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KALUZA-KLEIN FRW RENYI HOLOGRAPHIC DARK ENERGY MODEL IN SCALAR-TENSOR THEORY OF GRAVITATION

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This work examines the dark energy phenomenon by studying the Renyi Holographic Dark Energy (RHDE) and pressure-less Dark Matter (DM) within the frame-work of Saez-Ballester (SB) scalar-tensor theory of gravitation(Phys. Lett. A113, 467:1986). To achieve a solution, we consider the viable deceleration parameter (DP), which contributes to the average scale factor $a = e^{\frac{1}{\gamma}\sqrt{2\gamma t+c_1}}$, where γ, and c_1 are respectively arbitrary, and integration constants. We have derived the field equations of SB scalar-tensor theory of gravity with the help of Kaluza-Klein FRW Universe. We have investigated cosmological parameters namely, DP (q), energy densities (ρ_M) and (ρ_R) of DM and RHDE, scalar field (ϕ), and equation of state parameter (ω_R). The physical debate of these cosmological parameters are investigated through graphical presentation. Moreover, the stability of the model are studied through squared sound speed (v_s^2) and the well-known cosmological plane $\omega_R - \omega_R^{'}$ and all energy conditions and also, density parameters are analyzed through graphical representation for our model.

Keywords: *Kaluza-Klein FRW Universe; RHDE; Energy Conditions; Saez-Ballester theory*

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1. INTRODUCTION

The cosmological observations like, type Ia Supernovae [1, 2] have provided the convincing evidence that our Universe is dominated by two dark sectors containing dark matter and dark energy. The behavior of Dark Matter (DM) and Dark Energy (DE) is one of the most important issues today in modern cosmology. The present Planck data says that there is 68.3% DE of the total energy contents of the Universe. The approaches to answer this DE problems and cosmic acceleration issues fall into two categories: (i) To introduce dynamical DE models in the RHS of Einstein's field equations in the background of general relativity, (ii) Modify the LHS of Einstein's field equations, it leads to modified theories of gravity. We refer [3, 4, 5] for the detail investigation of DE models and modified theories of gravity.

The holographic DE model (HDE) suggests, this model is originated from holographic principle and its energy density can be expressed by $\rho_{de} = \frac{3C^2M_p^2}{L^2}$, here C^2 is a numerical constant, M_p^2 is the reduced Planck mass and L denotes the size of the current Universe such as the Hubble scale [6, 7]. In addition, the holographic DE has some problems and cannot explain the time line of a flat FRW Universe [8, 9]. One of the proposed solutions for the HDE problems is the consideration of various entropies. One of the considered entropy is Tsallis entropy which has been used in many papers [10, 11, 12, 13]. In recent years, various entropy formalism have been used to discuss the gravitational and cosmological setups. Also, some new holographic DE models are constructed such as Tsallis HDE [14]. In literature [15, 16, 17] ADE models are available which deal with various aspects of the evolution of the Universe. Wei and Cai [18] have proposed a new version of this model referred as New Agegraphic Dark Energy (NADE) model, by replacing the cosmic age T with the cosmic conformal age η for the time scale. Hence, in this model, the dark energy density is of the form $\rho_a = 3n^2m_p^2\eta^{-2}$, where the conformal time η is defined as $\eta = \int_0^t \frac{1}{a}dt = \int_0^a \frac{1}{a^2H}da$ here a is the average scale factor of the Universe and Hubble parameter $H = \frac{\dot{a}}{a}$, and overdot (.) represents derivative with respect to the cosmic time (t) only. Sheykhi and Setare [19] have explore NADE model with viable gravitational constant G in a non-flat Universe. And also,they have generalized to viscous the NADE model in the presence of interacting term between dark sectors.

A new DE model based on the holographic hypothesis, inspired by a Q generalized entropy, suggested by Renyi [20], called RHDE [21], has proposed with IR cutoff as the Hubble radius. Maradpour et al. [22] have investigated thermodynamic approach to HDE and the Renyi entropy. Some other researchers [23, 24, 25] have analyzed based on Renyi entropy to investigate various cosmological phenomena. Dixit [26] have investigated RHDE models in FRW Universe with two IR cutoffs with redshift parametrization. Chunlen and Rangdeey [27] have discussed exploring the RHDE model with the future and the particle Horizons as IR cutoff. Sharma and Dubey [28] have discussed cosmological behavior of interacting RHDE models. Sarfraz et al. [29] have analyzed the Study of RHDE model in framework of Chern Simons Modified Gravity.

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Sharma and Dubey [30] have discussed statefinder diagnostic for the RHDE. Saha et al. [31] have investigated RHDE in higher dimension cosmology. Sharma and Dubey [32] have studied RHDE model in the framework of Brans-Dicke (BD) scalar tensor theory of gravity. Divya and Aditya [33] have investigated anisotropic RHDE models in background of general relativity. Recently, Bhattacharjee [34] has investigated interacting Tsallis and Renyi HDE with hybrid expansion law. Santhi and Sobhanbabu [35] have studied Bianchi type-III Tsallis holographic dark energy model in Saez–Ballester theory of gravitation. Sobhanbabu and Santhi [36] have investigated Kantowski–Sachs Tsallis holographic dark energy model with sign-changeable interaction. Divya and Aditya [37] have studied observational constraints on RHDE model in anisotropic Kantowski Sachs Universe. recently, Sobhanbabu and Santhi [38] have studied Bianchi type-III RHDE models a in scalar tensor theory. Very recently, Sobhanbabu et al. [39] have analyzed Kantowski–Sachs Barrow holographic dark energy model in the frame-work of SB theory of gravitation.

In this paper, inspired by the above investigations, we have considered the KK FRW Universe for the RHDE model with the frame-work of scalar-tensor theory of gravity. This model also, provides the DE model for clear and easy cosmological evolution. The paper is organized as follows: In the next Section, we present the field equations and obtained their solution of RHDE model in the frame-work of SB theory. In Section. 3, we study the solution of the field equations and cosmological parameters are investigated to RHDE model. In the last Section, we have presented some conclusions.

