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# EFFECTIVE SEMICLASSICAL EVOLUTION OF BOSE EINSTEIN CONDENSATES

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In this work we analyze the effective evolution of a one dimensional Bose-Einstein Condensate (BEC) within a semi-classical description of quantum systems based on expectation values of quantum dispersions and physical observables, known as momentous quantum mechanics. We show that the most prominent features and physical parameters of the system can be determined from the dynamics of the corresponding semiclassical system, consisting of an extended phase space including original classical observables and quantum dispersions, and we also show that particle trajectories for expectation values of observables are a particular characteristic in this framework. We also demonstrate that interactions with several potentials can be implemented in an intuitive way.

**Keywords:** *Effective quantum mechanics; Bose-Einstein condensate; Semiclassical evolution*

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## 1. INTRODUCTION

Around 100 years ago, Einstein and Bose made a groundbreaking prediction that a system of noninteracting bosons, under certain conditions, would undergo a phase transition to a state with a macroscopic population in the ground state, even at finite (low) temperatures [1]. This remarkable phenomenon, known as Bose-Einstein condensation (BEC), has since captured the attention of the physics community, and has laid the foundation for exploring the quantum world on a macroscopic scale. However, it was only in recent years that experimental techniques, such as laser and evaporative cooling [2], and the development of novel traps [3], have allowed for the unambiguous observation of BEC in weakly interacting atomic Bose gases within laboratory settings [4].

On the theoretical front, the dynamics of dilute trapped Bose-Einstein gases have been effectively described by mean-field theories, with the Gross-Pitaevskii equation (GPE), a nonlinear Schrödinger equation (NLSE), proving to be a highly accurate representation of the BEC ground state and its excitation spectrum at or near absolute zero temperature [5]. Most theoretical work has primarily focused on the Thomas-Fermi limit, where the nonlinearity of the GPE dominates the bare trap excitation energies, corresponding to situations with a large number of particles [6]. Yet, the NLSE and GPE have also found applications beyond the realm of BEC, with the NLSE appearing in various fields such as optics, acoustics, and materials science [5]. In particular, the NLSE serves as a universal equation in three-dimensional problems, where analytical solutions are challenging, and numerical simulations typically demand substantial computational resources. An effective approach to the study of such systems could offer a powerful method for gaining valuable insights into the behavior of BEC models.

Effective techniques in quantum mechanics enable us to approximate solutions for complex systems. Notably, the momentous quantum mechanics formulation transforms a quantum system into a semiclassical counterpart, where the system's dynamics are determined via an effective-semiclassical Hamiltonian [7]. This method reintroduces the concept of particle trajectories, which is absent in traditional quantum mechanics. These trajectories depict the evolution of expectation values of position and momentum operators, and of quantum dispersions as well, obtained as an average of these values across an infinite number of quantum ensembles. This interpretation is akin to the evolution obtained in the Bohm description of quantum mechanics [9]. The versatility of this approach has made it applicable to the study of a wide range of quantum systems, spanning from relatively simple models such as quantum tunneling [10] to more intricate models within the domain of quantum cosmology.

In this work we explore a semiclassical analysis of a Bose-Einstein condensate in one dimension, attempting to bridge the gap between the theoretical foundations of BEC, the NLSE, and the practical applications of these principles in quantum physics and beyond.

As experimental advances continue to unlock new possibilities for the study of BEC, it is crucial to investigate the collective dynamics of these macroscopic ensembles, particularly in the context of one-dimensional systems. Our analysis aims to provide valuable insights and analytical solutions that complement numerical

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studies, further enhancing our understanding of BEC and its broader implications in various domains of physics and engineering.

## 2. THE GROSS-PITAEVSKII EQUATION AS A NLSE

In this first approach we analyze a one dimensional BEC as a semiclassical system, for which we obtain its dynamical evolution, displaying the effectiveness of the method, and discuss the broad spectrum of generalizations to more complex systems, including higher dimensional ones. This section follows closely [11].

We start considering a gas of bosons with a fixed average number of particles, $N$, confined by a potential well trap. The ground state of $N$ noninteracting bosons confined by the potential is obtained by putting all the particles in their lowest energy state, its normalized single-particle wave function is

$$\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{\frac{-m\omega}{2\hbar}x^2}. \tag{1}$$

The density distribution is $n(x) = N|\psi_o|^2$. At finite temperatures, there are particles occupying the lowest energy level (condensate), for which we require their occupation number to be large and close to the total number $N$, and others occupying higher levels (thermal component) $N_T$, so the condition for BEC occurrence is then $N_T < N$, satisfied for $T$ sufficiently small. The critical temperature at which this happens is

$$k_B T_c = 0.94\hbar\omega n^{1/3}. \tag{2}$$

Typical values of these parameters in available experiments are $N \sim 10^4 - 10^7$, and $T_c \sim (10^2 - 10^3)$nK, values very well in agreement with experimental results.

Non-interacting boson particles are well described by the expressions above, and have also been corroborated by experiments [12]. However, this non-interacting picture is simplistic: the gas in this case has infinite compressibility, and one would expect interaction between the particles to drastically change its behavior, even for very dilute samples. Therefore, and interacting system should be considered, and the conditions under which Bose-Einstein condensation is attained further studied.

The dynamic of a gas of interacting boson particles is well described by the NLSE, the Gross-Pitaevskii equation or GPE [11]

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{r}) + g|\psi|^2\right]\psi \tag{3}$$

where

$$g = \frac{4\pi\hbar^2 a}{m} \tag{4}$$

modulates the interaction and is defined in terms of the ground-state scattering length $a$. $N$ is fixed from the normalization condition for its macroscopic wave function $N = \int \psi^2 d^3\mathbf{r}$. We will consider a harmonic trapping potential $V_{\text{ext}}$.

We will analyze the evolution of this macroscopic system under an effective description, under the physical conditions discussed above, and the average distance between particles much larger than the scattering length $a$.

For a one dimensional BEC, in eq. ( 3) we take $\nabla \to \frac{\partial}{\partial x}$. Several methods and analysis were implemented over the past years attempting to solve, analytically, approximately and numerically the GP equation, in 1, 2 and 3 dimensions, for harmonic and more general potentials (Thomas-Fermi limit [6], numerical [13], variational analytical [14]). We describe now the momentous quantum mechanics method.

## 3. EFFECTIVE DESCRIPTION OF BOSE-EINSTEIN CONDENSATES

### 3.1. Momentous quantum mechanics

Momentous quantum mechanics is an effective formulation describing the semiclassical evolution of quantum systems provided by a Hamiltonian defined in an extended phase space, with expectation values of observables and quantum dispersions as classical variables [7]. The dynamical system so obtained has (in general) an infinite number of degrees of freedom. Expectation values of quantum dispersion (termed "quantum variables"), for one degree of freedom, are defined as follows

$$\Delta(x^a p^b) = \langle(\hat{x} - x)^a(\hat{p} - p)^b\rangle_{\text{Weyl.}} \tag{5}$$

where $p = \langle\hat{p}\rangle, q = \langle\hat{q}\rangle$, and Weyl refers to a totally symmetrical ordering. Similar expressions apply for more than one degree of freedom.

The quantum effective Hamiltonian, defined as $H_Q = \langle \hat{H} \rangle$, is given explicitly as

$$H_Q = H(x,p) + \sum_{n=2}^{\infty} \sum_{a=0}^{n} \frac{1}{n!} \binom{n}{a} \frac{\partial^n H(x,p)}{\partial x^a \partial p^{n-a}} \Delta(x^a p^{n-a}). \tag{6}$$

$H(q,p)$ is the corresponding classical Hamiltonian of the system. The semiclassical dynamics can be obtained from the Hamiltonian (6) in the usual way, that is, $\dot{f} = \{q, H\}$, and for quantum variables we use $i\hbar \{q, H\} = \left\langle \left[ \hat{q}, \hat{H} \right] \right\rangle$.

This formulation for the effective dynamics is valid for general quantum systems, even for those that cannot be expressed with the usual kinetic and potential terms for the Hamiltonian, that is, $H = K + U$, although one may need to implement consistent truncations to the series for complex systems, as we show below. There is an alternate description in terms of canonical Casimir-Darboux variables [8].

The variables defined in (5) are not canonical order by order in the Hamiltonian (6). It is possible to get a canonical pair of variables, and a Darboux Casimir, for one degree of freedom, by means of the following transformation

$$s = \sqrt{\Delta(x^2)}, \quad p_s = \frac{\Delta(xp)}{\sqrt{\Delta(p^2)}}, \quad U = \Delta(x^2)\Delta(p^2) - \Delta(xp)^2, \tag{7}$$

for which we get $\{s, p_s\} = 1$, and $\{s, U\} = \{p_s, U\} = 0$.

