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PILGRIM DARK ENERGY BIANCHI TYPE-I *f*(*T*) GRAVITY MODEL

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In this work, we have analyzed Bianchi type-I space-time (spatially homogeneous and anisotropic), using an interacting two fluid – dark matter (DM) and Pilgrim dark energy (PDE) in the framework of *f*(*T*) gravity by taking into consideration the infrared (IR) cut-off as a candidate of Hubble's horizon $(L = 1/H)$. We have also performed the state-finder diagnostics and in addition, energy conditions are discussed to verify accelerating expansion of the universe.
Keywords: *Pilgrim dark energy; Dark matter; f(T) Gravity; Bianchi type-I space-time; Cosmology*
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INTRODUCTION

It is assumed that dark energy (DE) is responsible for accelerated expansion of the universe and also there is increasing evidence of DE over the last few years. Same has been confirmed by various observational experiments [1-3]. Despite the remarkable success of the standard cosmology, there are some issues which remain unresolved including the search for best DE candidate. Various approaches have been adopted for the same such as dynamical DE models and modified gravities.

Several modified theories of gravitation have been proposed to investigate accelerated expansion of the Universe, such as $f(R)$ gravity theory, $f(T)$ gravity theory, $f(R,T)$ gravity theory, $f(G)$ gravity theory, etc. As far as modified $f(R)$ gravity is concerned, Ricci scalar *R* is replaced by an arbitrary function of *R* in the Einstein-Hilbert action. Recently, Wankhade et al. [4] have investigated Renyi holographic dark energy (RHDE) with Hubble's IR cut-off in the framework of $f(R)$ gravity. Some other authors [5-8] have also recently worked on the same theory. In modified $f(R,T)$ gravity theory, the gravitational action takes in an arbitrary function of the Ricci scalar *R* along with trace of the stress energy momentum tensor *T*. Pradhan et al. [9], Singh and Kumar [10], Shaikh and Wankhade [11], Dagwal et al. [12] have investigated different aspects of $f(R,T)$ gravity.

The $f(T)$ theory of gravity [13] is the generalized teleparallel gravity, where T is the torsion scalar. This theory has attracted many people to explore it in different scenarios. Cai et al. have investigated various torsional constructions in the paradigm of $f(T)$ gravity [14]. Zubair and Waheed have studied different energy conditions in $f(T)$ gravity theory, with non-minimal torsion-matter [15]. Karami and Abdolmaleki have investigated the validity of the generalized second law of gravitational thermodynamics in the framework of $f(T)$ gravity [16]. Jamil et al. have attempted to resolve the dark matter (DM) problem in $f(T)$ gravity [17]. Bhatti et al. have investigated role of $f(T)$ gravity on the evolution of collapsing stellar model [18]. Dagwal and Pawar have worked on two fluid sources namely matter field and radiation field in the framework of $f(T)$ theory of gravity [19]. Karimzadeh and Shojaee have investigated phantom-like behavior in modified teleparallel gravity [20]. Chirde and Shekh have investigated holographic dark energy (HDE) cosmological model in modified $f(T)$ theory of gravity [21]. Bhoyar et al. have used hybrid expansion law to investigate the stability of accelerating universe in $f(T)$ gravity theory with linear equation of state [22]. Shekh and Chirde have examined certain aspects of the anisotropic accelerating Bianchi type-I model where two non-interacting fluids—one regular string and one DE—are present, in $f(T)$ gravity [23]. Shaikh et al. have studied LRS Bianchi type-I domain walls cosmological models in $f(T)$ gravity using volumetric expansions laws for the depiction model [24]. Mandal and Sahoo have investigated evolution of particle production in $f(T)$ gravity [25].

The search for viable DE model is the basic key leading to reconstruction phenomenon in modified theories of gravity. Holographic dark energy (HDE) models are having significant place in discussing the accelerated expansion of universe. In PDE model [26] it is considered that a repulsive force that is accelerating the Universe is phantom type with $(\omega_{DE} < -1)$ and it is so strong that it prevents formation of the black hole. The energy density of PDE has the form [27]

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$$\rho_\Lambda = 3\xi^2 m_p^{4-u} L^{-u} , \tag{1}$$

where ξ and u are dimensionless constants. By taking $m_p = 1$, here we consider Hubble horizon $L = 1/H$ as the IR cutoff to find $f(T)$ model using equation (1).

