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FLRW COSMOLOGY WITH HYBRID SCALE FACTOR IN $f(R, L_m)$ GRAVITY

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In this paper, we aim to describe the cosmic late-time acceleration of the Universe in $f(R, L_m)$ gravity framework proposed by Harko (2010) with the help of an equation of state for strange quark matter. To achieve this, we adopt a specific form of $f(R, L_m)$ gravity as $f(R, L_m) = \frac{R}{2} + L_m^n$, where n is arbitrary constants. Here we utilize a hybrid scale factor to resolve the modified field equations in the context of $f(R, L_m)$ gravity for an isotropic and homogeneous Friedmann–Lemaître–Robertson–Walker (FLRW) metric in presence of strange quark matter (SQM). Also, we analyze the dynamics of energy density, pressure and the state finder parameters and explained the distinctions between our model and the current dark energy models in the presence of SQM. We observed a transition from an accelerating to a decelerating phase in the Universe, followed by a return to an accelerating phase at late times. Also, we analyzed the state finder diagnostic as well equation of state parameter and found that the model exhibited quintessence-like behavior. The conclusion drawn from our investigation was that the proposed $f(R, L_m)$ cosmological model aligns well with recent observational studies and effectively describes the cosmic acceleration observed during late times.

Keywords: *FLRW Cosmological Model; $f(R, L_m)$ gravity; Strange Quark Matter; Hybrid Scale Factor*

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1. INTRODUCTION

In the twentieth century, the late-time accelerated expansion of the Universe has been a puzzling cosmic mystery that has caused controversy among researchers. Some astronomical and cosmological observations, including Supernovae Ia (SN Ia) [1, 2, 3, 4, 5], Cosmic Microwave Background (CMB) [6, 7], Baryon Acoustic Oscillations (BAO) in galaxy clustering [8, 9], Large Scale Structure (LSS) [10, 11] and Wilkinson Microwave Anisotropy Probe (WMAP) [12], all point to the conclusion that the Universe is presently experiencing accelerated expansion. This contradicts the prevailing theory of General Relativity (GR) on a large scale. During the early stages of the Big Bang, radiation played a significant role in driving the expansion of the Universe. As Universe expanded and cooled, matter took over, leading to a matter-dominated phase. However, recent observations suggest that we are now in a new era where "Dark Energy" (DE) is the dominant force influencing the expansion of the Universe. The exact mechanism behind this faster expansion is still under debate. To overcome this problem, researchers have proposed various alternative theories with one of the most common being modified gravity theories (MGT). These MGTs provide alternative explanations for the cosmic acceleration and serve as potential substitutes for GR.

Buchadahl (1970) introduced most favorite modified gravity as $f(R)$ gravity which provided a way to extend Einstein's universal theory of relativity. This theory was developed to explain the rapid expansion of the Universe and formation of its structures. Some $f(R)$ models were considered in [13, 14] for their ability to pass regional tests and incorporate concepts of dark energy and inflation. Additionally, it was speculated that $f(R)$ gravity models could potentially describe galactic dynamics of large test particles without the need for dark matter [15, 16, 17, 18, 19].

Harko and Lobo (2010) recently proposed the $f(R, L_m)$ gravity theory, where $f(R, L_m)$ is a function of Lagrangian matter density (L_m) and Ricci scalar (R). This theory represents the most general form of Riemann-space gravitational theories. In $f(R, L_m)$ gravity, test particles experience non-geodesic motion with additional forces perpendicular to their four-velocity vectors [20]. Some researchers found that the $f(R, Lm)$ gravity models open up new possibilities that extend beyond the algebraic structure observed in the Hilbert-Einstein action [21]. The energy conditions in $f(R, Lm)$ are broad and versatile, encompassing both the familiar energy conditions in General Relativity and $f(R)$ gravity. These conditions allow for arbitrary couplings, non-minimal couplings, and non-couplings between matter and geometry [22]. The mass-radius relationship is explored within the non-minimal geometry-matter coupling $f(R, Lm)$ gravity through investigating the simplest case as

