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THERMAL PROPERTIES AND MASS SPECTRA OF HEAVY MESONS IN THE PRESENCE OF A POINT-LIKE DEFECT

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In this research, the radial Schrödinger equation is solved analytically using the Nikiforov-Uvarov method with the Cornell potential. The energy spectrum and the corresponding wave function are obtained in close form. The effect of Topological Defect on the thermal properties and mass spectra of heavy mesons such as charmonium and bottomonium are studied with the obtained energy spectrum. It is found that the presence of the Topological Defect increases the mass spectra and moves the values close to the experimental data. Our results agreed with the experimental data and are seen to be improved when compared with other works.
Keywords: *Schrödinger equation; Nikiforov-Uvarov method; Cornell Potential; Mass Spectra; Topological Defect*
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I. INTRODUCTION

Recent years have seen an increase in interest among academics in the study of mass spectra, thermal properties, energy spectra, information theory, and expectation values of a quantum system [1-5]. This is achieved by solving the Schrodinger equation (SE) or the Klein- Gordon equation. In order to determine a system's eigenvalues and eigenfunctions, potential models like the Yukawa potential [6], Eckart potential [7], Hellmann potential [8], Kratzer potential [9], and so on are utilized. Analytical techniques are used to solve these equations with the potential of choice. The majority of analytical techniques used include the Nikiforov-Uvarov (NU) method [10-20], the Nikiforov-Uvarov Functional Analysis (NUFA) method [21-26], the series expansion method (SEM) [27-29], Laplace transformation method (LTM) [30], the Exact Quantization Rule [31,32], WKB approximation method [33-36] and others [37-41]. The Schrodinger equation can be used to study the interactions of the heavy quarkonium system (HQS) [42]. In defining the mass spectra (MS) of the heavy quarkonium system, such as bottomonium and charmonium, the solutions of the Schrodinger equation with spherically symmetric potentials are of great importance [43,44]. Confining-type potentials, such as the Cornell potential (CP) with two terms, a confining and a Coulomb interaction term, are typically utilized to analyze this system [45]. Several authors have been interested in the investigation of the heavy quarkonium system with the Cornell potential [46-49]. For instance, Kumar et al.,[50] solved the Schrodinger equation with the generalized Cornell potential and the result was used to calculate the mass spectra of the heavy mesons (HMs). Vega and Flores, [51] studied the mass spectra of the heavy mesons using the Cornell potential. Also, Mutuk [52] solved the Schrodinger equation with the Cornell potential using a neural network approach. The bottomonium, charmonium, and bottom-charmed spectra were calculated. Additionally, Hassanabadi et al. [53], used the variational method to solve the Schrodinger equation with the Cornell potential. The eigenvalues were used to calculate the mesonic wave function. Furthermore, the thermal properties (TPs) of the heavy mesons have been studied recently with extended Cornell potential and exponential-type potentials [54-58]. Researchers have been interested in the study of quantum systems interacting with a single particle in a specific potential with a topological defect (TD). It is believed that the early universe phase transition occurs when topological defect was formed [59,60]. Researchers have recently become interested on how topological defect affects the dynamics of both relativistic and non-relativistic systems, including screw dislocation [61], bound electron eigenstates, and holes to a declination. Furtado et al., [62] examined the Landau levels in the presence of a topological defect. Moreover, Hassanabadi and Hosseinpour [63] examined how topological defect affected the hydrogen atom's position in curve-space time. The topological defect plays a vital role in altering the physical properties of many quantum systems, such as condensed matter physics, which it appears as monopoles and strings [64-66]. A linear defect in an elastic medium, such as a dislocation or crack, causes a change in the topology of the medium, which has an effect on the medium's physical characteristics [67]. In light of these observations, no researcher is yet to document how topological defect affects the thermal and mass spectra of the heavy mesons to the best of our knowledge. So, the purpose of this work is to examine how topological defect affects the mass spectra and

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thermal properties of the heavy mesons by employing the Nikiforov-Uvarov approach to solve the Schrodinger equation with the Cornell potential. For convenience, we have assumed that our heavy mesons are spinless particles [42,57, 68-70]. This is because most potentials with spin addition cannot be solved analytically, necessitating the employment of numerical methods like the Runge-Kutte approximation [71], Numerov matrix approach [72], Fourier grid Hamiltonian method [73], and so on [74].

