nu^2\sigma \\
-11\alpha^2n^2 - 3\alpha^2n^3 - 12\alpha^2n + 6w(\alpha)(\tfrac{d}{d\alpha}w(\alpha))\alpha\mu n^3\nu^2\sigma + 22w(\alpha)(\tfrac{d}{d\alpha}w(\alpha))\alpha\mu n^2\nu^2\sigma \\
+24w(\alpha)(\tfrac{d}{d\alpha}w(\alpha))\alpha\mu n\nu^2\sigma + 8w(\alpha)(\tfrac{d}{d\alpha}w(\alpha))\alpha\mu^3\sigma - 3(w(\alpha))^2\mu n^3\nu^2\sigma - 11(w(\alpha))^2\mu n^2\nu^2\sigma \\
-12(w(\alpha))^2\mu n\nu^2\sigma + 3\mu b\alpha^{2+2n}n^2 + 2c\mu\alpha^{2+3n}n^2 + 6n^2\alpha^{2+n}a\mu + 8\mu b\alpha^{2+2n}n + 6c\mu\alpha^{2+3n}n \\
+10n\alpha^{2+n}a\mu - 11(w(\alpha))^2\mu^3n^2\sigma - 12(w(\alpha))^2\mu^3n\sigma - 4(w(\alpha))^2\mu\nu^2\sigma - 3(w(\alpha))^2\mu^3n^3\sigma - 4\alpha^2
\end{gathered}$$

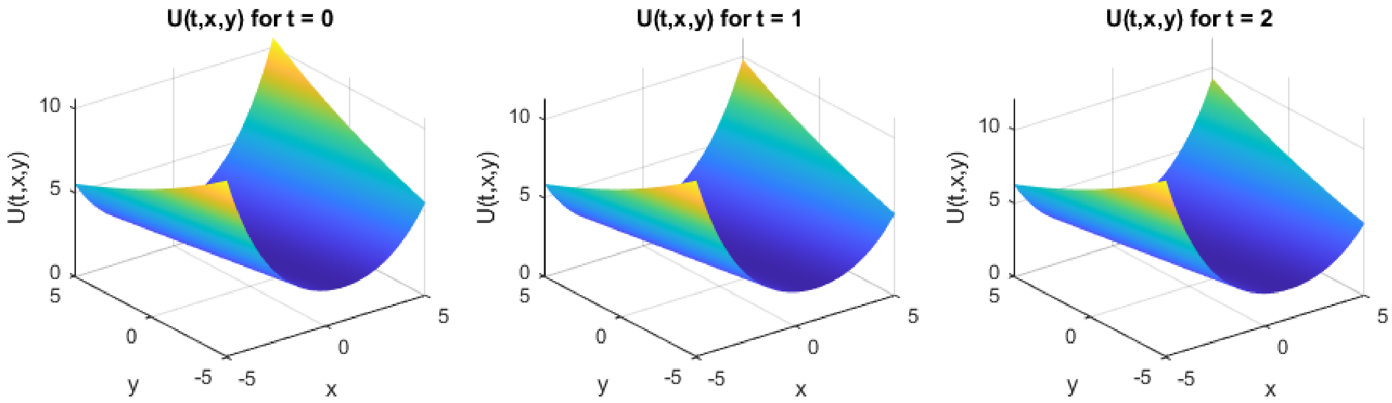

Figure 2. Travelling wave $u(t,x,y)$ over time

$$+4\mu b\alpha^{2+2n} - 4(w(\alpha))^2\mu^3\sigma + 4\alpha^{2+n}a\mu = 0, \tag{36}$$

under the similarity variables $w(\alpha) = \frac{\mathrm{d}}{\mathrm{d}\zeta}U(\zeta)$ and $\alpha = U(\zeta)$. Using Lie symmetry, we get a single reduction as symmetry reduces second order ODE (27) to first order ODE (36) in $w(\alpha)$. Solving (36) yield us the invariant solution given by

$$w(\alpha) = 1/18\frac{\sqrt{-648\alpha AC(\frac{1}{9}Cc\mu B(D)\alpha^{3+3n} + A(\frac{1}{4}b\mu(D)\alpha^{2n+3} + (\Psi)B)E)(\mu^2+\nu^2)E\mu\sigma B(D)}}{\sigma\mu\alpha(\mu^2+\nu^2)ABE(D)C},$$

where,

$$\frac{2}{3} + n = A, 2 + 3n = 3A, n + \frac{1}{2} = B, 2n + 1 = 2B,$$
$$1 + n = C, 1 + n = C, 2 + n = \mathrm{D}, n + \frac{1}{3} = E, 3n + 1 = 3E,$$
$$\alpha^{n+3}a\mu - \frac{1}{2}\alpha(2+n)(1+n)\left(\alpha\mu^3\sigma c_1 + \alpha\mu\nu^2\sigma c_1 + \alpha^2 - 2k_2\right) = \Psi.$$

Using similarity variable, the invariant solution can be written as

$$U(\zeta) = w(\alpha)\zeta + c_1.$$

The invariant solution $U(\zeta)$ is given by

$$\zeta - \int^{U(\zeta)} 18\frac{\sigma\mu\Omega ABE(D)C(\mu^2+\nu^2)}{\sqrt{-648(\frac{1}{9}C\mu cB(D)\Omega^{3A} + (\frac{1}{4}b\mu(D)\Omega^{2+2n} + (\Psi)B)AE)C\mu B(\mu^2+\nu^2)\Omega^2 A\sigma E(D)}}d\Omega + c_2 = 0,$$

where

$$\frac{2}{3} + n = A, 2 + 3n = 3A, n + \frac{1}{2} = B, 2n + 1 = 2B,$$
$$1 + n = C, 1 + n = C, 2 + n = \mathrm{D}, n + \frac{1}{3} = E, 3n + 1 = 3E,$$
$$a^{2+n}a\mu - \frac{1}{2}(2+n)(1+n)\left(a\mu^3\sigma c_1 + a\mu\nu^2\sigma c_1 + a^2 - 2k_2\right) = \Psi.$$

The ODE (27) is complex and its solution even for $n = 1$ and for particular value of $k_2 = 0$ is challenging to find as presented

$$\int^{U(\zeta)} \frac{\sqrt{30}(\mu^2+\nu^2)\sigma\mu}{\sqrt{(\mu^2+\nu^2)\sigma\mu(-3c\Omega^4\mu + 30c_1\mu^3\sigma + 30c_1\mu\nu^2\sigma - 5b\Omega^3\mu - 10\Omega^2a\mu + 30\Omega)\Omega}}d\Omega - \zeta - c_2 = 0.$$

For ODE (27), we will find the solution of the form given by

$$U(\zeta) = G(\zeta)e^{\lambda\zeta},$$

where $G(\zeta)$ is to be determined, and λ is a constant. This assumption reduces the ODE (27) to a manageable one.