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$f = R + L_m + RL_m$, where the gravitational field is coupled to the matter field and the coupling constant [23]. The $f(R, L_m)$ cosmological model concurs with present observations and effectively predicts late-time cosmic acceleration for the FRW metric [24]. Some researchers [25] investigate a transit dark energy cosmological model in $f(R, Lm)$ gravity and using observational constraints, they establish a significant relationship between energy density parameters. The anisotropic nature of the Universe has been explored in $f(R, L_m)$ gravity for spatially homogeneous and isotropic FRW cosmological model and determine the present phase of the Universe [26]. The FRW metric solutions in $f(R, L_m)$ gravity successfully evade the Big-Bang singularity and can predict cosmic acceleration without relying on a cosmological constant owing to the geometry-matter coupling terms in the Friedmann-like equations [27]. Incorporating bulk viscosity, the $f(R, L_m)$ cosmological model offers a robust explanation for recent observations, effectively capturing the cosmic expansion scenario [28]. Certain scholars have explored wormhole solutions within the framework of $f(R, L_m)$ gravity and derived the field equations for the general $f(R, L_m)$ function, considering the static as well spherically symmetric Morris-Thorne wormhole metric [29].

The hybrid scale factor plays a crucial role in achieving viable cosmic dynamics without relying on any specific relationship between the pressure and energy density of the Universe in teleparallel gravity[30, 31]. Some authors [32] have obtained the exact solutions for LRS Bianchi-I metric in presence of holographic dark energy using hybrid expansion law. Some scholars used a hybrid scale factor to investigate the shearing, non-rotating and expanding character of the cosmos, which approaches anisotropy over large values of time t [33]. The $(n + 2)$ dimensional flat FRW Universe in the framework of general theory of relativity has been investigated utilizing hybrid expansion law with thick domain wall and bulk viscous fluid [34]. Certain scholars successfully tackled the Einstein field equations for strange quark matter to explore a 5-dimensional cosmological model discussed the dynamic aspects of concerning solution [35]. The Kantowski-Sachs cosmological model has been explored in the $f(R)$ theory of gravity with quark and strange quark matter and found that the spatial volume V is finite at $t = 0$, expands as t increases and becomes infinitely large as $t \to \infty$ [36]. Certain authors have successfully derived the solution for gravitational field equations using a power law relationship between the metric potentials and equation of state (EoS) [37].

Building upon the aforementioned investigations, many scholars have extensively studied FRW cosmological models, investigating their behavior concerning different energy sources and using hybrid expansion law as well strange quark matter in various modified gravity scenarios. These dedicated authors have sought to unveil the dynamic and cosmological properties of our Universe [38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48].

2. FORMALISM OF $f(R, L_m)$ GRAVITY

The $f(R, Lm)$ gravity model proposed by Harko and Lobo (2010) [20] is a generalization of the $f(R)$ gravity whose action is given by

$$S = \int f(R, L_m)\sqrt{-g}d^4x \tag{1}$$

Here $f(R, L_m)$ is a function of Ricci scalar R and Lagrangian of the matter density L_m.
One can acquire the Ricci scalar R by contracting the Ricci tensor R_{ij} as

$$R = g^{ij}R_{ij} \tag{2}$$

where the Ricci-tensor is given by

$$R_{ij} = \partial_\kappa \Gamma^\kappa_{ij} - \partial_j \Gamma^\kappa_{\kappa i} + \Gamma^\lambda_{ij}\Gamma^\kappa_{\lambda\kappa} - \Gamma^\kappa_{j\lambda}\Gamma^\lambda_{\kappa i} \tag{3}$$

Here $\Gamma^i_{j\kappa}$ represents the components of well-known Levi-Civita connection as indicated by

$$\Gamma^i_{j\kappa} = \frac{1}{2}g^{i\lambda}\left[g_{\kappa\lambda,j} + g_{\lambda j,\kappa} - g_{j\kappa,\lambda}\right] \tag{4}$$

The corresponding field equations of $f(R, L_m)$ gravity can be derived by varying the action (1) with respect to g_{ij} as,

$$f_R R_{ij} + (g_{ij}\Box - \nabla_i\nabla_j) f_R - \frac{1}{2}(f - f_{L_m}L_m) g_{ij} = \frac{1}{2}f_{L_m}T_{ij} \tag{5}$$

