East European Journal of Physics. 1. 20-28 (2025)
DOI:10.26565/2312-4334-2025-1-02 ISSN 2312-4334

ACCELERATING THE COSMOLOGICAL MODEL WITH ZERO-MASS SCALAR FIELD IN LYRA'S GEOMETRY

M. Krishna[a], K. Sobhan Babu[b], R. Santhikumar[c]*
[a]Raghu Engineering College, Visakhapatnam, Andhra Pradesh, India
[b]University Collage of Engineering, Narasaraopeta, Andhra Pradesh, India
[c]Aditya Institute of Technology and Management, Tekkali, Srikakulam Dist. Andhra Pradesh-India
**Corresponding author email: skrmahanthi@gmail.com*
Received October 10, 2024; revised December 15, 2024; in final form December 26, 2024; accepted January 8, 2025

Examining Bianchi's type-III cosmological model involves incorporating a zero-mass scalar field in the context of Lyra's geometry. The source of energy-momentum tensor is supposed to be a bulk viscous fluid. A barotropic equation of state is applied to characterize the Pressure and density, seeking a specific solution to the field equations. This solution is derived using the distinctive variation principle for Hubble's parameter proposed by [M.S. Berman, Il Nuovo Cimento B, 74, 182 (1983)]. The ensuing analysis delves into the physical properties inherent in this model.
Keywords: *Accelerating; Cosmology; Lyra's geometry*
PACS: 4.50.Kd, 03.30.+p, 98.80.Bp, 66.20.-d, 98.80Cq

INTRODUCTION

The advancement of general relativity was profoundly influenced by the groundbreaking contributions of Einstein and Hilbert [2–4], leaving an indelible mark on cosmology, physics, and mathematics. The works of both Hilbert and Einstein extensively employed Riemannian geometry [5], a framework where space-time is equipped with a metric and an affine structure. These key components are delineated by $g_{\mu\nu}$ and a connection represented by $\Gamma^{\alpha}_{\mu\nu}$, respectively.

The geometric and gravitational characteristics of space-time find expression in the curvature tensor $R^{\mu}{}_{\nu\sigma\lambda}$ and its contraction, providing the foundation for the construction of the Einstein tensor.

Several modifications to Riemannian geometry have been proposed to unify the universe's gravitation, the electromagnetic field, and other fundamental interactions. Weyl [6] made one such attempt by trying to unify gravitation and electromagnetism within a single space-time geometry. However, Weyl's theory faced criticism due to its reliance on the non-integrability of length transfer. Later, Lyra [7] introduced a further modification to Riemannian geometry by incorporating a gauge function into a less manifold structure, thereby eliminating the issue of non-integrability in length transfer. This modification naturally gave rise to a displacement vector. Building on Lyra's work, Sen [8] and Sen and Dunn [9] developed a new scalar-tensor theory of gravitation and formulated an analogy of the Einstein field equations based on Lyra's geometry. Halford [10] noted that the constant vector displacement field ϕ_i Lyra's geometry functions similarly to the cosmological constant Λin conventional general relativity. Furthermore, Halford [11] demonstrated that the scalar-tensor theory derived from Lyra's geometry yields predictions consistent with observational limits, matching the results of Einstein's theory.

Cosmological models of Bianchi [12] exhibit both homogeneity and anisotropy, providing a framework for investigating the gradual isotropization of the universe over time. Additionally, from a mathematical and theoretical perspective, anisotropic universes offer greater generality than isotropic Friedmann-Robertson-Walker (FRW) models.

Exploring interacting fields, particularly involving a zero-mass scalar field, is essential to address the unresolved challenge of unifying gravitational and quantum theories. This study delves into the intricate problem of reconciling these fundamental aspects of physics.

Furthermore, examining viscous mechanisms in cosmology is pivotal in elucidating the high entropy observed at present. This investigation contributes valuable insights into understanding the thermodynamic properties and evolution of the cosmos.

This paper is framed within the context of previous research studies. Reddy et al. [13-14] investigated Bianchi type-III models incorporating bulk viscous coefficients. Katore et al. [15] explored solutions for zero-mass cosmological models with bulk viscous coefficients within the Lyra geometry. Halford [16] provided an overview of Lyra's geometry, and Singh [17] further delved into the same topic. Santhikumar [18] focused on accelerating cosmological models, while Santhikumaret al. [19] explored Lyra's geometry heat flow cosmological models. Krishna [20] also examined plane-symmetric cosmological models within Lyra's geometry. Numerous authors have extended their research within Lyra's geometry, laying the foundation for future research. By Motivation from these studies, this paper presents a novel contribution: a Bianchi type-III cosmological model incorporating a zero-mass scalar field and perfect fluid and bulk viscous effects in Lyra's geometry.

Cite as: M. Krishna, K.S. Babu, R. Santhikumar, East Eur. J. Phys. 1, 20 (2025), https://doi.org/10.26565/2312-4334-2025-1-02

The structure of this paper is organized as follows: Section 2 examines the metric and field equations within the Bianchi type-III cosmological model, incorporating a zero-mass scalar field in Lyra's Geometry. Section 3 is dedicated to deriving the solutions to the field equations. In Section 4, we focus on explaining the physical properties of the models. Section 5 provides a detailed discussion, and Section 6 concludes the paper.

