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TWO-DIMENSIONAL HYDRODYNAMICS AS A CLASS OF SPECIAL HAMILTONIAN SYSTEMS

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The paper defines a class of Hamiltonian systems whose phase flows are exact solutions of the two-dimensional hydrodynamics of an incompressible fluid. The properties of this class are considered. An example of a Lagrangian one-dimensional system is given, which after the transition to the Hamiltonian formalism leads to an unsteady flow, that is, to an exact solution of two-dimensional hydrodynamics. The connection between these formalisms is discussed and the Lagrangians that give rise to Lagrangian hydrodynamics are introduced. The obtained results make it possible to obtain accurate solutions, such as phase flows of special Hamiltonian systems.

Keywords: *Hamiltonian; Lagrangian; Exact solutions; Two-dimensional hydrodynamics; Phase flow*

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1. INTRODUCTION

The interest in the formulation of hydrodynamic systems of equations in Hamiltonian form arose quite a long time ago (see, for example, the review [1, 2]). Already G. Lamb has realized in his paper that the Klebsch variables are canonically conjugate variables. This makes it possible to give incompressible fluid equations a Hamiltonian form with a Hamiltonian, the role of which is played by the total kinetic energy of the incompressible fluid. From a general point of view such a problem reduces to generalization of classical mechanics to field systems with an infinite number of freedom degrees. One of the difficulties was the choice of canonically conjugate variables, which did not always take a physically transparent content. Today this problem has been overcome and has lost its relevance. The usefulness of the Hamiltonian approach is especially noticeable when using the Hamiltonian formalism, for example, to develop a general theory of waves in nonlinear media. The foundations of such an approach were laid in [3, 4].

After discovery of a method for nonlinear equations solution using the inverse scattering problem method, it has been found that such equations, in a certain sense, are field analogues of the Hamiltonian equations of classical mechanics [5]. This contributed to using Hamiltonian formalism in the direction of integrating nonlinear equations. Also, the Hamiltonian approach turned out to be useful in searching for invariants of hydrodynamic media [6] (see also [7]).

Infinite-dimensional groups of diffeomorphisms are closely related by hydrodynamic systems [8, 9]. In fact, they are the configuration space for many hydrodynamic equations. Thus, groups of volume-preserving diffeomorphisms are closely related to the Euler equations of incompressible fluid, compressible fluid and ideal magnetohydrodynamics (see, for example, [10]). The approach using Poisson brackets to the description of hydrodynamic systems has also been intensively developed. Mathematical achievements in the Hamiltonian description of hydrodynamic systems can be found in [11, 14]. The development of the formalism of Poisson brackets led to the classification of Poisson brackets of the hydrodynamic type within the framework of the differential geometric approach [11, 12, 13, 14]. Another aproach arose earlier from the analogy of barotropic flow of an ideal fluid in a potential force field with Hamiltonian systems [15]. In this paper [15] an attempt was made to extend the analogy between hydrodynamics and Hamiltonian mechanics to the case of arbitrary Hamiltonian systems.

In this paper the interest is rather opposite and is directed to identification of properties of finite-dimensional Hamiltonian systems, which provide a connection with hydrodynamic systems. The reason is that for an arbitrary Hamiltonian system the phase flows do not correspond to hydrodynamic flows. In other words, we are interested in what class of Hamiltonian systems generates hydrodynamic flows. Such Hamiltonian systems are finite-dimensional and in the case of two-dimensional ideal hydrodynamics the dimensionality of the phase space is equal to 2. In this paper we consider two-dimensional hydrodynamics, which has an infinite number of conservation laws [16]. The conditions under which the phase flow of a Hamiltonian system generates the velocity field of two-dimensional hydrodynamics of an incompressible fluid are obtained. An example

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of the corresponding flow is given. A special class of one-dimensional Hamiltonian systems associated with hydrodynamic flows is identified.