Where, $f_R = \frac{\delta f(R,L_m)}{\delta R}$, $f_{L_m} = \frac{\delta f(R,L_m)}{\delta L_m}$, $\Box = \nabla_i\nabla^j$ and T_{ij} is the Stress-energy momentum tensor for perfect fluid, given by

$$T_{ij} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{ij}} \tag{6}$$

By contracting the field Eq. (5), we find the relationship between Ricci scalar R, matter Lagrangian density L_m and T trace of the stress-energy-momentum tensor T_{ij} as

$$Rf_R + 3\Box f_R - (f - f_{L_m} L_m) = \frac{1}{2} f_{L_m} T \tag{7}$$

where $\Box F = \frac{1}{\sqrt{-g}} \partial_i \left(\sqrt{-g} g^{ij} \partial_j F\right)$ for any function of F
Now by taking covariant derivative of Eq. (5), one can acquire the following result as

$$\nabla^i T_{ij} = 2\nabla^i ln(f_{L_m}) \frac{\partial L_m}{\partial g^{ij}} \tag{8}$$

3. THE MOTION EQUATIONS IN $f(R, L_m)$ GRAVITY

According to the most recent observations of Planck collaboration (2013) [49], our Universe is isotropic and homogeneous at larger scales. Therefore, to explore the current cosmological model we consider the flat Friedman-Lematre-Robertson-Walker (FLRW) metric of the form,

$$ds^2 = -dt^2 + a^2 \left(dx^2 + dy^2 + dz^2\right) \tag{9}$$

Where $a(t)$ is the scale factor that signifies the expanding nature of the Universe at a time t.

For metric (9), the non-zero components of Christoffel symbols are

$$\Gamma^0_{ij} = a\dot{a}\delta_{ij}, \Gamma^k_{0j} = \Gamma^k_{j0} = \frac{\dot{a}}{a}\delta^k_j, \quad for \;\; i, j, k = 1, 2, 3 \tag{10}$$

With the help of (4), the non zero components of Ricci tensor are given by

$$R_{00} = -3\frac{\ddot{a}}{a}, \qquad R_{11} = R_{22} = R_{33} = a\ddot{a} + 2\dot{a}^2 \tag{11}$$

Hence, the Ricci-scalar (R) associated with line element (9) is found as

$$R = 6\left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right] = 6\left[\dot{H} + H^2\right] \tag{12}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter.

As the quark gluon plasma served as a perfect fluid, the energy momentum tensor (EMT) for strange quark matter (SQM) is given by

$$T_{ij} = (\rho_{sq} + p_{sq})\, u_i u_j + p_{sq} g_{ij} \tag{13}$$

Here ρ_{sq} and p_{sq} are energy density and thermodynamic pressure of the SQM. and $u^i = (1, 0, 0, 0)$ components of co-moving four velocity vectors in cosmic fluid with $u_i u^j = 0$ and $u_i u^i = -1$.
.

$$\rho_{sq} = \rho_q + B_c \qquad p_{sq} = p_q - B_c \tag{14}$$

Following the assumption that quarks are non-interacting and massless particles, an the pressure is approximated by an EoS of the form

$$p_q = \frac{\rho_q}{3} \tag{15}$$

Therefore, $p_{sq} = \frac{1}{3}(\rho_{sq} - \rho_0)$ is the linear EoS for SQM with ρ_0 is the density at zero pressure. If $\rho_0 = 4B_c$, the EoS for strange quark matter in the bag is reduced to

$$p_{sq} = \frac{\rho_{sq} - 4B_c}{3} \tag{16}$$

where, B_c is the bag constant.

The modified Friedmann equations which describe the dynamics of Universe in $f(R, L_m)$ gravity are given by

$$3H^2 f_R + \frac{1}{2}(f - Rf_R - f_{L_m} L_m) + 3H\dot{f_R} = \frac{1}{2} f_{L_m} \rho_{sq} \tag{17}$$

and

$$\dot{H} + 3H^2 f_R - \ddot{f_R} - 3H\dot{f_R} + \frac{1}{2}(f_{L_m} L_m - f) = \frac{1}{2} f_{L_m} p_{sq} \tag{18}$$

The overhead dot (.) depicts the derivative corresponding to time t.