METRIC AND FIELD EQUATIONS

The Bianchi type-III metric is

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)e^{-2x}dy^2 - C^2(t)dz^2, \tag{1}$$

where $A, B\&C$ are cosmic scale factors.

The field equations in standard gauge for Lyra's geometry, as obtained by Sen [8], are

$$\left(R_{ij} - \frac{1}{2}g_{ij}R\right) + \left(\frac{3}{2}\phi_i\phi_j - \frac{3}{4}g_{ij}\phi_k\phi^k\right) = \kappa T_{ij} + \left(\psi_{;i}\psi_{;j} - \frac{1}{2}g_{ij}\psi_{,k}\psi^{,k}\right). \tag{2}$$

In the Einstein field equations, κ(kappa) is a constant related to the gravitational constant 'G' and the speed of light 'C' Specifically, $\kappa = \frac{8\pi}{c^4}G$, in natural units, where c=1 κ reduces to 8π, so $\kappa = 8\pi G$. It simplifies the Einstein field equations to avoid explicitly carrying around the constant, making focusing on the functional relationships between variables easier. We consider $\kappa = 1$, (*the n*atural units $G = 1$, $8\pi G = 1$)).
Here $\phi_i = (0,0,0,\beta(t))$ is the displacement vector,
Consider

$$\bar{T}_{ij} = \left(\frac{3}{2}\phi_i\phi_j - \frac{3}{4}g_{ij}\phi_k\phi^k\right) = \begin{cases} \frac{3}{4}\beta^2(t), for\ i = j = 0\ (Time - time) \\ -\frac{3}{4}g_{ij}g^{00}\beta^2(t)\ , for\ i = j\ \neq 0\ (spatial\ components for\ i = j = 1,2,3) \end{cases}, \tag{3}$$

so, we have

$$\bar{T}_0^0 = \frac{3}{4}\beta^2(t), \bar{T}_1^1 =, \bar{T}_2^2 = \bar{T}_3^3 = -\frac{3}{4}\beta^2(t) \tag{4}$$

$$\bar{\Psi}_{ij} = \left(\psi_{;i}\psi_{;j} - \frac{1}{2}g_{ij}\psi_{,k}\psi^{,k}\right) = \begin{cases} \frac{1}{2}\dot{\psi}^2\ , for\ i = i = 0\ (Time - time) \\ -\frac{1}{2}g_{ij}\dot{\psi}^2, for\ i = j \neq 0\ (spatial\ components for\ i = j = 1,2,3) \end{cases}, \tag{5}$$

so, we have

$$\bar{\Psi}_0^0 = \frac{1}{2}\dot{\psi}^2\ , \bar{\Psi}_1^1 = \bar{\Psi}_2^2 = \bar{\Psi}_3^3 = -\frac{1}{2}\dot{\psi}^2 \tag{6}$$

T_{ij} is the energy-momentum tensor for bulk viscous and zero-mass scalar fields as

$$T_{ij} = (\rho + \bar{p})u_i u_j - \bar{p}g_{ij} \tag{7}$$

Together with

$$u^i u_i = 1, \bar{p} = p - \eta u^i_{,i} = p - 3\eta H \tag{8}$$

Where u_i is the four-velocity vector of the distribution,
p is the Pressure,$\bar{p}$ is the adequate Pressure,
η is the bulk viscosity coefficient, and ψ is the zero-mass scalar field.
The non-vanishing energy-momentum tensor components are
Here

$$T_0^0 = \rho, T_1^1 = T_2^2 = T_3^3 = -\bar{p}. \tag{9}$$

Introducing a barotropic equation of state into the discussion

$$p = (\gamma - 1)\rho, 0 \le \gamma \le 2 \tag{10}$$

Employing co-moving coordinates, the field equations (1) – (8)

$$-\frac{\dot{A}\dot{C}}{AC} - \frac{\dot{B}\dot{C}}{BC} - \frac{\dot{A}\dot{B}}{AB} + \frac{1}{A^2} + \frac{3}{4}\beta^2 = \left(\rho + \frac{1}{2}\dot{\psi}^2\right), \tag{11}$$

$$\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\frac{\dot{B}\dot{C}}{BC}+\frac{3}{4}\beta^2=\left(\overline{p}+\frac{1}{2}\dot{\psi}^2\right) \tag{12}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{C}}{C}+\frac{\dot{A}\dot{C}}{AC}+\frac{3}{4}\beta^2=\left(\overline{p}+\frac{1}{2}\dot{\psi}^2\right) \tag{13}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}-\frac{1}{A^2}+\frac{3}{4}\beta^2=\left(\overline{p}+\frac{1}{2}\dot{\psi}^2\right) \tag{14}$$

$$\frac{\dot{A}}{A}-\frac{\dot{B}}{B}=0 \tag{15}$$

The scalar field ψ complies with the following equation.

$$\psi^{i}_{;i}=0 \tag{16}$$

Hence, the semi-colon $(;)$ indicates covariant differentiability

By using the equations(14), which yields that

$$\ddot{\psi}+\dot{\psi}\left[\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right]=0 \tag{17}$$

