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ON THE STABILITY OF PLANETARY MOTIONS DURING STELLAR APPROACHES

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The problem of the spatial motion of a passively gravitating body during an to the central body of a perturbing body – a test star – is considered. Using the exact expression of the force function, an integral invariant relationship – a quasi-integral – was found. Due to the quasi-integral, the regions of possible motion of the passively gravitating body, the surfaces of minimal energy (a generalization of the zero velocity surfaces), and the singular points of these surfaces were determined. The stability of planetary motion according to Hill during the approach of a test star to the Solar System was investigated. Criteria for the possibility, as well as the impossibility of capturing the passively gravitating body by the test star, were established. According to the Hill stability criteria, critical values of the orbital parameters of the test star were established, at which the planets of the Solar System either become satellites of the test star or leave the bounds of the Solar System.
Keywords: *Celestial mechanics; Restricted three-body problem; Jacobi function analog; Quasi-integral; Law of energy conservation; Surfaces of minimal energy; Singular points; Hill stability*
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1. PROBLEM FORMULATION. AN ANALOGUE OF JACOBIAN FUNCTION

In the works of A.G. Mamedov [4, 5, 6], the evolution of planetary orbits during stellar approaches to the Solar System is explored within the framework of the planar averaged parabolic three-body problem. It has been shown that with a moderate approach of the perturbing body to the central body, the size and shape of the orbit of the perturbed body remain constant, with only its orientation changing. A test star of solar mass was used as the perturbing body, and the orbits of the planets during its approach to the Sun at a distance of 50 au were studied. The results are presented in the form of figures and tables.

In the work of Kholshevnikov and Mishchuk [13], the restricted hyperbolic three-body problem was considered, and an assessment was made of the influence of a test star of solar mass on the orbits of the planets during its approach to the Sun from a distance of 100 au. to 1152 au. It has been shown that during a moderate approach of such a star to the Sun, the sizes of the planetary orbits do not undergo any changes. When the test star approaches the Sun to a distance of 100 au, the inclination, eccentricity, longitude of ascending node, and argument of pericenter change very little.

In this study, the motion of the passively gravitating body M is examined in a rotating and pulsating coordinate system [1,2] within the framework of the restricted three-body problem. The actively gravitating bodies are: the central body M_0 with mass m_0, and the perturbing body M' with mass m', where $m_0 \le m'$. In this coordinate system, the origin coincides with the barycenter G_0 of the actively gravitating bodies, the G_0xy plane aligns with the plane of motion of these bodies, and the G_0x axis aligns with the line connecting bodies M_0 and M'. The true anomaly of the perturbing body v' is used as the independent variable. Consequently, the equations of motion for body M in this coordinate system take a simple form [1,2].

$$\frac{d^2x}{dv'^2} - 2\frac{dy}{dv'} = \frac{\partial\Omega}{\partial x}, \quad \frac{d^2y}{dv'^2} + 2\frac{dx}{dv'} = \frac{\partial\Omega}{\partial y}, \quad \frac{d^2z}{dv'^2} = \frac{\partial\Omega}{\partial z} \tag{1}$$

We will refer to equation system (1) as SHAPNER's equations - an acronym formed from the surnames Scheibner [9], Petr and Nechvil [10, 11 and Rein 12]. In equation system (1), the force function $\Omega = \Omega(v', x, y, z)$ is analogous to the Jacobi function in the circular problem and is defined by equality

$$\Omega = \rho'\left[\frac{1}{2}\left(x^2 + y^2 - e'z^2\cos v'\right) + p'^3\left(\frac{\mu}{r_1} + \frac{1-\mu}{r_2}\right)\right]. \tag{2}$$

Here, the dimensionless quantity ρ' is defined below, e' and p' are the eccentricity and the focal parameter of the perturbing body M' orbit relative to the central M_0, and $1-\mu$ and μ are the relative masses of the main bodies M' and M_0:

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$$1-\mu=\frac{m'}{m_0+m'},\quad \mu=\frac{m_0}{m_0+m'}\quad \left(\mu<\frac{1}{2}\right) \tag{3}$$

respectively. The distances of the passively gravitating body from the main bodies r_1 and r_2 are determined by equalities

$$r_1^2=\left(x-p'\mu+p'\right)^2+y^2+z^2,\quad r_2^2=\left(x-p'\mu\right)^2+y^2+z^2. \tag{4}$$

In this context, the distance between the main bodies r' equals

$$r'=p'\rho',\quad \rho'=\frac{1}{1+e'\cos v'},\quad p'=q'\left(1+e'\right), \tag{5}$$

where q' is the minimum distance (perihelion distance in the Solar System) of the perturbing body from the central one. The equality in (5) for r' practically defines the orbit of the perturbing body: for $e'<1$ it is an elliptical orbit, for $e'>1$ it is a hyperbolic orbit, and for $e'=1$ it is a parabolic orbit. Additionally, the range of variation of the true anomaly is assumed to be

$$-\pi\le v'\le\pi,\ \left(e'\le 1\right),\quad \pi-\arccos\left(\frac{1}{e'}\right)<v'<\pi+\arccos\left(\frac{1}{e'}\right),\ \left(e'>1\right), \tag{6}$$

where the first interval corresponds to changes in v' during elliptical and parabolic motions of the perturbing body, and the second interval corresponds to hyperbolic motion.