2. METRIC AND FIELD EQUATIONS OF RHDE IN SB THEORY

We consider the non-Ricci, non-compact five-dimensional FRW type KK Universe in the form

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + sin^2\theta d\phi^2\right) + \left(1-kr^2\right)d\xi^2\right], \tag{1}$$

where $R(t)$ is the five-dimensional scale factor of the model and $k = 0,\ 1,\ -1$ represents the curvature parameter for flat, closed and open Universe. We define the following physical parameters for KK FRW Universe: Volume $V = R^3$, Hubble parameter $H = \frac{\dot{R}}{R}$, scalar expansion $\theta = 3\frac{\dot{R}}{R}$, and DP $q = -\frac{R\ddot{R}}{\dot{R}^2}$, here R is average scale factor.

The SB field equations for matter and RHDE distribution are given by [40]

$$G_{\mu\nu} - w\phi^n\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}\right) = -(T_{\mu\nu} + \bar{T}_{\mu\nu}), \tag{2}$$

and the scalar field ϕ satisfies the following equation

$$2\phi^n\phi^{;\mu}_{,\mu} + n\phi^{n-1}\phi_{,\lambda}\phi^{,\lambda} = 0, \tag{3}$$

where $G_{\mu\nu}$ represents the Einstein tensor and $T_{\mu\nu}$ & $\bar{T}_{\mu\nu}$ are energy momentum tensors for pressure-less dark matter and RHDE respectively. For physical interpretation, the energy momentum tensors for matter and RHDE can be written as

$$T_{\mu\nu} = diag[1,0,0,0]\rho_M, \tag{4}$$

and

$$\bar{T}_{\mu\nu} = diag[1,-\omega_R,-\omega_R,-\omega_R]\rho_R, \tag{5}$$

where ρ_R, ρ_M are energy densities of RHDE and matter and p_R is the pressure of RHDE. $\omega_R = \frac{p_R}{\rho_R}$ is an equation of state (EoS) parameter. So, the field equations for the discussed metric can be written as SB field Eq.(2), for KK FRW Universe Eq.(1) with the help of Eq.(4), and (5) can be written as

$$6\frac{\dot{R}^2}{R^2} + 6\frac{k}{R^2} + \frac{w\phi^n\dot{\phi}^2}{2} = \rho_M + \rho_R, \tag{6}$$

$$3\left(\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{a^2}\right) - \frac{w\phi^n\dot{\phi}^2}{2} = -\omega_R\rho_R, \tag{7}$$

$$\ddot{\phi} + 4\frac{\dot{R}\dot{\phi}}{R\phi} + \frac{n\dot{\phi}^2}{2\phi^2} = 0, \tag{8}$$

From Eq.(6), we get the continuity equation is

$$\dot{\rho_R} + \dot{\rho_M} + 4\left((1+\omega_R)\rho_R + \rho_M\right)\frac{\dot{R}}{R} = 0, \tag{9}$$

where the overhead dot indicates differentiation with respect to time t.

3. SOLUTION AND THE MODEL

The field Eqs.(6)-(8) form a system of three highly non-linear equations with R, ϕ, ρ_M, ρ_R, and ω_R five (5) unknowns. So, we need three more physical conditions to get consistency solution. For this reason we take the following conditions: The DP is taking as linear function of the average scale factor as [41, 42]

$$q = -\frac{R\ddot{R}}{\dot{R}^2} = k_2 + \gamma\frac{\dot{R}}{R}, \tag{10}$$

where k_2 and γ is an arbitrary constants. For $k_2 = -1$, we get the solution of Eq.(10),

$$R = e^{\frac{1}{\gamma}\sqrt{2\gamma t + c_1}}, \tag{11}$$

where c_1 is an integrating constant. Hence, KK FRW Universe Eq.(1), can be written as

$$ds^2 = dt^2 - e^{\frac{2}{\gamma}\sqrt{2\gamma t + c_1}}\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + sin^2\theta d\phi^2\right) + \left(1-kr^2\right)d\xi^2\right] \tag{12}$$

The Volume V, Hubble parameter H for our model found to be

$$V = e^{\frac{3}{\gamma}\sqrt{2\gamma t + c_1}}, \tag{13}$$

$$H = \left(2\gamma t + c_1\right)^{-\frac{1}{2}}, \tag{14}$$

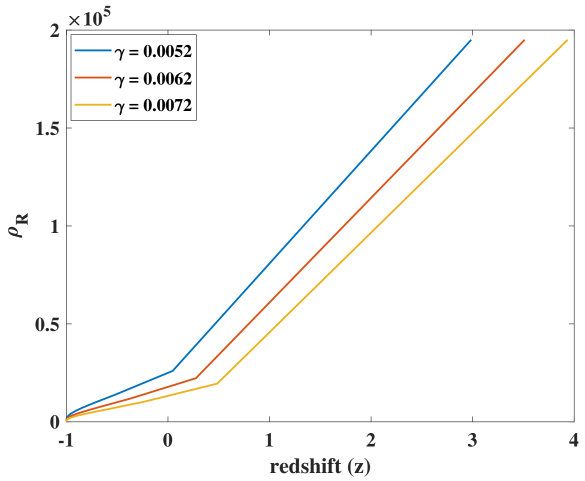

Figure 1. The behavior of energy density (ρ_R) of RHDE versus redshift (z) for $\gamma = 0.0052$, $\gamma = 0.0062$, $\gamma = 0.0072$ and $c_1 = 0.000016$.