And conservation of L.H.S of Eq.(2) leads that

$$\left(R^{j}_{i}-\frac{1}{2}g^{j}_{i}R\right)_{;j}+\left[\frac{3}{2}(\phi_i\phi^j)_{;j}-\frac{3}{4}\left(\phi_k\phi^k g^{j}_{i}\right)_{;j}\right]=0 \tag{18}$$

$$\frac{3}{2}\phi_i\left[\frac{\partial\phi^j}{\partial x^j}+\phi^l\Gamma^{\mathrm{j}}_{\mathrm{lj}}\right]+\frac{3}{2}\phi^j\left[\frac{\partial\phi_i}{\partial x^j}-\phi_l\Gamma^{\mathrm{l}}_{\mathrm{ij}}\right]-\frac{3}{4}g^{j}_{i}\phi_k\left[\frac{\partial\phi^k}{\partial x^j}+\phi^l\Gamma^{\mathrm{k}}_{\mathrm{lj}}\right]-\frac{3}{4}g^{j}_{i}\phi^k\left[\frac{\partial\phi_k}{\partial x^j}-\phi_l\Gamma^{\mathrm{l}}_{\mathrm{kj}}\right]=0. \tag{19}$$

Eq. (19) leads that

$$\frac{3}{2}\beta\dot{\beta}+\frac{3}{2}\beta^2\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)=0 \tag{20}$$

The Solution to The Field Equations

Integrating eq. (13), we get

$$B=kA \tag{21}$$

Take $k=1$, without loss of generality, So we have

$$B=A \tag{22}$$

By using Equation (16) in Equations (9)-(12) reduced to

$$-2\frac{\dot{A}\dot{C}}{AC}-\left(\frac{\dot{A}}{A}\right)^2+\frac{1}{A^2}+\frac{3}{4}\beta^2=\left(\rho+\frac{1}{2}\dot{\psi}^2\right), \tag{23}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{C}}{C}+\frac{\dot{A}\dot{C}}{AC}+\frac{3}{4}\beta^2=\left(\overline{p}+\frac{1}{2}\dot{\psi}^2\right), \tag{24}$$

$$2\frac{\ddot{A}}{A}+\left(\frac{\dot{A}}{A}\right)^2-\frac{1}{A^2}+\frac{3}{4}\beta^2=\left(\overline{p}+\frac{1}{2}\dot{\psi}^2\right). \tag{25}$$

The relation between average scale factor 'a' and Volume 'V' are

$$V=\sqrt{-g}=A(t)B(t)C(t)e^{-x} \text{and } a(t)=(V)^{1/3}=\left(A(t)B(t)C(t)e^{-x}\right)^{1/3}. \tag{26}$$

The average Hubble parameter is

$$H=\frac{1}{3}\left(H_x+H_y+H_z\right)=\frac{\dot{a}}{a} \tag{27}$$

where $H_x=\frac{\dot{A}}{A}, H_y=\frac{\dot{B}}{B}, H_z=\frac{\dot{C}}{C}$.

The scalar expansion is

$$\theta = u^i_{;i} = 2\frac{\dot{A}}{A} + \frac{\dot{C}}{C} \tag{28}$$

The shear scalar is

$$\sigma^2 = \frac{1}{2}\sigma_{ik}\sigma^{ik} = \frac{1}{2}\left[2\left(\frac{\dot{A}}{A}\right)^2 + \left(\frac{\dot{C}}{C}\right)^2 - \frac{1}{3}\left(2\frac{\dot{A}}{A} + \frac{\dot{C}}{C}\right)^2\right] \tag{29}$$

The mean anisotropic parameter is

$$A_\alpha = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{\Delta H_i}{H}\right)^2 = \frac{1}{3H^2}\left[{H_x}^2 + {H_y}^2 + {H_z}^2 - \frac{1}{9}\left(H_x + H_y + H_z\right)^2\right] \tag{30}$$

where $\Delta H_i = H_i - H$, for $i = 1,2,3$

Since Eqs.$(23) - (25)$equations are highly non-linear equations. Hence, to derive a definitive solution, it is imperative to consider the following requisite conditions.

(i) Utilizing the variation of Hubble's parameter proposed by Berman [1], we obtain models of the universe characterized by the constant decelerating parameter

$$q = \frac{-a\ddot{a}}{\dot{a}^2} = \text{constant.} \tag{31}$$

The solutions of Eq. (31) yields that

$$a = [k_1 t + k_2]^{\frac{1}{(q+1)}} \tag{32}$$

This equation indicates that the criterion for accelerated expansion is$(1 + q) > 0$.

(ii) Since, $\theta^2 \infty \sigma^2$Collin et al. [21]
which gives us

$$A = C^n, \text{ for } n \neq 1 \tag{33}$$

By Equations (22), (31), (32) and (33)
We obtain metric coefficients, which are

$$A = B = [k_4 t + k_5]^{\frac{3n}{(1+q)(2n+1)}} \tag{34}$$

$$C = [k_4 t + k_5]^{\frac{3}{(1+q)(2n+1)}}. \tag{35}$$

Using Eq. (22) and (33), the Eqs. (23)-(25) reduces to

$$-(2n + n^2)\left(\frac{\dot{C}}{C}\right)^2 + C^{-2n} + \frac{3}{4}\beta^2(t) = \rho + \frac{1}{2}\dot{\psi}^2, \tag{36}$$

$$(n + 1)\frac{\ddot{C}}{C} + n^2\left(\frac{\dot{C}}{C}\right)^2 + \frac{3}{4}\beta^2(t) = \bar{p} + \frac{1}{2}\dot{\psi}^2 \tag{37}$$

$$2n\frac{\ddot{C}}{C} + n(2n - 1)\left(\frac{\dot{C}}{C}\right)^2 - C^{-2n} + \frac{3}{4}\beta^2(t) = \bar{p} + \frac{1}{2}\dot{\psi}^2. \tag{38}$$