II. THE MODEL

For space time with a point-like global monopole (PGM), the line element that explains it, takes the form [75]

$$ds^2 = -c^2dt^2 + \frac{dr^2}{\alpha^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \tag{1}$$

Where α is the parameter related to the PGM which depends on the energy scale $0 \prec \alpha = 1 - 8\pi G\eta_0^2 \prec 1$. Furthermore, Eq. (1) portrays a space time with scalar curvature. On inserting the potential under consideration, we have

$$R = R_\mu^\mu = \frac{2\left(1-\alpha^2\right)}{r^2} \tag{2}$$

In this way, the Schrodinger equation takes the form

$$-\frac{\hbar^2}{2\mu}\nabla_{LB}^2\psi(\vec{r},t) + V(r,t)\psi(\vec{r},t) = i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} \tag{3}$$

where μ is the particle's mass, $\nabla_{LB}^2 = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right)$ with $g = \det(g_{ij})$, is the Laplace-Beltrami operator and $V(r,t) = V(r)$ is GMP(1). Thereby, the Schrodinger equation for the GMP in a medium with the presence of the PGM(1) is

$$-\frac{\hbar^2}{2\mu r^2}\begin{bmatrix}\alpha^2\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) \\ +\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\end{bmatrix}\psi(r,\theta,\varphi,t) + V\psi(r,\theta,\varphi,t) = i\hbar\frac{\partial\psi(r,\theta,\varphi,t)}{\partial t} \tag{4}$$

Where the $V(r) = w_1 r - w_2 / r$ is the Cornell potential employed for modeling the quarkonium interaction [70]. This model has been greatly utilized for this purpose in recent past by numerous researchers.

Here, let us consider a particular solution to Eq. (5) given in terms of the eigenvalues of the angular momentum operator $\hat{L}^2$ as

$$\psi(r,\theta,\varphi,t) = e^{-\frac{E_{nl}t}{\hbar}}\frac{U(r)}{r}Y_{l,m}(\theta,\varphi) \tag{5}$$

where $Y_{l,m}(\theta,\varphi)$ are spherical harmonics and $R(r)$ is the radial wave function.

On substitution of Eq. (6) into Eq. (4), the radial part of the Schrodinger equation for the Cornell potential in the presence of TD is obtained as follows

$$\frac{d^2R_{nl}(r)}{dr^2} + \left[\frac{2\mu E_{nl}}{\alpha^2\hbar^2} - \frac{2\mu w_1 r}{\alpha^2\hbar^2} + \frac{2\mu w_2}{\alpha^2\hbar^2 r} - \frac{l(l+1)}{\alpha^2 r^2}\right]R_{nl}(r) = 0 \tag{6}$$

Eq. (6) is not solvable in its present form, Eq. (6) needs to be transformed from $r \to x$ coordinate using the following $x = 1/r$, on application, Eq. (6) is rewritten as follows:

$$\frac{d^2R_{nl}(x)}{dx^2} + \frac{2}{x}\frac{dR_{nl}(x)}{dx} + \frac{1}{x^4}\left[\frac{2\mu E_{nl}}{\alpha^2\hbar^2} - \frac{2\mu w_1}{\alpha^2\hbar^2 x} + \frac{2\mu w_2 x}{\alpha^2\hbar^2} - \frac{l(l+1)x^2}{\alpha^2}\right]R_{nl}(x) = 0 \tag{7}$$

The approximation scheme on the term w_1/x is introduced by assuming that there is a characteristic radius r_0 of the meson. The approximation scheme is achieved by the expansion of w_1/x in a power series around r_0; i.e. around $\delta \equiv\sim 1/r_0$,up to the second order [56]. By setting $y = x - \delta$ and around $y = 0$ we have;

$$\frac{d^2R_{nl}(x)}{dx^2}+\frac{2x}{x^2}\frac{dR_{nl}(x)}{dx}+\frac{1}{x^4}\left[-\tilde{\varepsilon}+\tilde{\beta}_1x-\tilde{\beta}_2x^2\right]R_{nl}(x)=0 \tag{8}$$

where

$$-\tilde{\varepsilon}=\frac{2\mu E_{nl}}{\alpha^2\hbar^2}-\frac{6\mu w_1}{\alpha^2\hbar^2\delta},\tilde{\beta}_1=\frac{2\mu w_2}{\alpha^2\hbar^2}+\frac{6\mu w_1}{\alpha^2\hbar^2\delta^2},and\,\tilde{\beta}_2=\frac{2\mu w_1}{\alpha^2\hbar^2\delta^3}+\frac{l(l+1)}{\alpha^2}.$$

