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BIANCHI TYPE VI_0 GENERALIZED GHOST PILGRIMS DARK ENERGY COSMOLOGICAL MODEL IN SAEZ-BALLESTER THEORY OF GRAVITATION

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The Generalized Ghost Pilgrim Dark Energy (GGPDE) in the Saez-Ballester Theory of Gravitation (SBTG) and the Bianchi type VI_0 space-time framework serve as the foundation for this work. We used Mishra and Dua's [Astrophys. Space Sci. 366, 6 (2021)] straightforward parameterization of average scale factor $a(t) = exp\{(\alpha t + \beta)^p\}$ to find precise solutions to the field equations. We have looked into the GGPDE and dark matter (DM), both when they interact and when they don't. For both models, some significant and well-known parameters are produced, including the Hubble parameter, the equation of state (EOS) parameter, the deceleration parameter, etc. It is discovered that for both models, the deceleration parameter denotes an accelerated phase and the EOS parameter a cosmological constant. For both the non-interacting and interacting models, the stability analysis and energy conditions are examined.
Keywords: *Hubble parameter; EOS parameter; deceleration parameter; GGPDE; SBTG*
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1. INTRODUCTION

According to recent astronomical observations [2-4], we live in an expanding and accelerating Universe. These findings imply that the Universe is dominated by two dark components: dark matter (DM) and dark energy. Dark matter, a pressure less substance, is primarily utilized to explain galaxy curves and the formation of the Universe's structure, whereas DE, an exotic energy with a huge negative pressure, is used to explain the Universe's cosmic acceleration. Researchers are working hard to determine the nature of dark energy, and several ideas have been presented to define it. The cosmological constant with the EOS parameter $\omega = p/\rho = -1$, where p is the pressure andρ is the energy density of DE, is the simplest and most obvious choice for DE. However, it suffers from fine-tuning and cosmic coincidence issues [5]. To address the issue of DE, cosmologists have developed many DE models such as quintessence [6-7], phantom [7-8], tachyon [9], dilaton [10], and so on.

A type of DE known as Veneziano ghost DE has been postulated [11-13] to explain the Universe's current rapid stage. When H is used in place of the Hubble parameter, the energy density of the vacuum ghost field is proportional to 3 QCDH [14-15]. QCDH stands for the QCD mass scale. Because this GDE's energy density DE relies linearly on the Hubble parameter H, as in DE = H, and is connected to the QCD (Quantum Chromo dynamics) mass scale, it has attracted the interest of researchers. The robust interaction in nature is described by QCD. In QCD, the Veneziano ghost field's general vacuum energy has the form $H + O(H^2)$[16]. In the early Universe's evolution, which serves as the early DE, the term H^2has a key place [17]. In comparison to the standard GDE, also known as generalized ghost dark energy (GGDE), one can provide better agreement with observational data by taking the term H^2into consideration [18]. The generalized model's energy density is given by$\rho_{DE} = \tau H + \eta H^2$ where η is a constant. Based on the hypothesis that the strong repulsive force of the type of DE can prevent black hole (BH) creation, Wei [19] presented a new dark energy model dubbed pilgrim DE (PDE). In terms of PDE, GGDE has been changed as$\rho_{DE} = (\tau H + \eta H^2)^u$, where u is a PDE parameter [20].

Scientists have come up with different models to try and understand dark energy. They use these models to explain how the universe is expanding and what might be causing it. One of these models is called GGPDE in the Bianchi type I Universe, which was explored by Santhi et al. [21]. Another model, investigated by Jawad [22], is called GGPDE in the context of a non-flat FRW Universe. Gravitation theory based on Saez-Ballester was used by Garg et al. [23] to study GGPDE. With the aid of a straightforward parameterization of the average scale factor *a(t)*, Mishra and Dua [1] estimated the FLRW Universe in the Brans-Dicke theory. One of the simplest models with an anisotropic background to describe the early phases of the universe's evolution is the Bianchi type model. Bianchi space-times are helpful in creating models of spatially homogenous and anisotropic cosmologies due to the simplicity of the field equations and relative ease of solutions.

We propose a study of the GGPDE in SBTG within the context of Bianchi type VI_0 space-time, which is motivated by the aforementioned recent efforts of various authors [24-26]. The following is the manuscript's structure: Section 2

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discusses metric and field equations. We found the answers to the field equations in Section 3. Sections 4 and Section 5 address the non-interactive and interacting models, respectively. Sections 6 and 7 include descriptions of the stability analysis and energy conditions, respectively. In Section 8, many parameters are illustrated and explained. Section 9's final observations bring the paper to a close.

2. METRIC AND FIELD EQUATIONS:

The spatially homogeneous and anisotropic Bianchi type VI_0 space-time is given by

$$ds^2 = -A^2dx^2 - B^2e^{2x}dy^2 - C^2e^{-2x}dz^2 + dt^2, \tag{1}$$

where A, B, C are the gravitational potentials which are functions of cosmic time *t*.

The Saez-Ballester field equations are given by

$$G_j^i - w\phi^n\left(\phi^{,i}\phi_{,j} - \frac{1}{2}\delta_j^i\phi^{,k}\phi_{,k}\right) = \left(T_j^i + \bar{T}_j^i\right), \tag{2}$$

where G_j^i is Einstein tensor and T_j^i,$\bar{T}_j^i$ are energy momentum tensors of dark matter and GGPDE respectively.
The scalar field ϕ satisfies the equation

$$2\phi^n\phi_{;i}^{;i} + n\phi^{n-1}\phi_{,k}\phi^{,k} = 0. \tag{3}$$

The energy momentum tensor of dark matter (DM) is given by

$$T_j^i = diag[\,0,0,0,\rho_m]. \tag{4}$$

The energy momentum tensor of GGPDE is given by

$$\bar{T}_j^i = diag[\,-p_{DE}, -p_{DE}, -p_{DE}, \rho_{DE}]. \tag{5}$$

Here scalar field ϕ and the energy momentum tensors components depend only on cosmic time.

