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GENERALIZED GHOST PILGRIM DARK ENERGY IN BRANS–DICKE THEORY

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This paper aims to investigate, how the Bianchi Type-V cosmological model can be solved using the generalized ghost pilgrim dark energy postulated by the Brans-Dicke theory of gravitation (Phys. Rev.124, 925 1961). To discover the answers, we rely on the assumptions of (i) the correlation between metric potentials and (ii) the exponential relationship between scale factor and scalar field. The generalized ghost pilgrim dark energy model has been found to be correlated with the polytrophic gas dark energy model. A few physical quantities have been used to explain the solutions' physical behavior.
Keywords: *Brans-Dicke theory of gravitation; Bianchi Type-V cosmological model; Ghost Pilgrim Dark Energy*
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1. INTRODUCTION

The cosmos is expanding more quickly than previously assumed, which is the most remarkable finding in contemporary cosmology. High Red shift Supernova is provided the first indication of this expansionary behavior, which was evidence by a wide range of astronomers. According to Cosmic Microwave Background Radiation (CMBR) and supernovae surveys radiation, dark energy and dark matter are the three primary constituents of the cosmos. Dark matter (DM) is an unidentified form of matter that exhibits the same clustering features as ordinary matter, whereas dark energy (DE) is an unusual form of an unidentified repulsive force.

The Bianchi V model is a natural extension of the homogeneous and isotropic Friedmann-Robertson-Walker (FRW) models that allows for a certain form of anisotropy. This model implies homogeneity but not isotropy, making it a helpful tool for investigating how deviations from perfect isotropy may affect the universe's evolution. Because scalar fields are sensitive to the universe's expansion rate, an anisotropic model such as Bianchi V can show behaviors that differ from those observed in isotropic environments. Bianchi models, notably Bianchi V, enable researchers to evaluate the potential level of anisotropy and its impact on observable values. This model bridges the gap between idealized isotropic models and more realistic cosmological phenomena, allowing researchers to investigate how minor anisotropies and curvature influenced the early universe's history and current condition.

Among the different dynamic models for dark energy, the equation of state (EoS) parameter is seen as the best choice to serve as dark energy. The Quintessence K-essence is popular DE models. Phantom, quantum, tachyon, holographic DE, age graphic DE, two fluids DE, anisotropic DE and so on. The solution came from the phantom dark energy, which is a scalar field with negative kinetic energy. A phantom-like dark energy results in everything crashing as our universe approaches its big-rip ending. The pilgrim dark energy model (PDE) was proposed by Wei [1]. This idea was supported by the idea that the strong repulsive force of DE makes it possible to prevent the development of black holes (BHs). The investigations of Babichev et al. [2] support the same conclusion, namely that BH mass decreases as a result of the phantom accretion event. To create a black hole free phantom universe, several authors have suggested several methods. Sharif and Jawad [3,4] have researched the interacting PDE in universes with both flat and non-flat configurations and various IR cutoffs. Sharif and Rani [5] and Jawad and Debnath [6], as well as Jawad [7], have conducted research on PDE cosmological models within different modified theories of gravitation. The connection between PDE and scalar field models was explored by Jawad and Majeed [8]. Jawad et al. [9] conducted a study on the properties of several newly developed versions of PDE within the DGP braneworld. Feng and Shen [10], Movahed and Sheykhi [11], S.D. Katore, and D. V. Kapse [12], Zubair and Abbas [13], Fayaz and Hossienkhani [14], Honarvaryan and Moradpour [15] have explored the different elements of ghost and generalized ghost dark energy (GGDE) models. The GGDE density is referred to as generalized ghost pilgrim dark energy (GGPDE) in the terms of PDE.

The Brans and Dicke [16] developed a scalar-tensor theory of gravity to integrate Mach's principle into Einstein's theory of gravitation. The Brans and Dicke theory suggest that all types of matter interact equally with a scalar field that has dimensions inversely proportional to the gravitational constant. The Brans-Dicke theory is an important scalar-tensor theory that has been widely used in contemporary cosmology, as highlighted by Banerjee and Pavon [17] and Bertolami and Martins [18]. The extended chaotic inflation (Linde [19]), the potential challenge of a 'graceful exit' (Pimental [20]) and an inflationary universe (Johri and Mathiazhagan [21]), all stem from the Brans-Dicke scalar-tensor theory.

Setare [22] investigated the holographic dark energy in non-flat Brans–Dicke cosmology. Vagenas and Setare [23] have examined the cosmological dynamics of interacting holographic dark energy models. Kiran, Reddy and Rao [24]

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have recently conducted research on Brans-Dicke theory of gravitation, specifically focusing on minimally interacting holographic DE models. Regarding the HDE cosmological model, Rao [25] has taken into consideration the Brans-Dicke theory of gravitation. Because they are homogenous and anisotropic, cosmological models of the Bianchi type are crucial for understanding how the universe isochronizes over the course of cosmic history.

The idea behind the construction of cosmological models is the same as ours and Bhardwaj et al. [26]. The process differs slightly when energy momentum tensors are included. Furthermore, the assumptions used to obtain the solutions are different. They compared their results with observational data, but we compared our results using mathematical results. We are inspired by their work, yet our results are not the same.

In this paper, there are various astrophysical reasons to adopt anisotropic models in the current epoch. In general relativity, isotropic solutions (such as the Friedmann-Lemaître-Robertson-Walker metric) are not the sole universe-modeling possibilities. Anisotropic models, like Bianchi cosmologies, provide mathematically sound answers to Einstein's field equations. Studying these solutions broadens our understanding of the types of worlds that are theoretically feasible under general relativity, even if they do not appear to apply to our universe. Analyzing anisotropic models may help us understand how an initially anisotropic state might change into the isotropic cosmos we see today or whether there are still theoretically significant traces of this early anisotropy that are too faint to detect today.