Even more, under general arguments it can be shown that all the relevant qantum information in the system can be obtained from the canonical variable $s$, and the effective Hamiltonian can be written in the following way

$$H_Q(x,s) = \frac{p^2 + p_s^2}{2m} + V_{\text{eff}}(x,s), \tag{8}$$

where the effective, all-order potential is

$$V_{\text{eff}} = \frac{U}{2ms^2} + \frac{1}{2}[V(x+s) + V(x-s)], \tag{9}$$

where $V(x)$ is the classical interaction potential.

In this way we can generate the dynamics from this canonical effective Hamiltonian and its correspondent all-order potential in the usual way. We will analyze the effective quantum evolution for the BEC under both schemes described above.

### 3.2. Effective GPE

The prescription to obtain the quantum Hamiltonian described above can be implemented for the one-dimensional GPE (3). Particularly, since this is a non-linear Schrödinger equation, we can deduce the corresponding classical Hamiltonian in the following way

$$\begin{aligned} \hat{H} &\equiv -\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(x) + g|\psi|^2 \rightarrow \\ H_{\text{class}} &\equiv \frac{p^2}{2m} + V_{ext}(x) + g|\psi|^2, \end{aligned} \tag{10}$$

considering the term $g|\psi|^2$ in $H_Q$ as part of the effective interacting potential, as we will describe below.

By far, the most interesting trapping potential $V_{ext}$ is harmonic, since it represents the most common experimental implementation for BEC's, and we will use this in the following.

As for the interpretation of the interacting, non-linear potential $\psi(x,t)$ at the classical level, we point out that the *classical* system is a starting reference upon which the effective analysis will be built, although its energy may be taken from experimental settings. The bridge between these two regimes, the quantum and the classical, is the wave function for the non interacting boson gas discussed in section 2, so we propose considering a generic squeezed coherent state of the form

$$\psi(x,t) = \frac{1}{(2\pi\rho^2)^{1/4}} \exp\left[ -\frac{\alpha}{4\rho^2}(x - \langle x \rangle)^2 + ip(x - \langle x \rangle) \right]. \tag{11}$$

with $\alpha = 1 - i\langle \Delta x \Delta p + \Delta p \Delta x \rangle$. From this the interacting potential reads (taking $\langle x \rangle = 0$)

$$g|\psi|^2 = \frac{g}{(2\pi\rho^2)^{1/2}} \exp\left[ -\frac{(x - \langle x \rangle)^2}{2\rho^2} \right] = \frac{g}{(2\pi\rho^2)^{1/2}} \exp\left[ -\frac{x^2}{2\rho^2} \right]. \tag{12}$$

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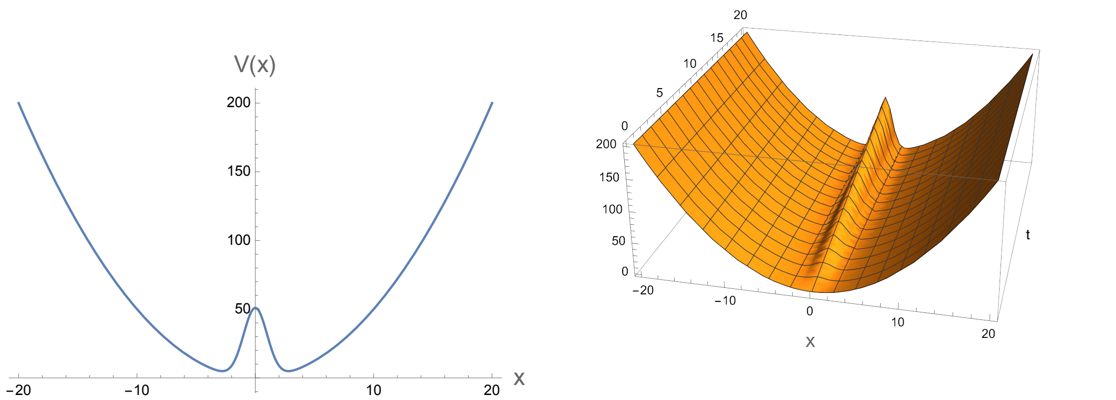
One can see this already at second order because there are three quantum variables, and they cannot conform a canonical system.

The classical Hamiltonian (10) then reads

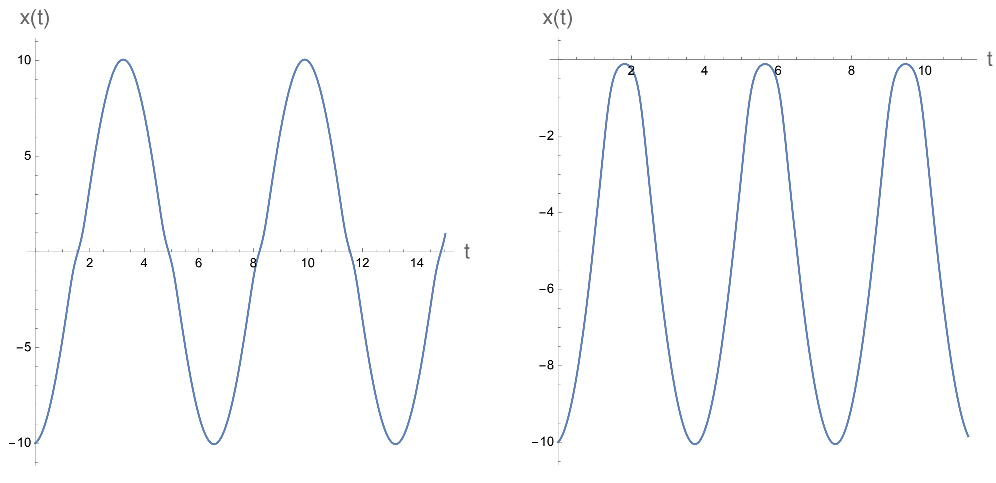
$$H(x,p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{ge^{-\frac{x^2}{2\rho^2}}}{(2\pi\rho^2)^{1/2}}. \tag{13}$$

We can determine the behavior of the 1 dimensional BEC as if it were a classical particle, driven by (13). In fig. (1) we show the "classical potential", and in fig. (2) the corresponding semiclassical trajectories, a particularly interesting feature of our effective treatment. The classical BEC particle is trapped inside the potential, and is confined in the right (left) valley, or the harmonic trap, depending on whether its energy $E$ is greater or less than the local potential maximum $\frac{a\hbar^2}{m\rho}$.




**(a)** Classical potential for $a$=10 **(b)** Classical potential as a function of $x$ and $a$.

**Figure 1.** Classical potential for the 1 dimensional BEC in a harmonic trap




**(a)** Classical trajectory for $E \geq \frac{a\hbar^2}{m\rho}$ **(b)** Classical trajectory for $E \leq \frac{a\hbar^2}{m\rho}$

**Figure 2.** Classical trajectories for the 1 dimensional BEC in a harmonic trap.

The classical Hamiltonian (13) shows the kinetic and potential terms, a form suited for the momentous effective treatment. We discuss the evolution in two parts: one where $\rho$ varies slowly and can be treated as a constant, and the general case.

## 4. EFFECTIVE EVOLUTION

We proceed to analyze the effective evolution for the BEC system, both in the order by order and the canonical descriptions.

As was discussed in the previous sections, the macroscopic quantum evolution of the condensate for non interacting particles ($g = 0$) can be determined from its wave function, which has an analytic expression given in (1); values of physical observables can be readily obtained. On the other hand, momentous quantum mechanics determines the dynamical evolution of expectation values of observables as a function of time: the macroscopic evolution of the BEC can be interpreted as an effective particle governed, at the quantum level, by the GPE (3).

For the effective quantum description of the general interacting picture we make use of the non interacting behavior, by choosing initial values of classical and quantum (dispersions) variables from their corresponding values of the later. That is, we use the expectation values obtained from the ideal BEC, and experimental parameters employed in the obtention of the condensate, as initial values for the evolution of the full quantum effective system. That is, we take $f(0) =< \hat{f} >_0$, for classical and quantum variables.

The total energy, described by the three terms in the Hamiltonian (10), $E = E_{\rm kin} + E_{\rm ho} + E_{\rm int}$ (kinetic, harmonic and interacting components), is restricted by the virial relation [11]

$$0 = 2E_{\rm kin} - 2E_{\rm ho} + 3E_{\rm int}. \tag{14}$$

Again, the initial values for each one of the components are obtained from the non interacting system. In this way, we can determine the effective dynamical evolution of the interacting BEC, as we show below.

### 4.1. Constant variance $\rho$

**4.1.1. Second order system.** As we mentioned above, we analyze first the case where $\rho$ is slowly varying, for which it can be considered constant. $\rho$ is the width of the distribution, so slow variation would represent the period of time during which stable evolution is attained, before the colapse and revival of the matter wave [15]. Its value can be controlled in experimental settings.