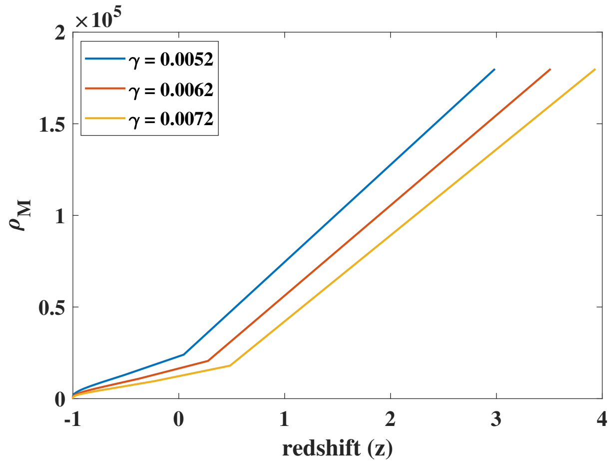

Figure 2. The behavior of energy density (ρ_R) of BHDE versus redshift (z) for $\gamma = 0.0052$, $\gamma = 0.0062$, $\gamma = 0.0072$ and $c_1 = 0.000016$.

In Figures 1 & 2, corresponding equations (15) & (16), we show the variation of the energy density (ρ_R) of RHDE & matter (ρ_M) with the Hubble's horizon cut-off with respect to the redshift (z) for the approprivate values of the model parameter respectively. It is observed that the both ρ_R & ρ_M are positive throughout evolution of the Universe and decreasing function of redshift and finally it reached to zero.
We consider Hubble horizon as a candidate for IR cutoff [43] i.e., $L = H^{-1}$ and $8\pi = 1$, we obtain energy density of RHDE as

$$\rho_R = \frac{3d^2H^2}{1+\pi\nu H^{-2}} = \frac{3d^2(2\gamma t + c_1)^{-1}}{1+\pi\delta(2\gamma t + c_1)} \tag{15}$$

From equations (6) & (15), we have the energy density of DM is

$$\rho_M = \frac{6}{2\gamma t + c_1} + 6ke^{\frac{-2\sqrt{2\gamma t + c_1}}{\gamma}} + \frac{w}{2}e^{\frac{-8}{\gamma}\sqrt{2\gamma t + c_1}} - \frac{3d^2}{(2\gamma t + c_1) + \pi\delta(2\gamma t + c_1)} \tag{16}$$

From equations (7), (11), (14) & (15), we have EoS parameter is

$$\omega_R = \left[\frac{w}{2H^2}e^{\frac{-8}{\gamma H}} - 3(1-\gamma H)H^{\frac{-3}{2}} - 3kH^{-2}e^{-\frac{2}{\gamma H}}\right]\left[\frac{1+\pi\delta H^{-2}}{3d^2}\right], \tag{17}$$

$H = (2\gamma t + c_1)^{\frac{-1}{2}}$, and $\dot{H} = -\frac{\gamma}{\sqrt{(2\gamma t+c_1)^3}}$

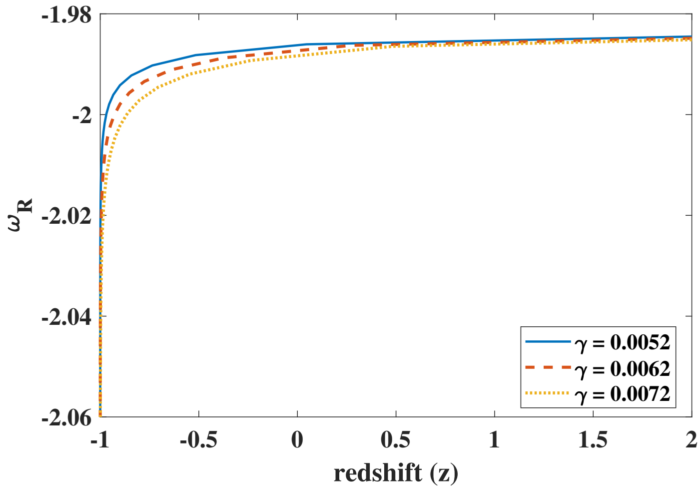

Figure 3. The behavior of equation of state parameter (ω_R) of BHDE versus redshift (z) for $\gamma = 0.0052$, $\gamma = 0.0062$, $\gamma = 0.0072$ and $c_1 = 0.000016$.

Variation of equation of state parameter (ω_R) against redshift (z) in Figure 3,, corresponding to Eq. (17) for the values of $\gamma = 0.0052, 0.0062, 0.0072$. It can be observed that the ω_R completely varies in aggressive phantom region ($\omega_R < -1$) only. If the value of γ increases the phantom region increases.

ω_R - $\omega_R^{'}$ plane:

In this section, $\omega_R^{'}$ is found

$$\left.\begin{aligned}\omega_R^{'} =& \frac{2\pi\delta\dot{H}}{3d^2}\left[3kH^{-2}e^{-\frac{-2}{\gamma H}} + 3(1-\gamma H)H^{-\frac{3}{2}} - \frac{w}{2}H^{-2}e^{-\frac{8}{\gamma H}}\right] + \left(\frac{1+\pi\delta H^{-2}}{3d^2H}\right)\\ &\left[w(\frac{4}{\gamma} - H)e^{-\frac{8}{\gamma H}}\frac{\dot{H}}{H^4} + \frac{9}{2}(1-\gamma H)H^{-\frac{5}{2}}\dot{H} + 3\gamma H^{-\frac{3}{2}}\dot{H} - \frac{3k\dot{H}}{2\gamma H^4}(1-\gamma H^5)e^{-\frac{2}{\gamma H}}\right]\end{aligned}\right\} \tag{18}$$

The evolution of the ω_R-$\omega_R^{'}$ plane is shown in Figure 4 for different values of γ. We observe that the ω_R-$\omega_R^{'}$ plane for our model is in the thawing region throughout evolution of the Universe ($\omega_R < 0$, and $\omega_R^{'} > 0$).