REMARK 1. The Jacobian function analog defined by equality (2) corresponds to the case where $m_0<m'$. If $m_0>m'$, then the Jacobian function analog, the relative masses μ and $1-\mu$, as well as the distances r_1 and r_2 should be determined by equalities

$$\Omega=\rho'\left[\frac{1}{2}\left(x^2+y^2-e'z^2\cos v'\right)+p'^3\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right)\right], \tag{7}$$

$$\mu=\frac{m'}{m_0+m'},\quad 1-\mu=\frac{m_0}{m_0+m'}\quad \left(\mu<\frac{1}{2}\right), \tag{8}$$

$$r_1^2=\left(x+p'\mu\right)^2+y^2+z^2,\quad r_2^2=\left(x+p'\mu-p'\right)^2+y^2+z^2. \tag{9}$$

2. QUASI-INTEGRAL AND THE LAW OF ENERGY CONSERVATION.

In the restricted circular ($e'=0$) three-body problem, the equations of motion (1) for SHAPNER admit a Jacobi integral

$$V^2-2\Omega=2C,\quad 2C=V_0^2-2\Omega_0=const, \tag{10}$$

where the zero subscript denotes the values of the velocity V of the passively-gravitating body and the Jacobian function Ω at some initial value of the true anomaly v_0', and C is the constant of the Jacobi integral.

It is clear that in the non-circular ($e'\neq 0$) restricted three-body problem, such a first integral as the Jacobi integral (10) does not exist. This is due to the fact that the force function Ω explicitly depends on the independent variable v'. Indeed, if we multiply the first equation of system (1) by dx/dv', the second by dy/dv', and the third by dz/dv', summing the resulting equations and integrating over v', we obtain

$$\frac{V^2}{2}-\frac{V_0^2}{2}=\int_{v_0'}^{v'}\left(\frac{\partial\Omega}{\partial x}\frac{dx}{dv'}+\frac{\partial\Omega}{\partial y}\frac{dy}{dv'}+\frac{\partial\Omega}{\partial z}\frac{dz}{dv'}\right)dv'=\int_{v_0'}^{v'}\left(\frac{d\Omega}{dv'}-\frac{\partial\Omega}{\partial v'}\right)dv', \tag{11}$$

or

$$\frac{V^2}{2}-\Omega=\frac{V_0^2}{2}-\Omega_0-\int_{v_0'}^{v'}\frac{\partial\Omega}{\partial v'}dv'\ . \tag{12}$$

The relation obtained (12) is not a first integral of the motion equations (1): it should be considered as an integral invariant relation, or quasi-integral [2]. This can be rewritten in the form of

$$\frac{V^2}{2}-\Omega+u(v')=h, \quad h=\frac{V_0^2}{2}-\Omega_0+u(v_0'), \quad (13)$$

if an unknown antiderivative function $u(v')$:

$$u(v')-u(v_0')=\int_{v_0'}^{v'}\frac{\partial\Omega}{\partial v'}dv', \quad (14)$$

is introduced. In the quasi-integral (13), *h* represents a constant energy and depends on the value of the unknown function $u(v_0')$, i.e., it takes different values on different trajectories of motion.

Note that the quasi-integral (13) in the case of circular motion $e'=0$ of the perturbing body transforms into the Jacobi integral (10), since in this case $\partial\Omega/\partial v'=0$, i.e. $u(v')\equiv 0$. Thus, the obtained quasi-integral (13) in the non-circular ($e'\neq 0$) restricted three-body problem represents the law of conservation of energy of the passively gravitating body: the total energy of the body *M*, consisting of the Jacobi energy $V^2/2-\Omega$ and the additional energy $u(v')$, is a constant quantity, depending only on the initial values of the coordinates and velocities of the body M. The quantity *h* can be considered as the constant energy, having its specific value on each trajectory [2]. Additionally, the Jacobi energy $V^2/2-\Omega$ reaches its maximum value at the pericenter of the orbits of the main bodies M_0 and M', and its minimum value at the apocenter. Therefore, during $0\leq v'\leq\pi$, the double inequality

$$\frac{V_a^2}{2}-\Omega_a\leq\frac{V^2}{2}-\Omega\leq\frac{V_p^2}{2}-\Omega_p, \quad (15)$$

holds, where the indices "*a*" and "*p*" correspond to apocenter and pericenter. Moreover, the additional energy $u(v')$ monotonically increases as the main bodies M_0 and M' move away from the pericenter, and the passively gravitating body *M* gains additional (potential) energy from them. Conversely, as the main bodies move towards the pericenter, the additional energy $u(v')$ decreases, and the body *M* transfers energy to the main bodies [2].