The effective Hamiltonian (6), together with (13) reads in this case

$$\begin{aligned} H_Q = \quad & \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + \frac{ge^{-\frac{x^2}{2\rho^2}}}{(2\pi\rho^2)^{1/2}} + \frac{\Delta(p^2)}{2m} + \frac{1}{2}m\omega^2\Delta(x^2) + \\ + \quad & \frac{ge^{-\frac{x^2}{2\rho^2}}}{(2\pi\rho^2)^{1/2}} \sum_{n=2}^{\infty} \frac{1}{n!}\left(-\frac{1}{\sqrt{2}\rho}\right)^n H_n\left(\frac{x}{\sqrt{2}\rho}\right)\Delta(x^n), \end{aligned} \tag{15}$$

where $H_n(z)$ is the Hermite polynomial of degree $n$. As can be seen in this expression, the total Hamiltonian has an infinite number of terms, rendering an impossible system to treat in full, so we truncate the series to lowest orders in dispersions.

The Hamiltonian $H_Q$ up to second order in momenta is

$$\begin{aligned} H_Q = \quad & \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + \frac{ge^{-\frac{x^2}{2\rho^2}}}{(2\pi\rho^2)^{1/2}} \\ + \quad & \frac{\Delta(p^2)}{2m} + \frac{\Delta(x^2)}{2}\left\{m\omega^2 + \frac{ge^{-\frac{x^2}{2\rho^2}}}{(2\pi\rho^2)^{1/2}}\left(\frac{x^2}{\rho^4} - \frac{1}{\rho^2}\right)\right\}. \end{aligned} \tag{16}$$

Equations of motion for classical variables follow

$$\begin{aligned} \dot{x} &= \frac{p}{m}, \\ \dot{p} &= -m\omega^2x + \frac{ge^{-\frac{x^2}{2\rho^2}}}{(2\pi\rho^2)^{1/2}}\left\{\frac{x}{\rho^2} + \frac{\Delta(x^2)}{2}\left(\frac{x^3}{\rho^6} - \frac{3x}{\rho^4}\right)\right\}, \end{aligned} \tag{17}$$

whereas for quantum variables [16]) we get

$$\begin{aligned} \frac{\mathrm{d}\Delta(x^2)}{\mathrm{d}t} &= 2\frac{\Delta(xp)}{m}, \\ \frac{\mathrm{d}\Delta(xp)}{\mathrm{d}t} &= \frac{\Delta(p^2)}{m} - \Delta(x^2)\left\{m\omega^2 + \frac{ge^{-\frac{x^2}{2\rho^2}}}{(2\pi\rho^2)^{1/2}}\left(\frac{x^2}{\rho^4} - \frac{1}{\rho^2}\right)\right\}, \\ \frac{\mathrm{d}\Delta(p^2)}{\mathrm{d}t} &= -2\Delta(xp)\left\{m\omega^2 + \frac{ge^{-\frac{x^2}{2\rho^2}}}{(2\pi\rho^2)^{1/2}}\left(\frac{x^2}{\rho^4} - \frac{1}{\rho^2}\right)\right\}. \end{aligned} \tag{18}$$

**4.1.2. Canonical system.** Now, for the canonical formulation (8) the effective Hamiltonian (13) reads

$$H_Q(x,s) = \frac{p^2+p_s^2}{2m} + \frac{U}{2ms^2} + \frac{1}{2}m\omega^2(x^2+s^2) + \frac{ge^{-\frac{(x^2+s^2)}{2\rho^2}}}{(2\pi\rho^2)^{1/2}}\cosh\left(\frac{xs}{\rho^2}\right). \tag{19}$$

The corresponding equations of motion are as follows

$$\begin{aligned}
\dot{x} &= \frac{p}{m},\\
\dot{s} &= \frac{p_s}{m},\\
\dot{p} &= -m\omega^2 x - \frac{ge^{-\frac{x^2+s^2}{2\rho^2}}}{\rho^2(2\pi\rho^2)^{1/2}}\left[s\sinh\left(\frac{xs}{\rho^2}\right) - x\cosh\left(\frac{xs}{\rho^2}\right)\right],\\
\dot{p_s} &= \frac{U}{ms^3} - m\omega^2 s - \frac{ge^{-\frac{x^2+s^2}{2\rho^2}}}{\rho^2(2\pi\rho^2)^{1/2}}\left[x\sinh\left(\frac{xs}{\rho^2}\right) - s\cosh\left(\frac{xs}{\rho^2}\right)\right].
\end{aligned} \tag{20}$$

### 4.2. General variance $\rho^2 = \Delta(x^2)$

For the general case for which $\rho$ is dynamical we need to modify the dynamics obtained in the previous section. Though more general, it is interesting to contrast this evolution with the one in the previous subsection, to determine under which regimes, or parameter values, the constant variance analysis is sufficient.

**4.2.1. Second order system** In this case, where $\rho^2 = \Delta(x^2)$, the Hamiltonian (13) gives the effective Hamiltonian, up to second order in momenta

$$\begin{aligned}
H_Q = {}& \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{ge^{-\frac{x^2}{2\Delta(x^2)}}}{[2\pi\Delta(x^2)]^{1/2}}\\
&+ \frac{\Delta(p^2)}{2m} + \frac{\Delta(x^2)}{2}\left\{m\omega^2 + \frac{ge^{-\frac{x^2}{2\Delta(x^2)}}}{[2\pi\Delta(x^2)]^{1/2}}\left(\frac{x^2}{\Delta(x^2)^2} - \frac{1}{\Delta(x^2)}\right)\right\},
\end{aligned} \tag{21}$$

or

$$H_Q = \frac{p^2+\Delta(p^2)}{2m} + \frac{m\omega^2}{2}\left[x^2+\Delta(x^2)\right] + \frac{ge^{-\frac{x^2}{2\Delta(x^2)}}}{2[2\pi\Delta(x^2)]^{1/2}}\left(1+\frac{x^2}{\Delta(x^2)}\right). \tag{22}$$

Equations of motion for classical variables are

$$\begin{aligned}
\dot{x} &= \frac{p}{m},\\
\dot{p} &= -m\omega^2 x + \frac{ge^{-\frac{x^2}{2\Delta(x^2)}}}{2[2\pi\Delta(x^2)]^{1/2}}\left(\frac{x^3}{\Delta(x^2)^2} - \frac{x}{\Delta(x^2)}\right),
\end{aligned} \tag{23}$$

while for quantum variables read

$$\frac{\mathrm{d}\Delta(x^2)}{\mathrm{d}t} = 2\frac{\Delta(xp)}{m},$$

$$\begin{aligned}
\frac{\mathrm{d}\Delta(xp)}{\mathrm{d}t} = \frac{\Delta(p^2)}{m} - m\omega^2\Delta(x^2) + \frac{ge^{-\frac{x^2}{2\Delta(x^2)}}}{2[2\pi\Delta(x^2)]^{1/2}}\Bigg\{ & 1 - \frac{x^2}{\Delta(x^2)^{1/2}}\\
&+ x^2\left(\frac{3}{\Delta(x^2)} - \frac{x^2}{\Delta(x^2)^2}\right)\Bigg\},
\end{aligned}$$

$$\frac{\mathrm{d}\Delta(p^2)}{\mathrm{d}t} = -2m\omega^2\Delta(xp) + \frac{ge^{-\frac{x^2}{2\Delta(x^2)}}}{[2\pi\Delta(x^2)]^{1/2}}\left\{\frac{\Delta(xp)}{\Delta(x^2)} + 2x^2\frac{\Delta(xp)}{\Delta(x^2)^2} - x^4\frac{\Delta(xp)}{\Delta(x^2)^3}\right)\Bigg\}. \tag{24}$$

Equations of motion for higher order truncations can be readily obtained. In the appendix we show the third order system for general $\rho$.

**4.2.2. Canonical system for general variance** Since, from (7) we have $\rho^2 = \Delta(x^2) = s^2$, the canonical all-order Hamiltonian (19) is now

$$H_Q(x,s) = \frac{p^2 + p_s^2}{2m} + \frac{U}{2ms^2} + \frac{1}{2}m\omega^2(x^2+s^2) + \frac{g e^{-\frac{(x^2+s^2)}{2s^2}}}{(2\pi s^2)^{1/2}} \cosh\left(\frac{x}{s}\right). \tag{25}$$

It is evident that the equations of motion for this case will be modified. We get

$$\begin{aligned}
\dot{x} &= \frac{p}{m}, \\
\dot{s} &= \frac{p_s}{m}, \\
\dot{p} &= -m\omega^2 x - \frac{g e^{-\frac{(x^2+s^2)}{2s^2}}}{(2\pi)^{1/2}s^3}\left[s\sinh\left(\frac{x}{s}\right) - x\cosh\left(\frac{x}{s}\right)\right], \\
\dot{p_s} &= \frac{U}{ms^3} - m\omega^2 s - \frac{g e^{-\frac{(x^2+s^2)}{2s^2}}}{(2\pi)^{1/2}s^4}\left[(x^2-s^2)\cosh\left(\frac{x}{s}\right) - xs\sinh\left(\frac{x}{s}\right)\right].
\end{aligned} \tag{26}$$

## 5. NUMERICAL EVOLUTION

We present the evolution for each one of the cases discussed in section 4. Being the dynamics governed by a system of nonlinear coupled differential equations we analyzed their solution numerically.