Stability Analysis

We consider an important parameter to verify the stability of the RHDE model. If squared speed sound v_s^2 is ($v_s^2 > 0$) positive then the model is stable whereas v_s^2 is ($v_s^2 < 0$) negative the model is unstable. For our RHDE model v_s^2 is given by

$$\left.\begin{aligned}v_s^2 =& \left[\frac{w}{2H^2}e^{\frac{-8}{\gamma H}} - 3(1-\gamma H)H^{\frac{-3}{2}} - 3kH^{-2}e^{-\frac{2}{\gamma H}}\right]\left[\frac{1+\pi\delta H^{-2}}{3d^2}\right] + \frac{1+\pi\delta H^{-2}}{2\dot{H}\left(1+\pi\delta H^{-2}+\pi\delta H^2\right)}\\ &\left[\frac{2\pi\delta\dot{H}}{3d^2}\left[3kH^{-2}e^{-\frac{-2}{\gamma H}} + 3(1-\gamma H)H^{-\frac{3}{2}} - \frac{w}{2}H^{-2}e^{-\frac{8}{\gamma H}}\right] + \left(\frac{1+\pi\delta H^{-2}}{3d^2H}\right)\left[w(\frac{4}{\gamma} - H)e^{-\frac{8}{\gamma H}}\frac{\dot{H}}{H^4}\right.\right.\\ &\left.\left. + \frac{9}{2}(1-\gamma H)H^{-\frac{5}{2}}\dot{H} + 3\gamma H^{-\frac{3}{2}}\dot{H} - \frac{3k\dot{H}}{2\gamma H^4}(1-\gamma H^5)e^{-\frac{2}{\gamma H}}\right]\right]\end{aligned}\right\} \tag{19}$$

Figure 5 shows that the sound speed c_s^2 is decreasing function of redshift (z) and it is negative ($c_s^2 < 0$) throughout history of the Universe for the varios values of γ, and $c_1 = 0.000016$ which describes our RHDE model is unstable.

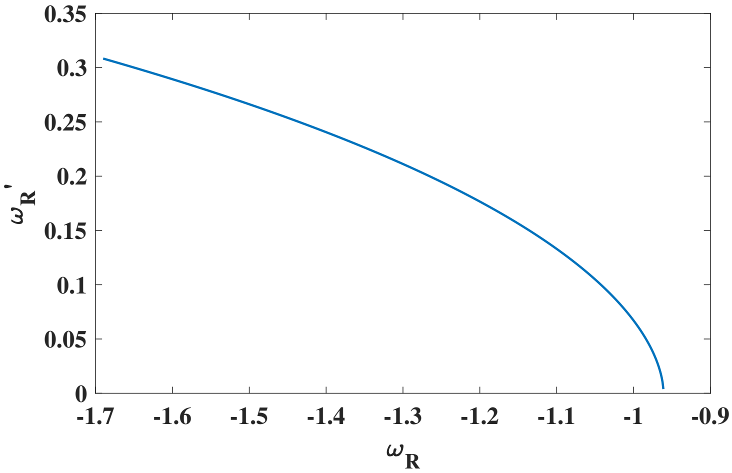

Figure 4. The behavior of ω_R versus ω'_R for $\gamma = 0.0052$ and $c_1 = 0.000016$.

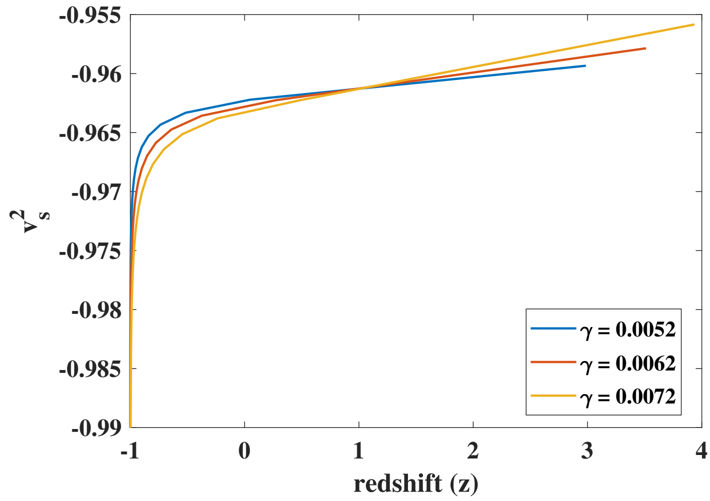

Figure 5. The behavior of squared speed of sound v_s^2 (km/h) versus redshift (z) for $\gamma = 0.0052$, $\gamma = 0.0062$, $\gamma = 0.0072$ and $c_1 = 0.000016$.

Density Parameters

Now we define dimensionless density parameters of dark energy as

$$\Omega_M = \frac{\rho_M}{3H^2}, \quad and \;\; \Omega_R = \frac{\rho_R}{3H^2} \tag{20}$$

By substituting the expressions of ρ_M and ρ_R and Hubble parameter H in the above equations, we get the density parameter of dark matter (ρ_M) and dark energy (ρ_R) for our RHDE model and analyzed its behavior through graphical representation for the various values of γ. Figure 6 & 7 shows the behavior of Ω_M and Ω_R versus redshift (z). The density parameter of DM observed that it increases as the universe evolves. The density parameter of RHDE observed that it decreses as the universe evolves. Also, we have observed that the RHDE density parameter Ω_R meets the [44] values which exhibits consistent results with the recent observations for different values of γ.

Energy Conditions

The study of the energy conditions came into existence from the Raychaudhuri equations which play significant role in any discussion of the congruence of null and time like geodesics. The bounce behavior of cosmological model is also realized using these conditions (energy conditions) as mentioned in references [45, 46, 47, 48, 49, 50, 51].
Null Energy Condition (NEC): $\rho_R(1 + \omega_R) \geq 0$,
Weak Energy Condition (WEC): $\rho_R \geq 0$, $\rho_R(1 + \omega_R)$,

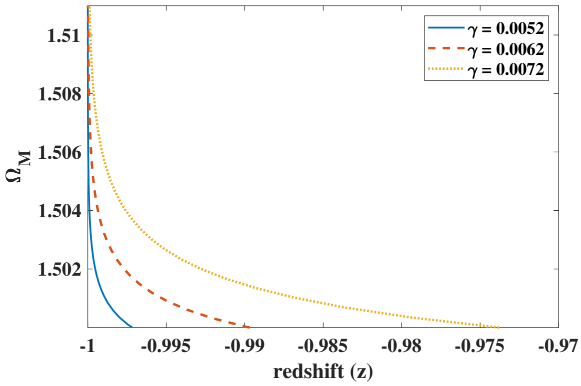

Figure 6. The behavior of matter density parameter (Ω_M) versus redshift (z) for $\gamma = 0.0052$, $\gamma = 0.0062$, $\gamma = 0.0072$ and $c_1 = 0.000016$.