The equation above is expressed in the form solvable by the Nikiforov-Uvarov formalism. The major equation closely related with this method is given in the following form;

$$P''(x)+\frac{\tilde{\tau}(x)}{\sigma(x)}P'(x)+\frac{\sigma(x)}{\sigma^2(x)}P(x)=0. \tag{9}$$

The following is obtained; $\tilde{\tau}(x)=2x,\sigma(x)=x^2,and\ \tilde{\sigma}(x)=-\tilde{\varepsilon}+\tilde{\beta}_1x-\tilde{\beta}_2x^2,$ which shows explicitly that our Eq. (8) satisfies the requirement of the Nikiforov-Uvarov approach. It is worthy to point out also that $\tilde{\sigma}(x),and\ \sigma(x)$ are polynomials of at most second degree, and $\tilde{\tau}(x),$ is at most a polynomial of first degree. The Nikiforov-Uvarov method is a really popular method amongst mathematical scientist and related discipline. Several authors have used this method to solve similar problems of interest [10-20]. Even through the method is quite popular, it will be useful to highlight some details, so as to make our paper self-contained. For this reason, this will be detailed in the appendix. Following the steps outlined in the appendix (Eqs. (A1-A7), the energy equation and radial wave function are obtained as follows

$$E_{nl}=\frac{3w_1}{\delta}-\frac{\alpha^2\hbar^2}{8\mu}\left[\frac{\frac{6\mu w_1}{\alpha^2\delta^2\hbar^2}+\frac{2w_2}{\alpha^2\hbar^2}}{n+\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{2\mu w_1}{\alpha^2\delta^3\hbar^2}+\frac{l(l+1)}{\alpha^2}}}\right]^2 \tag{10}$$

and

$$R_{nl}=N_{nl}x^{-\frac{\tilde{\beta}_1}{2\sqrt{\tilde{\varepsilon}}}}e^{-\frac{\tilde{\varepsilon}}{x\tilde{\varepsilon}}}L_n^{\frac{\tilde{\beta}_1}{\sqrt{\tilde{\varepsilon}}}}\left(\frac{2\tilde{\varepsilon}}{x\sqrt{\tilde{\varepsilon}}}\right) \tag{11}$$

THERMAL PROPERTIES OF THE CORNELL POTENTIAL IN THE PRESENCE OF DEFECT

We introduce the partition function *Z(β)*, which provides a measure of thermally accessible states, to explore the thermal properties of the heavy mesons. It can be determined by adding together all possible energy states. Following the Boltzmann-Gibbs distribution, *Z(β)* is given by the relation [76];

$$Z(\beta)=\sum_{n=0}^{n_{\max}}e^{-\beta E_{nl}}. \tag{12}$$

Where $\beta=1/kT$ and with k is the Boltzmann constant. Substituting Eq. (10) in Eq. (12), summing over all accessible energy levels, we obtain the partition function as follows:

$$Z(\beta,\alpha,n_{\max})=\left(x_1e^{\frac{A_4\beta}{x_1^2}}-e^{\frac{A_4\beta}{x_2^2}}x_2-\sqrt{A_4}\sqrt{\pi}\sqrt{\beta}erf\left(\frac{\sqrt{A_4}\sqrt{\beta}}{x_1}\right)+\sqrt{A_4}\sqrt{\pi}\sqrt{\beta}erf\left(\frac{\sqrt{A_4}\sqrt{\beta}}{x_2}\right)\right). \tag{13}$$