Note that the law of conservation of energy (13) at the moment the primary bodies pass through the periapses of their orbits can be represented as

$$\frac{V^2}{2}-\Omega+u(v')=h_p, \quad h_p=\frac{V_p^2}{2}-\Omega_p+\frac{\tilde{\Omega}_{\min}}{1+e'}, \quad (16)$$

where the index "*p*" signifies the values of the Jacobian function analog Ω and the velocity V of the passively gravitating body, calculated at the moment the perturbing body passes through the pericenter, i.e. at $v'=0$. Furthermore, $\tilde{\Omega}_{\min}$ denotes the minimum value of the function

$$\tilde{\Omega}=\frac{1}{2}\left(x^2+y^2+z^2\right)+p'^3\left(\frac{\mu}{r_1}+\frac{1-\mu}{r_2}\right)\geq 0, \quad (17)$$

which is related to the Jacobian function analog Ω by equality [2]

$$\frac{\partial\Omega}{\partial v'}=\tilde{\Omega}\frac{e'\sin v'}{\left(1+e'\cos v'\right)^2}=\tilde{\Omega}\frac{d\rho'}{dv'}. \quad (18)$$

Such a value of the function $\tilde{\Omega}$ exists on the circle [2]

$$x=-\frac{p'}{2}\left(1-2\mu\right), \quad y^2+z^2=\frac{3p'^2}{4}, \quad (19)$$

and is equal to

$$\tilde{\Omega}_{\min}=\frac{p'^2}{2}\left(3-\mu+\mu^2\right). \quad (20)$$

3. REGIONS OF POSSIBLE MOTION. SURFACES OF MINIMUM ENERGY AND THEIR CRITICAL POINTS

The conservation of energy at the pericenter (16) can be rewritten in the form

$$V^2+2u(v')-2\tilde{\Omega}_{\min}\cdot\rho'=2\Omega-2\tilde{\Omega}_{\min}\cdot\rho'+2h_p\geq 0, \quad (21)$$

from which we can identify the regions of possible motion

$$2\Omega - 2\tilde{\Omega}_{\min} \cdot \rho' \geq C, \quad C = -2h_p = 2\Omega_p - V_p^2 - \frac{2\tilde{\Omega}_{\min}}{1+e'}, \tag{22}$$

where C is the equivalent of the Jacobi constant, and $\tilde{\Omega}_{\min}$ has been defined previously. The boundary of the region (22)

$$2\Omega - 2\rho'\tilde{\Omega}_{\min} \geq C, \tag{23}$$

is referred to as surfaces of minimum energy, the equation of which we write in the form [2]

$$H \equiv x^2 + y^2 - e'z^2 \cos v' + 2p'^3\left(\frac{\mu}{r_1} + \frac{1-\mu}{r_2}\right) - 2\tilde{\Omega}_{\min} = C\left(1 + e'\cos v'\right). \tag{24}$$

It's clear that the function $H \equiv H\left(x, y, z, p', e', v', \mu, C\right)$, meaning the family of surfaces of minimum energy (24) depends not only on the coordinates *x, y, z* but also on five parameters: p', e', v', μ, C. The Jacobi constant equivalent C characterizes the energy of the passively-gravitating body M, and the focal parameter p' represents the linear scale of the surfaces. With given values of these parameters p', e', v', μ and C, the body M cannot move beyond the surface defined by equation (24). When $e' = 0$, the surfaces of minimum energy (24) transform into the zero velocity surfaces of the restricted circular three-body problem. Moreover, from equation (24), it follows that the family of minimum energy surfaces given values of the parameters p', e', μ and C for all true anomaly values v' within the range $[-v_a, v_a]$ is located between two surfaces [2]

$$x^2 + y^2 - e'z^2 \cos v' + 2p'^3\left(\frac{\mu}{r_1} + \frac{1-\mu}{r_2}\right) - 2\tilde{\Omega}_{\min} = C\left(1 + e'\right), \tag{25}$$

and

$$x^2 + y^2 - e'z^2 \cos v_a + 2p'^3\left(\frac{\mu}{r_1} + \frac{1-\mu}{r_2}\right) - 2\tilde{\Omega}_{\min} = C\left(1 + e'\cos v_a\right), \tag{26}$$

Furthermore, the singular points of the family (24) at fixed values of the parameters p', e', v' and μ are the points where it is impossible to construct a unique tangent plane. Therefore, the singular points of the family (24) are determined by algebraic equations

$$\begin{aligned}
\frac{\partial H}{\partial x} &= 2\left[x - p'^3\frac{\mu}{r_1^3}\left(x + p' - p'\mu\right) - p'^3\frac{1-\mu}{r_2^3}\left(x - p'\mu\right)\right] = 0,\\
\frac{\partial H}{\partial y} &= 2y\left(1 - p'^3\frac{\mu}{r_1^3} - p'^3\frac{1-\mu}{r_2^3}\right) = 0,\\
\frac{\partial H}{\partial z} &= -2z\left(e'\cos v' + p'^3\frac{\mu}{r_1^3} + p'^3\frac{1-\mu}{r_2^3}\right) = 0,
\end{aligned} \tag{27}$$

which coincide with the same equations used to determine libration points in the restricted three-body problem [1,2].