$$\begin{aligned}&3\mu(n+\tfrac{2}{3})(2+n)b(G(\zeta)e^{\lambda\zeta})^{2+2n}+6(\tfrac{1}{3}c\mu(2+n)(G(\zeta)e^{\lambda\zeta})^{2+3n}+(a\mu(G(\zeta)e^{\lambda\zeta})^{2+n}\\&+\tfrac{1}{2}(e^{\lambda\zeta})^2(2G(\zeta)\mu\sigma(\mu^2+\nu^2)\tfrac{d^2}{d\zeta^2}G(\zeta)-\mu\sigma(\mu^2+\nu^2)(\tfrac{d}{d\zeta}G(\zeta))^2+2G(\zeta)\mu\sigma\lambda(\mu^2+\nu^2)\tfrac{d}{d\zeta}G(\zeta)\\&+(G(\zeta))^2(\lambda^2\mu^3\sigma+\lambda^2\mu\nu^2\sigma-1))(2+n))(n+\tfrac{2}{3}))(1+n)=k_2(6n^3+22n^2+24n+8)\end{aligned} \tag{37}$$

By factoring out $e^{\lambda\zeta}$ from the above equation, comparing nonlinear terms of $G(\zeta)$ and by simplifying, we get the following condition

$$3\ (1+n)\ (n+2/3)\ (2+n)\left(-1+\mu\,\sigma\left(\mu^2+\nu^2\right)\lambda^2\right)=0, \tag{38}$$

that yields the values of λ given by

$$\lambda_1=\frac{1}{\sqrt{\mu^3\sigma+\mu\nu^2\sigma}},\quad \lambda_2=-\frac{1}{\sqrt{\mu^3\sigma+\mu\nu^2\sigma}}. \tag{39}$$

Finally, it provide us the solutions of the form $U(\zeta)=Ge^{\lambda\zeta}$, where $G(\zeta)=c_1\zeta+c_2$ is a constant, as we received $G''(\zeta)=0$. The solutions are given by

$$U(\zeta)=(c_1\zeta+c_2)e^{\frac{1}{\sqrt{\mu^3\sigma+\mu\nu^2\sigma}}\zeta}+(c_1\zeta+c_2)e^{-\frac{1}{\sqrt{\mu^3\sigma+\mu\nu^2\sigma}}\zeta} \tag{40}$$

Thus, the solution of the ZK equation (1) can be obtained in original variables by substituting the value of $\zeta=\mu x+\nu y-t$ and $u(t,x,y)=U(\zeta)$ presented by

$$u(t,x,y)=[c_1(\mu x+\nu y-t)+c_2]e^{\frac{1}{\sqrt{\mu^3\sigma+\mu\nu^2\sigma}}(\mu x+\nu y-t)}+[c_1(\mu x+\nu y-t)+c_2]e^{-\frac{1}{\sqrt{\mu^3\sigma+\mu\nu^2\sigma}}(\mu x+\nu y-t)}$$

The solution is presented in Figure-3.

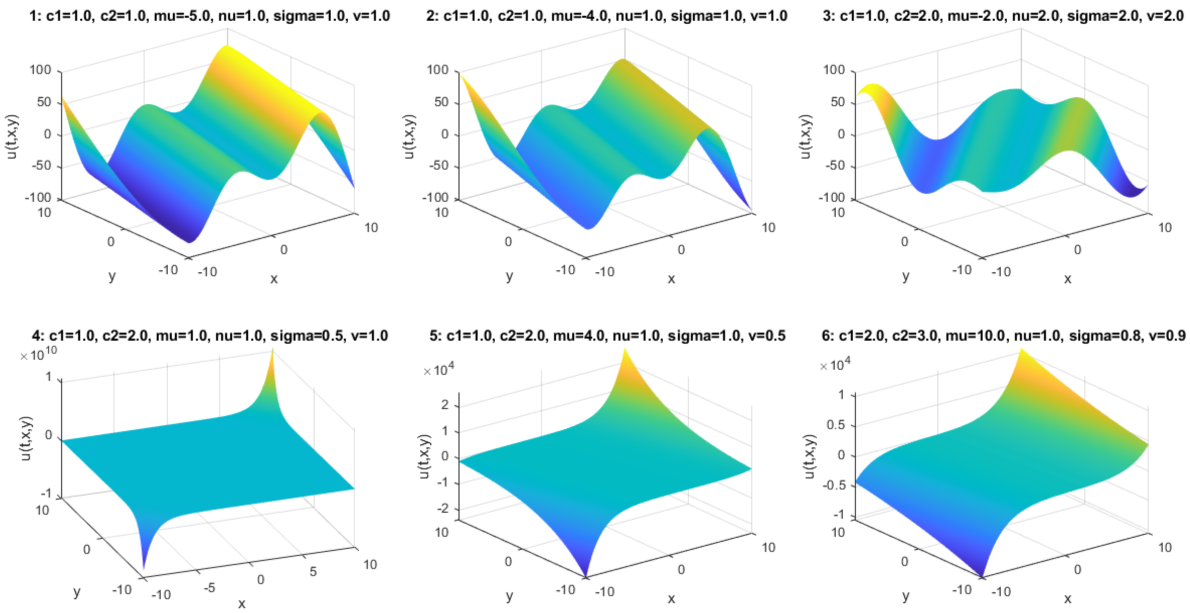

Figure 3. Travelling wave $u(t,x,y)$ over time

This solution represent travelling wave solution and its behaviour over time for different values of parameters.

4. A COMPARISON BETWEEN SYMMETRY REDUCTION AND MULTI-REDUCTION VIA MULTIPLIERS

In this section, we study the symmetry classification and possible reductions of ZK equation (1) based on the parameters involved in function $G(u)=au^n+bu^{2n}+cu^{3n}$ containing triple power non-linearities. The ZK equation (1) has three translation symmetries given by

$$X_1=D_t,\quad X_2=D_x,\ \text{and}\ X_3=D_y, \tag{41}$$

which has corresponding single reduction from $(1+2)$ ZK equation (1) to $(1+1)$ second order PDE in (α,β)-space given by respectively

$$\begin{gathered}(\tfrac{\partial}{\partial\alpha}F(\alpha,\beta))a(F(\alpha,\beta))^n+(\tfrac{\partial}{\partial\alpha}F(\alpha,\beta))b(F(\alpha,\beta))^{2n}+(\tfrac{\partial}{\partial\alpha}F(\alpha,\beta))c(F(\alpha,\beta))^{3n}\\ +\tfrac{\partial^3}{\partial\alpha^3}F(\alpha,\beta)+\tfrac{\partial^3}{\partial\beta^2\partial\alpha}F(\alpha,\beta)=0,\\ \text{with similarity variables } [\{U(x,y,t)=F(x,y)\},\{\alpha=x,\beta=y\}].\end{gathered}\tag{42}$$

,

$$\begin{gathered}\tfrac{\partial}{\partial\alpha}F(\beta,\alpha)=0,\\ \text{with similarity variables } [\{U(x,y,t)=F(y,t)\},\{\alpha=t,\beta=y\}].\end{gathered}\tag{43}$$

and

$$\begin{gathered}\tfrac{\partial}{\partial\alpha}F(\beta,\alpha)+(\tfrac{\partial}{\partial\beta}F(\beta,\alpha))a(F(\beta,\alpha))^n+(\tfrac{\partial}{\partial\beta}F(\beta,\alpha))b(F(\beta,\alpha))^{2n}\\ +(\tfrac{\partial}{\partial\beta}F(\beta,\alpha))c(F(\beta,\alpha))^{3n}+\tfrac{\partial^3}{\partial\beta^3}F(\beta,\alpha)\}=0\\ \text{with similarity variables } [\{U(x,y,t)=F(x,t)\},\{\alpha=t,\beta=x\}].\end{gathered}\tag{44}$$