2. LAGRANGIAN AND HAMILTONIAN FORMALISM

We will discuss the transition from the Lagrangian to the Hamiltonian formalism in two-dimensional hydrodynamics. This transition has unusual features. The equations of motion of Lagrangian particles in Lagrangian variables in two-dimensional hydrodynamics can be easily written as

$$\ddot{x} = -\frac{\partial P(x,y,t)}{\partial x}$$

$$\ddot{y} = -\frac{\partial P(x,y,t)}{\partial y} \tag{1}$$

Here, for simplicity, we have chosen a density $\rho = const$, which simply renormalizes the pressure $P(x,y,t)$. This system of equations can be obtained from the Lagrange variational principle from the Lagrangian $L = \dot{x}^2/2 + \dot{y}^2/2 - P(x,y,t)$. Pressure plays the role of potential energy. However, this system must be supplemented by the incompressibility condition $div\vec{V} = 0$. In incompressible hydrodynamics, this condition additionally determines the unknown pressure. The problem is an inadequate type of this condition for the Lagrangian formalism. Availability of a velocity field determing trajectories of Lagrangian particles is an essential property. Enter $\dot{x} = V_1\,(x,t)\,,\;\; \dot{y} = V_2\,(x,y,t)$, then the incompressibility condition takes the form

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} = 0 \tag{2}$$

Entering Lagrangian variables and a velocity field depending only on coordinates assumes consistency with both the incompressibility condition (2) and the equations of motion (1). The incompressibility condition implies the existence of a function such that

$$\dot{x} = \frac{\partial H\,(x,y,t)}{dy}$$

$$\dot{y} = -\frac{\partial H\,(x,y,t)}{\partial x} \tag{3}$$

In other words, the incompressibility condition means the Hamiltonian description of Lagrangian particles. However, the occurence of the Hamiltonian description is not connected with the Lejandre Lagrangian transformation of two-dimensional hydrodynamics $L = \dot{x}^2/2 + \dot{y}^2/2 - P(x,y,t)$. This feature of the appearance of the Hamiltonian formalism in hydrodynamic systems has been observed many times. Thus, for example, the Hamiltonian equations of motion of point vortices cannot be formulated in Lagrangian form. The principal difference between the initial Hamiltonian formalism obtained by the Lejandre Lagrangian transformation and the Hamiltonian formalism from the incompressibility condition lies in the difference in the dimensionality of phase spaces. In the first it is 4-dimensional, in the second it is 2-dimensional phase space. It is clear that the transition to lower dimensionality is associated with the conservation of volume $dxdy$. We will discuss this reduction of the initial Hamiltonian formalism a little later.Thus, the main problem is to determine the type of the Hamiltonian and the constraints on it following from the equations of motion (1). Let us proceed to the analysis of these constraints. Differentiating (3) with respect to time, we express $\ddot{x}, \ddot{y}$ through $\dot{y}, \dot{x}$ and H.

$$\ddot{x} = \frac{\partial^2 H}{\partial x\,\partial y}\dot{x} + \frac{\partial^2 H}{\partial y^2}\dot{y} + \frac{\partial^2 H}{\partial y\,\partial t}$$

$$\ddot{y} = -\frac{\partial^2 H}{\partial x^2}\dot{x} - \frac{\partial^2 H}{\partial x\,\partial y}\dot{y} - \frac{\partial^2 H}{\partial x\,\partial t} \tag{4}$$

Thus, the second derivatives are expressed through the first ones and the coordinates of the Lagrangian particle. Taking into account this relation, let us write the equations of motion (1) in the form solved with respect to $\dot{x}$ and $\dot{y}$. This is easy to do by considering it as a system of linear algebraic equations relatively to derivatives of coordinates with respect to time.