The solutions to the algebraic equations (27) are the collinear singular points $L_1 = L_1\left(x_1, 0, 0\right)$, $L_2 = L_2\left(x_2, 0, 0\right)$, $L_3 = L_3\left(x_3, 0, 0\right)$ and two pairs of coplanar (triangular) singular points: $L_4 = L_4\left(x_4, y_4, 0\right)$, $L_5 = L_5\left(x_4, -y_4, 0\right)$ in the plane z = 0 and $L_6 = L_6\left(x_6, 0, z_6\right)$, $L_7 = L_7\left(x_6, 0, -z_6\right)$ in the plane $y = 0$ (see below). The collinear singular point L_1 is located to the left of the main body M_0 of lesser mass, i.e., $x_1 < \overline{x}_1 = -p'(1-\mu) < 0$, L_2 is located between the main bodies, i.e., $\overline{x}_1 < x_2 < 0 < \overline{x}_2 = p'\mu$, and L_3 is to the right of the main body M' of greater mass, i.e., $x_3 > \overline{x}_2$. The triangular singular points L_4 and L_5 are located in the left half-plane $x < 0$, closer to the main body M_0 of lesser mass and for them $x_4 < 0$. Furthermore, if the masses of the main bodies are equal $m_0 = m'$, then $\mu = 1/2$ and the singular point L_2 will be located at the center of mass of the main bodies – at the origin, i.e., $x_2 = 0$.

Now, let's determine the triangular singular points L_4 and L_5, in the plane $z = 0$ from the system of algebraic equations (27), in which the third equation is absent, and the first equation is rewritten in another form:

$$x\left[1-p'^3\frac{\mu}{r_1^3}-p'^3\frac{1-\mu}{r_2^3}\right]-p'^4\mu(1-\mu)\left(\frac{1}{r_1^3}-\frac{1}{r_2^3}\right)=0, \tag{28}$$

$$y\left(1-p'^3\frac{\mu}{r_1^3}-p'^3\frac{1-\mu}{r_2^3}\right)=0\ .$$

It is evident that the system (28) is consistent only when $r_1 = r_2$. Therefore, there exists a unique real analytical solution at $y \neq 0$ in the form of $r_1 = r_2 = p'$. Let's express the found solution in coordinates x and y, where in the expression (4) for r_1 and r_2, $z = 0$ should be set. This gives us

$$x = x_4 = -\frac{p'}{2}+p'\mu\ , \quad y = y_4 = \frac{\sqrt{3}}{2}p', \quad z = 0\ . \tag{29}$$

It is clear that the x_4-coordinate (x_4) of the singular point L_4 or L_5 depends on the focal parameter p' of the test star's orbit and its mass through μ, while the y_4-coordinate (y_4) depends only on p'.

Next, from the system of equations (27) with $y = 0$, i.e., from the system of two equations

$$x\left[1-p'^3\frac{\mu}{r_1^3}-p'^3\frac{1-\mu}{r_2^3}\right]-p'^4\mu(1-\mu)\left(\frac{1}{r_1^3}-\frac{1}{r_2^3}\right)=0$$

$$z\left(e'\cos v'+p'^3\frac{\mu}{r_1^3}+p'^3\frac{1-\mu}{r_2^3}\right)=0\ , \tag{30}$$

we find two symmetric coplanar solutions $L_6 = L_6(x_6, 0, z_6)$ and $L_7 = L_7(x_6, 0, -z_6)$ relative to the x-axis. It is clear that $x_6 = x_6(v')$ and $z_6 = z_6(v')$, and from the second equation of the system (30), it follows that real solutions L_6 and L_7 can only exist when $\cos v' < 0$. It should be noted that at $v' = \pm\frac{\pi}{2}$, the coplanar singular points will also include two infinitely distant singular points $L_8 = L_8(0,0,+\infty)$ and $L_9 = L_9(0,0,-\infty)$, known for the circular problem [2]. At $e' \to 0$, the infinitely distant libration points tend towards the infinitely distant singular points. Moreover, the equations (30) also have an analytical solution in the plane $y = 0$ for parabolic motion ($e' = 1$) of the test star and $\cos v' = -1$. This solution also has the form $r_1 = r_2 = p'$ and in coordinates, similar to (29), is written as follows:

$$x = x_6 = -\frac{p'}{2}+p'\mu\ , \quad y = 0, \quad z = z_6 = \frac{\sqrt{3}}{2}p' \quad \left(\cos v' = -1, e' = 1\right)\ . \tag{31}$$

Therefore, in the parabolic motion of the test star and at $\cos v' = -1$, the x-coordinate of the singular point L_6 or L_7 depends on the focal parameter p' of the orbit of the test star and its mass through μ, while the z-coordinate z_6 depends only on p'. In other cases, the coordinates of the coplanar singular points L_6 or L_7 are determined only numerically.

Thus, in the restricted circular three-body problem, there are a total of seven libration points including the infinitely distant ones, whereas in the non-circular problem we are considering, the number of singular points is greater.

In conclusion, let us note some differences between singular points and libration points. In each singular point, there is a bifurcation of the minimum energy surfaces, i.e., a transition from one state to another. Bifurcation also occurs at the values of the true anomaly $v' = \pm\frac{\pi}{2}$ [2]. Unlike the coordinates of libration points, the coordinates of coplanar singular points are not stationary particular solutions of the SHAPNER equations (1), as they do not satisfy these equations. Libration points are conical singular points, while coplanar points are singular points of the "center" type.

REMARK 2. The existence of a Jacobian integral analogue in the restricted elliptical, parabolic, and hyperbolic three-body problems was denied. However, work [2] has proven that such an analogue – a quasi-integral does exist. From this discovered quasi-integral, an analogue of zero-velocity surfaces – minimum energy surfaces – is derived. These surfaces also allow for the existence of satellite-type motions, i.e., there is Hill stability at certain parameter values, which will be discussed in the next section.