Where the following non-dimensional parameters have been defined for simplicity;

$$x_1=A_3,x_2=A_3+n_{\max},A_1=\frac{3w_1}{\delta^2},A_2=\frac{6w_1\mu}{\alpha^2\delta}+\frac{2w_2\mu}{\alpha^2\delta}.A_3=\frac{1}{2}+\sqrt{\frac{2w_1\mu}{\hbar^2\alpha^2\delta^3}+\frac{l(l+1)}{\alpha^2\delta}+\frac{1}{4}},and\ A_4=\frac{A_2^2\hbar}{8\mu}A_3.$$

On successful evaluation of the partition function, several other thermodynamic variables can be obtained by using the following;

Internal Energy: $U(\beta)=-\frac{d\ln Z(\beta)}{d\beta}$ [76], The Helmholtz free energy: $F(\beta)=-\frac{1}{\beta}\ln Z(\beta)$ [76], heat capacity; $C(\beta)=\beta^2\frac{d^2\ln Z(\beta)}{d\beta^2}$ [76] and entropy; $S(\beta)=\ln Z(\beta)-\beta\frac{dInZ(\beta)}{d\beta}$ [76].

DISCUSSION AND RESULTS

The prediction the mass spectra (MS) of HQS such as charmonium and bottomonium is carried out using the following relation [77,78].

$$M = 2\mu + E_{nl} \tag{14}$$

where μ is quarkonium mass and $E_{n\ell}$ is energy eigenvalues. Substituting Eq. (10) into Eq. (14) gives,

$$M = 2\mu + \frac{3w_1}{\delta} - \frac{\alpha^2\hbar^2}{8\mu}\left[\frac{\frac{6\mu w_1}{\alpha^2\delta^2\hbar^2} + \frac{2w_2}{\alpha^2\hbar^2}}{n + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\mu w_1}{\alpha^2\delta^3\hbar^2} + \frac{l(l+1)}{\alpha^2}}}\right]^2 \tag{15}$$

The numerical values of bottomonium and charmonium masses are $\mu_b = 4.823GeV$ and $\mu_c = 1.209GeV$, and the corresponding reduced mass are $\tilde{\mu}_b = 2.4115GeV$ and $\tilde{\mu}_c = 0.6045GeV$ respectively [79]. The potential parameters were fitted with experimental data (ED) [80]. This was achieved by solving a simultaneous equation for α equals to 0.1,0.2 and 1 respectively. The mass spectra of the heavy mesons were predicted in the absent and present of the topological defect for different quantum states. In the case of charmonium predictions for 1S and 2S states we noticed that the prediction were accurate with the experimental data in the present and absent of the topological defect. In 3S and 4S, it was noticed that as the topological defect increased to 0.2 the value of the mass spectra was very close to the experimental data. A similar trend was noticed in 1P,2P,1D and 2D states when the topological defect was introduced and the predicted values were close to the experimental data and was seen to be improved from works reported by [27, 30,42] as shown in Table 1.

Table I. Mass spectra of Charmonium in (GeV). The following fitting parameters has been employed; $m_C = 1.209GeV$, $\mu = 0.6045GeV$, $W_1 = 0.20GeV$, $W_2 = 1.244GeV$, $\delta = 0.231GeV$, and $\hbar = 1$.

State	α	Our result	AIM [42]	LTM [30]	SEM[27]	Experiment [80]
1S	0.1	3.096	3.096	3.0963	3.095922	3.096
	0.2	3.096	3.096	3.0963	3.095922	3.096
	1.0	3.096	3.096	3.0963	3.095922	3.096
2S	0.1	3.686	3.686	3.5681	3.685893	3.686
	0.2	3.686	3.686	3.5681	3.685893	3.686
	1.0	3.686	3.686	3.5681	3.685893	3.686
3S	0.1	4.040	4.275	4.0400	4.322881	4.040
	0.2	4.040	4.275	4.0400	4.322881	4.040
	1.0	4.040	4.275	4.0400	4.322881	4.040
4S	0.1	4.261	4.865	4.5119	4.989406	4.263
	0.2	4.262	4.865	4.5119	4.989406	4.263
	1.0	4.269	4.865	4.5119	4.989406	4.263
1P	0.1	3.515	3.214	3.5687	-	3.525
	0.2	3.524	3.214	3.5687	-	3.525
	1.0	3.255	3.214	3.5687	-	3.525
2P	0.1	3.771	3.773	3.5687	3.756506	3.773
	0.2	3.772	3.773	3.5687	3.756506	3.773
	1.0	3.779	3.773	3.5687	3.756506	3.773
1D	0.1	3.734	3.412	4.0407	-	3.770
	0.2	3.769	3.412	4.0407	-	3.770
	1.0	3.504	3.412	4.0407	-	3.770
2D	0.1	4.149	-	-	-	4.159
	0.2	4.158	-	-	-	4.159
	1.0	4.146	-	-	-	4.159