### 5.1. Constant variance $\rho$, second order in momenta

In this case we obtained the following initial conditions for momenta from (5) and (11): $\Delta(x^2) = \frac{1}{2\rho^2}, \Delta(p^2) = \frac{\rho^2}{2}, \Delta(xp) = 0$, and parameters: $\hbar = m = \omega = 1, a = 3$.

In Figure 3 we show the effective potential, displaying its dramatic departure from the classical one (Figure 1) due to quantum back reaction. We may also show its evolution as a function of $a$, although it is not so critical as $\rho$. The semiclassical particle evolves according to the effective potentials depicted, and we display some interesting trajectories in Figure 4: the behavior is affected by the value of $\rho$, which is the variance of the state considered.

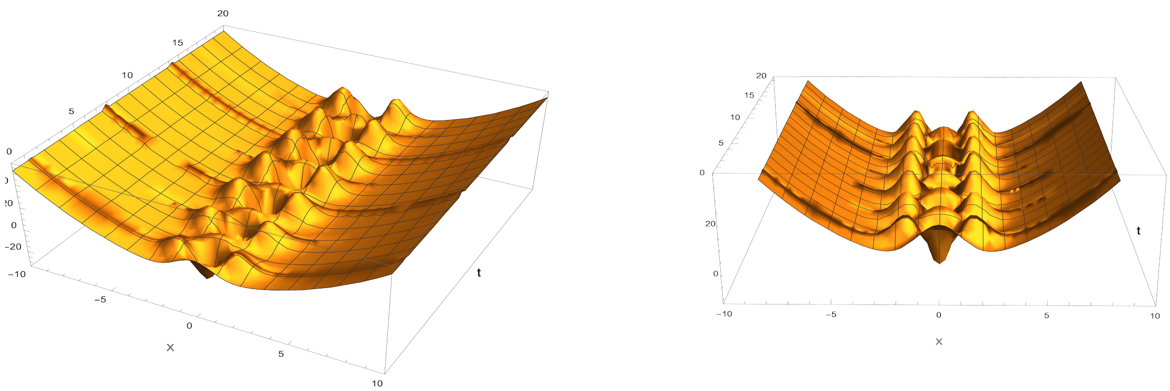
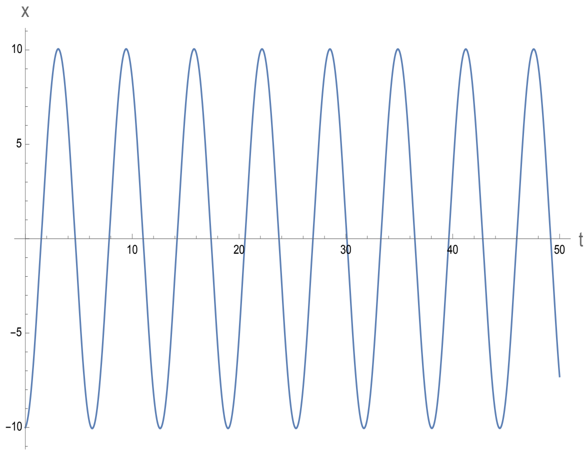



**Figure 3.** Effective potential for the BEC with constant variance $\rho = 2.5$, $a = 3$.

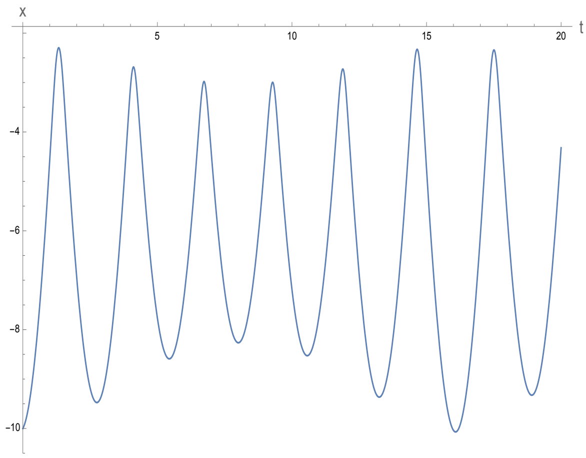
This system was obtained by truncating the Hamiltonian (6) up to second order in momenta, $\Delta(x^2), \Delta(xp), \Delta(p^2)$, and this could be extended to higher orders to take into account higher order dispersions, as shown in Appendix (B) for example. There is a generalization, though, where we do not need to make such truncations, which we analyze next.

### 5.2. Constant variance $\rho$, all-order potential

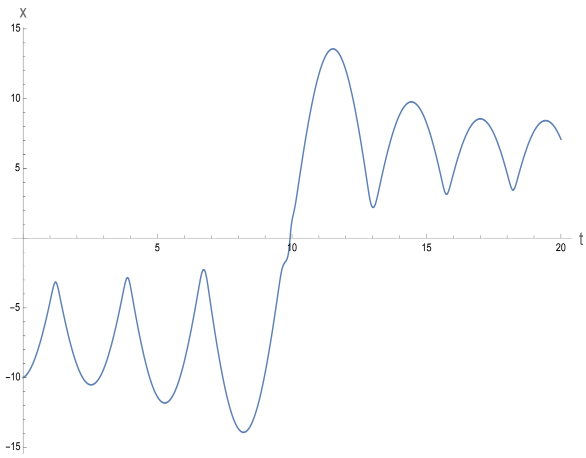
We analyze now the system for $\rho$ constant, employing the canonical formulation of the effective formulation described in subsection 4.1.2. We expect this description to be more general than the one in the previous subsection, particularly because now we have no truncations in the potential. Employing the same initial conditions and parameters as before, with the obvious modifications for variables (7) we obtain the potential shown in fig. 5