4. CRITERIA FOR HILL STABILITY OF MOTION.

In the restricted three-body problem, the motion of a passively gravitating body is considered Hill stable if it remains confined within a certain closed region around one of the primary bodies. In other words, if the passively gravitating body, at any values of the true anomaly v', maintains a satellite-type motion around one of the primary bodies and remains within a restricted area, its motion is deemed Hill stable. The concept of Hill stability is intimately linked to the value of the Jacobian constant analog *C*, calculated at the special point L_2, which lies between the primary bodies M_0 and M' and corresponds to a satellite-type motion. The value of the Jacobian constant analog from the family of minimal energy surfaces corresponding to the special point $L_2(x_2,0,0)$ is denoted by C_2, i.e., from equation (24) we set

$$C_2 = H(x_2,0,0) = x_2^2 + 2p'^3\left(\frac{\mu}{\sqrt{(x_2+p'-p'\mu)^2}} + \frac{1-\mu}{\sqrt{(x_2-p'\mu)^2}}\right) - p'^2\left(3-\mu+\mu^2\right). \quad (32)$$

Then, for any values of the true anomaly v', chosen as the independent variable, the inequality

$$C = 2\Omega_p - V_p^2 - \frac{2\tilde{\Omega}_{\min}}{1+e'} \geq \frac{C_2}{1+e'\cos v'}, \quad (33)$$

where Ω_p, V_p^2 and $\tilde{\Omega}_{\min}$ are defined earlier, is satisfied.

Using inequality (33), the criteria for stability, instability, and conditional stability of the motion of a small mass body are determined in the restricted elliptical, hyperbolic, and parabolic three-body problems.

In the case of the restricted elliptical three-body problem, the sufficient condition – a criterion for Hill stability of the motion of a small mass body *M* – takes the form of equation in reference [2],

$$C \geq \frac{C_2}{1-e'}, \quad (34)$$

where the values of the Jacobian constant analog C_2 and *C* are determined by equations (32) and (33).

The opposite inequality (34)

$$C < \frac{C_2}{1-e'} \quad (35)$$

defines the criterion for the instability of the motion of the body *M* according to Hill. In this case, there will be values of v' for which the inequality $C < C_2$ will hold.

For the parabolic or hyperbolic restricted three-body problem, one should use the inequality

$$\frac{1}{1+e'\cos v'} < \infty,$$

from which it follows that the criterion for the stability of the motion of body *M* according to Hill is asymptotically fulfilled, i.e., when $C \to \infty$. Therefore, stability of the motion of body *M* according to Hill in the restricted parabolic and hyperbolic three-body problems can never be achieved. Indeed, for any arbitrarily chosen C, there will be a positive value of the true anomaly $v'_p \to 0$ such that for any $|v'| > v'_p$, the inequality $C < C_2$ will hold.

When the instability criterion (35) is satisfied, there will be such a value of $v'_p > 0$ that for any $|v'| \leq v'_p$ the inequality $C \geq C_2$ is met. In such cases, the motion of the passively gravitating body is referred to as conditionally stable according to Hill [2].

In the case of the restricted elliptical three-body problem, the criterion for conditional stability of the motion of body *M* according to Hill is the fulfillment of the double inequality

$$\frac{C_2}{1+e'} \leq C < \frac{C_2}{1-e'} \quad (36)$$

and for the parabolic and hyperbolic restricted three-body problems, conditional stability of the motion of body *M* according to Hill is achieved under condition

$$\frac{C_2}{1+e'} \leq C < \infty. \quad (37)$$

It should be noted that conditional stability of the motion of a small mass body *M* according to Hill occurs when the instability criterion (35) and inequality [2]

$$\cos v' |\geq \cos v'_a = \frac{1}{e'}\left(\frac{C_2}{C} - 1\right), \quad |v'| \leq v'_a , \tag{38}$$

are met.

The criterion for absolute instability of the motion of body *M* according to Hill is the inequality

$$C < \frac{C_2}{1+e'} \quad , \tag{39}$$

which ensures that the inequality $C < C_2$ is satisfied at any v'.

The criterion for stability of the motion of a passively gravitating body in the restricted elliptical three-body problem according to Hill within a certain bounded area encompassing both primary bodies is the fulfillment of inequality

$$C \geq \frac{C_3}{1-e'}, \quad \left(\frac{C_3}{1+e'} \leq C < \frac{C_3}{1-e'}\right). \tag{40}$$

Here $C_3 = H(x_3, 0, 0)$ is the value from the family of minimal energy surfaces corresponding to the special point $L_3(x_3, 0, 0)$, located to the right of the primary body of greater mass M'. Additionally, the brackets indicate the criterion for conditional stability according to Hill for motion in this area. The inequality opposite to (40) represents the criterion for instability of the motion of body *M* in this area.

For brevity, the stability of motion according to Hill, associated with the value C_2 of the Jacobian constant analogue, i.e., meeting criterion (34), will be referred to as *first-type stability*, while the stability of motion according to Hill when criterion (40) with the value C_3 is met will be referred to as *second-type stability*.

Similarly, criteria for stability, conditional stability, instability, and absolute instability according to Hill for the motion of a passively-gravitating body in the restricted elliptical three-body problem associated with other special points can be established. For brevity, these criteria are not presented here.