In the case of bottomonium, it was observed that for 1S, and 2S quantum states the mass spectra were all equal to the experimental data and works reported by [27, 30,42] as shown in Table 2. It was noticed that for 3S and 4S states, a significant change in the mass spectra was observed when the topological defect was set to 0.1 and 0.2. A similar trend was observed with other predicted states when the topological defect was increased as shown in Table 2.

Table II. Mass spectra of Bottomonium in (GeV). The following fitting parameters has been employed; $m_b = 4.823\ GeV$, $\mu = 2.4115\ GeV$, $W_1 = 0.20GeV$, $W_2 = 1.569\ GeV$, $\delta = 0.378\ GeV$, and $\hbar = 1$.

State	α	Our result	AIM [42]	LTM [30]	SEM[27]	Experiment [80]
1S	0.1	9.460	9.460	9.745	9.515194	9.460
	0.2	9.460	9.460	9.745	9.515194	9.460
	1.0	9.460	9.460	9.745	9.515194	9.460

State	α	Our result	AIM [42]	LTM [30]	SEM[27]	Experiment [80]
2S	0.1	10.023	10.023	10.023	10.01801	10.023
	0.2	10.023	10.023	10.023	10.01801	10.023
	1.0	10.023	10.023	10.023	10.01801	10.023
3S	0.1	10.355	10.585	10.302	10.44142	10.355
	0.2	10.355	10.585	10.302	10.44142	10.355
	1.0	10.355	10.585	10.302	10.44142	10.355
4S	0.1	10.570	11.148	10.580	10.85777	10.580
	0.2	10.577	11.148	10.580	10.85777	10.580
	1.0	10.567	11.148	10.580	10.85777	10.580
1P	0.1	9.879	9.492	10.025	-	9.899
	0.2	9.789	9.492	10.025	-	9.899
	1.0	9.619	9.492	10.025	-	9.899
2P	0.1	10.230	10.038	10.303	10.09446	10.260
	0.2	10.124	10.038	10.303	10.09446	10.260
	1.0	10.114	10.038	10.303	10.09446	10.260
1D	0.1	9.994	9.551	10.303	-	10.164
	0.2	9.999	9.551	10.303	-	10.164
	1.0	9.864	9.551	10.303	-	10.164

We observed that the results obtained from the prediction of the mass spectra of charmonium and bottomonium for different quantum states are in agreement with the experimental data and are improved with the reports of [27, 30,42]. The thermal properties for charmonium are plotted as shown in Fig. 1(a-e).