Sharif and Rani have discussed pilgrim dark energy (PDE) model by taking the Hubble horizon as the IR cutoff in the framework of $f(T)$ gravity [27]. Also, cosmological evolution of PDE has been studied by Sharif and Zubair [28]. Jawad et al. have used Hubble's cut-off, Granda–Oliveros cut-off and generalized ghost cut-off to discuss the cosmological implications of interacting PDE models with cold dark matter (CDM) in fractal cosmology by taking the flat universe [29]. Sharif and Nazir [30] have discussed evolution (cosmological) of generalized ghost PDE in modified $f(T)$ theory of gravity. Jawad and Rani [31] have worked on cosmological evolution of PDE in $f(G)$ gravity. Myrzakulov et al. [32] have recreated two instances of interacting fluid scenario – ghost and PDE with pressure less DM in the framework of $f(Q)$ gravity.

f(*T*) GRAVITY FORMALISM AND PILGRIM DARK ENERGY

Here, we provide an overview of the $f(T)$ gravity with thorough derivation of its field equations. From here onwards, let us define the notations of the Latin subscripts or superscripts as related to the tetrad field, whereas Greek notations are allied to the space-time coordinates. The line element for a general space-time metric can be described as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2}$$

It can be transformed into the tetrad, Minkowski's description of the transformation, in the following way:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ij} \theta^i \theta^j \tag{3}$$

$$dx^\mu = e_i^\mu \theta^i , \; \theta^i = e_\mu^i dx^\mu \tag{4}$$

where η_{ij}, a metric on Minkowski space-time is given by $\eta_{ij} = diag[1,-1,-1,-1]$ and $e_i^\mu e_\nu^i = \delta_\nu^\mu$ or $e_i^\mu e_\mu^j = \delta_i^j$. Also, $\sqrt{-g} = \det\left[e_\mu^i\right] = e$ is the root of determinant of metric. The components of the Weitzenbocks connection for a manifold – where, as per the contribution of the Levi-Civita connection, the Riemann tensor part with no torsion terms is null and the only non-zero torsion terms exist – are defined as

$$\Gamma_{\mu\nu}^\alpha = e_i^\alpha \partial_\nu e_\mu^i = -e_\mu^i \partial_\nu e_i^\alpha \tag{5}$$

This has a zero curvature; however, the torsion is nonzero. Through this connection, various components of the torsion tensors can be defined as

$$T_{\mu\nu}^\alpha = \Gamma_{\mu\nu}^\alpha - \Gamma_{\nu\mu}^\alpha = e_i^\alpha \left(\partial_\mu e_\nu^i - \partial_\mu e_\mu^i\right) \tag{6}$$

The con-torsion tensor is a space-time tensor from the difference between Weitzenbock and the Levi-Civita connections:

$$K_\alpha^{\mu\nu} = \left(-\frac{1}{2}\right)\left(T^{\mu\nu}{}_\alpha + T^{\nu\mu}{}_\alpha - T_\alpha^{\mu\nu}\right) \tag{7}$$

Another tensor $S_\alpha^{\mu\nu}$ can be defined from the constituents of the torsion and con-torsion tensors as follows to make the description of the Lagrangian and the equation of motion easier:

$$S_\alpha^{\mu\nu} = \left(\frac{1}{2}\right)\left(K^{\mu\nu}{}_\alpha + \delta_\alpha^\mu T^{\beta\nu}{}_\beta - \delta_\alpha^\nu T_\beta^{\beta\mu}\right) \tag{8}$$

The torsion scalar T is

$$T = T_{\mu\nu}^\alpha S_\alpha^{\mu\nu} \tag{9}$$

Now we define action by generalizing the Tele-parallel Theory i.e. $f(T)$ theory as

$$S = \int\left[T + f(T) + L_{matter}\right] e \, d^4x \tag{10}$$

Here $f(T)$ indicates an algebraic function of the torsion scalar T. We obtain the following equation of motion by functionally varying the action in equation (9) with regard to the tetrads:

$$S_{\mu}^{\nu\rho}\partial_{\rho}T\left(f_{TT}\right)+\left[e^{-1}e_{\mu}^{i}\partial_{\rho}\left(ee_{i}^{\alpha}S_{\alpha}^{\nu\rho}\right)+T^{\alpha}{}_{\lambda\mu}S_{\alpha}^{\nu\lambda}\right]\left(1+f_{T}\right)+\frac{1}{4}\delta_{\mu}^{\nu}\left(T+f\right)=T_{\mu}^{\nu} \tag{11}$$

where T_{μ}^{ν} is the energy momentum tensor and $f_T = df(T)/dT$. The field equation (11) is written in terms of tetrads and their partial derivatives; and appears very different from Einstein's equation. But by setting $f(T)=a_0=\text{constant}$, this is dynamically equivalent to the GR.