5. SATELLITE EXCHANGE BETWEEN PRIMARY BODIES

Let us now consider the problem of exchanging a satellite between the primary bodies M_0 and M', which are approaching each other along elliptical, hyperbolic, and parabolic orbits. The theory outlined above regarding the criteria for the stability of the motion of body M by Hill [2] allows us to establish: a) the necessary condition for the exchange or capture to take place, b) the sufficient condition for the impossibility of exchange, c) the range of values of the true anomaly v' during which a satellite exchange is possible.

Let's first consider the restricted elliptical three-body problem. When inequality (34) – the condition for the stability of the motion of body M by Hill – is met, an exchange of satellites between the primary bodies in elliptical motion is not possible. Therefore, inequality (34) can be considered as a sufficient condition for the impossibility of satellite exchange in the case of elliptical motion, as the energy of the satellite in this case is so low that it cannot detach from its parent body. However, the necessary condition for satellite exchange in elliptical motion is the fulfillment of the instability criterion by Hill, i.e., inequality (35). This inequality, along with the surfaces of minimal energy, facilitates the numerical search for the satellite's trajectory during exchange. Thus, when inequality (35) is met and the initial conditions are appropriately chosen, the satellite may either leave the vicinity of its parent body and become the satellite of the second body (exchange), transform into an independent celestial body (ejection), or remain the satellite of its parent body. Furthermore, when the criterion for conditional stability (36) is met, the conditions for the possibility of satellite exchange in elliptical motion should be clarified. The exchange can then only occur in interval $|v'| > v'_a$, i.e., in the vicinity of the most distant point of the primary bodies' orbit. It is precisely in this interval of true anomaly values, according to inequality (38), that the loss of conditional stability by Hill occurs.

In the case of parabolic and hyperbolic motion of the primary bodies, the necessary condition for exchange, i.e., the instability criterion (35) for the motion of body M by Hill, is always met. In this case, instability by Hill occurs at any energy level when $|v'|$ is sufficiently large. Furthermore, in the case of violation of the conditions for conditional stability (38) by Hill, i.e., in interval $|v'| > v'_a$, satellite exchange can occur in both elliptical and in parabolic and hyperbolic motions of the primary bodies.

Under the criterion of absolute instability (39), satellite exchange is theoretically possible at any value of the true anomaly v'. It is worth noting that such conditions were previously unknown, and the possibility or impossibility of satellite exchange was checked by intuitively selecting initial conditions and parameters [2].

For example, Hill [8], within the framework of the restricted circular three-body problem of the Sun-Earth-Moon, established that the satellite motion of the Moon relative to Earth is stable according to Hill, as the sufficient condition – inequality (34) is met when $e' = 0 : C = 2.2544 > C_2 = 2.00092$. As shown in the study [2], the Moon's motion remains stable according to Hill in the restricted elliptical three-body problem as well – the condition $C = 2.2544 > C_2 / (1 - e')$ =2.03501 is met, where e' = 0.016751 is the eccentricity of Earth's orbit. If e' were seven times larger, then the motion of the Moon would become conditionally stable according to Hill, i.e., inequality (38) would be met: $C = 2.2544 > C_2 / (1 + 7e') = 1.7909$, and inequality (34) would not be satisfied: $C = 2.2544 < C_2 / (1 - 7e') = 2.2667$.

6. ON SOME APPLICATIONS OF THE RESTRICTED THREE-BODY PROBLEM TO ASTRONOMICAL OBJECTS

In the study [3], within the framework of the restricted three-body problem, the motion of a star in a close binary system (CBS) with conservative mass transfer was investigated. Unlike the well-known Paczynski-Huang model, a new model defining the relative motion of the star in the CBS along an elliptical orbit was used. The third body in this scenario is the mass stream flowing from the donor star to the accretor star. The elliptical motion of the star takes into account the mutual attraction of the stars, reactive forces, and the gravitational force of the stars on the flowing stream. Changes in the semi-major axis and eccentricity of the second star were identified, showing that CBS does not form a closed mechanical system, i.e., a system that allows for the conservation of linear momentum and angular momentum. Moreover, the classical law of conservation of energy does not apply, but there exists an analogue of the conservation law in the form of a quasi-integral. This can be confirmed based on the general equations of motion by Meshcherskiy for a two-body problem with variable masses. Therefore, the use of the Paczynski-Huang model, which assumes that CBS forms a closed mechanical system, is not appropriate for this problem. The model proposed in the study [3] was subsequently named the Luk'yanov's model.

In the work [7], the problem of the motion of a rotating star in a close binary system (CBS) with conservative mass transfer was considered. Using the Luk'yanov's model [3], the relative motion of the star in close binary systems along an elliptical orbit was determined. The elliptical motion of the star accounts for the mutual attraction of the stars, reactive forces, the gravitational force of the stars on the flowing stream, and disturbances from the rotational movement of the accretor star. Changes in the semi-major axis, eccentricity, and angular velocity of the accretor star's orbit were defined depending on the parameter q – the mass ratio of the stars. The results were applied to the star system BF Aurigae (in the constellation Auriga) and presented in the form of diagrams. The Luk'yanov model is also applicable in studying the motion of stars in CBS with non-conservative mass exchange.