Lie symmetries provide only single reduction upto a independent variable and we received $(1+1)$ reduced PDE from $(1+2)$ PDE. In order to get invariant solutions, it require further analysis to study the problem, which is even challenging to solve one dimensional PDE. Even reduction under symmetry generators

$$X^1=X_1+X_2,\quad X^2=X_1+X_3,\ \text{and } X^3=X_1+X_2+X_3\tag{45}$$

provide us the single reduction to $(1+1)$ PDE in (α,β)-space given by respectively

$$\begin{gathered}\tfrac{\partial}{\partial\beta}F(\alpha,\beta)-(\tfrac{\partial}{\partial\beta}F(\alpha,\beta))a(F(\alpha,\beta))^n-(\tfrac{\partial}{\partial\beta}F(\alpha,\beta))b(F(\alpha,\beta))^{2n}\\ -(\tfrac{\partial}{\partial\beta}F(\alpha,\beta))c(F(\alpha,\beta))^{3n}-\tfrac{\partial^3}{\partial\beta^3}F(\alpha,\beta)-\tfrac{\partial^3}{\partial\beta\partial\alpha^2}F(\alpha,\beta)=0\\ \text{with similarity variables } [\{U(x,y,t)=F(y,-x+t)\},\{\alpha=y,\beta=-x+t\}].\end{gathered}\tag{46}$$

,

$$\begin{gathered}\tfrac{\partial}{\partial\beta}F(\alpha,\beta)+(\tfrac{\partial}{\partial\alpha}F(\alpha,\beta))a(F(\alpha,\beta))^n+(\tfrac{\partial}{\partial\alpha}F(\alpha,\beta))b(F(\alpha,\beta))^{2n}\\ +(\tfrac{\partial}{\partial\alpha}F(\alpha,\beta))c(F(\alpha,\beta))^{3n}+\tfrac{\partial^3}{\partial\alpha^3}F(\alpha,\beta)+\tfrac{\partial^3}{\partial\beta^2\partial\alpha}F(\alpha,\beta)=0\\ \text{with similarity variables } [\{U(x,y,t)=F(x,-y+t)\},\{\alpha=x,\beta=-y+t\}].\end{gathered}\tag{47}$$

and

$$\begin{gathered}\tfrac{\partial}{\partial\alpha}F(\beta,\alpha)-(\tfrac{\partial}{\partial\beta}F(\beta,\alpha))a(F(\beta,\alpha))^n-a(F(\beta,\alpha))^n\tfrac{\partial}{\partial\alpha}F(\beta,\alpha)-(\tfrac{\partial}{\partial\beta}F(\beta,\alpha))b(F(\beta,\alpha))^{2n}\\ -b(F(\beta,\alpha))^{2n}\tfrac{\partial}{\partial\alpha}F(\beta,\alpha)-(\tfrac{\partial}{\partial\beta}F(\beta,\alpha))c(F(\beta,\alpha))^{3n}-c(F(\beta,\alpha))^{3n}\tfrac{\partial}{\partial\alpha}F(\beta,\alpha)-2\,\tfrac{\partial^3}{\partial\beta^3}F(\beta,\alpha)\\ -4\,\tfrac{\partial^3}{\partial\beta^2\partial\alpha}F(\beta,\alpha)-3\,\tfrac{\partial^3}{\partial\beta\partial\alpha^2}F(\beta,\alpha)-\tfrac{\partial^3}{\partial\alpha^3}F(\beta,\alpha)=0\\ \text{with similarity variables } [\{U(x,y,t)=F(-x+y,-x+t)\},\{\alpha=-x+t,\beta=-x+y\}].\end{gathered}\tag{48}$$

So, multi-reduction using conservation laws analysis is better approach to study higher order and higher dimensional model.

5. MULTI-SOLITON SOLUTIONS VIA INVARIANCE TRANSFORMATION

The early history of solitons is presented in [23] and the forms of solitons we presented in this study is discussed in [24,25]. In this section we shall discuss soliton and multi-soliton solutions for ZK equation (1). The solutions can be expressed as $u(t,x,y)\to f(\alpha x+\beta y-vt)$, where the solitons propagates along the x and y directions with velocity v. As we discussed earlier the line travelling wave reductions for ZK equation (1) with a 2-dimensional symmetry algebra in (2+1)-dimensions such as line solitons via invariance under two commuting translations given by

$$\mathrm{X}_1=\left(\alpha^2+\beta^2\right)\partial_t+\alpha u\partial_x+\beta\partial_y,\quad \mathrm{X}_2=\beta\partial_x-\alpha\partial_y.\tag{49}$$

Reduction under above type of commuting translation symmetries is very common in application and yield interesting types of solutions called line travelling wave solutions or line soliton solutions, under similarity invariant transformations

$\zeta = \alpha x + \beta y - \gamma t$ and $u(t, x, y) = f(\zeta)$, where γ is the spped of. The physics of the line travelling wave is presented in [15] and yield the solutions correspond to the reduction

$$u(t, x, y) \to f(\alpha x + \beta y - \gamma t) \tag{50}$$

and satisfy the reduced ODE. Under the transformation

$$u(x, t, y) = f(\zeta),\ \zeta = \alpha x + \beta y - \gamma t$$

the ZK equation (1) reduces to ODE given by

$$\begin{aligned}\gamma \tfrac{\mathrm{d}}{\mathrm{d}\zeta} f(\zeta) + (a(f(\zeta))^n + b(f(\zeta))^{2n} + c(f(\zeta))^{3n})\alpha \tfrac{\mathrm{d}}{\mathrm{d}\zeta} f(\zeta) \\ +\sigma(\alpha^3 \tfrac{\mathrm{d}^3}{\mathrm{d}\zeta^3} f(\zeta) + \alpha\beta^2 \tfrac{\mathrm{d}^3}{\mathrm{d}\zeta^3} f(\zeta)) = 0\end{aligned} \tag{51}$$

It can be further simplified to

$$\alpha\sigma\left(\alpha^2 + \beta^2\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} f(\zeta)\right) + \frac{a\alpha f(\zeta)^{n+1}}{n+1} + \frac{b\alpha f(\zeta)^{2n+1}}{2n+1} + \frac{c\alpha f(\zeta)^{3n+1}}{3n+1} + \gamma f(\zeta) = 0$$

To determine the soliton, we look for a solution that decays to zero as $\zeta \to \pm\infty$. For the construction of solitons, one way to find solitary wave profile is to consider $sech$ function

$$f(\zeta) = A\,\mathrm{sech}^2(B\zeta)$$

where A, and B are constants to be determined in terms of wave velocity v, the coefficients a, b, c of the non-linearities u^n, u^{2n}, u^{3n} and dispersion's coefficient σ. The ODE under the solitary wave profile takes the form

$$\begin{aligned}6\sigma(\alpha^2 + \beta^2)A\,\mathrm{sech}(B\zeta)^2(\tanh(B\zeta)^2 - \tfrac{1}{3})B^2\alpha + \tfrac{a\alpha(A\,\mathrm{sech}(B\zeta)^2)^{n+1}}{n+1} \\ +\tfrac{b\alpha(A\,\mathrm{sech}(B\zeta)^2)^{2n+1}}{2n+1} + \tfrac{c\alpha(A\,\mathrm{sech}(B\zeta)^2)^{3n+1}}{3n+1} + \gamma A\,\mathrm{sech}(B\zeta)^2 = 0\end{aligned} \tag{52}$$