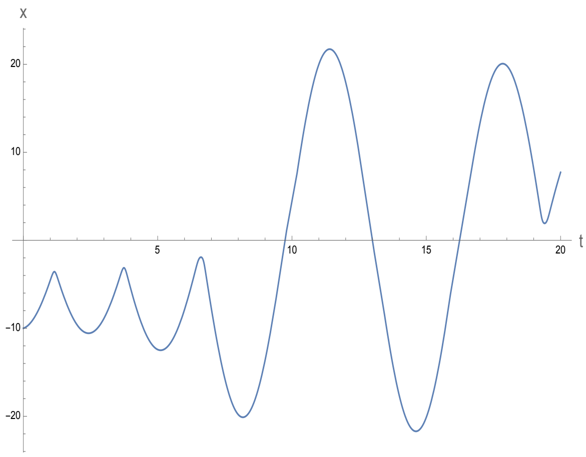

**(a)** Trajectory for $\rho = 1$




**(b)** Trajectory for $\rho = 7.5$

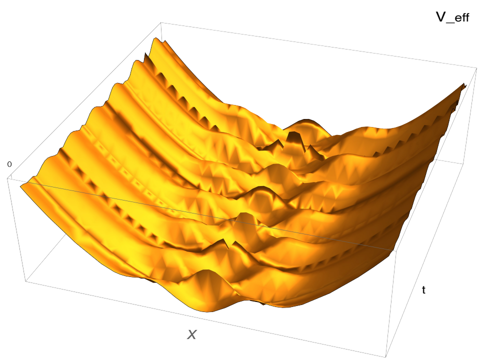



**(c)** Trajectory for $\rho = 15$

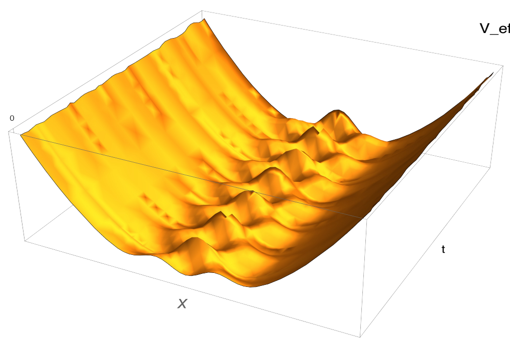



**(d)** Trajectory for $\rho = 30$

**Figure 4.** Effective trajectories with constant variance $\rho$ , $a = 3$.




**(a)** Potential for $\rho = 2.5$

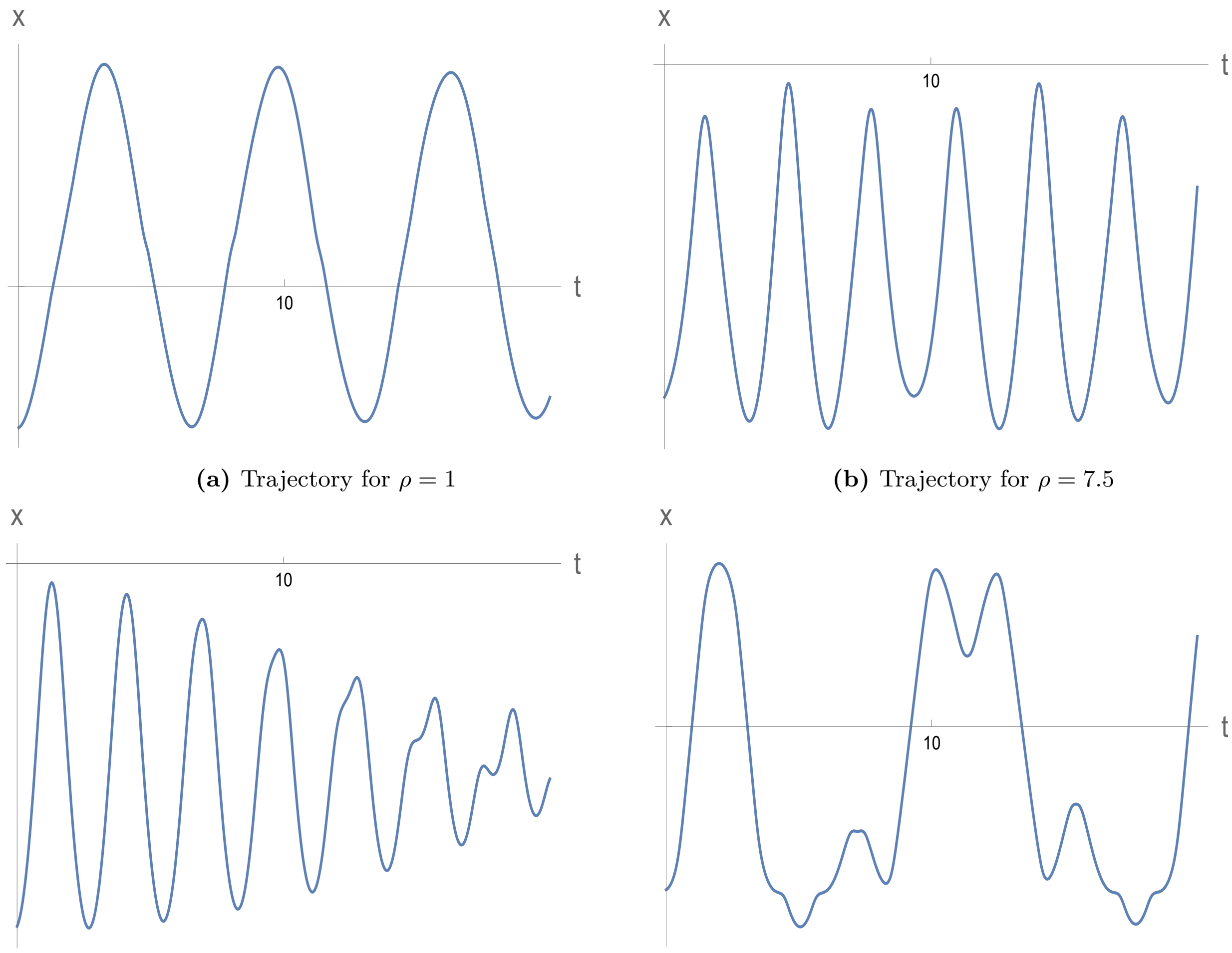



**(b)** Potential for $\rho = 1.5$

**Figure 5.** Effective potential for the BEC with constant variance $\rho = 2.5$ , canonical variable $s$.

This is the *all-orders effective potential* in (19). It is important to note that determining an effective potential, for which no truncations are needed, is the most interesting feature of our treatment, for one can analyze the entire quantum-effective evolution of the system from it. Effective trajectories are shown in fig. 6.

The behavior of the system can be analyzed from these trajectories, where one can modify the parameters and conditions according to experimental settings or phenomenological guidelines.




**(a)** Trajectory for $\rho = 1$

**(b)** Trajectory for $\rho = 7.5$

**(c)** Trajectory for $\rho = 15$

**(d)** Trajectory for $\rho = 30$

**Figure 6.** Effective trajectories with constant variance $\rho$ , $a = 3$, canonical variable $s$

### 5.3. General variance $\rho$, effective potential

We now study the evolution for the general variance $\rho$ as a dynamical variable. One can analyze several regions of interest in the physics of BEC, including interactions between two or more BEC's, evolution in the presence of external fields, among many other possibilities.

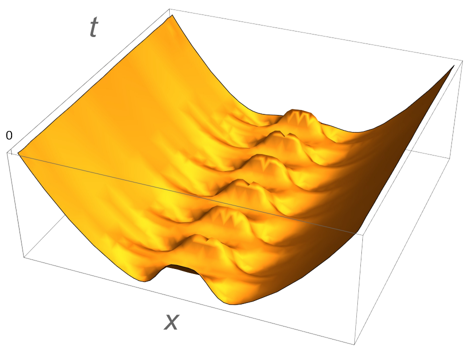
The dynamics is governed by the Hamiltonian (25). The last three terms in the r.h.s. of this expression is the effective potential for the system, for general variance. Its behavior is displayed in fig. 7, where the values for $\rho$ are the initial ones.

We can obtain the evolution of classical observables, and also determine the spreading $\Delta(x^2), \Delta(p^2)$ of the initial wave function (11), just as in [17]. Actually, this can be extended to more general interactions driven by different trapping potentials. We show trajectories in fig. 8.

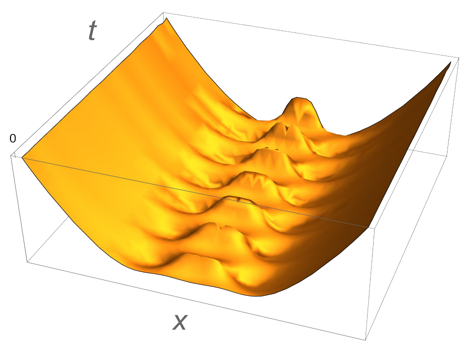
Interpretation of the dynamical evolution of the interacting BEC can be obtained from this effective treatment, particularly from the semiclassical potential and trajectories. For instance, while the "classical" particle, corresponding to the non interacting system, has a well defined behavior given the potential in fig. 1, in the interacting case, figures 5 and 7, depending on initial conditions and time of evolution, the BEC can *tunnel* from different regions in the trap. This corresponds to the collapse and revival of the matter wave discussed in [15]. By considering different trapping potentials, and even interacting BEC's, one can describe very interesting phenomena by applying this effective setting.

It is important to remember that the effective BEC is quantum in nature, so it should display its probabilistic nature. It is indeed the case, and the trajectories (and the whole evolution for that matter) should actually be displayed as $< \hat{x} > \pm\Delta x$, which is shown in fig. 9.

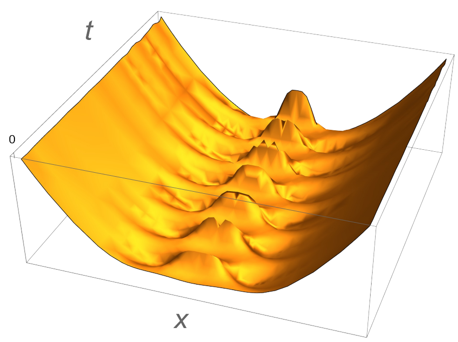
It is interesting to note that the trajectories evolve within a bounded dispersion.



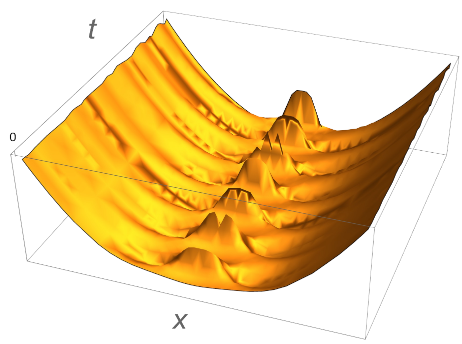

**(a)** Effective potential for $\rho_0 = 1$




**(b)** Potential for $\rho_0 = 1.8$




**(c)** Potential for $\rho_0 = 2$




**(d)** Potential for $\rho_0 = 2.5$

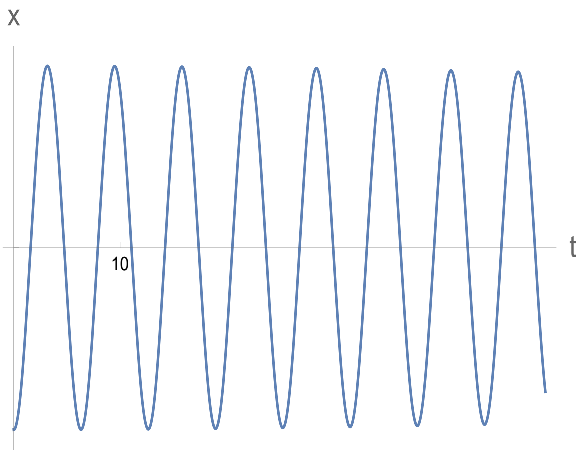
**Figure 7.** Effective potential with time-dependant variance $\rho$, in terms of the canonical variable $s$.