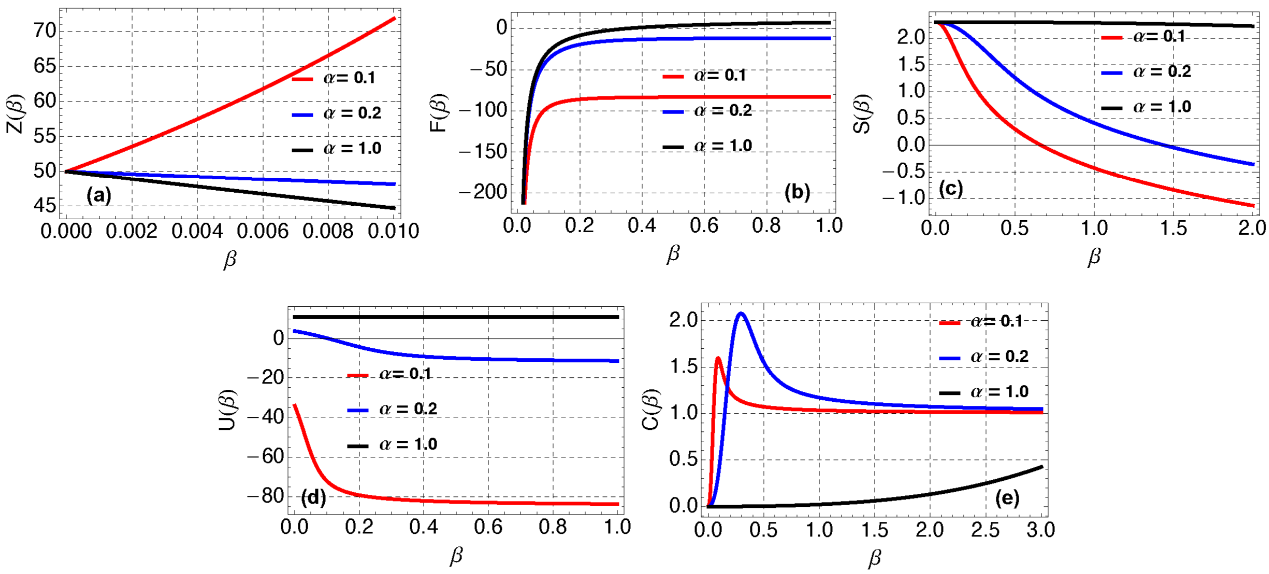

Figure 1. (a) Partition of Charmonium versus β. (b) Free Energy of Charmonium versus β. (c) Entropy of Charmonium versus β.(d) Mean Energy of Charmonium versus β. (e) Specific Heat Capacity of Charmonium versus β. For all five plots, α = 0.1 (red curve), α = 0.2 (blue curve) and α = 1.0 (red curve); all other parameters are the same as in Table (I).

In Fig. 1 (a), the partition function for topological defect = 0.1 shows a linear increase when the temperature is increased. When topological defect is equal to 0.2 and 1.0, the partition function is seen to decrease with an increase in temperature; same behavior is reported by Abu Shady et al., [55] and Kumar et al., [58]. In Fig. 1 (b), the free energy (FE) is plotted against temperature, we noticed that as the topological defect increases from 0.1 to 1.0, the free energy increases, which is in agreement with the experimental data. In Fig. 1 (c), the entropy of the system for charmonium is plotted. It is observed that when topological defect is equal to 0.1 and 0.2, the entropy is seen to decrease as the temperature increases, but when topological defect = 1.0, we noticed a steady entropy as the temperature is increased. In Fig.1 (d), the internal energy (IE) is plotted as a function of temperature. When topological defect = 1.0, a steady internal energy is noticed, but for topological defect = 0.1 and 0.2 an exponential decrease is observed followed by the internal energy of the system being steady when the temperature increases. Abu-Shady et al., [55] reported a decrease with increasing of temperature and maximum quantum number, our trend is on the expected line.

In Fig. 2 (a-e), the thermal properties of bottomonium is plotted as shown. In Fig.2 (a), the partition function is plotted as a function of temperature. When the topological defect = 0.1 a linear increase is noticed. For topological defect = 0.2 and 1.0, a slight increase and no increase on the partition function is seen respectively. In Fig. 2 (b), the free energy of the bottomonium is plotted against the temperature. We observed as the topological defect increases, the free energy is seen to increase. A similar observation was reported by Abu-Shady et al [55] and Kumar et al [58]. In Fig. 2 (c), the entropy is plotted against temperature for different values of topological defect, it was noticed that the entropy decreases with an increase in temperature. In [55,58] the authors found the entropy decreases with increasing temperature. In Fig. 2 (d), the inter energy plots shows that when topological defect = 1.0, no increase in the internal energy is noticed, but for topological defect = 0.1 and 0.2 a slight decrease is seen at the beginning then followed a constant value in the internal energy. In Fig.2 (e), the specific heat capacity is plotted against temperature. A sharp increase in specific heat capacity is noticed for topological defect = 0.1 and 0.2 and later converges at a point when the m specific heat capacity = 1. An exponential increase is noticed when topological defect = 1.