By substituting the values A, B and C in (2), we get

$$ds^2 = dt^2 - [k_4 t + k_5]^{\frac{6n}{(1+q)(2n+1)}}[dx^2 + e^{-2x}dy^2] - [k_4 t + k_5]^{\frac{6}{(1+q)(2n+1)}}dz^2. \tag{39}$$

Some Physical Properties of the Model

Eq. (39) describes the Bianchi type-III cosmological model featuring bulk viscous effects and a zero-mass scalar field under Lyra's geometry.
The Spatial volume is

$$V = (k_4 t + k_5)^{\frac{3(n+1)}{1+q}} e^{-x} \tag{40}$$

The Hubble's parameter is

$$H = \frac{k_4}{(1+q)(k_4 t + k_5)} \tag{41}$$

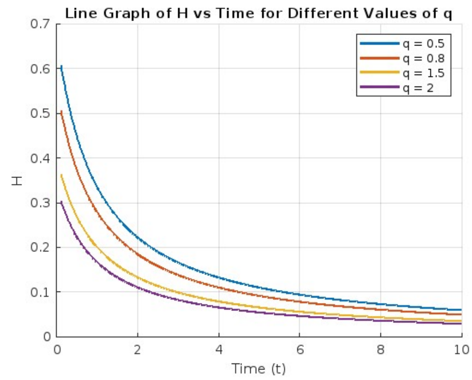

Figure 1. Hubble's Parameter H Vs. time t
Parameters used $k_4 = 1$; $k_5 = 1$

The scalar expansion is

$$\theta = \frac{3k_4}{(1+q)(k_4t+k_5)}. \tag{42}$$

The shear scalar is

$$\sigma^2 = 3(n^2+n-1)\left(\frac{k_4}{(2n+1)(1+q)(k_4t+k_5)}\right)^2. \tag{43}$$

Clearly

$$\lim_{t\to\infty}\frac{\sigma^2}{\theta^2} = \frac{(n^2+n-1)}{3(2n+1)^2} \neq 0. \tag{44}$$

Hence, the model approaches anisotropy for large values of t
The mean anisotropic parameter is

$$A_\alpha = \frac{(1-4n)}{3(2n+1)^2}. \tag{45}$$

By Solving Equation. (17) the zero-mass scalar field is

$$\psi(t) = C_3(k_4t+k_5)^{\left(\frac{q-2}{1+q}\right)} + C_2, \tag{46}$$

where $C_2 = \frac{C_1(1+q)}{k_4(1+q-3)}$, $C_1 and\ C_2$ are integration constants

By solving equation (20) with the help of Eqs. (34) and (35), we have
The displacement vector $\beta(t)$ is

$$\beta(t) = \frac{C_4}{(k_4t+k_5)^{\left(\frac{3}{1+q}\right)}}. \tag{47}$$

Where $C_4$is integration constant
Using Eqs. (34) and (35) in Eqs.(36)-(38) , we get
The density of the model is

$$\rho = \left(\frac{-9(n^2+2n)}{(2n+1)^2(1+q)^2}\right)\left(\frac{k_4}{k_4t+k_5}\right)^2 + \frac{1}{(k_1t+k_2)^{\frac{6n}{(1+q)(2n+1)}}} + \frac{3}{4}\left(\frac{C_4}{(k_4t+k_5)^{\left(\frac{3}{1+q}\right)}}\right)^2 - \frac{1}{2}\left(\frac{C_1}{(k_1t+k_2)^{\frac{3}{1+q}}}\right) \tag{48}$$

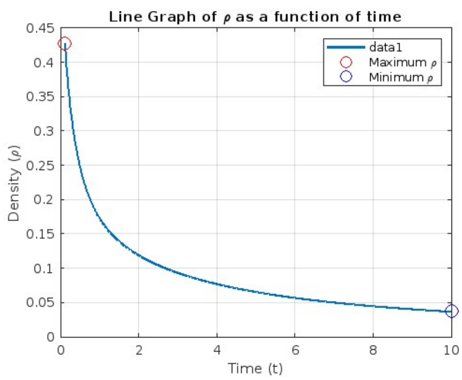

Figure 2. Density Vs. time
Parameters used $n = 2$; $q = 1$; $k_1 = 1$; $k_2 = 1$; $k_4 = 1$; $k_5 = 1$; $C_1 = 1$; $C4 = 1$

The Adequate Pressure of the model is

$$\bar{p} = \left(\frac{9(n^2+n+1)-3(2n+1)(1+q)}{(2n+1)^2(1+q)^2}\right)\left(\frac{k_4}{k_4t+k_5}\right)^2 + \frac{3}{4}\left(\frac{C_4}{(k_4t+k_5)^{\left(\frac{3}{1+q}\right)}}\right)^2 - \frac{1}{2}\left(\frac{C_1}{(k_1t+k_2)^{\frac{3}{1+q}}}\right). \tag{49}$$