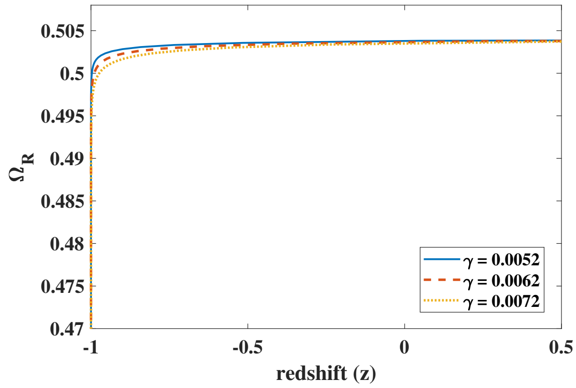

Figure 7. The behavior of density parameter (Ω_R) of RHDE versus redshift (z) for $\gamma = 0.0052$, $\gamma = 0.0062$, $\gamma = 0.0072$ and $c_1 = 0.000016$.

Strong Energy Condition (SEC): $\rho_R(1+\omega_R) \geq 0$, $\rho_R(1 \pm 3p) \geq 0$,
Dominant Energy Conditions (DEC): $\rho_R \geq 0$, $\rho(1 \pm \omega_R) \geq 0$.

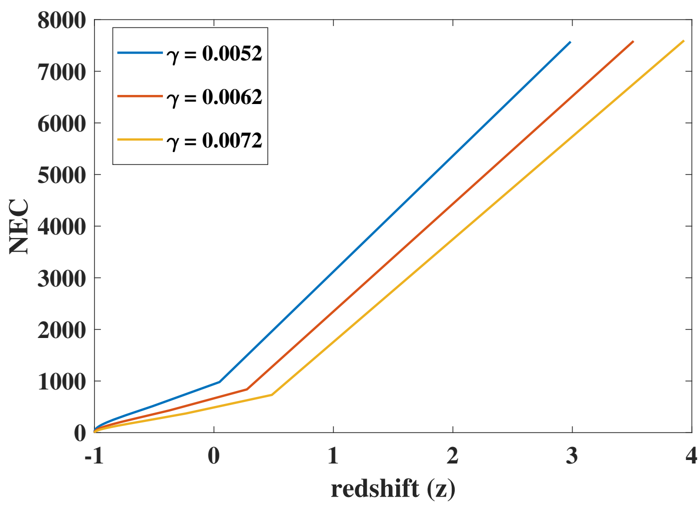

Figure 8. The behavior of NEC of RHDE versus redshift (z) for $\gamma = 0.0052$, $\gamma = 0.0062$, $\gamma = 0.0072$ and $c_1 = 0.000016$.

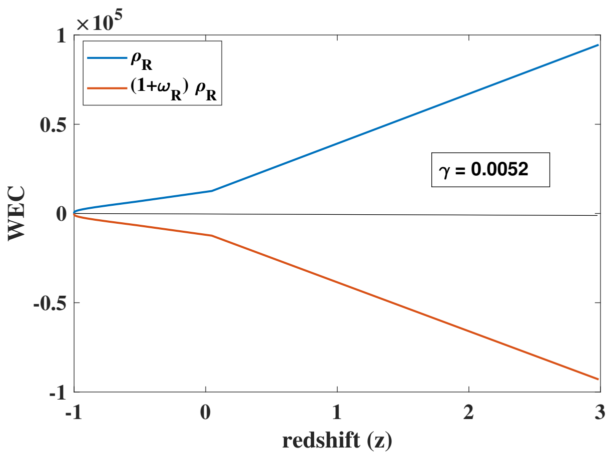

Figure 9. The behavior of WEC of RHDE versus redshift (z) for $\gamma = 0.0052$, and $c_1 = 0.000016$.

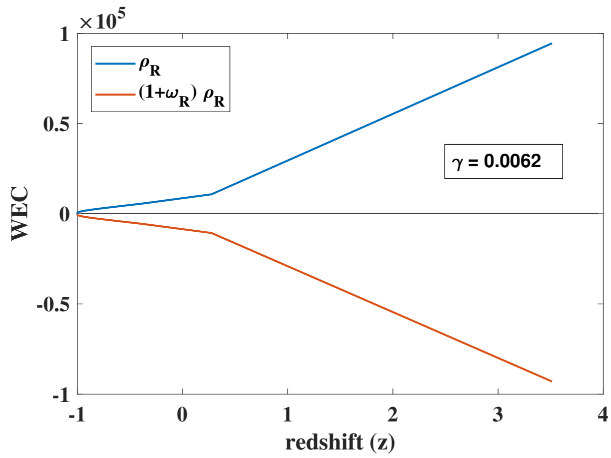

Figure 10. The behavior of WEC of RHDE versus redshift (z) for $\gamma = 0.0062$, and $c_1 = 0.000016$.

From Figure 8, we observe that the NEC ($(1 + \omega_R)\rho_R \geq 0$) are satisfied throughout evolution of the Universe for different values of $\gamma = 0.0052, 0.0062, 0.0072$.

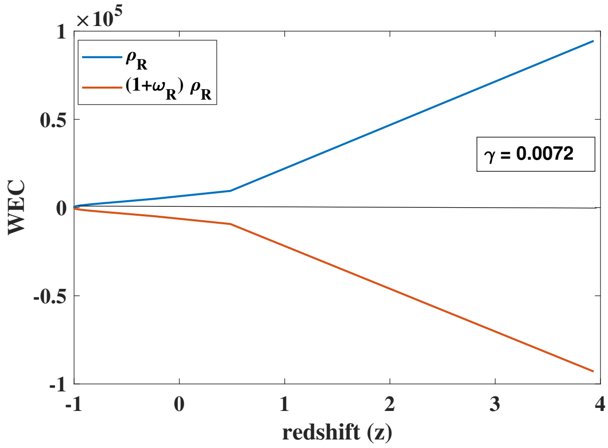

Figure 11. The behavior of WEC of RHDE versus redshift (z) for $\gamma = 0.0072$, and $c_1 = 0.000016$.