## 6. DISCUSSION

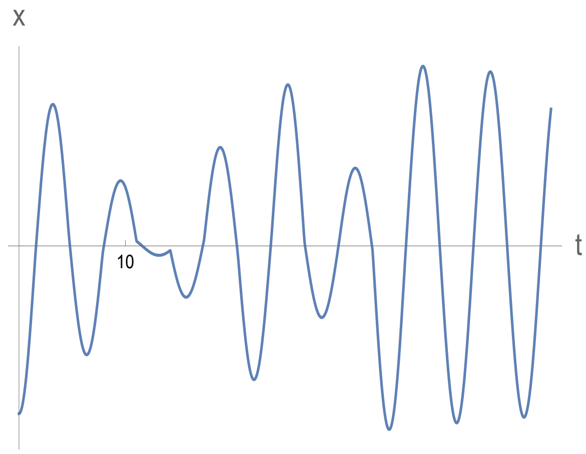
In this work we have presented a semiclassical effective treatment for a one dimensional Bose Einstein condensate in a harmonic trap with the typical interacting potential depending on the macroscopic wave function $\psi(x,t)$, whose dynamic is dictated by the Gross–Pitaevskii equation, a particular form of a non linear Schrödinger equation. We have shown that the physical information of the system is encoded in the expectation values of observables and quantum dispersions, acting as classical variables in an extended phase-space, with an effective Hamiltonian; their quantum evolution can be followed with one-particle trajectories, a very useful feature in the dynamical analysis of the system.

Our results match those obtained in different approaches of the GPE, particularly those in variational descriptions where the center and the width of the BEC cloud evolve as particles subjected to classical potentials (see [14]). Those results have been extended to the explanation of experimental results [18]. Since the setting for the momentous quantum mechanics is semiclassical, many generalizations to this one-dimensional system can be studied: two and three dimensional systems can be readily implemented, and the behavior nicely depicted as a one-particle one. Different interactions with, for example, traps or external fields can also be discussed, and the evolution can be interpreted from the effective quantum potential that we obtained, as mentioned in section 3. Moreover, the values taken for the parameters in the model come from phenomenological descriptions: experimental settings and constraints dictate the physically interesting values for quantum and effective variables.

The most remarkable feature of our analysis is the semiclassical characteristic of the condensate, allowing to treat it as a single particle (the most classical form of a matter wave) that inherently has a trajectory, something not present in usual quantum mechanics. We obtain an effective potential that controls the dynamics and evolution of the system and, from this, general features of the evolution can be obtained. As such, the question of quantum tunneling can be treated in a direct way [10], a very interesting phenomena currently under investigation. As discussed above, our results show a tunneling between regions in the trap, corresponding to

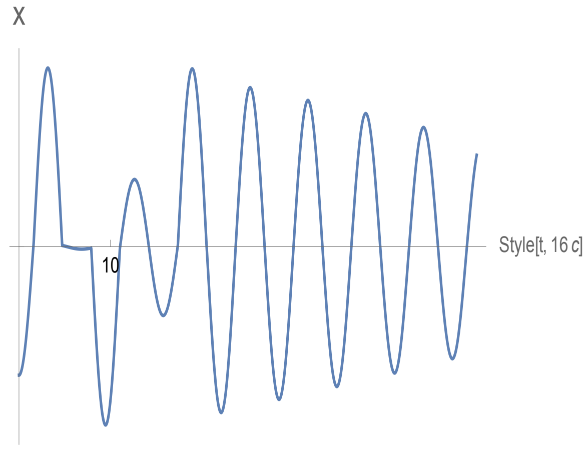



**(a)** Trajectory for $\rho_0 = 1$



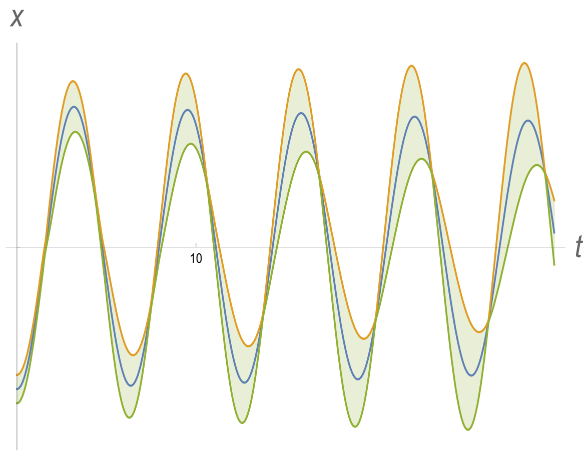

**(b)** Trajectory for $\rho_0 = 10$


**(c)** Trajectory for $\rho = 15$

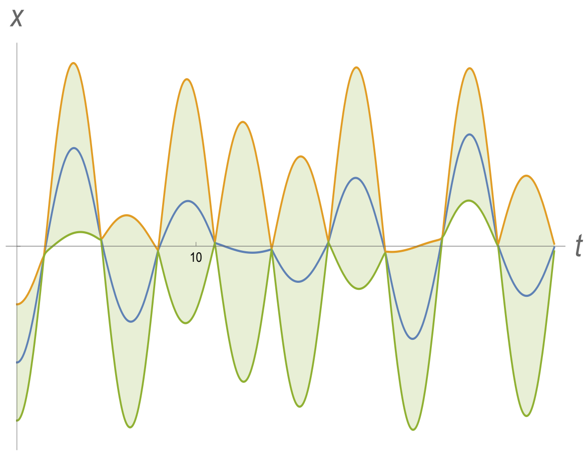



**(d)** Trajectory for $\rho_0 = 20$

**Figure 8.** Effective trajectories with time dependant variance $\rho$ , $a = 3$.




**(a)** Semiclassical trajectory with uncertainty for $\rho_0 = 2$




**(b)** Semiclassical trajectory with uncertainty for $\rho_0 = 10$

**Figure 9.** Effective trajectories with with quantum uncertainty, shaded region.

the collapse and revival of the matter wave described in [15]. Actual time of flight for condensates can also be estimated under our description.

Finally we presented two derivations for the evolution of the system, one in terms of (infinite) quantum momenta -hence the name-, and the other in terms of a all-order potential, mentioned above. We mentioned that the effective potential includes all the information that the moments do; however, the momenta expression is important in its own right because it is the one suited for systems whose Hamiltonian is not expressed in the usual kinetic and potential term.

Our methodology may serve as a valuable instrument for studying the dynamics of Bose-Einstein condensates. It offers a systematic approach to predicting and analyzing an several experimental scenarios, encompassing the influence of core-core interactions. Future applications to the investigation of interactions with radiation, expansions in BEC's and many other phenomena.

## A. GENERAL POISSON ALGEBRA FOR EXPONENTIAL TERMS

The interaction potential for the Bose-Einstein condensate includes the momenta inside an exponential function. To obtain the corresponding equations of motion we need to expand in a Taylor series, once in this polynomial form one computes the Poisson brackets among variables, and then switch back to the original expression. We perform this procedure explicitly.

First the Taylor series for the exponential

$$e^{-\frac{x^2}{2\Delta(x^2)}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!2^n}\frac{x^{2n}}{\Delta(x^2)^n}. \tag{27}$$

Poisson brackets have the following generic form

$$\left\{\Delta(x^a p^b), \frac{e^{-\frac{x^2}{2\Delta(x^2)}}}{[\Delta(x^2)]^{1/2}}\right\} = e^{-\frac{x^2}{2\Delta(x^2)}}\left\{\Delta(x^a p^b), \Delta(x^2)^{-1/2}\right\} + \left\{\Delta(x^a p^b), e^{-\frac{x^2}{2\Delta(x^2)}}\right\}\Delta(x^2)^{-1/2}. \tag{28}$$

The second term in the r.h.s. of this expression gives

$$\begin{aligned}\left\{\Delta(x^a p^b), \frac{e^{-\frac{x^2}{2\Delta(x^2)}}}{[\Delta(x^2)]^{1/2}}\right\} &= e^{-\frac{x^2}{2\Delta(x^2)}}\left\{\Delta(x^a p^b), \Delta(x^2)^{-1/2}\right\} \\ &\quad + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!2^n}x^{2n}\left\{\Delta(x^a p^b), \Delta(x^2)^{-n}\right\}\Delta(x^2)^{-1/2}. \end{aligned} \tag{29}$$

One can obtain generic formulae for any order momenta, however we obtain those for second and third order, that is, for momenta $\Delta(xp)$, $\Delta(p^2)$, $\Delta(x^2p)$, $\Delta(xp^2)$ y $\Delta(p^3)$.