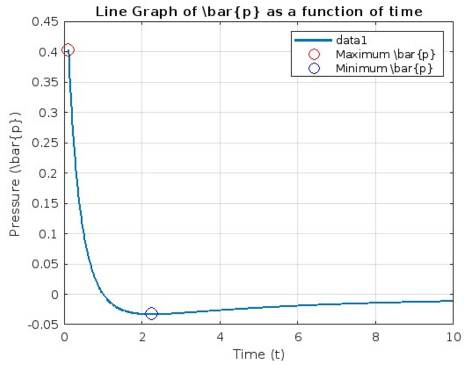

Figure 3. Adequate Pressure Vs. time
Parameters used n = 2; q = 1; k_1 = 1; k_2 = 1; k_4 = 1; k_5 = 1; C_1 = 1; C4 = 1

The Pressure of the model is

$$p = (\gamma-1)\left[\left(\frac{-9(n^2+2n)}{(2n+1)^2(1+q)^2}\right)\left(\frac{k_4}{k_4t+k_5}\right)^2 + \frac{1}{(k_1t+k_2)^{\frac{6n}{(1+q)(2n+1)}}} + \frac{3}{4}\left(\frac{C_4}{(k_4t+k_5)^{\left(\frac{3}{1+q}\right)}}\right)^2 - \frac{1}{2}\left(\frac{C_1}{(k_1t+k_2)^{\frac{3}{1+q}}}\right)\right]. \tag{50}$$

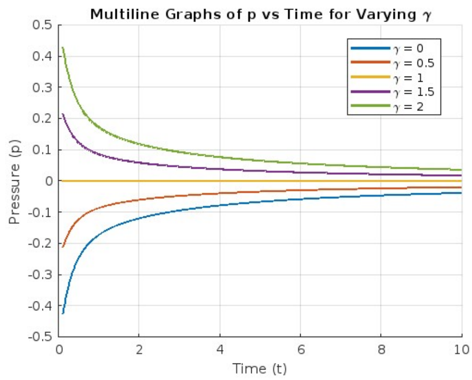

Figure 4. Pressure Vs. time
Parameters used n = 2; q = 1; k_1 = 1; k_2 = 1; k_4 = 1; k_5 = 1; C_1 = 1; C4 = 1, $0 \le \gamma \le 2$

The Coefficient of Bulk Viscosity of the model is

$$\eta = \frac{(1+q)(k_4t+k_5)}{3k_4}\left[\begin{array}{c}\left(\frac{-9(\gamma-1)(n^2+2n)-\left(\frac{9(n^2+n+1)-3(2n+1)(1+q)}{(2n+1)^2(1+q)^2}\right)}{(2n+1)^2(1+q)^2}\right)\left(\frac{k_4}{k_4t+k_5}\right)^2 \\ +\frac{(\gamma-1)}{(k_1t+k_2)^{\frac{6n}{(1+q)(2n+1)}}} + \frac{3(\gamma-2)}{4}\left(\frac{C_4}{(k_4t+k_5)^{\left(\frac{3}{1+q}\right)}}\right)^2 - \frac{1}{2}\left(\frac{(\gamma-2)C_1}{(k_1t+k_2)^{\frac{3}{1+q}}}\right)\end{array}\right]. \tag{51}$$

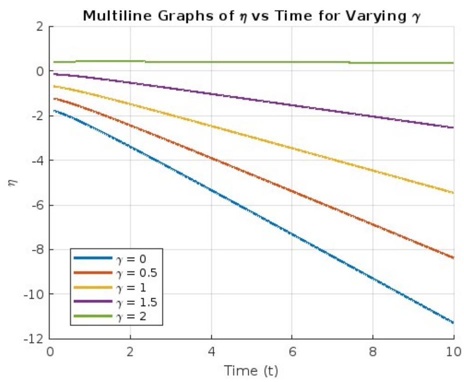

Figure 5. Bulk viscous Coefficient V_s. time t Parameters used n = 2; q = 1; k_1 = 1; k_2 = 1; k_4 = 1; k_5 = 1; C_1 = 1; C4 = 1, $0 \le \gamma \le 2$

The Density parameter of the model is

$$\Omega = \frac{1}{3}\left(\frac{(1+q)(k_4t+k_5)}{k_4}\right)^2\left[\left(\frac{-9(n^2+2n)}{(2n+1)^2(1+q)^2}\right)\left(\frac{k_4}{k_4t+k_5}\right)^2 + \frac{1}{(k_1t+k_2)^{\frac{6n}{(1+q)(2n+1)}}} + \frac{3}{4}\left(\frac{C_4}{(k_4t+k_5)^{\left(\frac{3}{1+q}\right)}}\right)^2 - \frac{1}{2}\left(\frac{C_1}{(k_1t+k_2)^{\frac{3}{1+q}}}\right)\right]. \quad (52)$$

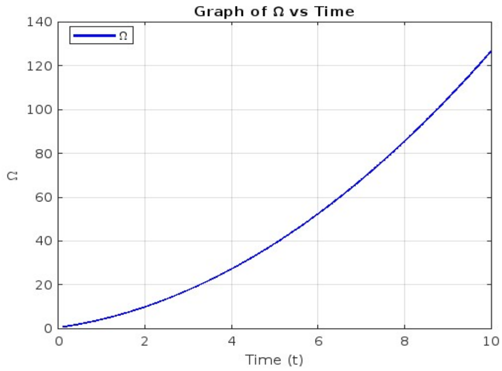

Figure 6. Density Parameter Vs. time t
Parameters used n = 2; q = 1; k_1 = 1; k_2 = 1; k_4 = 1; k_5 = 1; C_1 = 1; C4 = 1

From Eqs. (40) to (52), we observed that at t=0, the spatial volume and zero mass scalar are zero, increasing with cosmic time, showing the late-time accelerated expansion of the universe. Also, at t=0, the parameters $H, \theta, \rho, p, \bar{p}, \eta, \Omega$ are diverse while they vanish for infinitely large values of t. The mean anisotropic parameter is uniform throughout the whole evolution of the universe, which shows that the dynamics of the mean anisotropic parameter do not depend on cosmic time t. Also, since $\lim_{t\to\infty}\frac{\sigma^2}{\theta^2}$ It is constant; the model does not approach isotropy through the whole evolution of the universe. It may also be observed that the model Eq. (39) has no initial singularity.