The field equation (2) for the metric (1) using equations (4), (5) are obtained as

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} + \frac{1}{A^2} - \frac{1}{2}w\phi^n\dot{\phi}^2 = -p_{DE}, \tag{6}$$

$$\frac{\ddot{C}}{C} + \frac{\ddot{A}}{A} + \frac{\dot{C}\dot{A}}{CA} - \frac{1}{A^2} - \frac{1}{2}w\phi^n\dot{\phi}^2 = -p_{DE}, \tag{7}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{1}{A^2} - \frac{1}{2}w\phi^n\dot{\phi}^2 = -p_{DE}, \tag{8}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} - \frac{1}{A^2} + \frac{1}{2}w\phi^n\dot{\phi}^2 = \rho_m + \rho_{DE}, \tag{9}$$

$$\frac{\dot{B}}{B} - \frac{\dot{C}}{C} = 0. \tag{10}$$

From equation (3) we have

$$\ddot{\phi} + \dot{\phi}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) + \frac{n}{2}\frac{\dot{\phi}^2}{\phi} = 0. \tag{11}$$

By integrating equation (10) and assuming integration constant as unity, we get

$$B = C. \tag{12}$$

Now by using (12) in equations (6)-(9),(11) we get

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{1}{A^2} - \frac{1}{2}w\phi^n\dot{\phi}^2 = -p_{DE}, \tag{13}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{1}{A^2} - \frac{1}{2}w\phi^n\dot{\phi}^2 = -p_{DE}, \tag{14}$$

$$\frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}\dot{B}}{AB} - \frac{1}{A^2} + \frac{1}{2}w\phi^n\dot{\phi}^2 = \rho_m + \rho_{DE}, \tag{15}$$

$$\ddot{\phi} + \dot{\phi}(3H) + \frac{n}{2}\frac{\dot{\phi}^2}{\phi} = 0. \tag{16}$$

The energy conservation equation is

$$T_{j\;;i}^{i} + \bar{T}_{j\;;i}^{i} = 0. \tag{17}$$

From (17) we get

$$\dot{\rho}_m + \dot{\rho}_{DE} + 3H(\rho_m + \rho_{DE} + p_{DE}) = 0. \tag{18}$$

In this article, we have considered both interacting and non-interacting models. The continuity equations of DM and GGPDE through an interaction Q are

$$\dot{\rho}_m + 3H\rho_m = Q, \tag{19}$$

$$\dot{\rho}_{DE} + 3H(\rho_{DE} + p_{DE}) = -Q, \tag{20}$$

where $Q > 0$ shows energy flows from GGPDE to DM and $Q < 0$ means energy flows from DM to GGPDE and$Q = 0$ indicates non interaction model. Wei and Cai [27] proposed the interaction term Qas

$$Q = 3bH\rho_m, \tag{21}$$

where$b > 0$ is a coupling constant.

3. SOLUTIONS OF FIELD EQUATIONS:

Equations (13)-(16) are a system of four field equations in 6 unknowns $A, B, \rho_m, \rho_{DE}, p_{DE}$ and ϕ. To solve these field equations, we need two physical conditions. These are as follows:

(i) The energy momentum tensor of GGPDE is given by

$$\rho_{DE} = (\tau H + \eta H^2)^u, \tag{22}$$

where u is PDE parameter.

(ii) Mishra and Dua [1] proposed a simple parameterization of scale factor (see (31)) as

$$a(t) = exp\{(\alpha t + \beta)^p\}, \tag{23}$$

where $\alpha, \beta > 0$ and $0 < p < 1$ are arbitrary constants.

From equations (13) and (14), we get

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{k}{V} exp\left(\int \frac{\left(\frac{-2}{A^2}\right)}{\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right)} dt\right). \tag{24}$$

Where k is integration constant and V is the volume of the Universe (see (32))

Following Adhav [28], we assume

$$\frac{\dot{B}}{B} - \frac{\dot{A}}{A} = \frac{2}{A^2}. \tag{25}$$

Using equations (24) and (25), we get

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{k}{V} e^{-t}. \tag{26}$$

Integrating equation (25),

$$A = lB\, exp\left\{k \int \frac{e^{-t}}{[exp\{(\alpha t + \beta)^p\}]^3} dt\right\}, \tag{27}$$

where l is constant of integration.

Now from the above equations the metric potentials are obtained as

$$A = exp\{(\alpha t + \beta)^p\}\, l^{\frac{2}{3}}\, exp\left\{\frac{2k}{3} \int \frac{e^{-t}}{[exp\{(\alpha t + \beta)^p\}]^3} dt\right\}, \tag{28}$$

$$B = exp\{(\alpha t + \beta)^p\}\, l^{\frac{-1}{3}}\, exp\left\{\frac{-k}{3} \int \frac{e^{-t}}{[exp\{(\alpha t + \beta)^p\}]^3} dt\right\}. \tag{29}$$

From equation (16) the Saez-Ballester scalar field is obtained as

$$\phi(t) = \left[\frac{n+2}{2} \phi_0 \int exp(-3(\alpha t + \beta)^p)\, dt + \psi_0\right]^{\left(\frac{2}{n+2}\right)}, \tag{30}$$

where ϕ_0, ψ_0 are integration constants.