For the soliton solutions to hold, we match the coefficients of the non-linear terms and receive the following conditions

$$\begin{aligned}-2\sigma\left(\alpha^2 + \beta^2\right)B^2\alpha + \gamma = 0, \\ 6\sigma\left(\alpha^2 + \beta^2\right)A + \tfrac{1}{2}\tfrac{\gamma}{\sigma\left(\alpha^2+\beta^2\right)} = 0.\end{aligned}$$

that yield us the values of B and A respectively, in terms of parameters α, β, γ and σ given by

$$\begin{aligned}B = \pm\tfrac{1}{2}\tfrac{\gamma\sqrt{2}}{\sqrt{\alpha\sigma\left(\alpha^2+\beta^2\right)\gamma}}, \\ A = -\tfrac{1}{12}\tfrac{\gamma}{\sigma^2\left(\alpha^2+\beta^2\right)^2}.\end{aligned}$$

By substituting values of A and B back in soliton profile provides us the soliton solution of the form

$$u(t, x, y) = [-\frac{1}{12}\frac{\gamma}{\sigma^2\left(\alpha^2 + \beta^2\right)^2}]\,\mathrm{sech}^2(\frac{1}{2}\frac{\gamma\sqrt{2}}{\sqrt{\alpha\sigma\left(\alpha^2 + \beta^2\right)\gamma}}(\alpha x + \beta y - \gamma t))$$

The 3D plot of the soliton solution is presented in Figure-4.

Now, we can discuss the superposition of two solitons for the construction of multi-solitons structure of the form

$$u(t, x, y) = f_1\left(\alpha_1 x + \beta_1 y - \gamma_1 t\right) + f_2\left(\alpha_2 x + \beta_2 y - \gamma_2 t\right)$$

where the solitary wave profile for two-soliton solutions is

$$\begin{aligned}f_1\left(\zeta_1\right) = A_1\,\mathrm{sech}^2\left(B_1\zeta_1\right), \\ f_2\left(\zeta_2\right) = A_2\,\mathrm{sech}^2\left(B_2\zeta_2\right).\end{aligned}$$

and

$$\zeta_1 = \alpha_1 x + \beta_1 y - \gamma_1 t,$$

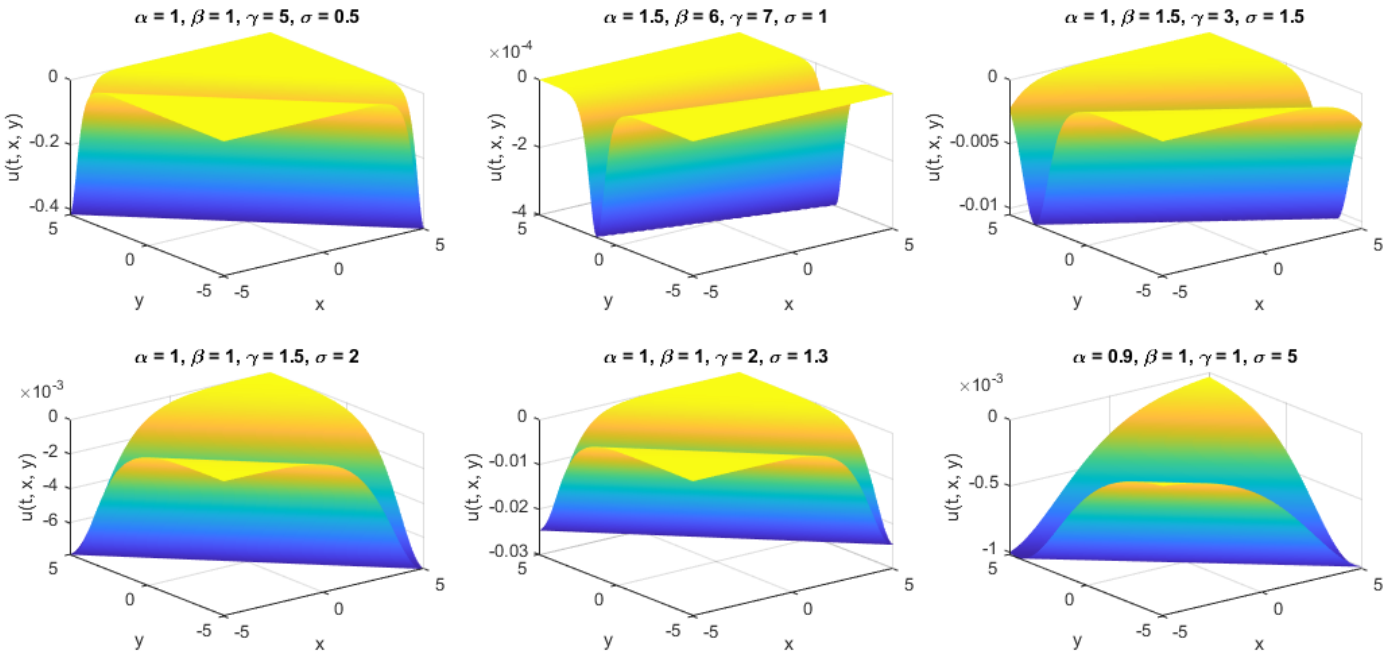

Figure 4. Soliton $u(t,x,y)$ with different parameters

$$\zeta_2 = \alpha_2 x + \beta_2 y - \gamma_2 t.$$

Each soliton can be determined by its own velocity, amplitude, and width based on the parameters $A_1, B_1, \alpha_1, \beta_1, \gamma_1$ for the first soliton and $A_2, B_2, \alpha_2, \beta_2, \gamma_2$ for the second soliton. By following the same ansatz, the two-soliton solutions can be presented by

$$u(t,x,y) = A_1 \operatorname{sech}^2 \left(B_1 \left(\alpha_1 x + \beta_1 y - \gamma_1 t\right)\right) + A_2 \operatorname{sech}^2 \left(B_2 \left(\alpha_2 x + \beta_2 y - \gamma_2 t\right)\right)$$

$$B_1 = B_2 = \frac{1}{2}\frac{\gamma\sqrt{2}}{\sqrt{\alpha\sigma\left(\alpha^2+\beta^2\right)\gamma}}, \quad A_1 = A_2 = -\frac{1}{12}\frac{\gamma}{\sigma^2\left(\alpha^2+\beta^2\right)^2}.$$

The two-soliton takes the form

$$u(t,x,y) = \sum_{i=1}^{2}[-\frac{1}{12}\frac{\gamma_i}{\sigma^2\left(\alpha_i^2+\beta_i^2\right)^2}] \operatorname{sech}^2\left(\frac{1}{2}\frac{\gamma_i\sqrt{2}}{\sqrt{\alpha_i\sigma\left(\alpha_i^2+\beta_i^2\right)\gamma_i}}\left(\alpha_i x + \beta_i y - \gamma_i t\right)\right).$$

The 3D plot of the two-soliton solution is presented in Figure-5.