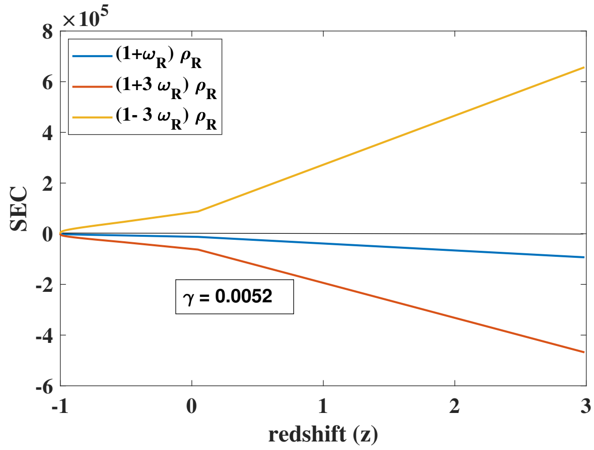

Figure 12. The behavior of SEC of RHDE versus redshift (z) for $\gamma = 0.0052$, and $c_1 = 0.000016$.

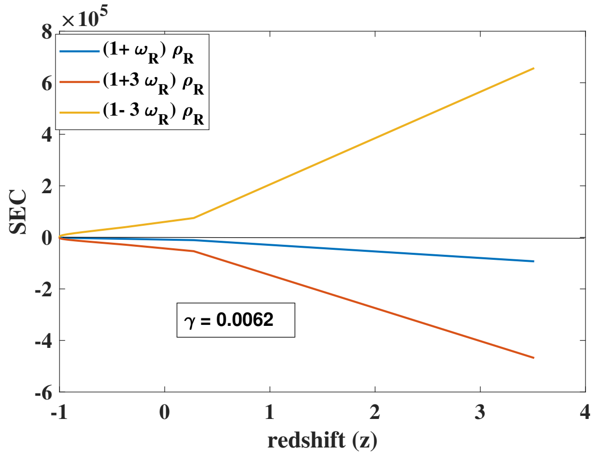

Figure 13. The behavior of SEC of RHDE versus redshift (z) for $\gamma = 0.0062$, and $c_1 = 0.000016$.

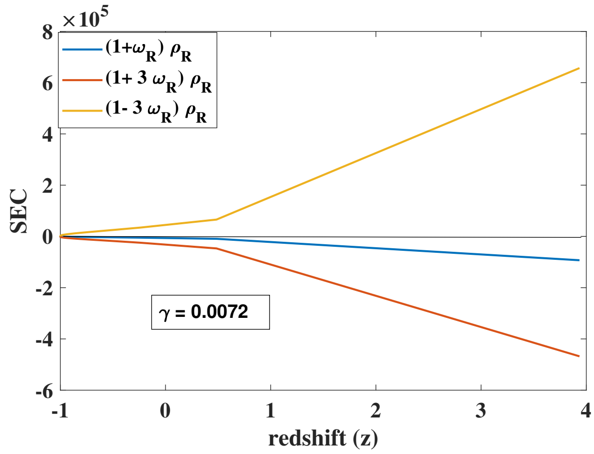

Figure 14. The behavior of SEC of RHDE versus redshift (z) for $\gamma = 0.0072$, and $c_1 = 0.000016$.

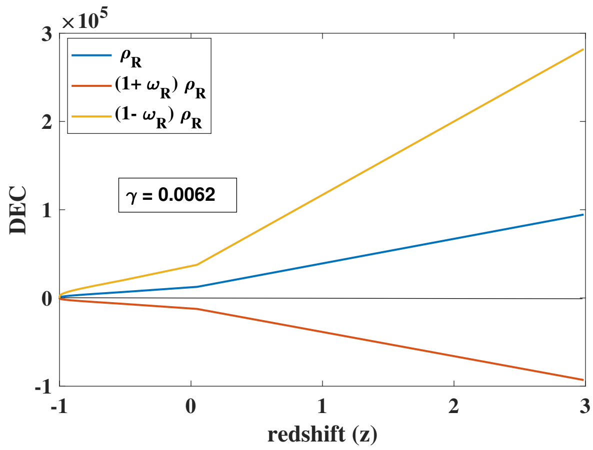

Figure 15. The behavior of DEC of RHDE versus redshift (z) for $\gamma = 0.0052$, and $c_1 = 0.000016$.

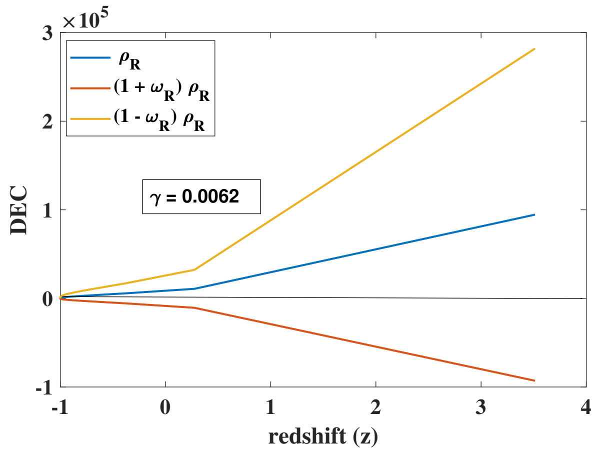

Figure 16. The behavior of DEC of RHDE versus redshift (z) for $\gamma = 0.0062$, and $c_1 = 0.000016$.