3. PHYSICAL AND KINEMATICAL PARAMETERS OF THE MODEL:

The parameters which play a vital role in the discussion of dynamics of the obtained model are as follows,

The average scale factor

$$a(t) = (AB^2)^{\frac{1}{3}} = exp((\alpha t + \beta)^p). \tag{31}$$

The volume V of the universe

$$V = \left(a(t)\right)^3 = AB^2 = exp((\alpha t + \beta)^{3p}). \quad (32)$$

From Fig. (1), the Volume increases as$t \to \infty$.It shows the spatial expansion of the universe.
The Hubble parameter

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\frac{\dot{V}}{V} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B}\right) = p\alpha(\alpha t + \beta)^{p-1}. \quad (33)$$

The scalar expansion of the universe

$$\theta = 3H = \left(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B}\right) = 3p\alpha(\alpha t + \beta)^{p-1}. \quad (34)$$

The shear scalar of the universe

$$\sigma^2 = \frac{1}{2}\left[\left(\frac{\dot{A}}{A}\right)^2 + 2\left(\frac{\dot{B}}{B}\right)^2\right] - \frac{\theta^2}{6} = k^2 \frac{e^{-2t}}{\exp\left((2(\alpha t+\beta)^{3p}\right)}. \quad (35)$$

From Figures (2), (3) and (4), observed that $H, \theta, \sigma^2$are diverge at $t = 0$ and tends small values as $t \to \infty$. Our model exhibits shear free universe at late time.

The average anisotropy parameter

$$A_h = \frac{1}{3H^2}\left(\frac{(H_1-H)^2+2(H_2-H)^2}{H^2}\right) = \frac{2ke^{-2t}}{9\alpha^4p^4(\alpha t+\beta)^{(4p-4)}[exp(\alpha t+\beta)^p]^6}, \quad (36)$$

where $H_1 = \frac{\dot{A}}{A}, H_2 = H_3 = \frac{\dot{B}}{B}$ are directional Hubble parameters**.**

From figure (5), clearly anisotropy exists at early time and decreases and tends to zero at late time. So, our model is anisotropic model and converges to an isotropic model at late time.
The deceleration parameter is

$$q = -\frac{a\ddot{a}}{(\dot{a})^2} = -1 - \left(\frac{p-1}{p}\right)(\alpha t + \beta)^{-p}. \quad (37)$$

Clearly, $q > 0$for $t < \frac{\left(\left(\frac{1}{p}-1\right)^{\frac{1}{p}}-\beta\right)}{\alpha}$and $q < 0$ for $t > \frac{\left(\left(\frac{1}{p}-1\right)^{\frac{1}{p}}-\beta\right)}{\alpha}$ and $q \to -1$ as $t \to \infty$.

So, the present model shows the transition from early deceleration to present acceleration phase of the Universe. From Figure (6), the decelerating parameter is positive (decelerating phase) initially and after some time it moves to negative (acceleration phase). So, our model exhibits both early deceleration and present-day acceleration of the Universe. This is coinciding with the present-day observations.

The jerk parameter is

$$j(t) = q + 2q^2 - \frac{\dot{q}}{H} = 1 + \frac{(p^2-3p+2)(\alpha t+\beta)^{-2p}+(3p^2-3p)(\alpha t+\beta)^{-p}}{p^2}. \quad (38)$$

From figure (7), the jerk parameter is positive throughout evolution of the universe. Cosmologists believe that the positive value of jerk parameter and negative value of decelerating parameter indicates the accelerating phase of expansion of the universe. So the obtained model denotes the present day accelerating phase of expansion of the Universe.

From (22) the energy density of GGPDE is

$$\rho_{DE} = [\tau\alpha p(\alpha t + \beta)^{p-1} + \eta\alpha^2 p^2(\alpha t + \beta)^{2p-2}]^u. \quad (39)$$

From Figure (8), observed that the energy density of GGPDE is diminishes w.r.t. cosmic time *t*.

4. NON-INTERACTING MODEL

The energy conservation equation for DM is

$$\dot{\rho}_m + 3H\rho_m = 0 \quad (40)$$

From (40) and (33), we get

$$\rho_m = \frac{\rho_0}{exp(3(\alpha t+\beta)^p)} \quad (41)$$

where ρ_0 is integration constant. Clearly it is decreasing as $t \to \infty$.
The energy conservation equation for GGPDE is

$$\dot{\rho}_{DE} + 3H(\rho_{DE} + p_{DE}) = 0. \tag{42}$$

From (42), (33), (39) and by using $\omega_{DE} = \frac{p_{DE}}{\rho_{DE}}$ the EoS parameter of GGPDE in non-interacting case is obtained as

$$\omega_{DE} = -1 - \frac{1}{3H}\frac{\dot{\rho}_{DE}}{\rho_{DE}},$$

$$\omega_{DE} = -1 - \frac{u(p-1)}{3p(\alpha t+\beta)^{p+1}}\left[\frac{\tau+2\eta p\alpha(\alpha t+\beta)^{p-1}}{\tau+\eta p\alpha(\alpha t+\beta)^{p-1}}\right]. \tag{43}$$

Clearly from figure (9), the non-interacting model denotes a quintessence universe and tends toΛCDM model at late time.i.e.,$\omega_{DE} \to -1$ as $t \to \infty$.

The pressure of GGPDE is obtained as

$$p_{DE} = \omega_{DE}\rho_{DE} = (\alpha t+\beta)^{p-1}\left(-1 - \frac{u(p-1)}{3p(\alpha t+\beta)^{p+1}}\left[\frac{\tau+2\eta p\alpha(\alpha t+\beta)^{p-1}}{\tau+\eta p\alpha(\alpha t+\beta)^{p-1}}\right]\right)\left([\tau\alpha p + \eta\alpha^2 p^2(\alpha t+\beta)^{p-1}]^u\right). \tag{44}$$

The overall density parameter of the non-interacting model is

$$\Omega = \Omega_m + \Omega_{DE} = \frac{1}{3H^2}(\rho_m + \rho_{DE}). \tag{45}$$

By using (41), (39) and (33)

$$\Omega = \frac{(\alpha t+\beta)^2\left[\frac{\rho_0}{e^{3(\alpha t+\beta)^p}} + \left(\frac{\tau(\alpha t+\beta)^p p\alpha}{\alpha t+\beta} + \frac{\eta\alpha^2p^2(\alpha t+\beta)^{2p}}{(\alpha t+\beta)^2}\right)^p\right]}{3\alpha^2p^2(\alpha t+\beta)^{2p}}. \tag{46}$$

Figure (10), displays that the overall density of the non-interacting model is increases w.r.t. expansion of the universe.