In this paper, we will be exploring the Bianchi type-V GGPDE model within the context of the Brans-Dicke theory of gravitation. Section 2 of the paper will present the metric and field equations for the Bianchi type-V within the Brans - Dicke theory of gravitation. Section 3 is devoted to presenting the solutions obtained from the field equations, along with a discussion of the calculations of different physical parameters. These solutions are the basis for our comprehension of the behavior and dynamics of the GGPDE model within the framework of the Bianchi type-V space-time cosmological model. Section 4 delves into the subject of sound speed. Section 5 discusses the analysis of the $\omega_G - \omega_G^{'}$ -plane. Section 6 offers conclusions, and Section 7 includes a detailed list of references cited in the paper.

2. METRIC AND FIELD EQUATIONS

The line element of the Bianchi type-V space-time cosmological model can be expressed as

$$ds^2 = dt^2 - D_1^2 dx^2 - D_2^2 e^{-2x} dy^2 - D_3^2 e^{-2x} dz^2 . \tag{1}$$

Where D_1, D_2 *and* D_3 are functions of time t only.

The action for Brans-Dicke theory is given by

$$s = \frac{1}{16\pi}\int \sqrt{-g}\left(\varphi R - \omega \frac{\varphi_{,l}\varphi^{,l}}{\varphi}\right) d^4 x + \frac{1}{16\pi}\int \sqrt{-g}\, L_m d^4 x . \tag{2}$$

Where L_m stands for Lagrangian matter field, φ denotes the Brans-Dicke scalar field and ω is the Brans-Dicke coupling constant.

The Brans-Dicke field equations can be expressed as follows

$$R_{uv} - \frac{1}{2} R\, g_{uv} = \frac{-8\pi}{\varphi} T_{uv} - \omega\varphi^{-2}\left(\varphi_{,u}\varphi_{,v} - \frac{1}{2} g_{uv}\varphi_{,l}\varphi^{,l}\right) - \frac{1}{\varphi}\left(\varphi_{u;v} - g_{uv}\varphi_{,l}^{\ ;l}\right), \tag{3}$$

and

$$\nabla\varphi = \varphi_{,l}^{l} = \frac{8\pi T}{3+2\omega}. \tag{4}$$

Here ω is constant, $G_{uv} = R_{uv} - \frac{1}{2} R\, g_{uv}$ is the Einstein tensor , R_{uv} is Ricci curvature tensor, R is Ricci scalar and T_{uv} is the energy momentum tensor of matter, φ is the Brans-Dicke scalar field, g_{uv} is the metric tensor of space time.

Brans-Dicke field equations can now be expressed as follows

$$G_u^{\ v} = \frac{-8\pi}{\varphi} T_u^{\ v} - \omega\varphi^{-2}\left(g^{uv}\varphi_{,u}\varphi_{,v} - \frac{1}{2}\delta_u^{\ v}\varphi_{,l}\varphi^{,l}\right) - \frac{1}{\varphi}\left(g^{uv}\varphi_{u;v} - \nabla\varphi\right). \tag{5}$$

Also, the energy conservation equation in the Brans-Dicke theory is

$$T_{;v}^{\ uv} = 0 . \tag{6}$$

The equation of the energy momentum tensors for matter and dark energy as follows

$$T_{uv} = T_{uv}^{m} + T_{uv}^{\wedge} . \tag{7}$$

Where T_{uv}^{m} *and* $T_{uv}^{\wedge}$ are represents the energy momentum tensor for dark matter and dark energy respectively and are defined as the energy momentum tensor for dark matter

$$T_{uv}^{m} = diag\,(1,0,0,0)\,\rho_m = diag\,(\rho_m, 0,0,0).$$

$$T_{11}^{m} = T_{22}^{m} = T_{33}^{m} = 0 \text{ and } T_{44}^{m} = \rho_m. \tag{8}$$

The energy momentum tensor for dark energy

$$T_{uv}^{\wedge} = diag\,(\rho_G, P_G, P_G, P_G),$$

$$T_{uv}^{\wedge} = diag\,(1, -\omega_G, -\omega_G, -\omega_G)\,\rho_G = diag\,(\rho_G, -\omega_G\rho_G, -\omega_G\rho_G, -\omega_G\rho_G). \tag{9}$$

Here the T_{44} component reflects the energy density in the stress-energy tensor. The diagonal spatial components (T_{11}, T_{22}, T_{33}) correspond to pressures in each spatial direction $(P_{xx}, P_{yy,} P_{zz})$.

$$T_{11}^{\wedge} = T_{22}^{\wedge} = T_{33}^{\wedge} = -\omega_G\rho_G \text{ and } T_{44}^{m} = \rho_G, \tag{10}$$

$$T_{uv} = T_{uv}^{m} + T_{uv}^{\wedge} = diag\,(\rho_m, 0,0,0) + diag\,(\rho_G, -\omega_G\rho_G, -\omega_G\rho_G, -\omega_G\rho_G),$$

$$T_{uv} = diag\,(\rho_G + \rho_m, -\omega_G\rho_G, -\omega_G\rho_G, -\omega_G\rho_G). \tag{11}$$

Where ρ_m *and* ρ_G are the energy densities of dark matter and the dark energy respectively and p_G is the pressure of the dark energy. ρ_m, ρ_G and p_G are dependent only on the variable time t.
Components of energy momentum tensors are

$$T_1^{\,1} = T_2^{\,2} = T_3^{\,3} = -\omega_G\rho_G \text{ and } T_4^{\,4} = \rho_G + \rho_m. \tag{12}$$

Substitute the values of Einstein tensor, metric tensor of Bianchi type-V space-time and energy momentum tensor in Brans-Dicke theory, we obtain the field equations of the model.