We consider the energy momentum tensor for interacting two fluids - dark matter (DM) and pilgrim dark energy (PDE), as

$$T_{\mu\nu}=\hat{T}_{\mu\nu}+\overline{T}_{\mu\nu} \tag{12}$$

where $\hat{T}_{\mu\nu}=\rho_m u_\mu u_\nu$ and $\overline{T}_{\mu\nu}=\left(\rho_\Lambda+p_\Lambda\right)u_\mu u_\nu-p_\Lambda g_{\mu\nu}$, with comoving coordinates $u^\mu=(0,0,0,1)$ and $u^\mu u_\nu=-1$, where u^μ is four velocity vector of fluid, p_Λ is pressure of PDE, ρ_m and ρ_Λ are energy densities of DM and PDE respectively.
For interacting DM and PDE, the continuity equation is satisfied by the total energy density as

$$\left(\dot{\rho}_m\right)+\left(\dot{\rho}_\Lambda\right)+3H\left(\rho_m+\rho_\Lambda+p_\Lambda\right)=0 \tag{13}$$

When the energy densities of DM and PDE do not conserve independently, the continuity equation of matter becomes

$$\left(\dot{\rho}_m\right)+3H\left(\rho_m\right)=Q \tag{14}$$

$$\left(\dot{\rho}_\Lambda\right)+3H\left(\rho_\Lambda+p_\Lambda\right)=-Q \tag{15}$$

where dot $(\cdot)$ denotes derivative with respect to time t, Q implies the collaboration between DM and PDE. For suitability, consider the interacting term as $Q=3\sigma H\rho_m$ [33], where σ is coupling constant.

METRIC AND COMPONENTS OF FIELD EQUATIONS

In our work, we consider the spatially homogeneous and anisotropic Bianchi type-I space-time as

$$ds^2 = dt^2 - A^2(t)\,dx^2 - B^2(t)\left[dy^2+dz^2\right], \tag{16}$$

where metric potentials A and B are the functions of cosmic time t only.
Now, the corresponding Torsion scalar T is given by

$$T=-2\left(2\frac{\dot{A}}{A}\frac{\dot{B}}{B}+\frac{\dot{B}^2}{B^2}\right) \tag{17}$$

Using the equation of motion in (10), Bianchi type-I space-time in (16), for the stress energy tensors (12), can be written as

$$\left(T+f\right)+4\left(1+f_T\right)\left\{\frac{\ddot{B}}{B}+\frac{\dot{B}^2}{B^2}+\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right\}+4\frac{\dot{B}}{B}\dot{T}f_{TT}=-p_\Lambda \tag{18}$$

$$\left(T+f\right)+2\left(1+f_T\right)\left\{\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{B}^2}{B^2}+3\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right\}+2\left\{\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right\}\dot{T}f_{TT}=-p_\Lambda \tag{19}$$

$$\left(T+f\right)+4\left(1+f_T\right)\left\{\frac{\dot{B}^2}{B^2}+2\frac{\dot{A}}{A}\frac{\dot{B}}{B}\right\}=\rho_m+\rho_\Lambda \tag{20}$$

So here we got three differential equations containing five unknowns — namely A, B, f, p and ρ.
We now define few kinematical quantities of space-time such as mean scale factor and volume respectively as

$$a^3=V=AB^2 \tag{21}$$

To express volumetric expansion rate of Universe, the mean Hubble parameter is defined as

$$H = \frac{1}{3}\left(H_1 + H_2 + H_3\right) \tag{22}$$

where H_1, H_2 and H_3 are the directional Hubble parameters in the directions of x, y and z axes respectively.
Anisotropy parameter, for discussing whether Universe approach isotropy or not, is defined as

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 \tag{23}$$

The expansion scalar and shear scalar are respectively defined as

$$\theta = \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B} \tag{24}$$

$$\sigma^2 = \frac{3}{2}H^2 A_m \tag{25}$$

EXACT MATTER DOMINATED SOLUTION

In order to solve the system of non-linear differential equations, we use the following physically plausible conditions. The latest findings of high red-shift type-Ia supernovae disclose that the universe is accelerating, contrary to the prediction of standard cosmology with standard matter and no cosmological constant that it is currently decelerating. Hence, the model with constant decelerating parameter have received considerable attention [34]. We extend the same results of [34] to solve the field equations by taking into consideration the variation of Hubble parameter as

$$a = \left(\varepsilon t + \varepsilon_1\right)^{k/\varepsilon}, \text{ for } \varepsilon \neq 0 \tag{26}$$

Now subtracting equation (18) from (19), we get,

$$\frac{d}{dt}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) + \left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)\frac{\dot{V}}{V} = 0 \tag{27}$$