$$\dot{x}\cdot\Delta(x,y,t) = \frac{\partial^2 H}{\partial y^2}\left(\frac{\partial^2 H}{\partial x\,\partial t} - \frac{\partial P}{\partial y}\right) - \frac{\partial^2 H}{\partial x\,\partial y}\left(\frac{\partial^2 H}{\partial y\,\partial t} + \frac{\partial P}{\partial x}\right)$$

$$\dot{y}\cdot\Delta(x,y,t) = -\frac{\partial^2 H}{\partial x^2}\left(\frac{\partial^2 H}{\partial y\,\partial t} + \frac{\partial P}{\partial x}\right) + \frac{\partial^2 H}{\partial x\,\partial y}\left(\frac{\partial^2 H}{\partial x\,\partial t} - \frac{\partial P}{\partial y}\right) \tag{5}$$

Here the notation is introduced $\Delta(x,y,t) \equiv \frac{\partial^2 H}{\partial x \partial y}\frac{\partial^2 H}{\partial x \partial y} - \frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial y^2}$. The condition of matching with (3) restricts the type of functions H and P. Substituting $\dot{x}$ and $\dot{y}$ through the derivatives of the Hamiltonian, we obtain

$$\frac{\partial H}{\partial y}\Delta(x,y,t) = \frac{\partial^2 H}{\partial y^2}\frac{\partial^2 H}{\partial x \partial t} - \frac{\partial^2 H}{\partial x \partial y}\frac{\partial^2 H}{\partial y \partial t} - \frac{\partial P}{\partial y}\frac{\partial^2 H}{\partial y^2} - \frac{\partial P}{\partial x}\frac{\partial^2 H}{\partial x \partial y}$$

$$\frac{\partial H}{\partial x}\Delta(x,y,t) = \frac{\partial^2 H}{\partial x \partial y}\frac{\partial^2 H}{\partial x \partial t} - \frac{\partial^2 H}{\partial x^2}\frac{\partial^2 H}{\partial y \partial t} - \frac{\partial P}{\partial x}\frac{\partial^2 H}{\partial x^2} - \frac{\partial P}{\partial y}\frac{\partial^2 H}{\partial x \partial y} \tag{6}$$

Solving this system of algebraic linear equations with respect to $\frac{\partial^2 H}{\partial x \partial t}$ and $\frac{\partial^2 H}{\partial y \partial t}$ after simple rearrangements, transformations and cancellation $\Delta(x,y,t) \neq 0$, we obtain a system of equations in the form

$$\frac{\partial^2 H}{\partial x \partial t} + \frac{\partial^2 H}{\partial x^2}\frac{\partial H}{\partial y} - \frac{\partial^2 H}{\partial x \partial y}\frac{\partial H}{\partial x} = \frac{\partial P}{\partial y}$$

$$\frac{\partial^2 H}{\partial y \partial t} + \frac{\partial^2 H}{\partial x \partial y}\frac{\partial H}{\partial y} - \frac{\partial^2 H}{\partial y^2}\frac{\partial H}{\partial x} = -\frac{\partial P}{\partial x} \tag{7}$$

In fact, this basic system of equations determines the unknown functions H and P. If these functions are known, then the trajectory of the Lagrangian particle is found by integrating the Hamiltonian system of equations. In other words, it reduces to a problem of classical mechanics. It is convenient to give this system a coordinate-free form using Poisson brackets. Let us introduce the Poisson bracket in the usual way

$$\{A,B\} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x}$$

Using this bracket let us write the system of equations (7) in more natural invariant form.

$$\frac{\partial}{\partial t}\frac{\partial H}{\partial x} + \left\{\frac{\partial H}{\partial x}, H\right\} = \frac{\partial P}{\partial y}$$

$$\frac{\partial}{\partial t}\frac{\partial H}{\partial y} + \left\{\frac{\partial H}{\partial y}, H\right\} = -\frac{\partial P}{\partial x} \tag{8}$$

or

$$\frac{\partial}{\partial t}\frac{\partial H}{\partial x} + \left\{\frac{\partial H}{\partial x}, H\right\} - \{x, P\} = 0$$

$$\frac{\partial}{\partial t}\frac{\partial H}{\partial y} + \left\{\frac{\partial H}{\partial y}, H\right\} - \{y, P\} = 0 \tag{9}$$

Note, that neither the Hamiltonian nor its first derivatives are conserved with time. However, excluding the pressure from this system it is easy to show that ΔH is conserved. Indeed, the equation for changing the Laplacian H has the form

$$\frac{\partial}{\partial t}\Delta H + \{H, \Delta H\} = 0 \tag{10}$$

In fact, it is the well-known equation for vorticity.