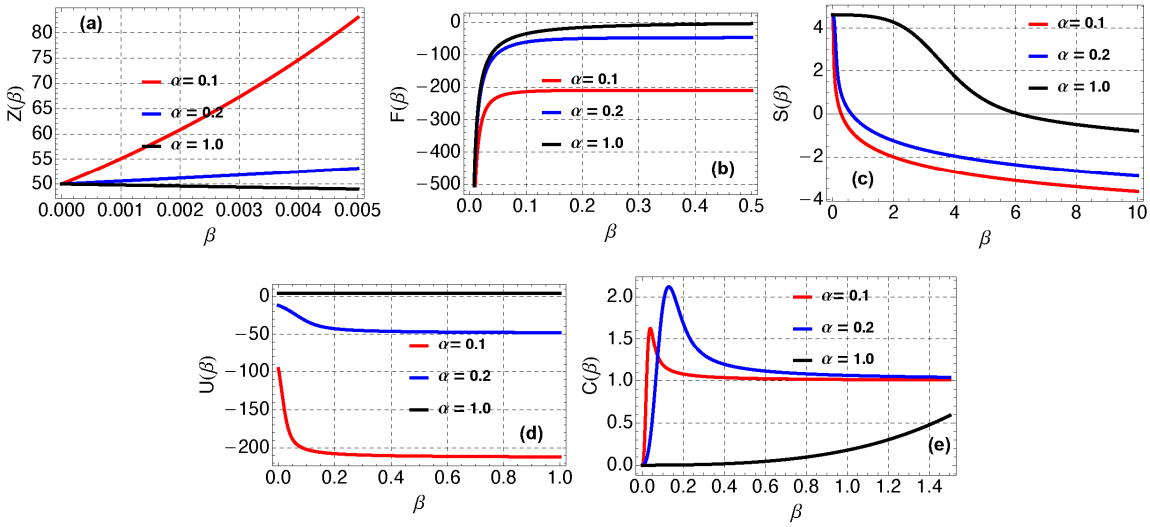

Figure 2. (a) Partition of Bottomonium versus β. (b) Free Energy of Bottomonium versus β. (c) Entropy of Bottomonium versus β.(d) Mean Energy of Bottomonium versus β. (e) Specific Heat Capacity of Bottomonium versus β. For all five plots, α = 0.1 (red curve), α = 0.2 (blue curve) and α = 1.0 (red curve); all other parameters are the same as in Table (II)

CONCLUSION

In this study, the effect of the topological defect on the mass spectra of heavy mesons is studied with the Cornell potential. The Schrodinger equation was solved analytically using the Nikiforov-Uvarov method. The approximate solutions of the energy spectrum and wave function in terms of Laguerre polynomials were obtained. We apply the present results to predict the mass spectra of heavy mesons such as charmonium and bottomonium in the present and absent of the topological defect for different quantum states and its thermal properties. We noticed that when the topological defect increases the mass spectra and moves closer to the experimental data. However, the results obtained showed an improvement when compared with the work of other researchers.

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Conflicts of interest/Competing interest

All the authors declared that there is no conflict of interest in this manuscript

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APPENDIX A: Review of Nikiforov-Uvarov (NU) method

In this section, the basic formalism of the Nikiforov-Uvarov method is reviewed. The relevant steps needed to arrive at the eigenvalue and eigenfunction are highlighted. This method was proposed by Nikiforov and Uvarov [81] to solving Hypergeometric-type differential equations of the form in Eq. (9). The solutions of Eq. (9) can be obtained by employing the trial wave function

$$P(x)=\phi(x)y_n(x), \tag{A1}$$

Which reduces Eq.(9) to an hypergeometric-type differential equation of the form ;

$$\sigma(x)y_n''"(x)+\tau(x)y_n'(x)+\lambda y_n(x)=0, \tag{A2}$$

The function $\phi(x)$ is defined as the logarithmic derivative [82-88]

$$\frac{\phi'(x)}{\phi(x)}=\frac{\pi(x)}{\sigma(x)} \tag{A3}$$

Where $\pi(x)$ is a polynomial of first–degree. The second term in Eq.(A1) is the hypergeometric function with its polynomial solution given by Rodrigues relation as follows;

$$y_n(x)=\frac{B_n}{\rho(x)}\frac{d^n}{dx^n}\left[\sigma^n\rho(x)\right] \tag{A4}$$

The term B_n is the normalization constant and $\rho(x)$ is known as the weight function which in principle must satisfy the condition given:

$$\frac{d}{dx}\left[\sigma(x)\rho(x)\right]=\tau(x)\rho(x). \tag{A5}$$

Where $\tau(x)=\tilde{\tau}(x)+2\pi(x)$.