Discussions for Physical Properties of the Model

Here, we can discuss the three physical models based on the value of $\gamma = 0, 2, \frac{4}{3}$ respectively

False Vacuum model

When $\gamma = 0$equals zero, the model embodies the false vacuum model with an Equation of State given by $p = -\rho$, characterizing both the 'false vacuum' and 'degenerate vacuum.' The explicit form and physical properties of this model are then delineated

$$p = -\rho = -\left[\left(\frac{-9(n^2+2n)}{(2n+1)^2(1+q)^2}\right)\left(\frac{k_4}{k_4t+k_5}\right)^2 + \frac{1}{(k_1t+k_2)^{\frac{6n}{(1+q)(2n+1)}}} + \frac{3}{4}\left(\frac{C_4}{(k_4t+k_5)^{\left(\frac{3}{1+q}\right)}}\right)^2 - \frac{1}{2}\left(\frac{C_1}{(k_1t+k_2)^{\frac{3}{1+q}}}\right)\right]. \quad (53)$$

$$\eta = \frac{(1+q)(k_4t+k_5)}{3k_4}\left[\left(\frac{-9(n^2+2n)-\left(\frac{9(n^2+n+1)-3(2n+1)(1+q)}{(2n+1)^2(1+q)^2}\right)}{(2n+1)^2(1+q)^2}\right)\left(\frac{k_4}{k_4t+k_5}\right)^2 + \frac{1}{(k_1t+k_2)^{\frac{6n}{(1+q)(2n+1)}}} - \frac{3}{2}\left(\frac{C_4}{(k_4t+k_5)^{\left(\frac{3}{1+q}\right)}}\right)^2 + \left(\frac{C_1}{(k_1t+k_2)^{\frac{3}{1+q}}}\right)\right] \quad (54)$$

The Equation of State parameter is

$$\omega = \frac{p}{\rho} = -1 \quad (55)$$

Zel'dovish fluid model (Stiff fluid model)

At $\gamma = 2$, $p = \rho$, representing a Zel'dovich fluid distribution. Then, the explicit form of the physical properties inherent in this model is detailed.

$$p = \rho = \left[\left(\frac{-9(n^2+2n)}{(2n+1)^2(1+q)^2}\right)\left(\frac{k_4}{k_4t+k_5}\right)^2 + \frac{1}{(k_1t+k_2)^{\frac{6n}{(1+q)(2n+1)}}} + \frac{3}{4}\left(\frac{C_4}{(k_4t+k_5)^{\left(\frac{3}{1+q}\right)}}\right)^2 - \frac{1}{2}\left(\frac{C_1}{(k_1t+k_2)^{\frac{3}{1+q}}}\right)\right] \quad (56)$$

$$\eta = \frac{(1+q)(k_4t+k_5)}{3k_4}\left[\left(\frac{-9(n^2+2n)-\left(\frac{9(n^2+n+1)-3(2n+1)(1+q)}{(2n+1)^2(1+q)^2}\right)}{(2n+1)^2(1+q)^2}\right)\left(\frac{k_4}{k_4t+k_5}\right)^2 + \frac{1}{(k_1t+k_2)^{\frac{6n}{(1+q)(2n+1)}}}\right] \quad (57)$$

$$\omega = \frac{p}{\rho} = 1 \quad (58)$$

Radiating model

When $\gamma = \frac{4}{3}$, then$p = \frac{\rho}{3}$. This representation corresponds to a matter distribution with disordered radiation, signifying the universe where the predominant portion of energy density exists, like radiation. Consequently, the model is termed a radiation-dominated universe or a radiating model. The explicit form of the physical properties inherent in this model is then elaborated.

$$p = \frac{1}{3}\rho = \frac{1}{3}\left[\left(\frac{-9(n^2+2n)}{(2n+1)^2(1+q)^2}\right)\left(\frac{k_4}{k_4t+k_5}\right)^2 + \frac{1}{(k_1t+k_2)^{\frac{6n}{(1+q)(2n+1)}}} + \frac{3}{4}\left(\frac{C_4}{(k_4t+k_5)^{\left(\frac{3}{1+q}\right)}}\right)^2 - \frac{1}{2}\left(\frac{C_1}{(k_1t+k_2)^{\frac{3}{1+q}}}\right)\right] \tag{59}$$