The metric's field equations (1) can be represented using equation (12) as shown

$$\frac{\ddot{D}_2}{D_2} + \frac{\ddot{D}_3}{D_3} + \frac{\dot{D}_2\dot{D}_3}{D_2D_3} - \frac{1}{D_1^2} + \frac{\omega}{2}\frac{\dot{\varphi}^2}{\varphi^2} + \frac{\dot{\varphi}}{\varphi}\left(\frac{\dot{D}_2}{D_2} + \frac{\dot{D}_3}{D_3}\right) + \frac{\ddot{\varphi}}{\varphi} = \frac{8\pi}{\varphi}T_1^{\,1} = -\frac{8\pi}{\varphi}\omega_G\rho_G, \tag{13}$$

$$\frac{\ddot{D}_1}{D_1} + \frac{\ddot{D}_3}{D_3} + \frac{\dot{D}_1\dot{D}_3}{D_1D_3} - \frac{1}{D_1^2} + \frac{\omega}{2}\frac{\dot{\varphi}^2}{\varphi^2} + \frac{\dot{\varphi}}{\varphi}\left(\frac{\dot{D}_1}{D_1} + \frac{\dot{D}_3}{D_3}\right) + \frac{\ddot{\varphi}}{\varphi} = \frac{8\pi}{\varphi}T_2^{\,2} = -\frac{8\pi}{\varphi}\omega_G\rho_G, \tag{14}$$

$$\frac{\ddot{D}_1}{D_1} + \frac{\ddot{D}_2}{D_2} + \frac{\dot{D}_1\dot{D}_2}{D_1D_2} - \frac{1}{D_1^2} + \frac{\omega}{2}\frac{\dot{\varphi}^2}{\varphi^2} + \frac{\dot{\varphi}}{\varphi}\left(\frac{\dot{D}_1}{D_1} + \frac{\dot{D}_2}{D_2}\right) + \frac{\ddot{\varphi}}{\varphi} = \frac{8\pi}{\varphi}T_3^{\,3} = -\frac{8\pi}{\varphi}\omega_G\rho_G, \tag{15}$$

$$\frac{\dot{D}_2\dot{D}_3}{D_2D_3} + \frac{\dot{D}_1\dot{D}_3}{D_1D_3} + \frac{\dot{D}_1\dot{D}_2}{D_1D_2} - \frac{3}{D_1^2} + \frac{\omega}{2}\frac{\dot{\varphi}^2}{\varphi^2} + \frac{\dot{\varphi}}{\varphi}\left(\frac{\dot{D}_1}{D_1} + \frac{\dot{D}_2}{D_2} + \frac{\dot{D}_3}{D_3}\right) = \frac{8\pi}{\varphi}T_4^{\,4} = \frac{8\pi}{\varphi}(\rho_G + \rho_m), \tag{16}$$

$$2\frac{\dot{D}_1}{D_1} = \frac{\dot{D}_2}{D_2} + \frac{\dot{D}_3}{D_3}, \tag{17}$$

$$\ddot{\varphi} + \frac{\dot{\varphi}}{\varphi}\left(\frac{\dot{D}_1}{D_1} + \frac{\dot{D}_2}{D_2} + \frac{\dot{D}_3}{D_3}\right) = \frac{8\pi}{3+2\omega}(\rho_m + \rho_G - 3\omega_G\rho_G). \tag{18}$$

Energy conservation equation is

$$\dot{\rho}_m + \rho_m\left(\frac{\dot{D}_1}{D_1} + \frac{\dot{D}_2}{D_2} + \frac{\dot{D}_3}{D_3}\right) + \dot{\rho}_G + \rho_G\left(\frac{\dot{D}_1}{D_1} + \frac{\dot{D}_2}{D_2} + \frac{\dot{D}_3}{D_3}\right) + \omega_G\rho_G\left(\frac{\dot{D}_1}{D_1} + \frac{\dot{D}_2}{D_2} + \frac{\dot{D}_3}{D_3}\right) = 0. \tag{19}$$

Where notation dot $(\cdot)$ representing ordinary differentiation with respect to time t.

From (13) to (18), the equations 13, 14 and 15 represents rr components and equation 16 represents 00 component of the field equations.

3. SOLUTIONS OF THE FIELD EQUATIONS

Now, equations (13)-(18) constitute a system of six distinct equations containing eight unknown variables $D_1, D_2, D_3, \rho_m, \omega_G, \rho_G$ *and* φ, P_G.

From equation (14), we have

$$D_1^2 = D_2 D_3 . \tag{20}$$

We assume that the metric potentials D_2 and D_3 have a relationship

$$D_2 = D_3^k . \tag{21}$$

Here k is a constant and k≠1.

A power-law relationship between scalar field ϕ and average scale factor a(t) of the form (Johri and Sudharsan [27]; Johri and Desikan [28]),

$$\varphi \propto [a(t)]^n .$$

Where n denotes a power index.

This form of scalar field has been the subject of investigation by numerous writers. We use the following assumption to simplify the system mathematically, taking into account the physical importance of the previous relationship.

$$\varphi = \varphi_0 [a(t)]^n ,$$

$$\varphi = [a(t)]^n \ where\ \varphi_0 = 1 . \tag{22}$$

Where n represents a constant that is greater than zero.

The scalar field and the scale factor are deeply interlinked in Brans-Dicke scalar-tensor theory. Their relationship determines the cosmic expansion rate, affects structure formation, and provides a pathway to test deviations from GR. This interplay makes the connection fundamental to understanding both early-universe and late-time cosmology.