Figures 9, 10, 11, 12, 13,14, 15, 16, 17 describes variation of WEC, DEC and SEC versus redshift (z) for the various values of γ. We observe that energy conditions WEC ($\rho_R \geq 0$, $\rho_R(1+\omega_R)$), DEC ($\rho_R(1+\omega_R) \geq 0$, $\rho_R(1 \pm 3p) \geq 0$) and SEC ($\rho_R \geq 0$, $\rho(1 \pm \omega_R) \geq 0$) are not satisfied for the various values of $\gamma = 0.0052, 0.0062, 0.0072$. The violation of the SEC condition represents the accelerated expansion of the Universe.

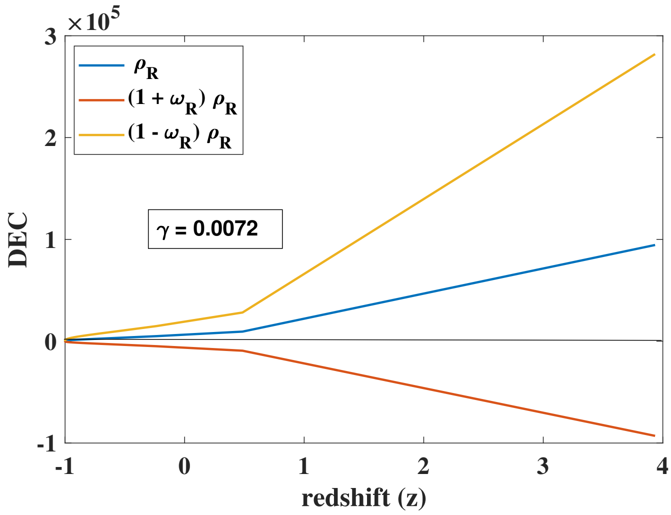

Figure 17. The behavior of DEC of RHDE versus redshift (z) for the values $\gamma = 0.0072$ and $c_1 = 0.000016$.

Deceleration Parameter

The signature of deceleration parameter (q) shows whether the model either accelerates or decelerates. If $q > 0$, the model exhibits decelerating expansion, the Universe exhibits accelerating expansion, for $q < 0$. The DP for our models, is given by

$$q = \frac{a\ddot{a}}{\dot{a}^2} = -1 + \gamma\left(2\gamma t + c_1\right)^{-\frac{1}{2}} \tag{21}$$

The evolution of the deceleration parameter (q) with redshift z is shown in Fig. 6. The DP q is observe that there is a

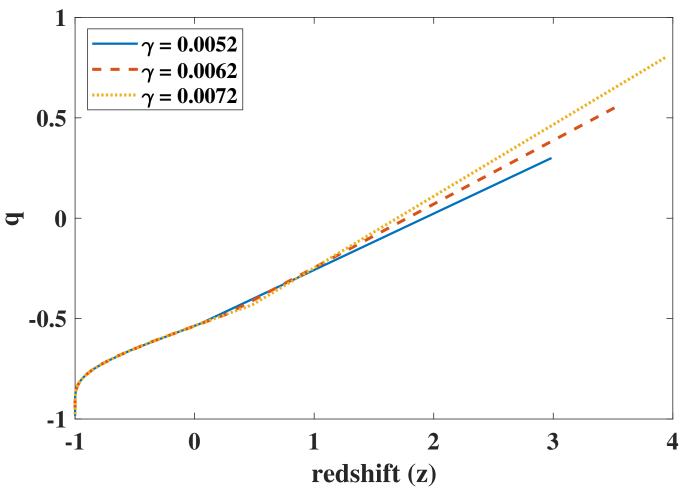

Figure 18. The behavior of deceleration parameter (q) versus redshift (z) for$\gamma = 0.0052$, $\gamma = 0.0062$, $\gamma = 0.0072$ and $c_1 = 0.000016$.

sign change in the trajectory of q from positive $(q > 0)$ to negative value $(q < 0)$. It represents that the Universe smooth transition from early decelerating region $(q > 0)$ to accelerating region $(q < 0)$ at late epochs. The present value of DP is consistent with the recent observational data [58].

Statefinder Parameters

The DE models have been proposed for explaining the accelerated expansion phenomenon of the universe. In order to check the viability of these models, statefinder parameters are widely used [52, 53, 54, 55, 56, 57]

$$r = \frac{\ddot{a}}{aH^3}, \quad s = \frac{r-1}{3(q-\frac{1}{2})} \tag{22}$$

The importment model described by the parameters $(r = 1, \ s = 1)$ shows CDM model, $(r = 1, s = 0)$ represents ΛCDM model, while $r < 1, \ s > 0$ indicate quintessence and phantom DE models.

In the present work, the trajectory statefinder $r - s$ plane for different values of γ is shown in Figure 20. It can be observed that the trajectory $r - s$ plane approaches to ΛCDM $(r = 1, \ s = 0)$ model for the value of $\gamma = 0.0052$. We also, observed that the trajectory of $r - s$ plane is initially lies in quintessence region $r < 1$, crosses the with quintessence and phantom regions and finally reached to ΛCDM in late times for tha values of $\gamma = 0.0062$ and 0.0072. We have also

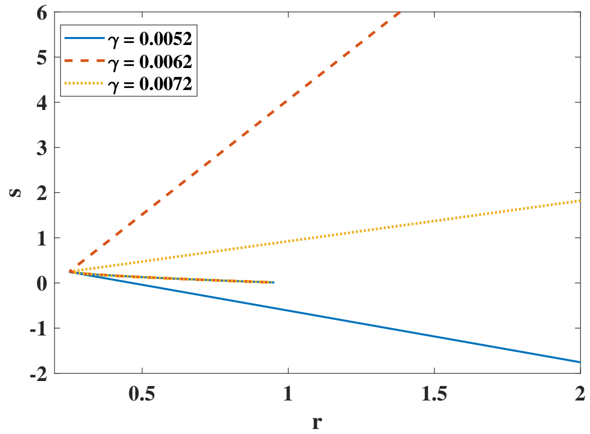

Figure 19. The behavior of statefinder $r - q$ plane for $\gamma = 0.0052$, $\gamma = 0.0062$, $\gamma = 0.0072$ and $c_1 = 0.000016$.