3. THE EXAMPLE OF FLOW OF 2-DIMENSIONAL HYDRODYNAMICS OF INCOMPRESSIBLE FLUID

Let's start with a simple example of a Hamiltonian system with a time-dependent Hamiltonian. This Hamiltonian is a particular solution of equation (10) and has the form

$$H = p^2 \sin^2(\omega t) + x^2 \cos^2(\omega t) \tag{11}$$

It is easy to see that it has special properties. Its main property, which we will discuss below, is the following

$$\Delta H = const \tag{12}$$

where, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial p^2}$ is the Laplacian in a phase space. The equations of motion for this system have the form

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = 2p \sin^2(\omega t)$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x} = -2x\cos^2(\omega t) \tag{13}$$

Let us note that equation (10) is satisfied for the Hamiltonian (11) and, therefore, the velocity field (12) satisfies the Euler equations and condition (2) provides an example of a spiral unsteady two-dimensional flow.

By excluding the variable p we obtain the equation for the variable x of the second order

$$\frac{\ddot{x}}{\sin^2(\omega t)} - 2\dot{x}\frac{\omega\cos(\omega t)}{\sin^3(\omega t)} = -4x\cos^2(\omega t) \tag{14}$$

This equation corresponds to Newton's equation and appears as the Lagrangian equation of motion of a one-dimensional physical system. Let us now discuss the Lagrangian system, which is generated by the Hamiltonian system (11). For this purpose we pass by means of the Lejandre transformation to the Lagrangian of the initial system.

$$\dot{x} = \frac{\partial H}{\partial p}$$

$$\pounds = \dot{x}p - H$$

For our particular case

$$\dot{x} = 2p\sin^2(\omega t)$$

and impulse is expressed through velocity according to

$$p = \frac{\dot{x}}{2\sin^2(\omega t)}$$

Note, that the Legendre transformation coincides with one of the equations of motion. Moreover, this transformation is not well determined everywhere. There are specific peculiarities at $\omega t = \pi k$, where $k = \pm 0, \pm 1, \pm 2, \ldots$. We won't pay attention to that for now. Then the Lagrangian type is determined as,

$$\pounds = \frac{\dot{x}^2}{4\sin^2(\omega t)} - x^2\cos^2(\omega t) \equiv \left(\frac{\dot{x}}{2\sin(\omega t)} + x\cos(\omega t)\right)\left(\frac{\dot{x}}{2\sin(\omega t)} - x\cos(\omega t)\right) \tag{15}$$

Lagrangian equations of motion have the form

$$\frac{d}{dt}\frac{\partial \pounds}{\partial \dot{x}} = \frac{\partial \pounds}{\partial x}$$

and in the case under consideration give the equation (14). Note that this equation describes the behavior of a one-dimensional Lagrangian system of rather exotic physical content. First of all, the particle is in a potential well with oscillating amplitude, which is affected by ¡¡friction¿¿ changed periodically with time. Note that two-dimensional flow of an incompressible fluid is associated with a special one-dimensional Lagrangian system.