5. INTERACTING MODEL

The energy conservation equation DM is

$$\dot{\rho}_m + 3H\rho_m = 3bH\rho_m. \tag{47}$$

By using (33) in (47),

$$\rho_m = \frac{\rho_1}{exp(3(1-b)(\alpha t+\beta)^p)}, \tag{48}$$

where ρ_1 is integration constant. Clearly the energy density of DM for interacting model decreases w.r.t. time t.

The energy conservation equation for GGPDE is

$$\dot{\rho}_{DE} + 3H(\rho_{DE} + p_{DE}) = -3bH\rho_m. \tag{49}$$

By using (33), (39),(48) in (49), the EoS parameter ω_{DE} of GGPDE is

$$\omega_{DE} = -1 - \frac{u(p-1)}{3p(\alpha t+\beta)^{p+1}}\left[\frac{\tau+2\eta p\alpha(\alpha t+\beta)^{p-1}}{\tau+\eta p\alpha(\alpha t+\beta)^{p-1}}\right] - \frac{b\rho_m}{\rho_{DE}}. \tag{50}$$

Figure (11) displays $-1 < \omega_{DE} < -1/3$initially and $\omega_{DE} \to -1$ for large values of *t.* So initially the interacting model denotes the quintessence model and it tends cosmological constant model (ΛCDMModel) for late time.

The pressure of GGPDE in Interacting case is

$$p_{DE} = \omega_{DE}\rho_{DE} = (\alpha t+\beta)^{p-1}\left(-1 - \frac{u(p-1)}{3p(\alpha t+\beta)^{p+1}}\left[\frac{\tau+2\eta p\alpha(\alpha t+\beta)^{p-1}}{\tau+\eta p\alpha(\alpha t+\beta)^{p-1}}\right] - \frac{b\rho_m}{\rho_{DE}}\right)\left([\tau\alpha p + \eta\alpha^2p^2(\alpha t+\beta)^{p-1}]^u\right). \tag{51}$$

The overall density parameter of the interacting model is

$$\Omega = \Omega_m + \Omega_{DE} = \frac{1}{3H^2}(\rho_m + \rho_{DE}). \tag{52}$$

By using (33),(48) and (39)in (52),

$$\Omega = \frac{(\alpha t+\beta)^2\left[\frac{\rho_1}{e^{(3-3b)(\alpha t+\beta)^p}} + \left(\frac{\tau(\alpha t+\beta)^p p\alpha}{\alpha t+\beta} + \frac{\eta\alpha^2p^2(\alpha t+\beta)^{2p}}{(\alpha t+\beta)^2}\right)^p\right]}{3\alpha^2p^2(\alpha t+\beta)^{2p}} \tag{53}$$

The overall density of interacting model increases with time *t* as universe expands. It is shown in figure (12). The both interacting and non-interacting models are tending to cosmological constant at late time and the overall density of both model increases with time.

6. STABILITY ANALYSIS

In this section the stability of both interacting and non-interacting models has discussed. In order to characterize the stability of models, the sign of $v_s^{\,2} = \frac{\dot{p}_{DE}}{\dot{\rho}_{DE}}$is crucial. If $v_s^{\,2} > 0$shows a table model and if $v_s^{\,2} < 0$ shows unstable model [29]. Also, the casualty condition must be satisfied, means that the speed of the sound less than the speed of the light.

The square speed of sound$v_s^{\,2}$ for non-interacting model is

$$v_s^{\,2} = -\frac{\left((\alpha t+\beta)\left(\frac{2p^3\eta^2\left(\left(\frac{3t\tau}{2}+\left(\left(u-\frac{1}{2}\right)p-u\right)\eta\right)\alpha+\frac{3\beta\tau}{2}\right)\alpha^2(\alpha t+\beta)^{2p-1}}{3} + p^4(\alpha t+\beta)^{-1+3p}\alpha^3\eta^3 + \frac{1}{2}\left(3\tau\left(\frac{\tau^2(\alpha t+\beta)^2\left((u-1)p-u\right)(\alpha t+\beta)^{-p}}{9} + \left(p^2(\alpha t+\beta)^{2p}\alpha^2\eta^2 + \frac{8p\left(\left(\frac{3t\tau}{2}+\left(\left(u-\frac{1}{2}\right)p-u-\frac{1}{8}\right)\eta\right)\alpha+\frac{3\beta\tau}{2}\right)\eta\alpha(\alpha t+\beta)^p}{9} + \frac{5\tau(\alpha t+\beta)\left(\left(\frac{3t\tau}{5}+\left(\left(u-\frac{3}{5}\right)p-u-\frac{1}{5}\right)\eta\right)\alpha+\frac{3\beta\tau}{5}\right)}{9}\right)p\right)\right)\right)\right)}{\left(p\left(\eta(\alpha t+\beta)^p p\alpha+\frac{\tau(\alpha t+\beta)}{2}\right)\left(\eta(\alpha t+\beta)^p p\alpha+\tau(\alpha t+\beta)\right)^2\right)} \tag{54}$$