In Bianchi V models, the metric can typically be expressed in terms of distinct directional scale factors, which represent the expansion along each spatial direction. These factors vary independently, leading to different expansion rates in each direction. The effective or average scale factor is then often defined as the geometric mean of these directional components:

$$a(t) = \left(D_1 D_2 D_3\right)^{\frac{1}{3}} . \tag{23}$$

This definition provides a single parameter for describing the average expansion rate, even though the expansion is anisotropic.

In the Bianchi Type-V cosmological model, the spatial volume can be described using the scale factors D_1, D_2 *and* D_3.

The Volume scale factor V can be written as

$$V = [a(t)]^3 = D_1 D_2 D_3 . \tag{24}$$

Using equations (20), (21), (22), (24) and (14), we get

$$D_3 = \left[\frac{(3+n)(k+1)}{2}\left(c_1 t + c_2\right)\right]^{\frac{2}{[(3+n)(k+1)]}} . \tag{25}$$

Where c_1 and c_2 are the integration constants.

From (21), we have

$$D_2 = D_3^k = \left[\frac{(3+n)(k+1)}{2}\left(c_1 t + c_2\right)\right]^{\frac{2k}{[(3+n)(k+1)]}} . \tag{26}$$

From equations (20), (25) and (26), we have

$$D_1 = \left[\frac{(3+n)(k+1)}{2}\left(c_1 t + c_2\right)\right]^{\frac{(k+1)}{[(3+n)(k+1)]}} . \tag{27}$$

Therefore, the corresponding metric can be expressed as

$$ds^2 = dt^2 - \left[\frac{(3+n)(k+1)}{2}\left(c_1 t + c_2\right)\right]^{\frac{(k+1)}{[(3+n)(k+1)]}} dx^2 - e^{-2x}\left[\frac{(3+n)(k+1)}{2}\left(c_1 t + c_2\right)\right]^{\frac{2k}{[(3+n)(k+1)]}} dy^2 - e^{-2x}\left[\frac{(3+n)(k+1)}{2}\left(c_1 t + c_2\right)\right]^{\frac{2}{[(3+n)(k+1)]}} dz^2 . \tag{28}$$

We determine the universe's scale factor using equations (24), (25), (26), and (27) as

$$a(t) = \left[\frac{(3+n)(k+1)}{2}\left(c_1 t + c_2\right)\right]^{\frac{(k+1)}{[(3+n)(k+1)]}} = \left[c_3\left(c_1 t + c_2\right)\right]^{\frac{k+1}{2C_3}} . \tag{29}$$

Where $c_3 = \dfrac{(3+n)(k+1)}{2}$.

Hence from equations (22) and (29), we have scalar field (φ) as

$$\varphi = \left[c_3\left(c_1 t + c_2\right)\right]^{\frac{(k+1)n}{2c_3}} . \tag{30}$$

From equation (29), we find the Hubble's parameter as

$$H = \frac{\dot{a}(t)}{a(t)} = \frac{c_1\left(k+1\right)}{2c_3\left(c_1 t + c_2\right)} . \tag{31}$$

According to Sharif and Nazir [29], as well as Santhi et al. [30], and Rao and Prasanthi [31], the GGPDE is defined as follows

$$\rho_G = \left(\alpha_1 H + \alpha_2 H^2\right)^{\gamma} . \tag{32}$$

Where γ is the dimensionless constant.

Therefore, the energy density of GGPDE is derived from equations (31) and (32).

$$\rho_G = \left(\frac{\alpha_1 c_1 (k+1)}{2c_3 (c_1 t + c_2)} + \frac{\alpha_2 c^2{}_1 (k+1)^2}{4c^2{}_3 (c_1 t + c_2)^2}\right)^{\gamma} . \tag{33}$$

We can determine the energy density of matter using equations (6), (14), (25), (26), (27), (30), and (32). Then

$$\rho_m = \frac{c_1{}^2\left[c_3\left(ct_1 + c_2\right)\right]^{\frac{n(k+1)}{2c_3}}}{8\pi c_3{}^2\left(c_1 t + c_2\right)^2}\left\{\frac{k^2+4k+1}{2} - \frac{3c_1^{-2}}{\left[c_3\left(c_1 t + c_2\right)\right]^{\frac{k+1-2c_3}{c_3}}} - \frac{\omega n^2\left(k+1\right)^2}{8} + \frac{3n\left(k+1\right)^2}{4}\right\} - \left(\frac{\alpha_1 c_1 (k+1)}{2c_3 (c_1 t + c_2)} + \alpha_2 \frac{c_1{}^2 (k+1)^2}{4c_2{}^2 (c_1 t + c_2)^2}\right)^{\gamma} \tag{34}$$