Now let us differentiate the equations (13) with respect to time and proceed to a system of higher order equations in time. After using the equations of motion (13) to eliminate the first derivatives in the right-hand sides of the system, we obtain

$$\frac{d^2x}{dt^2} = -4x\sin^2(\omega t)\cos^2(\omega t) + 4\omega p\sin(\omega t)\cos(\omega t)$$

$$\frac{d^2p}{dt^2} = -4p\sin^2(\omega t)\cos^2(\omega t) + 4\omega x\sin(\omega t)\cos(\omega t) \tag{16}$$

It is clear that the solutions of the initial system (13) are also solutions of the obtained system of higher order equations (16). The converse is not true, since new solutions can emerge due to differentiation.

The obtained system of equations (16) has a remarkable special property — it is a natural Lagrangian system, which can be written in the form

$$\frac{d^2x}{dt^2} = -\frac{\partial U(x,p,t)}{\partial x}$$

$$\frac{d^2p}{dt^2} = -\frac{\partial U(x,p,t)}{\partial p} \tag{17}$$

where the function $U = U(x, p, t)$ has the meaning of a potential energy in two-dimensional configuration space (x, p). For the example under consideration

$$U = 2(x^2 + p^2)\sin^2(\omega t)\cos^2(\omega t) - 4\omega px\sin(\omega t)\cos(\omega t) \tag{18}$$

Thus, differentiation of the Hamiltonian system (13) with respect to time transforms it into a Lagrangian system with Lagrangian

$$L = \frac{\dot{x}^2 + \dot{p}^2}{2} - U(x, p, t) \equiv \frac{\dot{x}^2 + \dot{p}^2}{2} - 2(x^2 + p^2)\sin^2(\omega t)\cos^2(\omega t) + 4\omega px\sin(\omega t)\cos(\omega t) \tag{19}$$

The Lagrangian dynamics in this special case corresponds not only to the initial Langrangian $L = \dot{x}^2/2 + \dot{y}^2/2 - P(x, y, t)$, but also to the condition of divergence-free velocity field. Obviously, this is due to specialization of the potential energy. Further we will be interested in the general properties that provide this.

The system of equations (17) can be obtained as,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{p}} = \frac{\partial L}{\partial p} \tag{20}$$

Just the property (17) distinguishes a special class of Hamiltonian systems. Next, we introduce such a class, determining its characteristic property in the general case. Now it is important that this class of Hamiltonian systems is not trivial, as the example (11) demonstrates.

4. CLASS OF HAMILTONIANS OF HYDRODYNAMIC SOLUTIONS

Let us now distinguish a class of Hamiltonian systems G with special properties.

Determination 1 We will consider the Hamiltonian as belonging to the class G, if the system of equations obtained by time differentiation is a natural Lagrangian system.

It is clear that the class of such systems is not empty. An example of such a system was discussed earlier. Let us consider what conditions the Hamiltonians belonging to the class G satisfy. To do this, consider an arbitrary Hamiltonian system. Its equations of motion have the form

$$\frac{dx}{dt} = \frac{\partial H(x, p, t)}{\partial p}$$

$$\frac{dp}{dt} = -\frac{\partial H(x, p, t)}{\partial x}$$

Let's perform time differentiation

$$\frac{d^2x}{dt^2} = \frac{\partial^2 H(x, p, t)}{\partial p \partial t} + \frac{\partial^2 H(x, p, t)}{\partial p \partial x}\frac{dx}{dt} + \frac{\partial^2 H(x, p, t)}{\partial p^2}\frac{dp}{dt}$$

$$\frac{d^2p}{dt^2} = -\frac{\partial^2 H(x, p, t)}{\partial x \partial t} - \frac{\partial^2 H(x, p, t)}{\partial x^2}\frac{dx}{dt} - \frac{\partial^2 H(x, p, t)}{\partial p \partial x}\frac{dp}{dt}$$