For instance

$$\begin{aligned}\left\{\Delta(xp), \Delta(x^2)\right\} &= \left\{\Delta(xp), \Delta(x^2)^{1/2}\Delta(x^2)^{1/2}\right\} \\ &= \Delta(x^2)^{1/2}\left\{\Delta(xp), \Delta(x^2)^{1/2}\right\} + \left\{\Delta(xp), \Delta(x^2)^{1/2}\right\}\Delta(x^2)^{1/2} \\ &= 2\Delta(x^2)^{1/2}\left\{\Delta(xp), \Delta(x^2)^{1/2}\right\} \\ &= -2\Delta(x^2), \end{aligned} \tag{30}$$

or

$$\left\{\Delta(xp), \sqrt{\Delta(x^2)}\right\} = -\sqrt{\Delta(x^2)}, \tag{31}$$

which agrees with [16]. Repeating $n$ times we obtain

$$\left\{\Delta(xp), \frac{1}{\Delta(x^2)^n}\right\} = \frac{2n}{\Delta(x^2)^n}. \tag{32}$$

Similar expression can be obtained for other momenta

$$\left\{\Delta(p^2), \frac{1}{\Delta(x^2)^n}\right\} = 4n\frac{\Delta(xp)}{\Delta(x^2)^{n+1}}, \tag{33}$$

$$\left\{\Delta(x^2p), \frac{1}{\Delta(x^2)^n}\right\} = 2n\frac{\Delta(x^3)}{\Delta(x^2)^{n+1}}, \tag{34}$$

$$\left\{\Delta(xp^2), \frac{1}{\Delta(x^2)^n}\right\} = 4n\frac{\Delta(x^2p)}{\Delta(x^2)^{n+1}}, \tag{35}$$

$$\left\{\Delta(p^3), \frac{1}{\Delta(x^2)^n}\right\} = 6n\frac{\Delta(xp^2)}{\Delta(x^2)^{n+1}}. \tag{36}$$

Writing back in (29) we get the final result

$$\left\{\Delta(xp), e^{-\frac{x^2}{2\Delta(x^2)}}\right\} = -\frac{x^2}{\Delta(x^2)}e^{-\frac{x^2}{2\Delta(x^2)}}, \tag{37}$$

$$\left\{\Delta(p^2), e^{-\frac{x^2}{2\Delta(x^2)}}\right\} = -2x^2\frac{\Delta(xp)}{\Delta(x^2)^2}e^{-\frac{x^2}{2\Delta(x^2)}}, \tag{38}$$

$$\left\{\Delta(x^2p), e^{-\frac{x^2}{2\Delta(x^2)}}\right\} = -x^2\frac{\Delta(x^3)}{\Delta(x^2)^2}e^{-\frac{x^2}{2\Delta(x^2)}}, \tag{39}$$

$$\left\{\Delta(xp^2), e^{-\frac{x^2}{2\Delta(x^2)}}\right\} = -2x^2\frac{\Delta(x^2p)}{\Delta(x^2)^2}e^{-\frac{x^2}{2\Delta(x^2)}}, \tag{40}$$

$$\left\{\Delta(p^3), e^{-\frac{x^2}{2\Delta(x^2)}}\right\} = -3x^2\frac{\Delta(xp^2)}{\Delta(x^2)^2}e^{-\frac{x^2}{2\Delta(x^2)}}. \tag{41}$$

## B. THIRD ORDER DYNAMICAL SYSTEM

The effective Hamiltonian (6) up to third order in momenta reads

$$H_Q = \frac{p^2+\Delta(p^2)}{2m} + \frac{m\omega^2}{2}\left[x^2+\Delta(x^2)\right] + \frac{ge^{-\frac{x^2}{2\Delta(x^2)}}}{2[2\pi\Delta(x^2)]^{1/2}}\left\{1+\frac{x^2}{\Delta(x^2)} - \frac{\Delta(x^3)}{3}\left(\frac{x^3}{\Delta(x^2)^3}-\frac{3}{\Delta(x^2)^2}\right)\right\} \tag{42}$$

Equations of motion for classical variables are

$$\dot{x} = \frac{p}{m}, \tag{43}$$

$$\dot{p} = -m\omega^2x + \frac{ge^{-\frac{x^2}{2\Delta(x^2)}}}{2[2\pi\Delta(x^2)]^{1/2}}\left\{\frac{x^3}{\Delta(x^2)^2}-\frac{x}{\Delta(x^2)} - \Delta(x^3)\left(\frac{x^4}{3\Delta(x^2)^4}-\frac{x^2}{\Delta(x^2)^3}-\frac{x}{\Delta(x^2)^3}\right)\right\} \tag{44}$$

As before, we compute the equations of motion for momenta using $i\hbar\{q,H\} = \left\langle\left[\hat{q},\hat{H}\right]\right\rangle$. The equations of motion for third order momenta are

$$\frac{\mathrm{d}\Delta(x^2)}{\mathrm{d}t} = 2\frac{\Delta(xp)}{m}, \tag{45}$$

$$\frac{\mathrm{d}\Delta(xp)}{\mathrm{d}t} = \frac{\Delta(p^2)}{m} - m\omega^2\Delta(x^2) + \frac{ge^{-\frac{x^2}{2\Delta(x^2)}}}{2[2\pi\Delta(x^2)]^{1/2}}\left\{1-\frac{x^4}{\Delta(x^2)^2}+\frac{3x^2}{\Delta(x^2)}-\frac{x^2}{\Delta(x^2)^{1/2}} + \frac{\Delta(x^3)}{3}\left(\frac{x^5}{\Delta(x^2)^4}-\frac{4x^3}{\Delta(x^2)^3}-\frac{3x^2}{\Delta(x^2)^3}+\frac{6}{\Delta(x^2)^2}\right)\right\}, \tag{46}$$

$$\begin{aligned}\frac{\mathrm{d}\Delta(p^2)}{\mathrm{d}t} = -2m\omega^2\Delta(xp) \quad &+ \quad \frac{ge^{-\frac{x^2}{2\Delta(x^2)}}}{[2\pi\Delta(x^2)]^{1/2}}\left\{\frac{\Delta(xp)}{\Delta(x^2)} + x^2\left(2\frac{\Delta(xp)}{\Delta(x^2)^2} - x^2\frac{\Delta(xp)}{\Delta(x^2)^3}\right)\right.\\ &+ \frac{x^3}{3}\left(x^2\frac{\Delta(x^3)\Delta(xp)}{\Delta(x^2)^5} + 3\frac{\Delta(x^2p)}{\Delta(x^2)^3} - 7\frac{\Delta(x^3)\Delta(xp)}{\Delta(x^2)^4}\right)\\ &+ \left. 5\frac{\Delta(x^3)\Delta(xp)}{\Delta(x^2)^3} - 3\frac{\Delta(x^2p)}{\Delta(x^2)^2} - x^2\frac{\Delta(x^3)\Delta(xp)}{\Delta(x^2)^4}\right\},\end{aligned} \tag{47}$$

$$\frac{\mathrm{d}\Delta(x^3)}{\mathrm{d}t} = 3\frac{\Delta(x^2p)}{m}, \tag{48}$$

$$\begin{aligned}\frac{\mathrm{d}\Delta(x^2p)}{\mathrm{d}t} = 2\frac{\Delta(xp^2)}{m} - m\omega^2\Delta(x^3) \quad &+ \quad \frac{ge^{-\frac{x^2}{2\Delta(x^2)}}}{2[2\pi\Delta(x^2)]^{1/2}}\left\{3 + \frac{\Delta(x^3)}{\Delta(x^2)} + 5\frac{\Delta(x^3)^2}{\Delta(x^2)^3} - 3\frac{\Delta(x^4)}{\Delta(x^2)^2}\right.\\ &+ x^2\left(2\frac{\Delta(x^3)}{\Delta(x^2)^2} - x^2\frac{\Delta(x^3)}{\Delta(x^2)^3} - \frac{\Delta(x^3)^2}{\Delta(x^2)^4}\right)\\ &+ \left.\frac{x^3}{3}\left(3\frac{\Delta(x^4)}{\Delta(x^2)^3} + x^2\frac{\Delta(x^3)^2}{\Delta(x^2)^5} - 7\frac{\Delta(x^3)^2}{\Delta(x^2)^4} - \frac{3}{\Delta(x^2)}\right)\right\},\end{aligned} \tag{49}$$