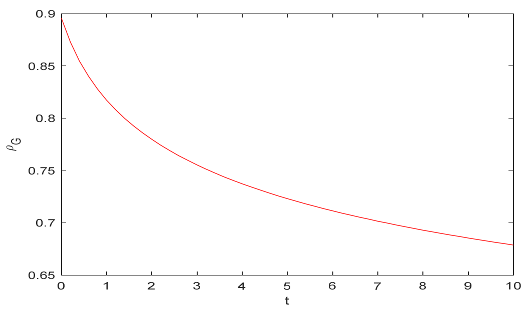

Figure 1. The energy density $\left(\rho_G\right)$ of GGPDE versus time t

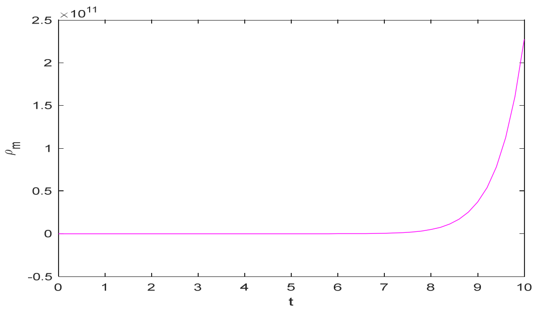

Figure 2. The energy density (ρ_m) of matter versus time t

Hence from equations (13) and (34), we have equation of EoS of GGPDE as

$$\omega_G = \frac{\left\{\left(k^2+k+1\right) - c_3\left(k+1\right) + \frac{\omega n^2\left(k+1\right)^2}{8} - \frac{c_1^{-2}}{\left[c_3\left(c_1 t + c_2\right)\right]^{\frac{k+1-2c_3}{c_3}}} + \frac{n\left(k+1\right)^2}{2} + \frac{n^2\left(k+1\right)^2 c_1\left[n\left(k+1\right) - 2c_3\right]}{4c_3\left(c_1 t + c_2\right)}\right\}}{} \; \frac{-\left[c_3\left(ct_1 + c_2\right)\right]^{\frac{n(k+1)}{2c_3}}}{8\pi\left(\frac{\alpha_1 c_1 (k+1)}{2c_3 (c_1 t + c_2)} + \frac{\alpha_2 c^2{}_1 (k+1)^2}{4c^2{}_3 (c_1 t + c_2)^2}\right)^{\gamma}} \frac{c_1{}^2}{c_3{}^2\left(c_1 t + c_2\right)^2} . \tag{35}$$

Hence from equations (18), (33) and (34), we find the pressure of GGPDE as

$$p_G = \frac{-\left[c_3\left(ct_1+c_2\right)\right]^{\frac{n(k+1)}{2c_3}}}{8\pi}\frac{c_1^{\,2}}{c_3^{\,2}\left(c_1t+c_2\right)^2}\left\{\begin{array}{l}\left(k^2+k+1\right)-c_3\left(k+1\right)+\dfrac{\omega n^2\left(k+1\right)^2}{8}-\dfrac{c_1^{\,-2}}{\left[c_3\left(c_1t+c_2\right)\right]^{\frac{k+1-2c_3}{c_3}}}+\dfrac{n\left(k+1\right)^2}{2}+\\ \dfrac{n^2\left(k+1\right)^2c_1\left[n\left(k+1\right)-2c_3\right]}{4c_3\left(c_1t+c_2\right)}\end{array}\right\}. \tag{36}$$

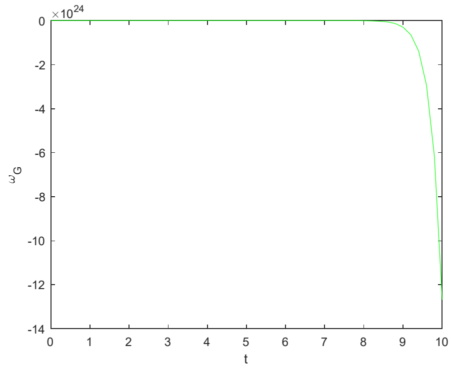

Figure 3. The EoS (ω_G) of GGPDE versus time

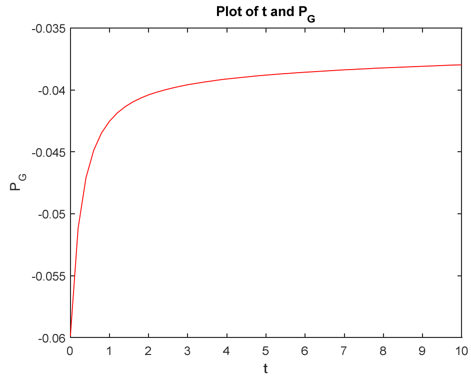

Figure 4. The pressure (P_G) of the D.E versus time t

In the cosmological model of Bianchi type-V, the Volume scale factor V can be written as

$$V=[a(t)]^3=\left[c_3\left(c_1t+c_2\right)\right]^{\frac{3(k+1)}{2c_3}}. \tag{37}$$

The Hubble parameter represented by H; the Hubble parameter H is explicated as the speed at which the scale factor a (t) evolves as time t progresses.

The Hubble's parameter can be calculated from equations (25) and (29) as follows

$$H=\frac{\dot{a}(t)}{a(t)}=\frac{c_1\left(k+1\right)}{2c_3\left(c_1t+c_2\right)} \tag{38}$$

The deceleration parameter, represented by q is another major parameter in cosmology. The value of q decides whether the cosmos is slowing down or speeding up. A positive value of q indicates the cosmos is slowing down while a negative value of q indicates the cosmos is speeding up.

The definition of the Deceleration parameter q is

$$q=\frac{-\ddot{a}(t)}{a(t)H^2}$$

From equations (29) and (38) as follows

$$q=\frac{-\ddot{a}}{aH^2}=-1+\frac{2c_3}{k+1}=-1+c_4 \tag{39}$$

where $c_4=\dfrac{2c_3}{k+1}$

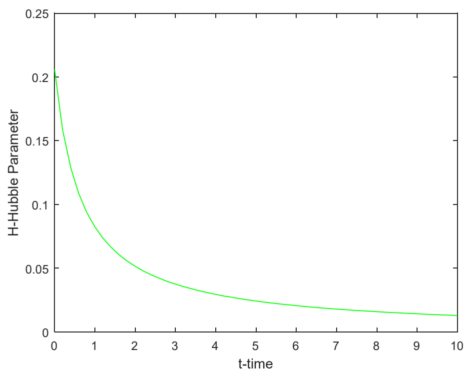

Figure 5. The Hubble's Parameter (H) versus time t

The statefinder Parameters (r,s): In order to differentiate between multiple D.E possibilities, Sahni and Varun [32] suggested a new parameters called the statefinder parameter (r,s).The pair consists of the calculation of r using the scale factor a(t) and its derivatives in relation to time (t) up to the third order, as well as combining 'r' with the deceleration parameter 'q' to create 's'.