Integrating above equation, it gives,

$$\frac{A}{B} = c_2 \exp\left[c_1 \int \frac{1}{V} dt\right] \tag{28}$$

where c_1 and c_2 are constants of integration.
By using equation (21), we get the metric potentials A and B in the form

$$A = M_1 V^{\frac{1}{3}} \exp\left[N_1 \int \frac{1}{V} dt\right], \tag{29}$$

$$B = M_2 V^{\frac{1}{3}} \exp\left[N_2 \int \frac{1}{V} dt\right], \tag{30}$$

where $M_1 = c_2 M_2$, $M_2 = \left(c_2\right)^{-1/3}$ and $N_1 = c_1 + N_2$, $N_2 = -\frac{c_1}{3}$. Also $M_i\left(i = 1,2\right)$ and $N_i\left(i = 1,2\right)$ satisfy the relations $M_1\left(M_2\right)^2 = 1$ and $N_1 + 2N_2 = 0$.
Now we consider average scale factor of the form.
Using equation (26) in equations (29) and (30), we get,

$$A = M_1\left(\varepsilon t + \varepsilon_1\right)^{\frac{k}{\varepsilon}} \exp\left[\frac{\varepsilon N_1}{\left(\varepsilon - 3k\right)}\left(\varepsilon t + \varepsilon_1\right)^{1-\frac{3k}{\varepsilon}}\right], \tag{31}$$

$$B = M_2\left(\varepsilon t + \varepsilon_1\right)^{\frac{k}{\varepsilon}} \exp\left[\frac{\varepsilon N_2}{\left(\varepsilon - 3k\right)}\left(\varepsilon t + \varepsilon_1\right)^{1-\frac{3k}{\varepsilon}}\right], \tag{32}$$

Therefore, using equations (31) and (32), space-time (16) filled with the fluid (12) in the framework of $f\left(T\right)$ gravity becomes

$$ds^2 = dt^2 - M_1{}^2 \tau^{\left(\frac{2k}{\varepsilon-3k}\right)} \exp\left[\frac{2\tau\varepsilon N_1}{(\varepsilon-3k)}\right] dx^2 - M_2{}^2 \tau^{\left(\frac{2k}{\varepsilon-3k}\right)} \exp\left[\frac{2\tau\varepsilon N_2}{(\varepsilon-3k)}\right]\left[dy^2 + dz^2\right] \tag{33}$$

where, $\tau = \left(\varepsilon t + \varepsilon_1\right)^{1-\frac{3k}{\varepsilon}}$.

DYNAMICAL PARAMETERS WITH PHYSICAL ACCEPTABILITY

Deceleration parameter is

$$q = -1 + \frac{\varepsilon}{k} \tag{34}$$

From equation (34), it is found that the deceleration parameter is constant throughout the expansion of the Universe.
Anisotropy parameter is

$$A_m = \frac{\varepsilon^2 \tau^2 \left(N_1{}^2 + 2N_2{}^2\right)}{3k^2} \tag{35}$$

From above equation (35), it is observed that the nature of the anisotropic parameter is varying with the evolution of the universe.
Expansion scalar is found to be

$$\theta = \frac{3k}{\left(\varepsilon t + \varepsilon_1\right)}, \tag{36}$$

From equation (36), it is observed that the expansion scalar is a decreasing function of time. At $t \to 0$, the expansion scalar is constant and as cosmic time increases, it decreases, which shows that as time increases, the universe is expanding but its rate of expansion is decreasing.
Shear scalar is

$$\sigma^2 = \frac{1}{2}\varepsilon^2 \left(N_1{}^2 + 2N_2{}^2\right) \tau^{\left(\frac{-6k}{\varepsilon-3k}\right)}, \tag{37}$$

It is found that the shear scalar is the inverse function of time. Initially it is constant and the model is shear free at an infinite expansion.
For PDE, using equation (1), the energy density is obtained as

$$\rho_\Lambda = 3\xi^2 k^u \tau^{\left(\frac{\varepsilon u}{3k-\varepsilon}\right)}, \tag{38}$$

From Figure 1, it is observed that energy density of PDE is always positive and as time increases, energy density decreases. i.e. as $t \to \infty$, $\rho_\Lambda \to 0$, this means that at infinite time, the Universe is empty.