The square speed of sound $v_s^{\,2}$ for interacting model is

$$v_s^{\,2} = -\frac{\left(3(\alpha t+\beta)\left(b e^{3(-1+b)(\alpha t+\beta)^p p^2 \rho_1}(\alpha t+\beta)^2 \left(3p\eta\tau^2\alpha(\alpha t+\beta)^{2p-1} + 3p^2\eta^2\tau\alpha^2(\alpha t+\beta)^{3p-2} + p^3(\alpha t+\beta)^{4p-3}\eta^3\alpha^3 + (\alpha t+\beta)^p\tau^3\right)(-1+b)\left(\frac{\left((\alpha t+\beta)^{2p}\alpha\eta p+\tau(\alpha t+\beta)^{(p+1)}\right)\alpha p}{(\alpha t+\beta)^2}\right)^{-u} + \frac{1}{9}\left(8(p-1)\left(\frac{\eta^2\alpha^2\left(\left(\eta\left(u-\frac{1}{2}\right)p+\frac{3t\tau}{2}-u\eta\right)\alpha+\frac{3\beta\tau}{2}\right)p^3(\alpha t+\beta)^{2p-1}}{2} + \frac{3p^4(\alpha t+\beta)^{-1+3p}\alpha^3\eta^3}{4} + \tau\left(\frac{9p^3(\alpha t+\beta)^{2p}\alpha^2\eta^2}{8} + \frac{\tau^2(\alpha t+\beta)^2\left((u-1)p-u\right)(\alpha t+\beta)^{-p}}{8} + \left(\eta\alpha p\left(\left(\eta\left(u-\frac{1}{2}\right)p+\left(-u-\frac{1}{8}\right)\eta+\frac{3t\tau}{2}\right)\alpha+\frac{3\beta\tau}{2}\right)(\alpha t+\beta)^p + \frac{5(\alpha t+\beta)\tau\left(\left(\eta\left(u-\frac{3}{5}\right)p+\left(-u-\frac{1}{5}\right)\eta+\frac{3t\tau}{5}\right)\alpha+\frac{3\beta\tau}{5}\right)}{8}\right)p\right)\right)u\right)\right)\right)}{\left(2\left(\eta(\alpha t+\beta)^p p\alpha+\frac{\tau(\alpha t+\beta)}{2}\right)(p-1)up\left(\eta(\alpha t+\beta)^p p\alpha+\tau(\alpha t+\beta)\right)^2\right)} \tag{55}$$

The graphs of square speed of sound for both interacting and non-interacting models are depicted in Figure (13) and Figure (14) respectively. In both models it is negative throughout the evolution of the universe. The negative sign of $v_s^{\,2}$denotes the unstableness of the model. So, the both interacting and non-interacting models are unstable models.

7. ENERGY CONDITIONS

In this section we discussed the energy conditions for interacting and non-interacting models.

The Energy conditions are given by

(1) $\rho_{DE} \geq 0$ (WEC)

(2)$\rho_{DE} + p_{DE} \geq 0$ (DEC)

(3) $\rho_{DE} + 3p_{DE} \geq 0$ (SEC)

The three energy conditions were plotted by using equations (39), (44) and (51) for both models.

From figure (15) and (16), it is observed that WEC and DEC are satisfied for both interacting and non-interacting models whereas SEC fails in both models. The violation SEC gives anti-gravitational effect for which universe gets jerk. So, the both models exhibit transition from early deceleration to present acceleration. So, the obtained models have good agreement with cosmological observations.

8. GRAPHICAL DISCUSSIONS

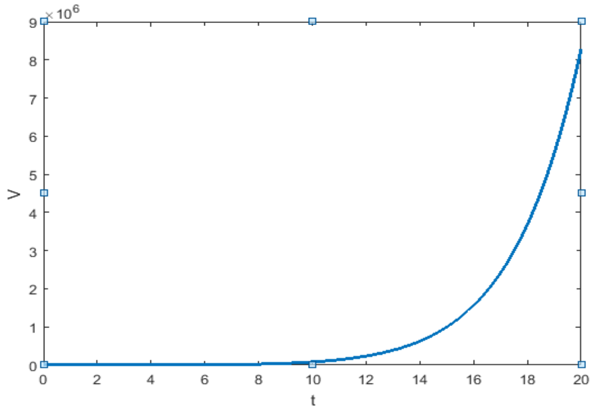

Figure 1.
The plot of V versus t
for $\alpha = 1.4, \beta = 0.2, p = 0.5$.
Clearly the spatial volume increases w.r.t. cosmic time t

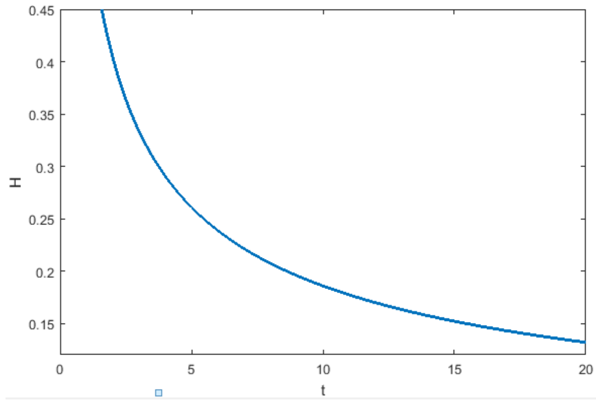

Figure 2.
The plot of H versus t
for $\alpha = 1.4, \beta = 0.2, p = 0.5$.
Clearly H decreases and tend to small value for late time

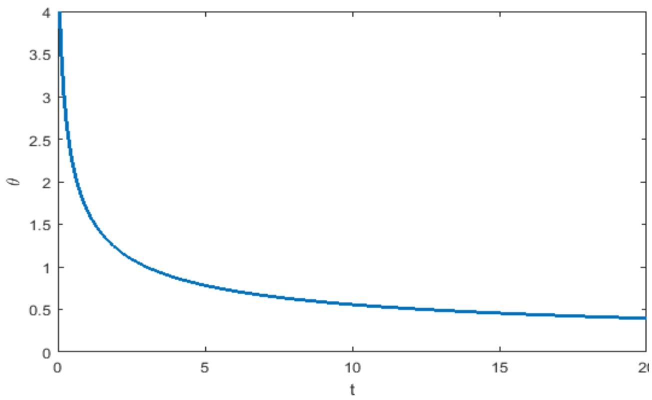

Figure 3.
The plot of θ versus t
for $\alpha = 1.4, \beta = 0.2, p = 0.5$.