It is imperative that we note here that the derivative of $\tau(x)$ should be $\tau(x)\prec 0$. The eigenfunctions and eigenvalues can be obtained using the expression defined by $\pi(x)$ and parameter λ, defined as follows

$$\pi(x)=\frac{\sigma'(x)-\tilde{\tau}(x)}{2}\pm\sqrt{\left(\frac{\sigma'(x)-\tilde{\tau}(x)}{2}\right)^2-\tilde{\sigma}(x)+k\sigma(x)}, and\, \lambda=k_-+\pi_-'(x) \tag{A6}$$

The value of k can be obtained by setting the discriminant in the square root in Eq. (A6) equal to zero. As such, the new eigenvalues equation can be given as

$$\lambda+n\tau'(x)+\frac{n(n-1)}{2}\sigma''(x)=0,(n=0,1,2,...) \tag{A7}$$

APPENDIX B: Solutions in Detail

Substituting $\tilde{\sigma}(x)=-\tilde{\varepsilon}+\tilde{\beta}_1x-\tilde{\beta}_2x^2$, into Eq. (A6) yields;

$$\pi(x)=\pm\sqrt{\tilde{\varepsilon}-\tilde{\beta}_1x+(\tilde{\beta}_2+k)x^2} \tag{B1}$$

The discriminant of the quadratic expression under the square root above is given as; $k=\dfrac{\tilde{\beta}_1^2-4\tilde{\beta}_2\tilde{\varepsilon}}{4\tilde{\varepsilon}}$, and substituting k into eq.(B1) yields; $\pi(x)=\pm\left(\dfrac{\tilde{\beta}_1 x}{2\sqrt{\tilde{\varepsilon}}}-\dfrac{\tilde{\varepsilon}}{\sqrt{\tilde{\varepsilon}}}\right)$ with derivative given as; $\pi'(x)=-\dfrac{\tilde{\beta}_1}{2\sqrt{\tilde{\varepsilon}}}$.

Recalling the expression for $\tau(x)$, we obtain the expression for $\tau(x)$, and its derivative respectively as follows;

$$\tau(x)=2x-\frac{\tilde{\beta}_1 x}{\sqrt{\tilde{\varepsilon}}}+\frac{\tilde{\varepsilon}}{\sqrt{\tilde{\varepsilon}}}\ \ and\ \ \tau'(x)=2-\frac{\tilde{\beta}_1}{\sqrt{\tilde{\varepsilon}}}. \tag{B2}$$

From Eq. (A7) and (A6), we have the following;

$$\frac{\tilde{\beta}_1^2-4\tilde{\beta}_2\tilde{\varepsilon}}{4\tilde{\varepsilon}}-\frac{\tilde{\beta}_1}{2\sqrt{\tilde{\varepsilon}}}=\frac{n\tilde{\beta}_1}{2\sqrt{\tilde{\varepsilon}}}-n^2-n \tag{B3}$$

Eq.(B3) yields the energy equation of the Cornell potential presented in Eq. (10).

ТЕПЛОВІ ВЛАСТИВОСТІ ТА СПЕКТРИ МАСИ ВАЖКИХ МЕЗОНІВ ЗА ПРИСУТНОСТІ ТОЧКОВОГО ДЕФЕКТУ

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У цьому дослідженні радіальне рівняння Шредінгера розв'язується аналітично за допомогою методу Нікіфорова-Уварова з потенціалом Корнела. Отримано в близькому вигляді енергетичний спектр і відповідну хвильову функцію. Вплив топологічного дефекту на теплові властивості та мас-спектри важких мезонів, таких як чармоній і боттоніум, вивчається за допомогою отриманого енергетичного спектру. Встановлено, що наявність топологічного дефекту збільшує мас-спектри та зміщує значення, близькі до експериментальних даних. Наші результати узгоджуються з експериментальними даними та вважаються кращими порівняно з іншими роботами.

Ключові слова: *рівняння Шредінгера; метод Нікіфорова-Уварова; потенціал Корнела; мас-спектри; топологічний дефект*