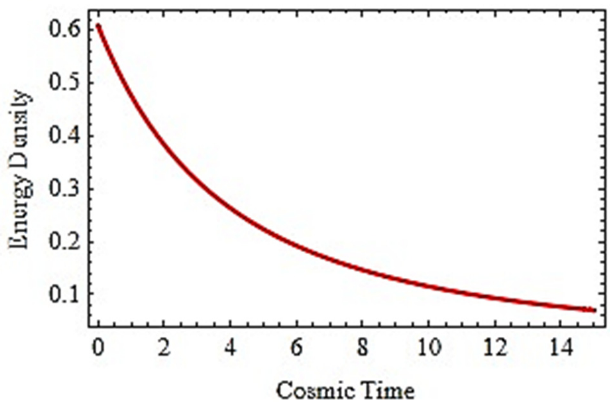

Figure 1. Graphical representation of energy density (ρ_Λ) of PDE versus cosmic time (t) by taking $\xi = 1.5$, $k = 0.3$, $u = 2$, $\varepsilon = 0.13$, $\varepsilon_1 = 1$.

Now we find exact solution of field equations using some physical quantities for depiction $f(T)$ model [35] which is,

$$f(T) = T^\eta \tag{39}$$

Isotropic pressure,

$$p_\Lambda = (-2)(\tau)^{\frac{-2\varepsilon}{\varepsilon-3k}}\left\{\tau_1\tau_2\tau_6 + 2\tau_4\left[1+\eta(-2)^{\eta-1}\tau_5\right]\right\}, \tag{40}$$

where, $\tau_1 = N_2\varepsilon\tau + k$, $N_3 = 2N_1 + N_2$, $\tau_2 = N_3\varepsilon\tau + 3k$, $\tau_3 = \tau_1\tau_2{}^2 - 2\varepsilon k\eta(\eta-1)\left[3\tau_1(1+N_3\tau)+\tau_2(1+3N_2\tau)\right]$, $\tau_4 = 3k\tau_1 - \varepsilon k(1+3N_2\tau)$, $\tau_5 = (\tau)^{\frac{2\varepsilon(1-\eta)}{\varepsilon-3k}}\tau_1{}^{\eta-1}\tau_2{}^{\eta-1}$ and $\tau_6 = -1-(-2)^{\eta-1}\tau_1{}^{\eta-2}\tau_2{}^{\eta-3}\tau_3$.

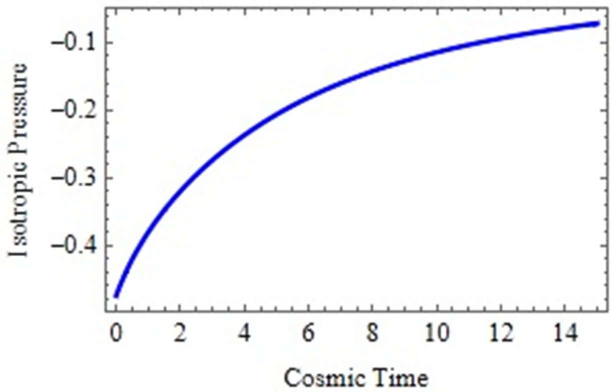

Figure 2. Graphical representation of isotropic pressure (p_Λ) of PDE versus cosmic time (t) by taking $c_1 = 0.5$, $k = 0.3$, $\varepsilon = 0.13$, $\varepsilon_1 = 1$, $\eta = 2$.

From Figure 2, one can observe that isotropic pressure of PDE model is always negative. Hence the Universe is filled with dark energy without baryonic matter.
Equation of state parameter,

$$\omega_\Lambda = \frac{-2}{3\xi^2 k^u}(\tau)^{\frac{\varepsilon(u-2)}{\varepsilon-3k}}\left\{\tau_1\tau_2\tau_6 + 2\tau_4\left[1+\eta(-2)^{\eta-1}\tau_5\right]\right\}, \tag{41}$$

Latterly, a considerable class of scalar field DE models has been given which includes Quintessence $(\text{if } \omega_\Lambda > -1)$, Phantom $(\text{if } \omega_\Lambda < -1)$ and Quinton — which can travel across Phantom to quintessence region. Also, the Quinton scenario of DE is designed to comprehend the nature of DE with ω_Λ across -1 . Setare and Saridakis [36] have examined DE models where equation of state parameter (ω_Λ) is across -1 , providing a tangible assertion to the Quinton paradigm.

Some other limits of ω_Λ — derived from observational findings obtained from SNe-Ia data and, SNe-Ia data combined with Cosmic Microwave Background (CMB) anisotropy and Galaxy clustering statistics; are respectively $-1.66 < \omega_\Lambda < -0.62$ and $-1.33 < \omega_\Lambda < -0.79$.

In the derived model, the equation of state parameter represents Quintessence region then it goes through a ΛCDM model and as time increases, it converts to a Phantom model. Same behavior can be observed from Figure 3.