The statefinder parameters are described as

$$r=\frac{\dddot{a}}{aH^3}=c_3^{\,3}\left[1-\frac{2c_3}{k+1}\right]\left[1-\frac{4c_3}{k+1}\right] \text{ and } s=\frac{r-1}{3\left(q-\frac{1}{2}\right)}=\frac{c_3^{\,3}\left[1-\frac{2c_3}{k+1}\right]\left[1-\frac{4c_3}{k+1}\right]-1}{3\left(\frac{-3}{2}+\frac{c_3}{k+1}\right)} \tag{40}$$

As the statefinder depends on the scale factor the statefinder parameters are analyzed geometrically. The Statefinder parameters are an essential tool in contemporary cosmology and is used to distinguish different models of dark energy. In this particular scenario, different paths in the {r, s} plane illustrate the time progression of various dark energy models. A popular region can be elucidated using the pair as follows: (r, s)=(1,0) which represents the Einstein-de Sitter limit, which is connected with a matter-dominated universe, (r, s)=(1,1) analogous to the ΛCDM limit, which is often associated with a universe controlled by dark energy with a cosmological constant, while s >0 and r<1 shows the parameter space associated with quintessence and phantom DE, the analysis of any deviation of a D.E model from these specific points is carried out in the {r, s} plane. Equation (40) display the diagnostic pairs (r, s) for the suggested model.

Density Parameter: Usually, many researchers depict the total density parameter as being close to one i.e. $\Omega\approx 1$. Determining the density parameter is essential in establishing if Ω> 1, Ω <1, or exactly Ω=1, since it ultimately dictates the fate of the cosmos. If the value Ω is greater than 1, this indicates that the cosmos is closed and will eventually stop expanding, collapsing in the future. If the value of $\Omega<1$, it indicates that the cosmos is open and is expected to keep expanding indefinitely. If the value is $\Omega=1$, it means that the cosmos is flat and has enough matter to slow down its expansion, but it may possibly have enough to cause it to collapse.

The dimensionless density parameter described as

$$\Omega=\Omega_m+\Omega_G \text{ where } \Omega_m=\frac{\rho_m}{3H^2}\ \textit{and}\ \Omega_G=\frac{\rho_G}{3H^2}. \tag{41}$$

From equations (34) and (38) we have

$$\Omega_m=\left[\begin{array}{l}\dfrac{c_1^2\left[c_3\left(ct_1+c_2\right)\right]^{\frac{n(k+1)}{2c_3}}}{8\pi c_3^2\left(c_1t+c_2\right)^2}\left\{\dfrac{k^2+4k+1}{2}-\dfrac{3c_1^{-2}}{\left[c_3\left(c_1t+c_2\right)\right]^{\frac{k+1-2c_3}{c_3}}}-\dfrac{\omega n^2\left(k+1\right)^2}{8}+\dfrac{3n\left(k+1\right)^2}{4}\right\}-\\ \left[\left(\dfrac{\alpha_1c_1(k+1)}{2c_3(c_1t+c_2)}+\alpha_2\dfrac{c^2{}_1(k+1)^2}{4c^2{}_3(c_1t+c_2)^2}\right)^{\gamma}\right.\end{array}\right]\frac{4c_3^2\left(c_1t+c_2\right)^2}{3c^2{}_1(k+1)^2}. \tag{42}$$

From equations (33) and (38) we have

$$\Omega_G=\left(\frac{\alpha_1c_1\left(k+1\right)}{2c_3\left(c_1t+c_2\right)}+\frac{\alpha_1c_1^2\left(k+1\right)^2}{4c_3^2\left(c_1t+c_2\right)^2}\right)\frac{4c_3^2\left(c_1t+c_2\right)^2}{3c_1^2\left(k+1\right)^2}. \tag{43}$$

From equations (42) and (43), we have density parameter defined as

$$\Omega=\frac{c_1^2\left[c_3\left(ct_1+c_2\right)\right]^{\frac{n(k+1)}{2c_3}}}{8\pi c_3^2\left(c_1t+c_2\right)^2}\frac{4c_3^2\left(c_1t+c_2\right)^2}{3c^2{}_1(k+1)^2}\left\{\frac{k^2+4k+1}{2}-\frac{3c_1^{-2}}{\left[c_3\left(c_1t+c_2\right)\right]^{\frac{k+1-2c_3}{c_3}}}-\frac{\omega n^2\left(k+1\right)^2}{8}+\frac{3n\left(k+1\right)^2}{4}\right\}. \tag{44}$$

4. STABILITY ANALYSIS

We use the squared speed of sound to assess the stability of our model. The parameter squared speed of sound, denoted as $v_s^{\,2}$,it is an essential factor in determining the stableness of the DE model. It is valuable in grasping the stability of dark energy models and is dependent on its sign.

The dark energy model is considered as stable if $v_s^{\,2}>0$ and the dark energy model can be classified as unstable $v_s^{\,2}<0$.

The formula for squared speed of the sound is $v_s{}^2 = \dfrac{\dot{p}_G}{\dot{\rho}_G}$.