$$\eta = \frac{(1+q)(k_4t+k_5)}{3k_4}\left[\left(\frac{-3(n^2+2n)-\left(\frac{9(n^2+n+1)-3(2n+1)(1+q)}{(2n+1)^2(1+q)^2}\right)}{(2n+1)^2(1+q)^2}\right)\left(\frac{k_4}{k_4t+k_5}\right)^2 + \frac{1}{3(k_1t+k_2)^{\frac{6n}{(1+q)(2n+1)}}} - \frac{1}{2}\left(\frac{C_4}{(k_4t+k_5)^{\left(\frac{3}{1+q}\right)}}\right)^2 + \frac{1}{3}\left(\frac{C_1}{(k_1t+k_2)^{\frac{3}{1+q}}}\right)\right] \tag{60}$$

$$\omega = \frac{p}{\rho} = \frac{1}{3} \tag{61}$$

Our observations show that the model described by equation (39) exhibits no singularity, specifically at t=0. The zero-mass scalar field displays divergence at t=0 but diminishes for larger t values. The spatial volume undergoes expansion with increasing t, as indicated by the positivity of 1+q, portraying accelerated universe expansion. Additionally, θ, σ^2, and H tend towards infinity at t=0 and converge towards zero for larger t values. The constancy of the average anisotropy parameter signifies its uniformity throughout the universe's evolution. However, since $\frac{\sigma^2}{\theta^2}$Remains unalter, indicating a sustained anisotropic nature. It is observed that for the closed universe, when ω_D It is a decreasing function of time and an increasing function of time for open and flat universes. Universe is Closed, open, and flat universes are varying in quintessence ($\omega_D > -0.5$), phantom ($-3 < \omega_D < -1$), and super phantom ($\omega_D < -0.3$) regions, respectively.

Scientific Comparison

Compared to the model proposed by B. Misra et al. (2015) [22], the current framework is more streamlined, with time dependence primarily expressed through power-law terms. It adopts a more phenomenological perspective on density evolution over time, featuring reduced complexity in its dependencies. This approach indicates an alternative or simplified cosmological paradigm, offering a distinct interpretation of ρ that could imply processes such as dissipation, decay, or energy loss. K.P. Singh et al. (2018) [23] explored cosmological models within the framework of Lyra's geometry, utilizing the Bianchi type III metric, with particular emphasis on the interaction between the Van der Waals fluid and Lyra's manifold, as well as its contribution to the generation of dark energy. In contrast, the current model focuses on deriving solutions involving a zero-mass scalar field and a bulk viscous fluid characterized by a barotropic equation of state. These two models adopt different approaches and interpretations, highlighting their distinct objectives and methodologies.

CONCLUSIONS

In this investigation, we explored the characteristics of a Bianchi type-III cosmological model incorporating a zero-mass scalar within Lyra's geometry, with the energy-momentum tensor sourced by bulk viscous fluid. We observed that at t = 0, the spatial volume and zero mass scalar are zero, increasing with cosmic time, showing the late-time accelerated expansion of the universe. Also, at t = 0, the parameters $H, \theta, \rho, p, \bar{p}, \eta, \Omega$ are diverse while they vanish for infinitely large values of t. The mean anisotropic parameter is uniform throughout the whole evolution of the universe, which shows that the dynamics of the mean anisotropic parameter do not depend on cosmic time t. Also, since $\lim_{t\to\infty}\frac{\sigma^2}{\theta^2}$ It is constant; the model does not approach isotropy through the whole evolution of the universe. It may also be observed that the model Eq. (39) has no initial singularity.

Notably, our findings reveal that the model is non-singular, exhibits shearing and non-rotating properties, and does not tend towards isotropy for large values of cosmic time t. The spatial volume displays an increasing trend with time (as 1+q > 0), suggesting the possibility of cosmic re-collapse in the finite future. This dynamic evolution entails phases of inflation, deceleration, and subsequent acceleration. Consequently, the model emerges as an accelerating cosmological model featuring a zero-mass scalar under Lyra's geometry. We have discussed the physical models corresponding to the False Vacuum, Stiff fluid, and radiating. These cosmological models are anisotropic and have no initial singularity. Hence,zero-mass scalar field and bulk viscosity are expected to play an essential role in the universe's early evolution. Therefore, the model presented here better understands the evaluation of the universe.

Acknowledgment

The authors wish to place on record their sincere thanks to the referee whose valuable comments and suggestions have helped improve this manuscript's quality.

ORCID
M. Krishna, https://orcid.org/0000-0003-1975-8930; **K. Sobhan Babu,** https://orcid.org/0000-0002-2991-7651
R. Santhikumar, https://orcid.org/0000-0001-5122-3800