For different velocities, when $\gamma_1 \neq \gamma_2$, the elastic collision occurs and pass through each other. Finally, we can generalise the solitons into multi-solitons by superposing more solitons and can be represented by

$$u(t,x,y) = \sum_{i=1}^{n}[-\frac{1}{12}\frac{\gamma_i}{\sigma^2\left(\alpha_i^2+\beta_i^2\right)^2}] \operatorname{sech}^2\left(\frac{1}{2}\frac{\gamma_i\sqrt{2}}{\sqrt{\alpha_i\sigma\left(\alpha_i^2+\beta_i^2\right)\gamma_i}}\left(\alpha_i x + \beta_i y - \gamma_i t\right)\right).$$

6. VORTEX SOLITON SOLUTIONS

We study the vortex type [26, 27] of solitons for ZK equation (1) in both spatial dimensions x and y that exhibit a vortex structure like a rotating wave. We seek the vortex solitons by using the transformation

$$u(t,x,y) \longrightarrow u(r,\theta,t),$$

where,

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = \sqrt{x^2+y^2},$$

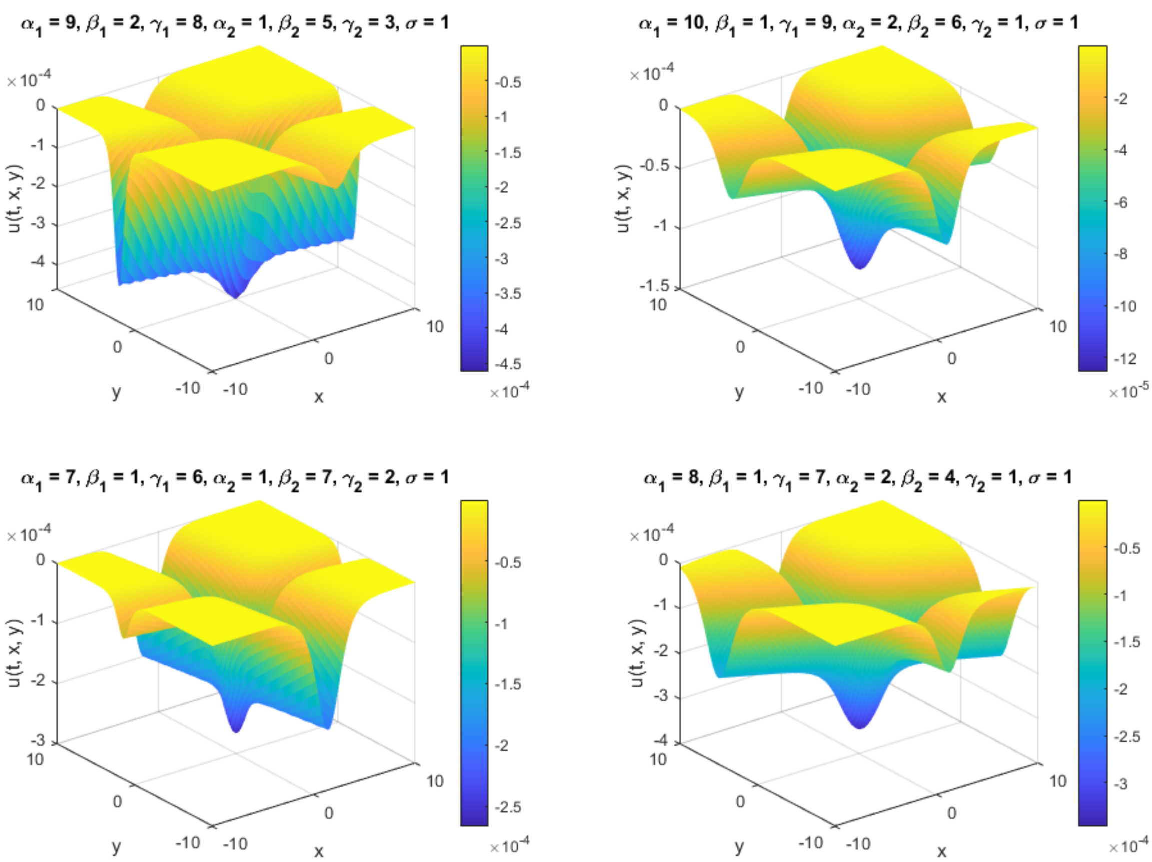

Figure 5. Soliton $u(t,x,y)$ with different parameters

r represents the radial distance and θ is the vortex angle. The Laplacian in polar coordinates provide us the Zakharov-Kuznetsov equation in polar coordinates given by

$$u_t + \left(au^n + bu^{2n} + cu^{3n}\right)\frac{\partial u}{\partial r} + \sigma\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)_r = 0 \tag{53}$$

Ansatz for vortex solitons can be applied to find solutions with a nontrivial phase of the form

$$u(r,\theta,t) = U(r)\mathrm{e}^{i\zeta}, \quad \zeta = m\theta - \omega t$$

Under the above transformation the ZK equation in ploar coordinates (53) reduces to ODE given by

$$-U(r)\omega + (aU(r)^n + bU(r)^{2n} + cU(r)^{3n})(\frac{\mathrm{d}}{\mathrm{d}r}U(r)) + \sigma(\frac{\mathrm{d}^3}{\mathrm{d}r^3}U(r) - \frac{m^2}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}U(r)) = 0 \tag{54}$$

Above ODE represents the radial profile of vortex solitons. The soliton must decay to zero as $r \longrightarrow \infty$, which ensures the localization of the solution. To find vortex solitons, we consider *sech* function

$$U(r) = Ar^m \,\mathrm{sech}^2(Br)$$

where A, and B are constants to be determined in terms of the coefficients a, b, c of the non-linearities u^n, u^{2n}, u^{3n} and dispersion's coefficient σ. The ODE under the solitary wave profile takes the form

$$\begin{aligned}&-Ar^m\,\mathrm{sech}(Br)^2\omega \\ &+\left(a\left(Ar^m\,\mathrm{sech}(Br)^2\right)^n + b\left(Ar^m\,\mathrm{sech}(Br)^2\right)^{2n} + c\left(Ar^m\,\mathrm{sech}(Br)^2\right)^{3n}\right)\left(\tfrac{\partial}{\partial r}\left(Ar^m\,\mathrm{sech}(Br)^2\right)\right)\omega \\ &+\sigma\left(\tfrac{\partial^3}{\partial r^3}\left(Ar^m\,\mathrm{sech}(Br)^2\right) - \frac{m^2\left(\frac{\partial}{\partial r}\left(Ar^m\,\mathrm{sech}(Br)^2\right)\right)}{r^2}\right) = 0\end{aligned} \tag{55}$$

For the soliton solutions to hold, we match the coefficients of the non-linear terms in r and receive the following conditions

$$\frac{1}{4}B^2m\sigma + 3\sigma m\left(m - \frac{2}{3}\right) = 0$$
$$\frac{1}{24}\omega + \frac{1}{4}B^2m\sigma = 0$$
$$16\sigma B^3 + 18B^2m\sigma + 16\sigma B\left(-\frac{1}{4}m^2 + \frac{3}{8}m\right) = 0$$
$$ABm^2\sigma - 5B^2\sigma - AB = 0$$

which respectively provide us the the values of B and A given by

$$B_1 = \pm 2\sqrt{-3m+2},$$
$$B_2 = \pm\frac{1}{6}\frac{\sqrt{-6\sigma m\omega}}{\sigma m},$$
$$B_3 = -\frac{9}{16}m + \frac{1}{16}\sqrt{145m^2 - 96m}.$$

$$A_1 = -\frac{10(3m-2)\sigma}{\sqrt{-3m+2}\left(m^2\sigma - 1\right)},$$
$$A_2 = -\frac{5\omega\sigma}{\sqrt{-6m\sigma\omega}\left(m^2\sigma - 1\right)},$$
$$A_3 = \frac{5}{16}\frac{\sigma\left(-9m + \sqrt{145m^2 - 96m}\right)}{m^2\sigma - 1}.$$