From equations (30), (33) we find the squared speed of sound

$$v_s{}^2 = \frac{\dfrac{c_1^3}{16\pi}\left[c_3\left(c_1t+c_2\right)\right]^{\frac{n(k+1)}{2c_3}-3}\left\{\left(k^2+k+1\right)-c_3\left(k+1\right)+\dfrac{\omega n^2\left(k+1\right)^2}{8}-\dfrac{c_1^{-2}}{\left[c_3\left(c_1t+c_2\right)\right]^{\frac{k+1-2c_3}{c_3}}}+\dfrac{n\left(k+1\right)^2}{2}+\dfrac{n^2\left(k+1\right)^2c_1\left[n\left(k+1\right)-2c_3\right]}{4c_3\left(c_1t+c_2\right)}\right\} - \dfrac{(k+1-2c_3)c_1}{8\pi}\left[c_3\left(c_1t+c_2\right)\right]^{\frac{(k+1)(m-2)-2c_3}{2c_3}}+\dfrac{n^2\left(k+1\right)^2c_1^4c_3\left[n\left(k+1\right)-2c_3\right]}{8\pi}\left[c_3\left(c_1t+c_2\right)\right]^{\frac{n(k+1)}{2c_3}-4}}{\dfrac{-\gamma c_1^2\left(k+1\right)}{2c_3\left(c_1t+c_2\right)^2}\left(\dfrac{\alpha_1c_1(k+1)}{2c_3(c_1t+c_2)}+\dfrac{\alpha_2c^2{}_1(k+1)^2}{4c^2{}_3(c_1t+c_2)^2}\right)^{\gamma-1}\left[\alpha_1+\dfrac{\alpha_2c_1\left(k+1\right)}{2c_3\left(c_1t+c_2\right)}\right]^{\gamma}} \tag{45}$$

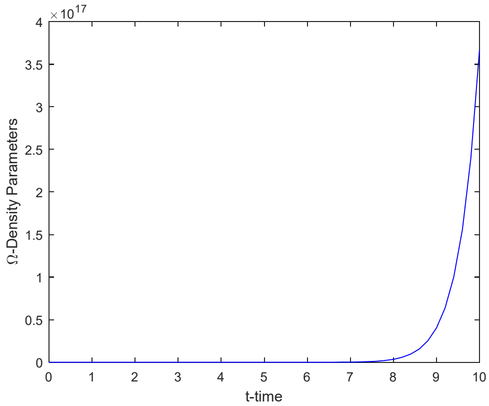

Figure 6. The Density Parameters Ω versus time t

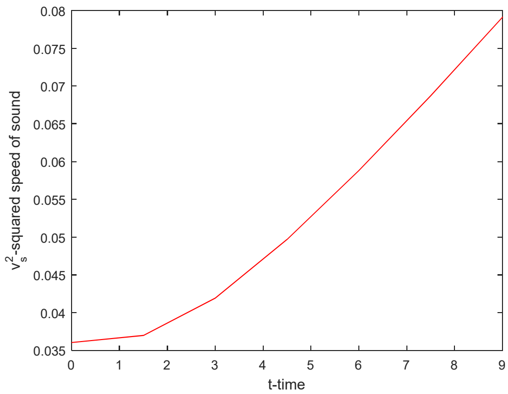

Figure 7. The squared speed of the sound versus time t

5. $\omega_G - \omega'_G$ PLANE

Caldwell and Linder [33] implemented a plane analysis of $\omega_G - \omega'_G$. The $\omega_G - \omega'_G$ plane investigation is utilized to analyze the dynamic characteristics of D.E models, where ω'_G is the derivative in terms of ln a. This method has been used with the quintessence model, resulting in two models of planes. The region's $\left(\omega_G<0, \omega'_G>0\right)$ interior represents the thawed region, whereas the area beneath the region $\left(\omega_G<0, \omega'_G<0\right)$ represents the frozen region. Calculate the derivative of the equation (35) when it comes to ln a, we obtain the expression for ω'_G as

$$\begin{aligned}
\omega'_G = &-\frac{c_1^2c_3}{4\pi(k+1)}\left[\frac{n(k+1)-4c_3}{2c_3}\right]\left[c_3\left(c_1t+c_2\right)\right]^{\frac{n(k+1)-4c_3}{2c_3}}\left[\frac{\alpha_1c_1(k+1)}{2c_3(c_1t+c_2)}+\frac{\alpha_2c^2{}_1(k+1)^2}{4c^2{}_3(c_1t+c_2)^2}\right]^{-\gamma} \\
&\left\{\begin{aligned}&\left(k^2+k+1\right)-c_3\left(k+1\right)+\frac{\omega n^2\left(k+1\right)^2}{8}-\frac{c_1^{-2}}{\left[c_3\left(c_1t+c_2\right)\right]^{\frac{k+1-2c_3}{c_3}}}+\frac{n\left(k+1\right)^2}{2}+\\ &\frac{n^2\left(k+1\right)^2c_1\left[n\left(k+1\right)-2c_3\right]}{4c_3\left(c_1t+c_2\right)}\end{aligned}\right\} \\
&-\frac{c_1^2\gamma}{8\pi}\left[\frac{\alpha_1c_1(k+1)}{2c_3(c_1t+c_2)}+\frac{\alpha_2c^2{}_1(k+1)^2}{4c_3^2(c_1t+c_2)^2}\right]^{-\gamma-1}\left[\frac{\alpha_1c_1}{(c_1t+c_2)}+\frac{\alpha_2c_1^2(k+1)}{c_3(c_1t+c_2)^2}\right]\left[c_3\left(c_1t+c_2\right)\right]^{\frac{n(k+1)-4c_3}{2c_3}} \\
&\left\{\left(k^2+k+1\right)-c_3\left(k+1\right)+\frac{\omega n^2\left(k+1\right)^2}{8}-\frac{c_1^{-2}}{\left[c_3\left(c_1t+c_2\right)\right]^{\frac{k+1-2c_3}{c_3}}}+\frac{n\left(k+1\right)^2}{2}+\frac{n^2\left(k+1\right)^2c_1\left[n\left(k+1\right)-2c_3\right]}{4c_3\left(c_1t+c_2\right)}\right\} \\
&-\frac{c_3\left(c_1t+c_2\right)}{4\pi(k+1)}\left\{\frac{(k+1-2c_3)c_3}{\left[c_3\left(c_1t+c_2\right)\right]^{\frac{k+1-c_3}{c_3}}}-\frac{n^2\left(k+1\right)^2c_1\left[n\left(k+1\right)-2c_3\right]}{4c_3\left(c_1t+c_2\right)^2}\right\}\left[c_3\left(c_1t+c_2\right)\right]^{\frac{n(k+1)-4c_3}{2c_3}} \\
&\left[\frac{\alpha_1c_1(k+1)}{2c_3(c_1t+c_2)}+\frac{\alpha_2c^2{}_1(k+1)^2}{4c^2{}_3(c_1t+c_2)^2}\right]^{-\gamma}
\end{aligned} \tag{46}$$