In the study [14], the problem of the stability of a planet's satellite motion was considered. Within the general three-body problem (Sun-planet-satellite), "Sundman surfaces" were constructed, based on which the concept of "stability by Sundman" was formulated. Special points of these surfaces were identified, possible motion regions were defined, and the stability of the special points by Sundman was established. The stability of the motion of all known natural satellites of planets was investigated, and it was shown that the motion of a number of natural satellites, stable by Hill, as well as some planet satellites stable by the Golubev method, turn out to be unstable by Sundman.

In work [15], within the framework of the restricted elliptical three-body problem, the criterion for stability by Hill was established. By virtue of this criterion, the stability of four exoplanets outside the solar system in a binary star system: CepheiAb, Gliese 86 Ab, HD 41004 Ab, and HD 41004 Bb was investigated.

7. INVESTIGATION OF HILL'S STABILITY OF PLANETARY MOTIONS DURING STELLAR APPROACHES

To investigate the stability of planetary motion in the Hill frame during the approach of a test star to the Solar System, moving along a hyperbolic (parabolic or elliptical) orbit, it is necessary to know its mass and orbital parameters. As an example, let us take a test star with mass m', heliocentric distance q' (in astronomical units), and eccentricity e', varying within the range

$$M_\odot \le m' \le 5M_\odot, \quad 50 \le q' \le 100, \quad 1 \le e' \le 5, \quad \left(0.1 \le e' \le 0.9\right), \tag{41}$$

where $M_\odot$ is the mass of the Sun, $e' = 1$ corresponds to parabolic motion of the test star, and the values of the eccentricity of its orbit are indicated in parentheses for elliptical motion. It should be noted that the relationship between the time t and the true anomaly v' of the test star depends on the type of its orbit. Thus, in the case of elliptical motion of the test star with orbit parameters $e' = 0.2$, $q' = 50$ au, and m' = 3 $M_\odot$, the change in v' in interval [0, 3π/4] corresponds to a change in time in the interval [0, 80.26] years. For the same values of eccentricity and mass of the test star, but at $q' = 75$ au, this interval of changes in v' corresponds to the interval [0, 147.45] years, and at $e' = 0.2$, $q' = 75$ au, and m' = 5 $M_\odot$, it corresponds to [0, 120.39] years.

In the case of parabolic motion ($e'=1$) of the test star at $q'=50$ au and $m'=3\,M_{\odot}$, the interval of changes [0, 3π/4] in the true anomaly v' corresponds to the time interval [0, 799.53] years, at $q'=75$ au and $m' = 3\,M_{\odot}$ – [0, 1468.83] years, and at $q'=75$ au and $m' = 5\,M_{\odot}$ – [0, 1199.29] years.

Finally, in the case of hyperbolic motion with $e'=1.15$, $q'=50$ au, and $m' = 3\,M_{\odot}$, the interval [0, 3π/4] of changes in v' corresponds to the time interval [0, 466.49] years, at $q'=75$ au and $m' = 3\,M_{\odot}$ – [0, 857.0] years, and at $q'=75$ au and $m' = 5\,M_{\odot}$ – [0, 699.74] years.

Table 1 presents the results of the study of Hill stability motion of three planets (Earth, Jupiter, and Saturn) depending on the focal parameter p' of the star's orbit and its mass m' during hyperbolic orbits, and in Table 2 - during parabolic orbits of the star. In these tables, C_p denotes the value of the analogue of the Jacobi constant when $v'=0$, and C_2 denotes its value computed at the special point L2. It turned out that in the case of a parabolic or hyperbolic star orbit, only conditional stability of planetary motion in the Hill frame occurs.

Table 1. The Hill stability of planetary motion in the restricted hyperbolic three-body problem: planet – Sun – star

Planets	p' (au)	m'	C_p	$\frac{C_2}{1+e'}$	Stability
Earth	107.5	$M_{\odot}$	11223.63801	6718.7500	Conditional stability
		$3M_{\odot}$	25583.32507	5687.60356	
		$5M_{\odot}$	47022.20069	4772.35271	
	161.25	$M_{\odot}$	20191.97684	15117.18750	
		$3M_{\odot}$	61105.18565	12797.10802	
		$5M_{\odot}$	95570.33302	10737.79359	
	215	$M_{\odot}$	33270.29723	26875.0	
		$3M_{\odot}$	112264.5806	22750.41426	
		$5M_{\odot}$	165109.6349	19089.41084	
Jupiter	107.5	$M_{\odot}$	13331.70751	6718.7500	
		$3M_{\odot}$	24861.51172	5687.60356	
		$5M_{\odot}$	49624.43491	4772.35271	
	161.25	$M_{\odot}$	21467.26449	15117.18750	
		$3M_{\odot}$	60262.03232	12797.10802	
		$5M_{\odot}$	95683.01105	10737.79359	
	215	$M_{\odot}$	33930.418233	26875.0	
		$3M_{\odot}$	111454.2512	22750.41426	
		$5M_{\odot}$	164324.6482	19089.41084	
Saturn	107.5	$M_{\odot}$	8727.26427	6718.7500	
		$3M_{\odot}$	27513.53878	5687.60356	
		$5M_{\odot}$	42004.00334	4772.35271	
	161.25	$M_{\odot}$	18716.70519	15117.18750	
		$3M_{\odot}$	63492.14497	12797.10802	
		$5M_{\odot}$	92061.73155	10737.79359	
	215	$M_{\odot}$	32919.55202	26875.0	
		$3M_{\odot}$	115125.8691	22750.41426	
		$5M_{\odot}$	162201.6013	19089.41084	