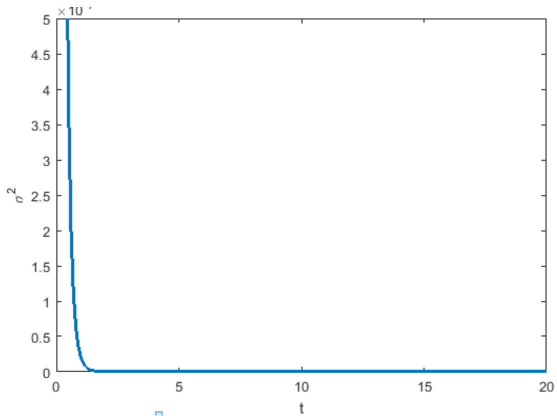

Figure 4.
The plot of σ^2 versus t
for $\alpha = 1.4, \beta = 0.2, p = 0.5, k = 0.03$.
It is observed that our universe is shear free at late time

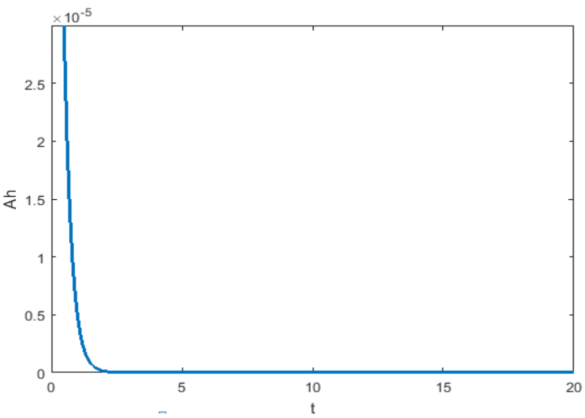

Figure 5.
The plot of $A_h$versus t
for $\alpha = 1.4, \beta = 0.2, p = 0.5, k = 0.03$.
It shows anisotropic nature in early stage and tends to zero (isotropic) at late lime

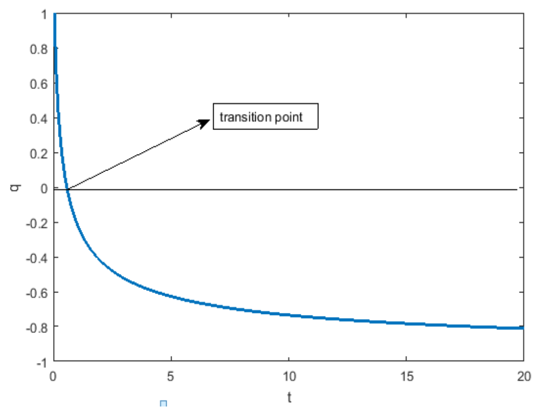

Figure 6.
The plot of q versus t
for $\alpha = 1.4, \beta = 0.2, p = 0.5$.
It is observed that q moves from positive region negative region.so our model denotes both decelerating and present accelerating phase

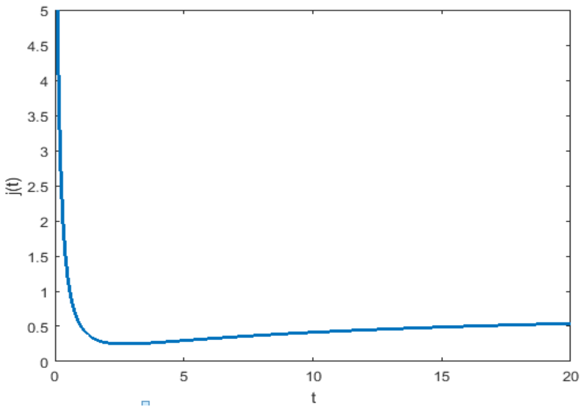

Figure 7.
The plot of j(t) versus t
for$\alpha = 1.4, \beta = 0.2, p = 0.5$.
It is observed that the jerk parameter is positive throughout the evolution of the universe.

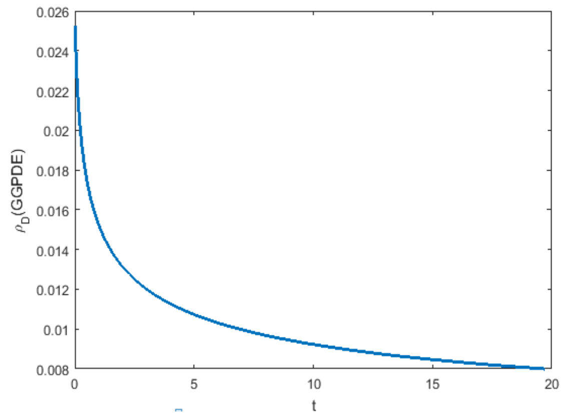

Figure 8.
The plot of ρ_{DE} versus t
for $\alpha = 1.4, \beta = 0.2, p = 0.5, \tau = 0.0004, \eta = 0.0005, u = 0.5$.
It is observed that the energy density of GGPDE is decreasing and tend to zero for large time t.