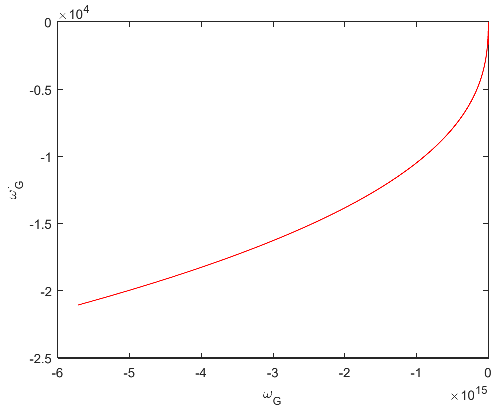

Figure 8. The plot of $\omega_G - \omega'_G$.

6. CONCLUSION

This paper discusses Bianchi Type-V cosmological model and the generalized ghost version of pilgrim dark energy in the Brans-Dicke scalar-tensor theory of gravitation (1961). The solutions to the field equations were obtained by making an assumption about the variation of Hubble's parameter. This paper delved into various physical aspects of the model. This portion provides a concise overview of the intriguing findings that were seen during the current investigation. We have taken $n = 1.8$, $k = 1.5$, $c_1 = 0.06$, $c_2 = 0.04$ and $c_3 = 9.1$ values to plot the graphs.

•In Figure 1, A graph is constructed showing the energy density of GGPDE as a function of time t. It demonstrates that the energy density of GGPDE stays positive throughout cosmic evolution, and it was also observed that the energy density of GGPDE diminishes as time passes.

•In Figure 2, A graph is created to illustrate how the energy density of matter changes over time. The graph indicates that the energy density starts at zero and then gradually moves towards the phantom region. As time increases, the energy density of matter is observed to also increase.

•In Figure 3, the EoS parameter for GGPDE versus time t is plotted. The model indicates that it starts in the matter dominated era and transitions into the phantom era.

•In Figure 4, we plotted the pressure of the dark energy against time t. The graph indicates that as time goes on, the pressure of the dark energy also goes up.

•At time t=0, the Hubble's parameter H is constant. In Figure 5, we plotted the Hubble's parameter H versus time t. It shows that the Hubble's parameter tends to zero as the time t tends to infinite per theoretical desire. It is also observed that the time t increases, the Hubble's parameter H as decreases.

•In Figure 6, we plotted the density parameter versus time t. It shows that the result is zero initially and then approaches towards the phantom region. As time increases, the density parameter is seen to increase.

•In Figure 7, the graph shows the square of the speed of sound as a function of time t. The square of the speed of sound increases with time, indicating positive values. The paths of the squared speed of the sound consistently exhibit positive behavior and stability in the model.

- In Figure 8, We graphed the $\omega_G - \omega'_G$ plane against time t and noticed that the trajectories of the $\omega_G - \omega'_G$ plane vary in the freezing region .It is concluded that the $\omega_G - \omega'_G$ plane Analysis of the current situation indicates that the accelerated expansion of the cosmos is consistently supported,
- The deceleration parameter q serves as an indicator of whether the model is expanding or not. A positive indication shows that the cosmos is decelerating, while a negative sign indicates that the universe is accelerating. In this paper is observed that,

For $0 < c_4$ <1, the negative deceleration parameter q indicates that the cosmos is experiencing acceleration during its evolution within this parameter range.

For $c_4 = 0$, the deceleration parameter q is less than zero, indicating that the universe is undergoing acceleration as it evolves in this condition.

For c_4 >1, if the deceleration parameter q is positive, it means that the cosmos is experiencing deceleration within this parameter range.

For $c_4 = 1$, the deceleration parameter q=0 indicating expansion with constant velocity. This means that the universe's expands when velocity remains constant over time under this specific condition.

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УЗАГАЛЬНЕНА GHOST PILGRIM ТЕМНА ЕНЕРГІЯ В ТЕОРІЇБ РАНСА–ДІКЕ

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Ця стаття має на меті дослідити, як космологічна модель Б’янкі типу V може бути розв’язана за допомогою узагальненої темної енергії привидів-пілігримів, постульованої теорією гравітації Бранса-Дікке (Phys. Rev.124, 925 1961). Щоб знайти відповіді, ми покладаємося на припущення про (i) кореляцію між метричними потенціалами та (ii) експоненціальний зв’язок між масштабним фактором і скалярним полем. Було виявлено, що узагальнена модель темної енергії примарного паломника корелює з моделлю темної енергії політрофічного газу. Кілька фізичних величин було використано для пояснення фізичної поведінки розчинів.

Ключові слова: *теорія гравітації Бранса-Дікке; космологічна модель Біанкі типу V; темна енергія Ghost Pilgrim*