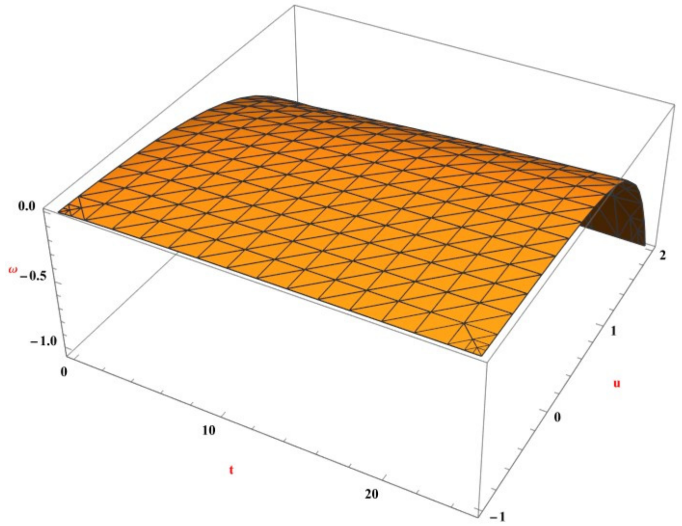

Figure 3. Graphical representation of Equation of State parameter (ω) of PDE versus cosmic time (t) by taking $\xi = 1.5$, $u = -1$ to $u = 2$, $c_1 = 0.5$, $k = 0.3$, $\varepsilon = 0.13$, $\eta = 2$, $\varepsilon_1 = 1$.

Stability factor,

$$\vartheta_s^{\,2}=\frac{2(\varepsilon-3k)}{3\xi^2uk^u}(\tau)^{\frac{\varepsilon(u-1)-3k}{\varepsilon-3k}}\left\{\begin{array}{l}(N_3\tau_1+N_2\tau_2)\tau_6+2\eta(\eta-1)(-2)^{(\eta-1)}\tau_4\tau_5\left[\dfrac{N_3}{\tau_2}+\dfrac{N_2}{\tau_1}-\dfrac{2}{(\varepsilon-3k)\tau}\right]\\ -\dfrac{2}{(\varepsilon-3k)\tau}\left\{\tau_1\tau_2\tau_6+2\tau_4\left[1+\eta(-2)^{\eta-1}\tau_5\right]\right\}\\ -(-2)^{\eta-1}\tau_1^{\,\eta-1}\tau_2^{\,\eta-2}\left\{\begin{array}{l}2N_3\tau_1\tau_2+N_2\tau_2^{\,2}+\tau_3\left[\dfrac{N_3(\eta-3)}{\tau_2}+\dfrac{(\eta-2)}{\tau_1}\right]\\ -2k\eta(\eta-1)\left\{N_3\left[3\tau_1+\varepsilon(3N_2\tau+1)\right]+3N_2\left[\tau_2+\varepsilon(N_3\tau+1)\right]\right\}\end{array}\right\}\end{array}\right\} \quad (42)$$

We should examine the physical acceptance to ensure that the appropriate solution in the current model is stable. First, the velocity of sound needs to be less than that of light in order for this to happen, i.e. within the range $0<\vartheta_s^{\,2}$. From Figure 4, it is observed that the stability factor for the present model is negative throughout the expansion of the Universe i.e. $\vartheta_s^{\,2}<0$ and hence the model is unstable throughout the expansion.

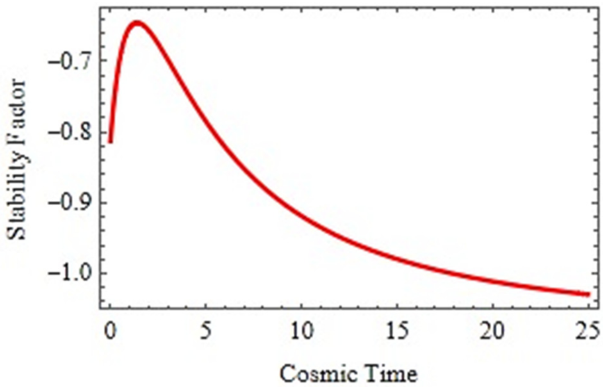

Figure 4. Graphical representation of stability factor of PDE versus cosmic time (t) by taking $\xi=1.5$, $u=2$, $c_1=0.5$, $k=0.3$, $\varepsilon=0.13$, $\varepsilon_1=1$, $\eta=2$.