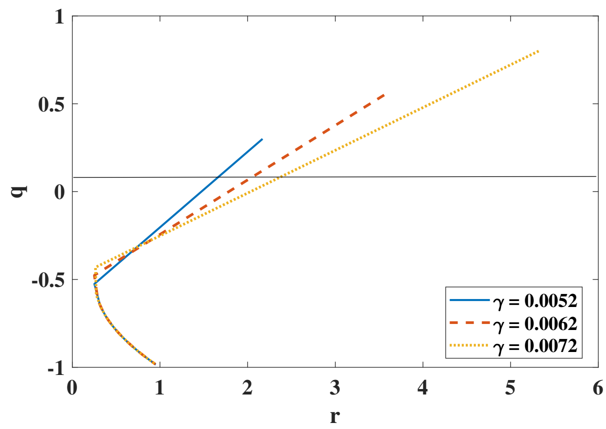

Figure 20. The behavior of Statefinder $r - q$ plane for $\gamma = 0.0052$, $\gamma = 0.0062$, $\gamma = 0.0072$ and $c_1 = 0.000016$.

shown the evolutionary trajectories of another statefinder $r - q$ plane for the RHDE model in Figure 20 for the best-fit values of γ. The evolutionary curve of the $r - q$ plane of RHDE model starts from the SCDM in the past and reaches above the de-Sitter expansion in the future, and it also shows the Chaplygin gas behavior throughout the evaluation. Since q changes its sign from positive to negative, it also represnts the recent phase transition of the Universe.

4. CONCLUSIONS

In this work, we have studied the Kaluza-Klein FRW RHDE model in SB scalar–tensor theory of gravitation. To obtain the deterministic model of the Universe we consider some physical plausible conditions, these conditions leads to a varying DP, which represents transition from the past decelerating Universe to the current accelerating Universe. The main conclusions of these two models are summarized as follows:

- The variation of the energy density (ρ_R) of RHDE & matter (ρ_M) with the Hubble's horizon cut-off observed that the both ρ_R & ρ_M are positive throughout evolution of the Universe and decreasing function of redshift and finally it reached to zero.

- Behavior of equation of state parameter (ω_R) completely varies in aggressive phantom region $(\omega_R < -1)$ only for the different values of $\gamma = 0.0052, 0.0062, 0.0072$. If the value of γ increases the phantom region increases.

- The evolution of the ω_R-ω_R' plane for our model is in the thawing region throughout evolution of the Universe.

- The sound speed c_s^2 is decreasing function of redshift (z) and it is negative $(c_s^2 < 0)$ throughout history of the Universe for the varios values of γ, and $c_1 = 0.000016$ which describes our RHDE model is unstable.

- The density parameter of DM observed that it increases as the universe evolves. The density parameter of RHDE observed that it decreses as the universe evolves. Also, we have observed that the RHDE density parameter Ω_R meets the [44] values which exhibits consistent results with the recent observations for different values of γ.

- We observe that energy conditions WEC, DEC and SEC are not satisfied for the various values of $\gamma = 0.0052$, 0.0062, 0.0072. The violation of the SEC condition represents the accelerated expansion of the Universe.

- The evolution of the deceleration parameter (q) is observe that there is a sign change in the trajectory of q from positive $(q > 0)$ to negative value $(q < 0)$. It represents that the Universe smooth transition from early decelerating region $(q > 0)$ to accelerating region $(q < 0)$ at late epochs. The present value of DP is consistent with the recent observational data [58].

- The trajectory $r-s$ plane approaches to ΛCDM $(r = 1,\ s = 0)$ model for the value of $\gamma = 0.0052$. We also, observed that the trajectory of $r-s$ plane is initially lies in quintessence region $r < 1$, crosses the with DE (quintessence and phantom) regions and finally reached to ΛCDM in late times for tha values of $\gamma = 0.0062$ and 0.0072. The evolutionary curve of the $r-q$ plane of RHDE model starts from the SCDM in the past and reaches above the de-Sitter expansion in the future, and it also shows the Chaplygin gas behavior throughout the evaluation. Since q changes its sign from positive to negative, it also represnts the recent phase transition of the Universe.

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ГОЛОГРАФІЧНА МОДЕЛЬ ТЕМНОЇ ЕНЕРГІЇ КАЛУЗА-КЛЕЙНА FRW РЕНЬЇ В СКАЛЯРНО-ТЕНЗОРНІЙ ТЕОРІЇ ГРАВІТАЦІЇ

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У цій роботі розглядається явище темної енергії шляхом вивчення голографічної темної енергії Реньї (RHDE) і темної матерії без тиску (DM) у рамках скалярно-тензорної теорії гравітації Саеза-Баллестера (SB) (Phys. Lett. A113). , 467:1986). Щоб знайти рішення, ми розглядаємо життєздатний параметр уповільнення (DP), який вносить внесок у середній масштабний коефіцієнт $a = e^{\frac{1}{\gamma}\sqrt{2\gamma t+c_1}}$, де γ і c_1 відповідно довільні та константи інтегрування. Ми вивели польові рівняння скалярно-тензорної теорії гравітації SB за допомогою Всесвіту Калуци-Клейна FRW. Ми досліджували космологічні параметри, а саме DP (q), густину енергії (ρ_M) і (ρ_R) DM і RHDE, скалярне поле (ϕ) і рівняння параметра стану (ω_R). Фізичні дебати цих космологічних параметрів досліджуються за допомогою графічного представлення. Крім того, стабільність моделі досліджується через квадрат швидкості звуку (v_s^2) і добре відому космологічну площину $\omega_R - \omega_R'$ і всі енергетичні умови а також параметри щільності аналізуються за допомогою графічного представлення нашої моделі.

Ключові слова: *Калуза-Клейн FRW Всесвіт; RHDE; енергетичні умови; теорія Саеза-Баллестера*