REFERENCES

[1] M.S. Berman, "A Special Law of Variation for Hubble's Parameter," Il Nuovo Cimento B, **74**, 182-186 (1983). http://dx.doi.org/10.1007/BF02721676
[2] A. Einstein, Preussische Akademie der Wissenschaften, Sitzungsberichte, (Berlin, 1915). pp. 831-839.
[3] D. Hilbert, "Nachrichten von der Gesellschaft der Wissenschaftenzu Göttingen," Math. Phys. Klasse, **3**, 395 (1915).
[4] A. Einstein, "The Foundation of the General Theory of Relativity," Annalen der Physik, **49,** 769-822 (1916) http://dx.doi.org/10.1002/andp.19163540702
[5] B. Riemann, Habilitationsschrift, Abhandlungen der Königlichen Gesellschaft der Wissenschaftenzu Göttingen, **13**(1), 1854 (1867).
[6] H. Weyl, *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, **465**, (1918). https://www.biodiversitylibrary.org/bibliography/42231
[7] G. Lyra, "Über eine Modifikation der Riemannschen Geometrie," Math. Z. **54**, 52-64 (1951). https://doi.org/10.1007/BF01175135
[8] D.K. Sen, "A Static Cosmological Model," Zeitschriftfur Physik A Hadrons and Nuclei, **149**(3), 311-323 (1957).
[9] D.K. Sen, and K.A. Dunn, J. Math. Phys. **12**, 578 (1971). https://doi.org/10.1063/1.1665623
[10] W.D. Halford, "Cosmological theory based on Lyra's geometry," Australian Journal of Physics, **23**(5), 863-870 (1970). https://doi.org/10.1071/PH700863
[11] W.D. Halford, "Scalar-tensor theory of gravitation in a Lyra manifold," J. Math. Phys. **13**, 1699–1703, (1972). https://doi.org/10.1063/1.1665894
[12] L. Bianchi, *Sugli spazí a tre dimensioni che ammettono un gruppo continuo di movimenti: memoria*, vol. **11**, (Tipografia della R. Academia dei Lincei, 1898).
[13] D.R.K. Reddy, R. Santhikumar, and T.V. Pradeepkumar, "Bianchi type-III Dark Energy Model in f(R,T) Gravity," Int. J. Theor. Physics. **52**, 1 239 (2013). https://doi.org/10.1007/s10773-012-1325-1
[14] D.R.K. Reddy, R. Santikumar, and R.L. Naidu, "Bianchi type-III cosmological model in f(R, T) theory of gravity," Astrophys. Space Sci. **342**, 249–252 (2012). https://doi.org/10.1007/s10509-012-1158-7
[15] K.S. Adhav, S.D. Katore, R.S. Rane, et al., "Zero mass scalar field with bulk viscous cosmological solutions in Lyra geometry," Astrophys. Space Sci. **323**, 87–90 (2009). https://doi.org/10.1007/s10509-009-0042-6
[16] W.D. Halford, "Cosmological Theory Based on Lyra's Geometry," Aust. J. Phys. **23**, 863-870 (1970). https://doi.org/10.1071/PH700863
[17] G.P. Singh, and K. Desikan, "A new class of cosmological models in Lyra geometry," Pramana – J. Phys. **49**, 205–212 (1997). https://doi.org/10.1007/BF02845856
[18] R.S. Kumar, and B. Satyannarayana, "Accelerating anisotropic cosmological model in f(R, T) theory of gravity," Indian J. Phys. **91**, 1293–1296 (2017). https://doi.org/10.1007/s12648-017-1017-z
[19] S. Rajamahanthi, S. Bora, S.P.S. Komu, *et al*., "Bianchi-type cosmological models with heat flow in Lyra's geometry," J. Pure Appl. Math. **7**(1), 37-44 (2023). https://doi.org/10.37532/2752-8081.23.7(1).01-09
[20] M. Krishna, S. Koppala, and R.S. Rajamahanthi, "Accelerating plane-symmetric cosmological model with bulk viscous and cosmic Strings in Lyra's geometry," Indian J. Phys. **98**, 3733–3740 (2024). https://doi.org/10.1007/s12648-024-03100-y
[21] C.B. Collins, E.N. Glass, and D.A. Wilkinson, "Exact Spatially Homogeneous Cosmologies," General Relativity and Gravitation, **12**, 805-823 (1980). http://dx.doi.org/10.1007/BF00763057
[22] B. Mishra, P.K. Sahoo, and Ch.B.S. Varma, "Dark Energy Cosmological Model for Bianchi Type III Space-Time with Perfect Fluid," International Journal of Pure and Applied Mathematics, **99**, 1 109-121 (2015). http://dx.doi.org/10.12732/ijpam.v99i1.9
[23] K.P. Singh, and M.R. Mollah, "Bianchi Type III Cosmological Model with Hybrid Scale Factor in the Presence of Van der Waals Fluid in Lyra Manifold," International Journal of Modern Physics A, **33**, 1850207 (2018). https://doi.org/10.1142/S0217751X1850207X

ПРИСКОРЕННЯ КОСМОЛОГІЧНОЇ МОДЕЛІ ЗІ СКАЛЯРНИМ ПОЛЕМ НУЛЬОВОЇ МАСИ У ГЕОМЕТРІЇ ЛІРИ

М. Крішна[a], К. Собхан Бабу[b], Р. Сантікумар[c]
[a]*Інженерний коледж Рагу, Вісакхапатнам, штат Андхра-Прадеш, Індія*
[b]*Університетський інженерний коледж, Нарасараопета, Андхра-Прадеш, Індія*
[c]*Інститут технології та менеджменту Адітя, Теккалі, округ Шрікакулам. Андхра-Прадеш-Індія*

Вивчення космологічної моделі III типу Б'янкі передбачає включення скалярного поля з нульовою масою в контексті геометрії Ліри. Джерелом тензора енергії-імпульсу вважається об'ємна в'язка рідина. Баротропне рівняння стану використовується для характеристики тиску та густини, шукаючи конкретний розв'язок рівнянь поля. Це рішення отримано з використанням принципу відмінної варіації для параметра Хаббла, запропонованого [M.S. Berman, Il Nuovo Cimento B, 74, 182 (1983)]. Подальший аналіз заглиблюється в фізичні властивості, притаманні цій моделі.

Ключові слова: *прискорення; космологія; геометрія Ліри*