Finally yields the vortex soliton solutions of the form

$$u(r,\theta,t) = Ar^m\,\mathrm{sech}^2(Br)\mathrm{e}^{i(m\theta-\omega t)},$$

Three vortex solutions obtained based on the values of $A1, A2, A3$ and $B1, B2, B3$.

$$u_1(r,\theta,t) = -\frac{10(3m-2)\sigma}{\sqrt{-3m+2}\left(m^2\sigma - 1\right)}r^m\,\mathrm{sech}^2(2r\sqrt{-3m+2})\mathrm{e}^{i(m\theta-\omega t)},$$
$$u_2(r,\theta,t) = -\frac{5\omega\sigma}{\sqrt{-6m\sigma\omega}\left(m^2\sigma - 1\right)}r^m\,\mathrm{sech}^2\left(\frac{1}{6}\frac{\sqrt{-6\sigma m\omega}}{\sigma m}r\right)\mathrm{e}^{i(m\theta-\omega t)},$$

The plot of the vortex solutions $u_1(r,\theta,t)$ is presented in Figure-6 and $u_2(r,\theta,t)$ in Figure-7.

$$u_3(r,\theta,t) = \frac{5}{16}\frac{\sigma\left(-9m + \sqrt{145m^2 - 96m}\right)}{m^2\sigma - 1}r^m\,\mathrm{sech}^2\left(\left(-\frac{9}{16}m + \frac{1}{16}\sqrt{145m^2 - 96m}\right)r\right)\mathrm{e}^{i(m\theta-\omega t)}.$$

Plot of solution $u_3(r,\theta,t)$ is presented in Figure-8 for different values of m.

For plotting solutions $u_1(r,\theta,t)$ and $u_2(r,\theta,t)$ we considered these values for amplitude $m = 6$, for scaling of the secant hyperbolic $\sigma = 1$, and $\omega = 1$, which determine the angular frequency of the oscillation. The contour plot for vortex solution $u_3(r,\theta,t)$ is presented in Figure-9.

In this study, we presented vortex solution $u_1(r,\theta,t)$, $u_2(r,\theta,t)$, and $u_3(r,\theta,t)$ of ZK equation (1) in polar coordinates includes a radial exponent r^m and a secant hyperbolic function $\mathrm{sech}(r)$ and a exponential term $\mathrm{e}^{i(m\theta-\omega t)}$, where m, σ, and ω are key parameters. For vortex solution $u_3(r,\theta,t)$ we presented four different graphs based on the values of amplitude m. The parameters involved affect the overall appearance and magnitude of the solutions. We can clearly see how parameters like radial exponent and secant profile affect the vortex solitons's decay and oscillation patterns over space.

7. DISCUSSION OF RESULTS IN PLASMA PHYSICS

The plasma waves generated in the model discussed here are stable. This is particularly the case related to travelling wave solutions. The vortex solutions are in line with a bell-shaped solitary wave solution which suggests that electric field potential, electric field and magnetic field are stable. Some of the figures indicate a nonlinear ion-acoustic solitary wave like behaviour of the wave. In all of the cases, the spread of the waves is stable with little or no chaotic type structure following the initial waves. The amplitudes of the waves, indicated by the magnitudes of the complex cases in some of the cases, display wave behaviours that are stable.

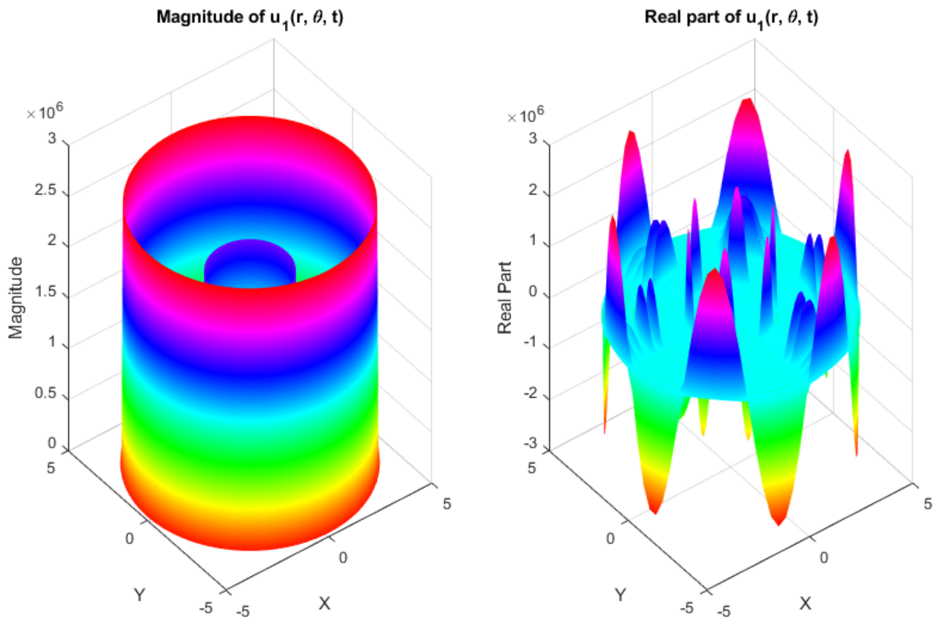

Figure 6. 3D plot of the real part and magnitude of $u_1(r, \theta, t)$

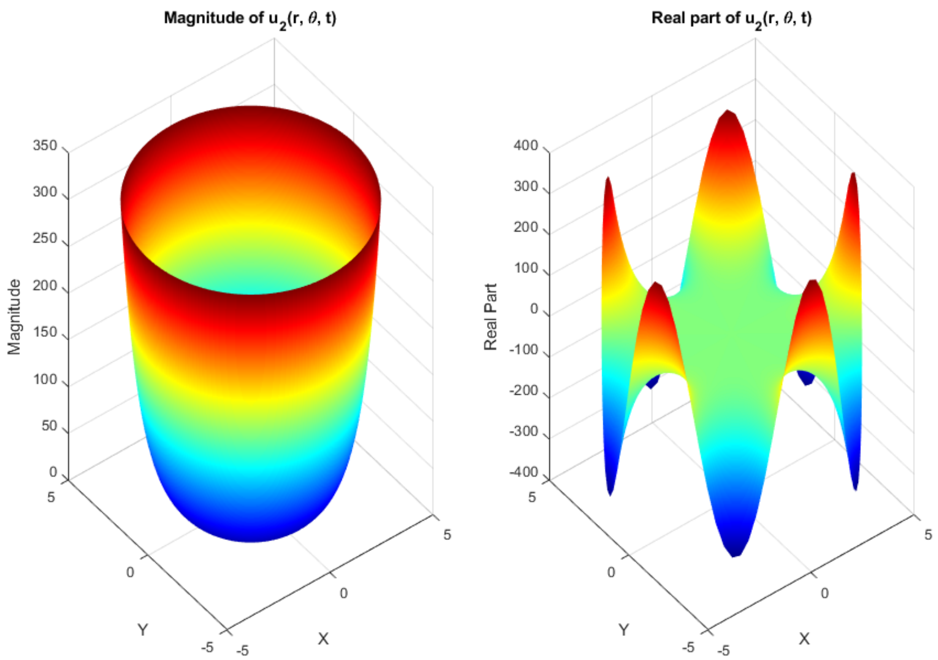

Figure 7. 3D plot of the real part and magnitude of $u_2(r, \theta, t)$

The analysis of the Zakharov-Kuznetsov (ZK) equation (1) with triple-power law nonlinearity reveals that the plasma waves generated in this model exhibit stability of the travelling wave solutions. These solutions, derived through double reduction approach and invariance transformations, demonstrate consistent and predictable propagation characteristics across two spatial dimensions and time. The stability of these waves is a critical finding, as it underscores their potential relevance in modeling multidimensional wave phenomena in plasma physics, such as ion-acoustic waves in magnetized plasmas.