Table 2. The Hill stability of planetary motion in the restricted parabolic three-body problem: planet – Sun – star

Planets	p' (au)	m'	C_p	$\frac{C_2}{1+e'}$	Stability
Earth	100	$M_{\odot}$	11219.60303	6250.0	Conditional stability
		$3M_{\odot}$	23482.33868	5290.79401	
		$5M_{\odot}$	44605.58035	4439.39787	
	150	$M_{\odot}$	19471.08396	14062.50	
		$3M_{\odot}$	56332.99884	11904.28652	
		$5M_{\odot}$	88706.78297	9988.64521	
	200	$M_{\odot}$	31408.15909	25000.0	
		$3M_{\odot}$	103775.4097	21163.17606	
		$5M_{\odot}$	152887.0666	17757.59148	
Jupiter	100	$M_{\odot}$	13530.15606	6250.0	
		$3M_{\odot}$	22763.67615	5290.79401	
		$5M_{\odot}$	48838.76295	4439.39787	
	150	$M_{\odot}$	20865.61324	14062.50	
		$3M_{\odot}$	55415.29318	11904.28652	
		$5M_{\odot}$	90292.22156	9988.64521	
	200	$M_{\odot}$	32241.85212	25000.0	
		$3M_{\odot}$	102759.3469	21163.17606	
		$5M_{\odot}$	153761.1559	17757.59148	
Saturn	100	$M_{\odot}$	94538.09535	6250.0	
		$3M_{\odot}$	169050.8402	5290.79401	
		$5M_{\odot}$	235037.3713	4439.39787	
	150	$M_{\odot}$	136655.9248	14062.50	
		$3M_{\odot}$	246167.2023	11904.28652	
		$5M_{\odot}$	337912.9926	9988.64521	
	200	$M_{\odot}$	186559.9802	25000.0	
		$3M_{\odot}$	337870.9594	21163.17606	
		$5M_{\odot}$	459756.0501	17757.59148	

8. CONCLUSION

The problem of the spatial motion of a passive-gravitating body during approach to the central body of a test star – the perturbing body, has been considered. The perturbing body - the star - may move along an elliptical, parabolic, or hyperbolic orbit. An exact expression of the force function without expansion into a series has been used. An integral invariant relationship - a quasi-integral, has been found. Due to the quasi-integral, regions of possible motion of the passive-gravitating body, surfaces of minimum energy, which generalize the surfaces of zero velocity, and the special points of these surfaces have been determined. The necessary condition has been established - the fulfillment of the Hill instability criterion for satellite exchange in the restricted elliptical three-body problem. It has been shown that in the case of parabolic or hyperbolic motion of the principal bodies, the necessary condition for satellite exchange is always satisfied. In the region of instability loss of Hill motion, satellite exchange can occur in both elliptical and parabolic or hyperbolic motions of the principal bodies. Exchange can only occur in the vicinity of the farthest point of the principal bodies' orbits. When the instability criterion is met and the initial conditions of the satellite are properly chosen, the satellite may either leave the vicinity of the parent body and become a satellite of the second body (exchange), or transform into an independent celestial body (ejection), or remain a satellite of the parent body.

To illustrate the obtained results, restricted hyperbolic, parabolic, and elliptical three-body problems have been considered as an example: Sun-planet-test star. In this case, the heliocentric distance q' of the test star and its mass m' oscillate within the range of 50 to 100 au and from one to five solar masses, respectively. According to the stability criteria of the first and second types, critical values of the orbit parameters of the test star have been established, at which the planets of the Solar System either become satellites of the test star or leave the boundaries of the Solar System.

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ПРО СТІЙКІСТЬ РУХІВ ПЛАНЕТ ПІД ЧАС ЗБЛИЖЕННЯ ЗІР

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Розглянуто задачу про просторовий рух пасивно гравітаційного тіла під час наближення до центрального тіла збурюючого тіла – пробної зірки. Використовуючи точний вираз силової функції, знайдено інтегральне інваріантне співвідношення – квазіінтеграл. За допомогою квазіінтеграла визначено області можливого руху пасивно гравітаційного тіла, поверхні мінімальної енергії (узагальнення поверхонь нульової швидкості) та особливі точки цих поверхонь. Досліджено стабільність руху планет за Хіллом під час наближення пробної зірки до Сонячної системи. Встановлено критерії можливості, а також неможливості захоплення тестовою зіркою пасивно гравітаційного тіла. Відповідно до критеріїв стійкості Хілла були встановлені критичні значення параметрів орбіти досліджуваної зірки, при яких планети Сонячної системи або стають супутниками досліджуваної зірки, або залишають межі Сонячної системи.

Ключові слова: *небесна механіка; обмежена задача трьох тіл; аналог функції Якобі; квазіінтеграл; закон збереження енергії; поверхні мінімальної енергії; особливі точки; стійкість Хілла*