4. COSMOLOGICAL SOLUTIONS FOR $f(R, L_m)$ GRAVITY

To examine the dynamics of Universe we employ the functional form of $f(R, L_m)$ gravity [26, 28, 29] of the form

$$f(R, L_m) = \frac{R}{2} + L_m^n \tag{19}$$

where n is any arbitrary constants and one can retain to standard Friedmann equations of GR for $n = 1$

For this particular functional form of $f(R, Lm)$ gravity, we have considered $Lm = \rho$ [50] and hence, for matter-dominated Universe, the Friedmann equations (17) and (18) yields,

$$2\dot{H} + 3H^2 = (n-1)\rho_{sq}^n - n\rho^{n-1}p_{sq} \tag{20}$$

and

$$3H^2 = (2n-1)\rho_{sq}^n \tag{21}$$

5. HYBRID SCALE FACTOR

The hybrid scale factor presents a transition in the cosmic evolution, shifting from early deceleration to late-time acceleration. In the early phase, the cosmic dynamics are dominated by power law behavior, while in the late phase, the exponential factor takes over. In the early stages of the Universe, the scale factor becomes zero, implying the absence of an initial singularity. Consequently, the chosen scale factor yields a time-dependent deceleration parameter, which effectively characterizes the transition of the Universe over time.

Therefore, to derive exact solutions for Friedmann equations (20) & (21) which involves three unknowns namely H, ρ_{sq} and p_{sq}, we employed the hybrid scale factor [51, 52, 53] as

$$a = e^{\alpha t} t^{\eta} \tag{22}$$

where, α and η

are positive constants. Also, when $\eta = 0$, the scale factor reverts to the exponential law and when $\alpha = 0$, the scale factor reduces to the power law.

So with this scale factor, we found the following time dependent kinematical properties as,

The Spatial Volume (V) given by

$$V = a^3 = \left(e^{\alpha t} t^{\eta}\right)^3 \tag{23}$$

The deceleration parameter (q) plays a crucial role in understanding the past and future evolution of the Universe. In cosmology, the deceleration parameter (q) is a measure of the rate at which expansion of the Universe is changing.For a universe dominated by matter and radiation, $q > 0$, indicating a decelerating expansion. On the other hand, if the Universe is dominated by dark energy with negative pressure, the $q < 0$, implying an accelerating expansion.

The Deceleration Parameter (q) is

$$q = -1 + \frac{\eta}{(\alpha t + \eta)^2} \tag{24}$$

The Hubble Parameter (H),

$$H = \frac{\alpha t + \eta}{t} \tag{25}$$

Scalar Expansion (θ)

$$\theta = 3H = 3\left(\frac{\alpha t + \eta}{t}\right) \tag{26}$$

Using Equ. (25) in Equ. (21) we obtained

Energy Density (ρ_{sq}) for SQM as

$$\rho_{sq} = \left[\frac{1}{(2n-1)}\left(\frac{\alpha t + \eta}{t}\right)^2\right]^{\frac{1}{n}} \tag{27}$$

Pressure (p_{sq}) for SQM is

$$p_{sq} = \frac{2}{n}\frac{\eta}{t^2}\left[\frac{1}{(2n-1)}\left(\frac{\alpha t + \eta}{t}\right)^2\right]^{\frac{n-1}{n}} - \frac{1}{(2n-1)}\left(\frac{\alpha t + \eta}{t}\right)^2 \tag{28}$$