The travelling wave solutions, represented as line solitons, for example (Figures 1-3) maintain their shape and speed over time, a hallmark of soliton behavior. This stability is attributed to the balance between the nonlinear effects introduced

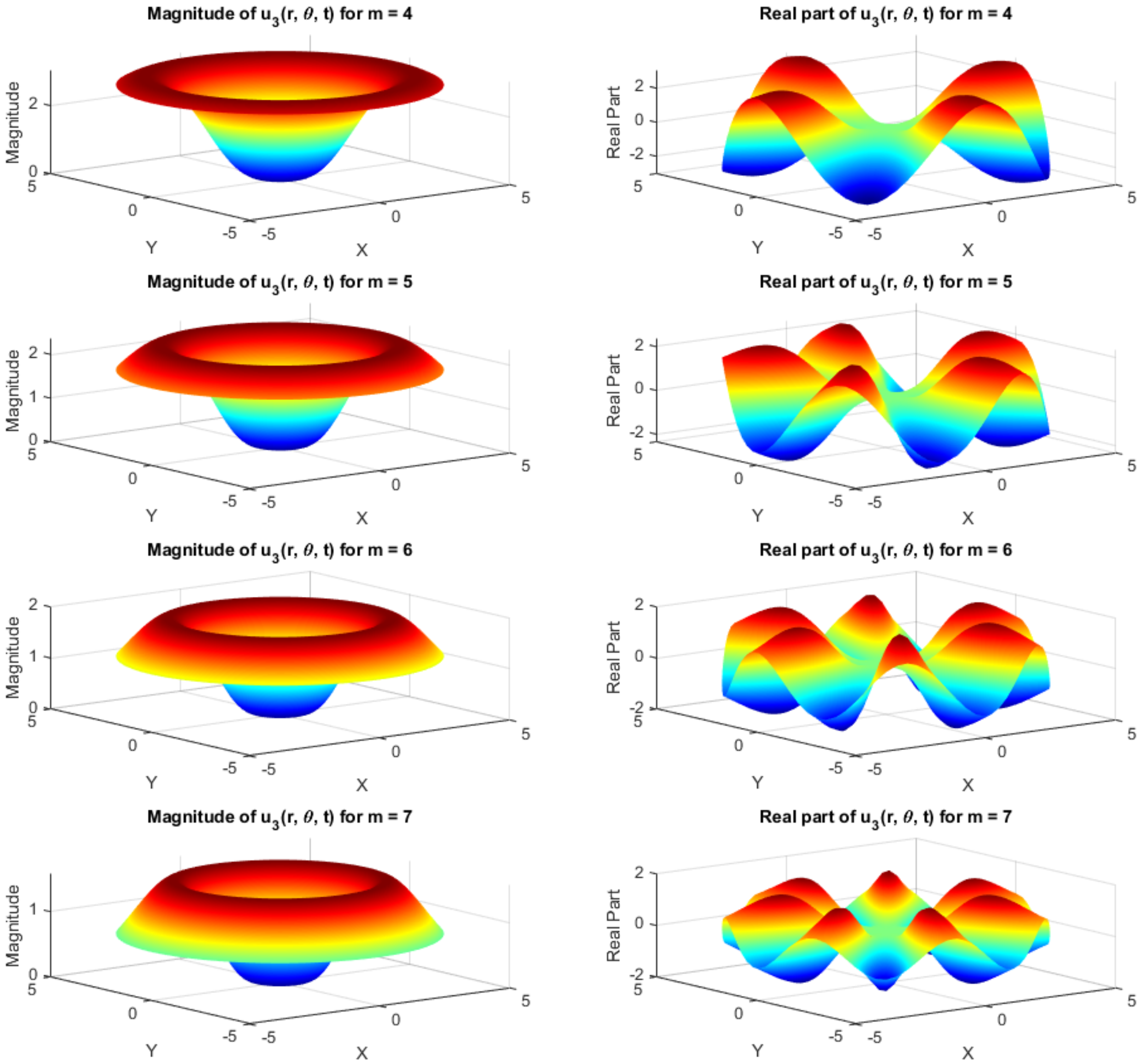

Figure 8. Real part and magnitude of $u_3(r, \theta, t)$

by the triple-power law terms, $G(u) = au^n + bu^{2n} + cu^{3n}$ and the dispersive effects captured by the terms $\sigma(u_{xx} + u_{yy})_x$ in the ZK equation. The absence of significant chaotic or turbulent structures following the initial wave propagation further reinforces the robustness of these solutions. This behavior aligns with the physical interpretation of ion-acoustic solitary waves, which are known to preserve their integrity in plasma environments under the influence of a magnetic field, as noted in prior studies [3, 5].

Similarly, the vortex soliton solutions (Figures 6-9) exhibit a stable, bell-shaped solitary wave profile, characterized by a radial decay governed by the sech^2 function and a phase-dependent oscillatory structure. This suggests that the associated electric field potential, electric field, and magnetic field components remain stable over time and space. The vortex solutions, expressed in polar coordinates as $u(r, \theta, t) = U(r)e^{i(m\theta - \omega t)}$, demonstrate a localized rotating wave pattern that decays to zero as $r \to \infty$, ensuring the confinement of energy and the absence of unbounded growth or instability. The stability of these vortex structures is particularly significant in the context of plasma physics, where such solutions can model coherent structures like ion-acoustic vortices in two-dimensional systems.