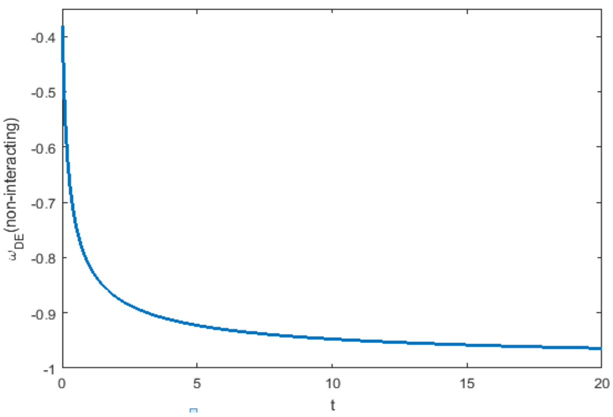

Figure 9.
The plot of EoS parameter ω_{DE} versus cosmic time of non-interacting model
for $\alpha = 1.4, \beta = 0.2, p = 0.5, \tau = 0.0004, \eta = 0.0005, u = 0.5$.
It shows the obtained model is quintessence model and tends to cosmological constant (ΛCDMmodel) as universe expands.

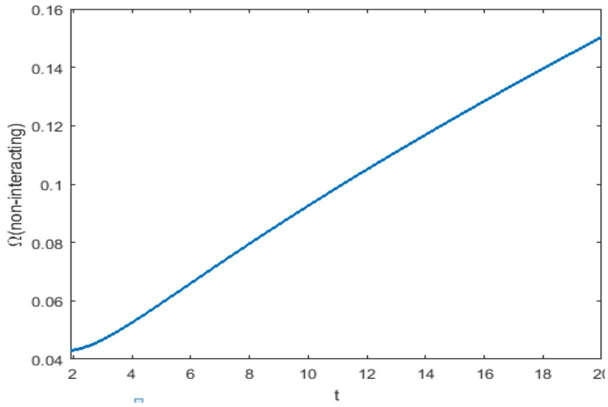

Figure 10.
The plot of overall density Ω versus time t for non-interacting model
for $\alpha = 1.4, \beta = 0.2, p = 0.5, \tau = 0.0004, \eta = 0.0005, u = 0.5, \rho_0 = 1$.
It displays the overall density of non – interacting model increases w.r.t. time t.

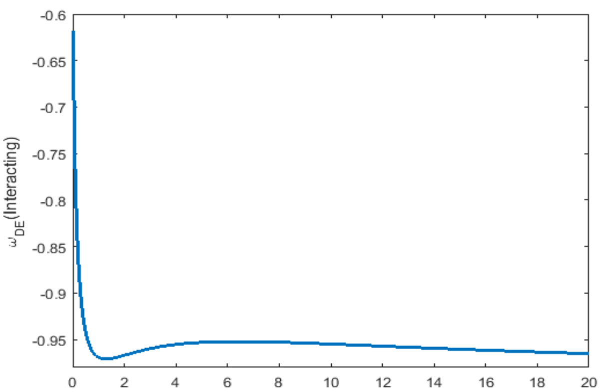

Figure 11.
The plot of EoS parameter ω_{DE} versus cosmic time of interacting model
for $\alpha = 1.4, \beta = 0.2, p = 0.5, \tau = 0.0004, \eta = 0.0005, u = 0.5, b = 0.5, \rho_1 = 0.04$.
It shows, initially $-1 < \omega_{DE} - 1/3$ and $\omega_{DE} \to -1$ as $t \to \infty$.
So the obtained interacting model is quintessence model andit tends to cosmological constant (ΛCDMmodel) as universe expands.

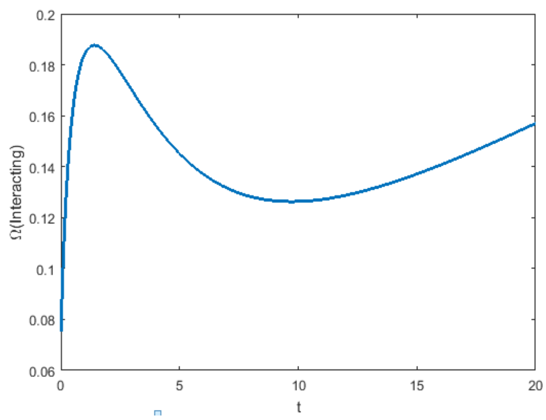

Figure 12.
The plot of overall density Ω versus time t for interacting model
for $\alpha = 1.4, \beta = 0.2, p = 0.5, \tau = 0.0004, \eta = 0.0005, u = 0.5, \rho_1 = 1, b = 0.5$.
The overall density of interacting model initially increases and after some time decreases and finally increases for large time t.

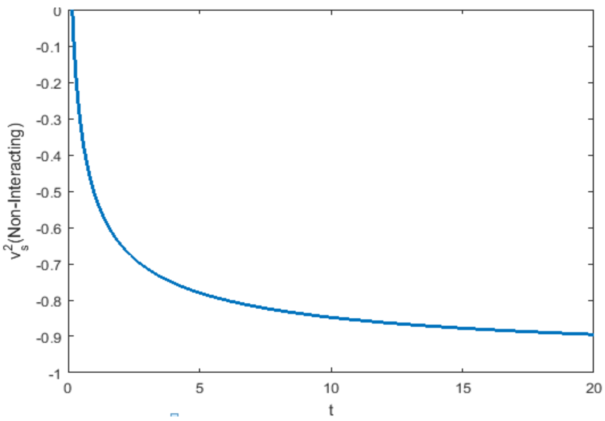

Figure 13.
The plot of $v_s{}^2$versus time t
for of non-Interactingmodel with $\alpha = 1.4, \beta = 0.2, p = 0.5, \tau = 0.0004, \eta = 0.0005, u = 0.5$.
It is observed that it is negative throughout the evolution of the universe.