We will use now the initial equations of motion and eliminate $\frac{dp}{dt}$ and $\frac{dx}{dt}$ in this system of equations

$$\frac{d^2x}{dt^2} = \frac{\partial^2 H(x, p, t)}{\partial p \partial t} + \frac{\partial^2 H(x, p, t)}{\partial p \partial x}\frac{\partial H(x, p, t)}{\partial p} - \frac{\partial^2 H(x, p, t)}{\partial p^2}\frac{\partial H(x, p, t)}{\partial x}$$

$$\frac{d^2p}{dt^2} = -\frac{\partial^2 H(x, p, t)}{\partial x \partial t} - \frac{\partial^2 H(x, p, t)}{\partial x^2}\frac{\partial H(x, p, t)}{\partial p} + \frac{\partial^2 H(x, p, t)}{\partial p \partial x}\frac{\partial H(x, p, t)}{\partial x}$$

It is convenient to write these equations using the Poisson bracket

$$\{A, B\} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x}$$

Then the system of equations takes the form

$$\frac{d^2x}{dt^2} = \frac{\partial}{\partial t}\frac{\partial H(x, p, t)}{\partial p} + \left\{\frac{\partial H(x, p, t)}{\partial p}, H(x, p, t)\right\} \tag{21}$$

$$\frac{d^2p}{dt^2} = -\frac{\partial}{\partial t}\frac{\partial H(x,p,t)}{\partial x} - \left\{\frac{\partial H(x,p,t)}{\partial x}, H(x,p,t)\right\} \quad (22)$$

In order that Hamiltonians to belong to the class G it is necessary and sufficient that the right parts of this system have the form

$$\frac{\partial}{\partial t}\frac{\partial H(x,p,t)}{\partial p} + \left\{\frac{\partial H(x,p,t)}{\partial p}, H(x,p,t)\right\} = \frac{\partial U(x,p,t)}{\partial x}$$

$$-\frac{\partial}{\partial t}\frac{\partial H(x,p,t)}{\partial x} - \left\{\frac{\partial H(x,p,t)}{\partial x}, H(x,p,t)\right\} = \frac{\partial U(x,p,t)}{\partial p}$$

The consistency conditions of this system of equations determine the class of Hamiltonians G. Equating the mixed derivatives of the right-hand sides we obtain the consistency condition,

$$\frac{\partial \Delta H(x,p,t)}{\partial t} + \{\Delta H(x,p,t), H(x,p,t)\} = 0 \quad (23)$$

which detemines the type of Hamiltonians belonging to class G. It follows that class G consists only of Hamiltonians that satisfy this equation and are significantly narrower than Hamiltonian systems. At a specified $H(x,p,t)$ the function $U(x,p,t)$ satisfies the equation

$$\Delta U(x,p,t) = 2\left\{\frac{\partial H(x,p,t)}{\partial p}, \frac{\partial H(x,p,t)}{\partial x}\right\}$$

Theorem 2 The class of Hamiltonians determines the velocity field of two-dimensional flows of an incompressible ideal fluid.

The proof is trivial, the condition (23) coincides with the equation for the stream function. This theorem leads to justification of singling out this class and the necessity of its study because of its physical importance for hydrodynamic problems.

The possibility of studying hydrodynamic flows exclusively as a special class of mechanical systems is an important ideological change following from the theorem.

5. LAGRANGIAN HYDRODYNAMICS

Let's start with some simple considerations that make sense in purely classical mechanics. Let us find an analogue of differentiation of motion equations in terms of classical mechanics. To do this, consider the classical system with the Lagrangian

$$L = \frac{1}{2}\left(\dot{x} - A(x,y,t)\right)^2 + \frac{1}{2}\left(\dot{y} - B(x,y,t)\right)^2 \quad (24)$$

An obvious feature of this Lagrangian is the achievement of a bare minimum of action on the equations of motion

$$\dot{x} = A(x,y,t)$$
$$\dot{y} = B(x,y,t) \quad (25)$$

For now, we will not take into account their Hamiltonian character in order to avoid cumbersome notation. This will simplify writing of subsequent equations. These two functions determine both the potential energy and other contributions linear in velocities. On the other hand, the motion equations for this Lagrangian have the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{\partial L}{\partial y}$$

For the Lagrangian under consideration

$$\ddot{x} = AA_x + BB_x + \dot{y}(A_y - B_x) + A_t$$

$$\ddot{y} = AA_y + BB_y - \dot{x}(A_y - B_x) + B_t \quad (26)$$

It is easy to see that in solutions (25) the equations coincide with the equations obtained by differentiation of motion equations (25).