The figures accompanying the travelling wave and vortex soliton solutions provide visual confirmation of this stability. For instance, the 3D plots of the travelling wave solutions (Figures 1-3) depict smooth, non-dispersive wave fronts that propagate without distortion, while the vortex soliton plots (Figures 6-9) illustrate a consistent magnitude and oscillatory pattern across varying parameters, such as the topological charge m. Notably, some figures indicate a non-linear ion-

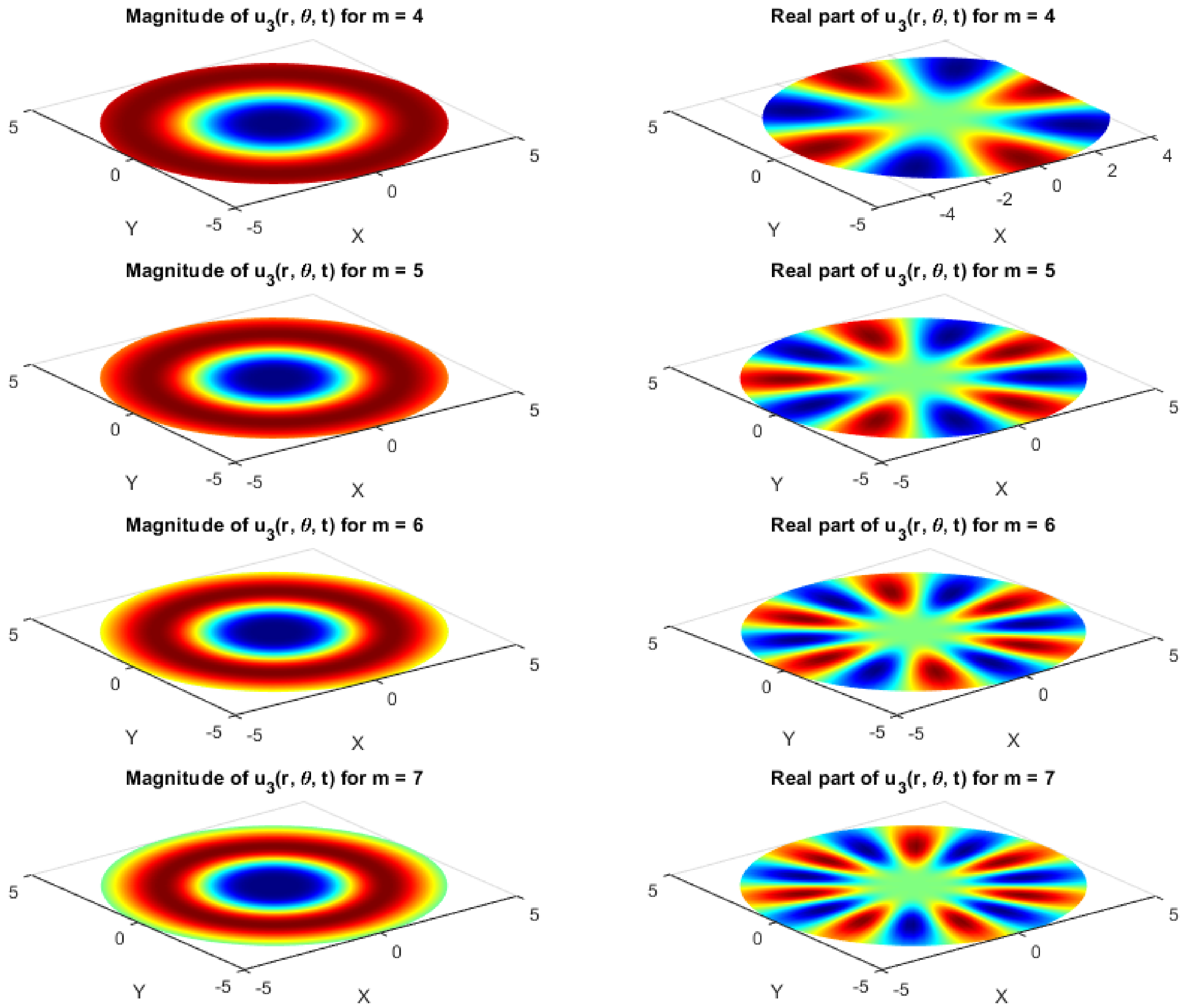

Figure 9. Real part and magnitude of $u_3(r, \theta, t)$

acoustic solitary wave-like behaviour, characterized by a steepened wave profile that remains stable over time, a feature consistent with experimental observations of ion-acoustic solitons in magnetized plasmas [8].

The amplitudes of the waves, particularly in the complex vortex soliton cases, further highlight their stable behaviour. The magnitude plots (Figures 6-9) show that the wave intensity remains bounded and predictable, with no evidence of amplification or dissipation that would suggest instability. This stability is likely a consequence of the symmetry-invariant conservation laws derived in the study, which impose constraints on the system that prevent chaotic divergence. The use of multipliers to construct these conservation laws facilitates the double reduction process, yielding reduced ODEs that admit stable, integrable solutions.

In all cases, whether travelling waves, multi-solitons, or vortex solitons, the spread of the waves remains controlled, with little to no chaotic structure emerging after the initial wave formation. This stability is particularly pronounced in the multi-soliton solutions (Figure 5), where elastic collisions between solitons occur without loss of form, a property indicative of integrability in the ZK system. Such behavior is consistent with the physical context of weak turbulence in plasma systems, where solitons maintain their coherence despite interactions, which is a important phenomenon of wave dynamics in space physics and fusion research [6, 7].

Overall, the stable wave behavior observed in this study suggests that the ZK equation with triple-power law nonlinearity provides a robust framework for modeling plasma waves in higher-dimensional settings. The findings have implications for understanding the stability and interaction of nonlinear waves in magnetized plasmas, potentially aiding in the design of experiments or simulations aimed at exploring ion-acoustic solitons and vortex structures in real-world plasma environments.

8. CONCLUSION

A large class of Zakharov-Kuznetsov (ZK) equation with the triple-power law non-linearity were studied. We have shown how a study of the relationship between symmetries and multipliers are attained and then utilised to obtain double reduction from (1+2) ZK equation to an ODE. We determined the invariance properties and constructed classes of conservation laws and discussed how the relationship leads to double reductions of the systems, ensuring stable solutions. Multi-solitons analysis is performed using invariance transformation and vortex soliton solutions. A comparison between symmetry and multi-reduction is presented, highlighting the advantage in producing integrable, stable outcomes. The physical interpretation of soliton solutions is also discussed in this study, emphasizing their stable propagation, evidenced by localized profiles, elastic collisions, and conserved quantities, which models coherent ion-acoustic and vortex waves in magnetized plasmas, offering valuable understanding of wave dynamics in space physics and fusion research.

Statement of Authors Participation All authors included above have contributed to the manuscript in some but significant way.

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ЗАСТОСУВАННЯ МУЛЬТИРЕДУКЦІЇ ТА МУЛЬТИСОЛІТОННОГО АНАЛІЗУ (2+1) РІВНЯННЯ ЗАХАРОВА-КУЗНЄЦОВА (ZK)

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Досліджено рівняння Захарова-Кузнєцова (З.К.) з триступеневою нелінійністю. Ми визначаємо властивості інваріантності та будуємо класи законів збереження та показуємо, як зв'язок призводить до подвійного скорочення систем, що дає стабільні рішення, такі як біжучі хвилі та солітони. Цей зв'язок визначається нещодавніми результатами, пов'язаними з «мультиплікаторами», які призводять до «загальних розбіжних систем». Мультисолітонний аналіз виконується за допомогою інваріантного перетворення, утворюючи стабільні багатосолітонні структури разом із вихровими солітонними рішеннями, які демонструють локалізовані дзвоноподібні профілі. Представлено порівняння між симетрією та множинним скороченням, підкреслюючи ефективність у досягненні інтегрованих результатів. У цьому дослідженні також обговорюється фізична інтерпретація солітонних розчинів, наголошується на їх стабільному поширенні та актуальності для моделювання когерентних іонно-звукових і вихрових хвиль у намагніченій плазмі.

Ключові слова: *подвійне скорочення; закони збереження; множники; рівняння Захарова-Кузнєцова; аналіз інваріантності; солітони; мультисолітони; вихрові солітони*