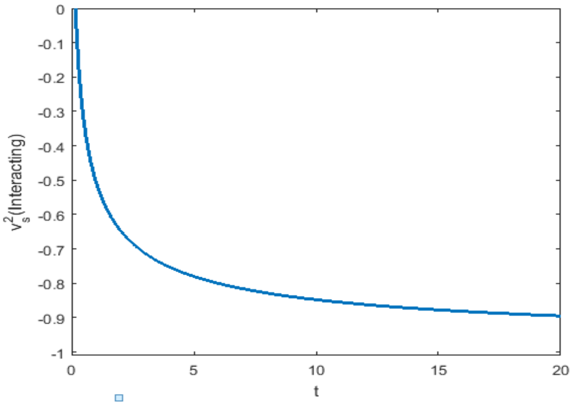

Figure 14.
The plot of $v_s{}^2$versus time t
for of Interactingmodel with $\alpha = 1.4, \beta = 0.2, p = 0.5, \tau = 0.0004, \eta = 0.0005, u = 0.5. \rho_1 = 0.04, b = 0.5$
It is observed that it is negative throughout the evolution of the universe.

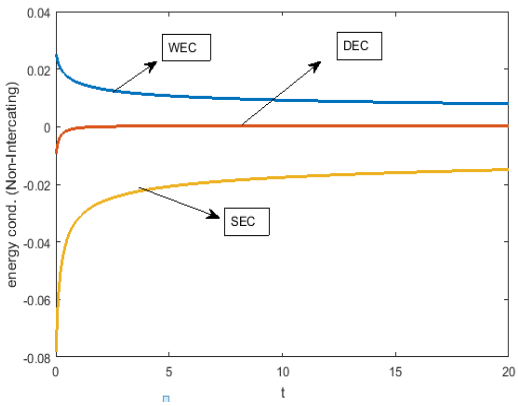

Figure 15.
The plots of energy conditions for non-Interacting model versus time t.

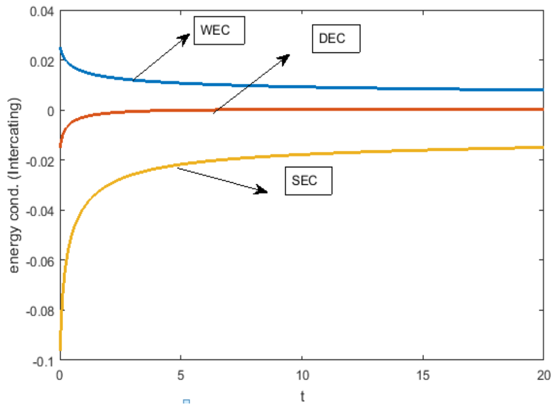

Figure 16.
The plots of energy conditions for Interacting model versus time t.

9. CONCLUSIONS

In this paper we have investigated the spatially homogeneous and anisotropic Bianchi type VI_0 space time with GGPDE in Saez-Ballester theory of gravitation. To obtain the solutions of field equations, the simple parametric form scale factor proposed by Mishra and Dua [1] is used. Both Interacting and non-interacting models have discussed. The findings of those models are given point wise as follows;

- The Spatial volume V is increasing with cosmic time *t.*
- The parameters H, θ, σ^2 are diminishes and approaches to zero as time evolves.
- From figure (5), the anisotropy parameter is diverging initially and decreasing with time and tend to zero at late time. This concludes the model is anisotropic in early universe and becoming isotropic model as $t \to \infty$.
- The decelerating parameter q is depicted in figure (6). The sign of decelerating parameter is changing from positive to negative. So, this model exhibits early deceleration and late time acceleration of the universe.
- The jerk parameter of the model is positive throughout the evolution of the universe. It can be observed from Figure (7).
- The EoS parameter for both models is presented in figures (9) and (11). For both models $\omega \to -1$ as time evolves. The both models behave like ΛCDM model at late time.
- The energy density of GGPDE is decreasing w.r.t. time and tends to a small value for large *t.*
- From the stability analysis of the models, it is observed that $v_s{}^2$ is negative for both models. So, the obtained interacting and non-interacting models are unstable.
- The overall density is increasing with time for both interacting and non-interacting models.
- The energy conditions were plotted for both the models. The energy conditions WEC, DEC are satisfied and SEC is violated for both models. So, the both models denote the accelerating expansion of the universe.

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УЗАГАЛЬНЕНА GHOST PILGRIMS КОСМОЛОГІЧНА МОДЕЛЬ ТЕМНОЇ ЕНЕРГІЇ Б'ЯНЧІ ТИПУ VI_0 В ТЕОРІЇ ГРАВІТАЦІЇ САЕЗА-БАЛЛЕСТЕРА

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Узагальнена Ghost Pilgrim темна енергія (GGPDE) у теорії гравітації Саеза-Баллестера (SBTG) і просторово-часова структура VI_0 типу Бьянчі слугують основою для цієї роботи. Ми використали просту параметризацію середнього масштабного коефіцієнта $a(t) = exp\{(\alpha t+\beta)^p\}$ Мішри та Дуа [Astrophys. Space Sci. 366, 6 (2021)], щоб знайти точні розв'язки рівнянь поля. Ми вивчили GGPDE і темну матерію (DM), як коли вони взаємодіють, так і коли вони не взаємодіють. Для обох моделей виробляються деякі важливі та добре відомі параметри, включаючи параметр Хаббла, параметр рівняння стану (EOS), параметр уповільнення тощо. Виявлено, що для обох моделей параметр уповільнення означає прискорену фазу, а параметр EOS — космологічну константу. Як для невзаємодіючих, так і для взаємодіючих моделей досліджуються аналіз стабільності та енергетичні умови.

Ключові слова: *параметр Хаббла; параметр EOS; параметр уповільнення; GGPDE; SBTG*