Theorem 3 If the Lagrangian of a mechanical system has a bare minimum, then the Lagrangian equations of motion of such a system coincide with the equations obtained by time differentiation of bare minimum coordinates.

Then the condition that this system is a natural mechanical system can be formulated as

$$\dot{y}(A_y - B_x) + A_t = \frac{\partial G}{\partial x}$$

$$-\dot{x}(A_y - B_x) + B_t = \frac{\partial G}{\partial y} \tag{27}$$

when considering the equations (25) they are reduced to a system of equations of the form

$$A_t + B(A_y - B_x) = \frac{\partial G}{\partial x}$$

$$B_t - A(A_y - B_x) = \frac{\partial G}{\partial y} \tag{28}$$

It is easy to check that these equations coincide with the previously obtained (21), (22) constraints to the Hamiltonian type (of course, when specializing to and). In this way we can understand the procedure for raising the order of the motion equations and the resulting constraints to the Lagrangian in terms of classical mechanics only. In fact, we obtain the equivalent equations of two-dimensional hydrodynamics in such a way. From a general point of view, these partial differential equations take place as matching conditions of some ordinary differential equations.

Besides, it is easy to take into account the divergence-free velocity field in this formalism. When specializing to stationary Hamiltonians

$$\dot{x} = A = -\frac{\partial H(x,y)}{\partial y}$$

$$\dot{y} = B = \frac{\partial H(x,y)}{\partial x}$$

motion equations of the Lagrangian system have the form

$$\ddot{x} = \frac{\partial P}{\partial x} - \dot{y}\Delta H$$

$$\ddot{y} = \frac{\partial P}{\partial y} + \dot{x}\Delta H \tag{29}$$

where, $P = \frac{1}{2}\left((\frac{\partial H}{\partial x})^2 + (\frac{\partial H}{\partial y})^2\right)$. It's easy to see that at $\Delta H = const$ the natural Lagrangian system is obtained automatically. This clarifies the original example analyzed in Section 3. Of course, in the more general case, if the condition $\Delta H = f(H)$ is met, these equations also coincide with the natural Lagrangian system.

6. CONCLUSIONS

In this paper a class of Hamiltonian systems G whose phase flows are exact solutions of two-dimensional hydrodynamics of incompressible fluid is identified. An unusual example of a Lagrangian one-dimensional system is given, which, upon transition to the Hamiltonian formalism, generates an unsteady two-dimensional flow. Lagrangian hydrodynamics as a consequence of a special choice of the Lagrangian is introduced. The obtained properties are useful in searching for exact solutions of two-dimensional hydrodynamics.

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ДВОВИМІРНА ГИДРОДИНАМИКА ЯК КЛАС СПЕЦІАЛЬНИХ ГАМІЛЬТОНОВИХ СИСТЕМ

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У роботі визначено клас гамільтонових систем, фазові потоки яких є точними рішеннями двовимірної гідродинаміки рідини, яка не стискується. Розглянуто властивості цього класу. Наведено приклад лагранжової одновимірної системи, яка після переходу до гамільтонового формалізму приводить до нестаціонарної течії, тобто до точного рішення двовимірної гідродинаміки. Обговорено зв'язок між цими формалізмами та введено лагранжиани, які породжують лагранжову гідродинаміку. Отримані результати дозволяють отримувати точні рішення, як фазові потоки спеціальних гамільтонових систем.

Ключові слова: *гамільтоніан; лагранжиан; точні рішення; двовимірна гідродинаміка; фазовий потік*