STATEFINDER PARAMETERS

Many models of DE have been developed in an attempt to comprehend its nature and provide an explanation for the Universe's accelerated expansion. Sahni et al. [37] have developed the crucial parameters known as Statefinder parameters to help differentiate between these models. The Statefinder parameters are associated to the third order derivatives of average scale factor. Different values of the pair of Statefinder parameters $\{r,s\}$ exhibit different DE models. In particular

- For ΛCDM, $(r=1, s=0)$,
- For SCDM, $(r=1, s=1)$,
- For HDE, $\left(r=1, s=\frac{2}{3}\right)$,
- For CG, $(r>1, s<0)$,
- For Quintessence, $(r<1, s>0)$.

The Statefinder parameters for our model are

$$r=\frac{(k-\varepsilon)(k-2\varepsilon)}{k^2} \quad (43)$$

and

$$s=\frac{2\left[(k-\varepsilon)(k-2\varepsilon)-k\right]}{3k(2\varepsilon-3k)} \quad (44)$$

With appropriate choice of constants $k = 0.3$ and $\varepsilon = 0.13$, it is clear that $r < 1$ and $s > 0$ which means that the model evolves around Quintessence region.

ENERGY CONDITIONS

Energy conditions are nothing but a set of certain conditions which characterize matter in the universe and are used in a variety of ways to comprehend how the universe has evolved. The objective of energy conditions in this work is to substantiate the accelerated expansion of Universe. These conditions can be obtained by the widely recognized Raychaudhury equations [38], [39], [40], whose forms are:

$$\frac{d\theta}{dT} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}u^\mu u^\nu \text{ and } \frac{d\theta}{dT} = -\frac{1}{2}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}n^\mu n^\nu, \tag{45}$$

where θ is the expansion factor, n^μ is null vector and, $\sigma^{\mu\nu}$ and $\omega_{\mu\nu}$ are shear and the rotation associated with the vector field u^μ, respectively. The following energy conditions are satisfied by the attractive gravity:

- Weak energy conditions (WEC) if $\rho \geq 0, \rho + p \geq 0$,
- Null energy condition (NEC) if $p + \rho \geq 0$,
- Dominant energy conditions (DEC) if $\rho \geq 0$, $\rho - |p| \geq 0$,
- Strong energy condition (SEC) if $\rho + 3p \geq 0$.

WEC (Energy density together with NEC):

$$\rho \geq 0, \rho + p \geq 0 \iff \begin{cases} 3\xi^2 k^u (\tau)^{\frac{-\varepsilon u}{\varepsilon - 3k}} \geq 0, \\ 3\xi^2 k^u (\tau)^{\frac{-\varepsilon u}{\varepsilon - 3k}} + (-2)(\tau)^{\frac{-2\varepsilon}{\varepsilon - 3k}} \left\{ \tau_1 \tau_2 \tau_6 + 2\tau_4 \left[1 + \eta (-2)^{\eta - 1} \tau_5 \right] \right\} \geq 0 \end{cases} \tag{46}$$

DEC:

$$\rho - |p| \geq 0 \iff 3\xi^2 k^u (\tau)^{\frac{-\varepsilon u}{\varepsilon - 3k}} - \left| (-2)(\tau)^{\frac{-2\varepsilon}{\varepsilon - 3k}} \left\{ \tau_1 \tau_2 \tau_6 + 2\tau_4 \left[1 + \eta (-2)^{\eta - 1} \tau_5 \right] \right\} \right| \geq 0 \tag{47}$$

SEC:

$$\rho + 3p \geq 0 \iff 3\xi^2 k^u (\tau)^{\frac{-\varepsilon u}{\varepsilon - 3k}} + 3(-2)(\tau)^{\frac{-2\varepsilon}{\varepsilon - 3k}} \left\{ \tau_1 \tau_2 \tau_6 + 2\tau_4 \left[1 + \eta (-2)^{\eta - 1} \tau_5 \right] \right\} \geq 0 \tag{48}$$

In the analysis of energy conditions, a violation of NEC gives rise to a violation of remaining other energy conditions, which depicts the reduction in energy density with expansion of Universe; furthermore, the violation of SEC depicts the accelerated expansion of the Universe.

Evolution of the energy conditions in the obtained Universe versus cosmic time (t), by proper choice of constants, is prescribed in Figure 5, Figure 6 and Figure 7. From these figures, along with Figure 1, we can observe that WEC $(\rho \geq 0)$ together with NEC $(p + \rho \geq 0)$ and DEC $(\rho - |p| \geq 0)$ are verified whereas SEC $(\rho + 3p \geq 0)$ violates. Hence, the violation of SEC gives rise to the accelerating expansion of the Universe.