$$\begin{aligned}\frac{\mathrm{d}\Delta(xp^2)}{\mathrm{d}t} = \frac{\Delta(p^3)}{m} \quad &- \quad 2m\omega^2\Delta(x^2p) + \frac{ge^{-\frac{x^2}{2\Delta(x^2)}}}{[2\pi\Delta(x^2)]^{1/2}}\left\{\frac{\Delta(x^2p)}{\Delta(x^2)} + 5\frac{\Delta(x^2p)\Delta(x^3)}{\Delta(x^2)^3} + 3\frac{\Delta(xp)}{\Delta(x^2)} - 3\frac{\Delta(x^3p)}{\Delta(x^2)^2}\right.\\ &+ x^2\left(2\frac{\Delta(x^2p)}{\Delta(x^2)^2} - x^2\frac{\Delta(x^2p)}{\Delta(x^2)^3} - \frac{\Delta(x^2p)\Delta(x^3)}{\Delta(x^2)^4}\right)\\ &+ \left.\frac{x^3}{3}\left(3\frac{\Delta(x^3p)}{\Delta(x^2)^3} + x^2\frac{\Delta(x^2p)\Delta(x^3)}{\Delta(x^2)^5} - 7\frac{\Delta(x^2p)\Delta(x^3)}{\Delta(x^2)^4} - 3\frac{\Delta(xp)}{\Delta(x^2)^2}\right)\right\},\end{aligned} \tag{50}$$

$$\begin{aligned}\frac{\mathrm{d}\Delta(p^3)}{\mathrm{d}t} = \quad &- \quad 3m\omega^2\Delta(xp^2) + \frac{ge^{-\frac{x^2}{2\Delta(x^2)}}}{[2\pi\Delta(x^2)]^{1/2}}\left\{3\frac{\Delta(xp^2)}{\Delta(x^2)} + 15\frac{\Delta(xp^2)\Delta(x^3)}{\Delta(x^2)^3} + 9\frac{\Delta(p^2)}{\Delta(x^2)}\right.\\ &- 9\frac{\Delta(x^2p^2)}{\Delta(x^2)^2} + \frac{3\hbar^2}{2\Delta(x^2)^2}\\ &+ x^2\left(6\frac{\Delta(xp^2)}{\Delta(x^2)^2} - 3x^2\frac{\Delta(xp^2)}{\Delta(x^2)^3} - 3\frac{\Delta(xp^2)\Delta(x^3)}{\Delta(x^2)^4}\right)\\ &+ \left. x^3\left(x^2\frac{\Delta(xp^2)\Delta(x^3)}{\Delta(x^2)^5} - 7\frac{\Delta(xp^2)\Delta(x^3)}{\Delta(x^2)^4} + 3\frac{\Delta(x^2p^2)}{\Delta(x^2)^3} - 3\frac{\Delta(p^2)}{\Delta(x^2)^2} - \frac{\hbar^2}{2\Delta(x^2)^3}\right)\right\}\end{aligned} \tag{51}$$

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## REFERENCES

[1] S.N. Bose. "Plancks gesetz und lichtquantenhypothese." Zeitschrift für Physik **26**, 178-181 (1924). Einstein. "Quantentheorie des einatomigen idealen gases, sitzungsberichte kgl." Preuss. Akad. Wiss **261** (1924). http://dx.doi.org/10.1007/BF01327326

[2] F. Schreck, and K. van Druten, "Laser cooling for quantum gases," Nature Phys. **17**(12), 1296-1304 (2021). https://doi.org/10.1038/s41567-021-01379-w

[3] B. Rhyno, N. Lundblad, D.C. Aveline, C. Lannert, S. Vishveshwara, "Thermodynamics in expanding shell-shaped Bose-Einstein condensates," Physical Review A, **104**6, 063310 (2021). https://doi.org/10.1103/PhysRevA.104.063310

[4] R.A. Dunlap, *Lasers and Their Application to the Observation of Bose-Einstein Condensates*, (Morgan and Claypool Publishers, 2019). https://doi.org/10.1088/2053-2571/ab2f2f

[5] J. Tempere, *Bose-Einstein Condensation, Superfluidity and Superconductivity*, (Universiteit Antwerpen, 2019).

[6] S. Stringari, "Collective excitations of a trapped Bose-condensed gas," Physical Review Letters, **77**, 2360 (1996). https://doi.org/10.1103/PhysRevLett.77.2360
F. Dalfovo, S. Giorgini, L. Pitaevskii, and S. Stringari," Rev. Mod. Phys. **71**, 463 (1999). https://doi.org/10.1103/RevModPhys.71.463

[7] M. Bojowald, and A. Skirzewski, "Effective equations of motion for quantum systems," Rev. Mathematical Phys. **18**(07), 713-745 (2006). https://doi.org/10.1142/S0129055X06002772

[8] B. Baytaş, M. Bojowald, and S. Crowe, "Effective potentials from semiclassical truncations," Physical Review A, **99**(4), 042114 (2019). https://doi.org/10.1103/PhysRevA.99.042114

[9] J.T. Cushing, A. Fine, and S. Goldstein, editors *Bohmian mechanics and quantum theory: an appraisal*. Vol. **184**, (Springer Science Business Media, 2013).

[10] L. Aragon-Muñoz, G. Chacón-Acosta, and H. Hernandez-Hernandez, "Effective quantum tunneling from a semiclassical momentous approach," International Journal of Modern Physics B, **34**(29), 2050271 (2020). https://doi.org/10.1142/S0217979220502719

[11] L. Pitaevskii, and S. Stringari, *Bose-Einstein condensation and superfluidity*, Vol. 164. (Oxford University Press, 2016).

[12] J.R. Ensher, et al., "Bose-Einstein condensation in a dilute gas: Measurement of energy and ground-state occupation," Physical Review Letters **77**, 4984 (1996). https://doi.org/10.1103/PhysRevLett.77.4984

[13] X. Antoine, W. Bao, and C. Besse, "Computational methods for the dynamics of the nonlinear Schrödinger/Gross–Pitaevskii equations," Computer Physics Communications, **184**(12), 2621-2633 (2013). https://doi.org/10.1016/j.cpc.2013.07.012

[14] V.M. Pérez-García, H. Michinel, J.I. Cirac, M. Lewenstein, and P. Zoller, "Dynamics of Bose-Einstein condensates: Variational solutions of the Gross-Pitaevskii equations," Physical Review A, **56**(2), 1424 (1997). https://doi.org/10.1103/PhysRevA.56.1424

[15] M. Greiner, O. Mandel, T.W. Hänsch, and I. Bloch, "Collapse and revival of the matter wave field of a Bose–Einstein condensate." Nature 419, **6902**, 51-54 (2002). https://doi.org/10.1038/nature00968

[16] B. Baytaş, M. Bojowald, and S. Crowe, "Faithful realizations of semiclassical truncations," Annals of Physics, **420**, 168247 (2020). https://doi.org/10.1016/j.aop.2020.168247

[17] L.C. Pereira, and V.A. do Nascimento. "Dynamics of Bose–Einstein Condensates Subject to the Pöschl–Teller Potential through Numerical and Variational Solutions of the Gross–Pitaevskii Equation," Materials, **13**(10), 2236 (2020). https://doi.org/10.3390/ma13102236

[18] V.M. Pérez-García, H. Michinel, J.I. Cirac, M. Lewenstein, and P. Zoller, "Low energy excitations of a Bose-Einstein condensate: A time-dependent variational analysis," Physical review letters, **77**(27), 5320 (1996). https://doi.org/10.1103/PhysRevLett.77.5320

## ЕФЕКТИВНА НАПІВКЛАСИЧНА ЕВОЛЮЦІЯ КОНДЕНСАТІВ БОЗЕ-ЕЙНШТЕЙНА

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У цій роботі ми аналізуємо ефективну еволюцію одновимірного конденсату Бозе-Ейнштейна (BEC) у напівкласичному описі квантових систем на основі очікуваних значень квантових дисперсій і фізичних спостережуваних, відомих як важлива квантова механіка. Ми показуємо, що найвидатніші особливості та фізичні параметри системи можна визначити з динаміки відповідної напівкласичної системи, що складається з розширеного фазового простору, включаючи оригінальні класичні спостережувані та квантові дисперсії, і ми також показуємо, що траєкторії частинок для очікуваних значень спостережувані є особливою характеристикою в цій структурі. Ми також демонструємо, що взаємодія з декількома потенціалами може бути реалізована інтуїтивно зрозумілим способом.

**Ключові слова:** *ефективна квантова механіка; конденсат Бозе-Ейнштейна; напівкласична еволюція*