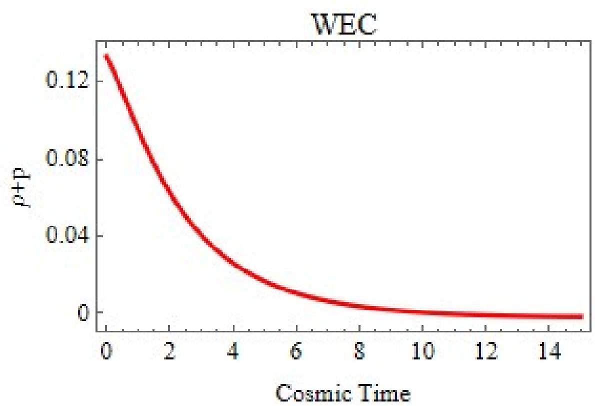

Figure 5. Graphical representation of evolution of WEC of the Universe versus cosmic time (t) by taking $\xi = 1.5$, $u = 2$, $c_1 = 0.5$, $k = 0.3$, $\varepsilon = 0.13$, $\varepsilon_1 = 1$, $\eta = 2$.

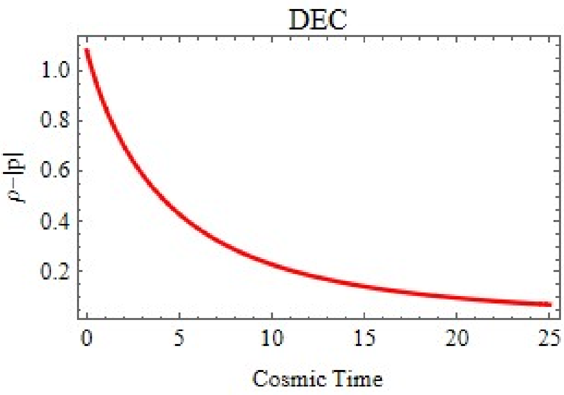

Figure 6. Graphical representation of evolution of DEC of the Universe versus cosmic time (t) by taking $\xi = 1.5$, $u = 2$, $c_1 = 0.5$, $k = 0.3$, $\varepsilon = 0.13$, $\varepsilon_1 = 1$, $\eta = 2$.

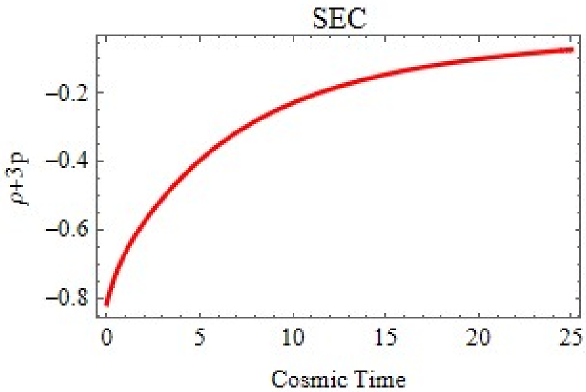

Figure 7. Graphical representation of evolution of SEC of the Universe versus cosmic time (t) by taking $\xi = 1.5$, $u = 2$, $c_1 = 0.5$, $k = 0.3$, $\varepsilon = 0.13$, $\varepsilon_1 = 1$, $\eta = 2$.

CONCLUSIONS

In present study, we considered a homogeneous and anisotropic Bianchi type-I universe with interacting DM and PDE in $f(T)$ gravity.

- It is observed that energy density of PDE is always positive and as time increases, energy density decreases [41-42].
- The equation of state parameter represents Quintessence region then it goes through a ΛCDM model and as time increases, it converts to Phantom model.
- The deceleration parameter is found to be constant throughout the expansion of the Universe.
- The stability factor for the present model is negative throughout the expansion of the Universe and hence the model is unstable [4].

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МОДЕЛЬ ГРАВІТАЦІЇ БІАНЧІ ТИПУ-I f(T) З ТЕМНОЮ ЕНЕРГІЄЮ PILGRIM

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У цій роботі ми проаналізували простір-час Б'янкі типу I (просторово однорідний та анізотропний), використовуючи дві взаємодіючі рідини – темну матерію (DM) і темну енергію Pilgrim (PDE) в рамках *f(T)* гравітації, враховуючи інфрачервоне (ІЧ) відсікання як кандидата на горизонт Хаббла $\left(L = 1/H\right)$. Ми також провели діагностику за допомогою шукача стану, а також обговорили енергетичні умови для перевірки прискореного розширення Всесвіту.

Ключові слова: *темна енергія Pilgrim; темна матерія; гравітація f(T); простір-час Біанчі типу I; космологія*