The equation of state (EoS) for SQM is

$$\omega = \frac{p_{sq}}{\rho_{sq}} = \frac{\frac{2}{n}\frac{\eta}{t^2}\left[\frac{1}{(2n-1)}\left(\frac{\alpha t+\eta}{t}\right)^2\right]^{\frac{n-1}{n}} - \frac{1}{(2n-1)}\left(\frac{\alpha t+\eta}{t}\right)^2}{\left[\frac{1}{(2n-1)}\left(\frac{\alpha t+\eta}{t}\right)^2\right]^{\frac{1}{n}}} \tag{29}$$

from Equ. (14) the density and pressure for quark matter is given by

$$\rho_q = \left[\frac{1}{(2n-1)}\left(\frac{\alpha t+\eta}{t}\right)^2\right]^{\frac{1}{n}} - B_c \tag{30}$$

and

$$p_q = \frac{2}{n}\frac{\eta}{t^2}\left[\frac{1}{(2n-1)}\left(\frac{\alpha t+\eta}{t}\right)^2\right]^{\frac{n-1}{n}} - \frac{1}{(2n-1)}\left(\frac{\alpha t+\eta}{t}\right)^2 + B_c \tag{31}$$

6. STATEFINDERS DIAGNOSTIC

Statefinder parameters are cosmological diagnostic tools used to study the expansion dynamics of the Universe. They were introduced as a way to probe the nature of dark energy, the mysterious force driving the accelerated expansion of the Universe. These statefinder parameters were proposed by Sahni et al. (2003) in their paper titled "Statefinder—a new geometrical diagnostic of dark energy" [54]. During this research they were discusses how these parameters can be useful in distinguishing between various dark energy models, including quintessence and cosmological constant models, by examining their trajectories in the $\{r, s\}$ plane. By measuring these statefinder parameters from observational data, cosmologists can gain insights into the nature of dark energy and fate of the Universe, helping to test and refine our understanding of the fundamental laws governing the cosmos.

The statefinder pair $\{r, s\}$ is defined as

$$r = \frac{\dddot{a}}{aH^3} \tag{32}$$

and

$$s = \frac{r-1}{3(q-\frac{1}{2})} \tag{33}$$

We analyze the statefinder parameters (r, s) for our cosmological $f(R, L_m)$ model. The values $(r, s) = (1, 0)$ and $(1, 1)$ are representative of the ΛCDM (Lambda Cold Dark Matter) and CDM (Cold Dark Matter) models, respectively. However, $s > 0$ and $r < 1$ correspond to dark energy (DE) models, such as the phantom and quintessence models. On the other hand, when $r > 1$ and $s < 0$, it reflects the behavior of the Chaplygin gas model.

With the help of Equ. (22), (24) and (25), above equations can be written as

$$r = 1 - \frac{3\eta}{(\alpha t+\eta)^2} + \frac{2\eta}{(\alpha t+\eta)^3} \tag{34}$$

$$s = \frac{2\eta[3\eta(\alpha t+\eta) - 2\eta]}{3(\alpha t+\eta)\,[3(\alpha t+\eta)^2 - 2\eta]} \tag{35}$$

When time (t) is zero, the statefinder pair attains the values

$\{r, s\} = \left\{1 - \frac{3\eta-2}{\eta^2}, \frac{2}{3\eta}\right\}$

From the figure we have observed that,

- Figure 1, clearly demonstrates that the average scale factor and spatial volume maintain a constant value at the initial time point $(t = 0)$. However, as time progresses, both parameters show a steady and consistent growth, eventually extending towards infinity for prolonged periods (t). This remarkable observation indicates an ongoing and continuous expansion of the Universe.

- From Figure 2, it is depict that the deceleration parameter (q) decreases as cosmic time increases and approaches -1 for large values of t, indicating the accelerating phase of the Universe which coincident with the observations of type Ia supernovae [2].

- In Figure 3, it is shown that both the Hubble parameter and scalar expansion parameters exhibit diversity during the early stages of the Universe. As time approaches infinity, these parameters tend to zero. The graph indicates that the Universe initially experienced rapid and infinite expansion, but later settled into a constant expansion rate during a later epoch.

- Figure 4 exhibits the variation of pressure from significantly large negative values to reaching zero intriguing negative pressure phenomenon is commonly referred to as dark energy (DE), playing a crucial role in driving the accelerated expansion of Universe.

- According to Figure 5, the energy density fallows an interesting trend, starting with a substantial value initially. However, as time progresses, the energy density gradually diminishes and eventually approaches zero as $t \to \infty$. This striking behavior strongly suggests the expansion of Universe.

- The Figure 6 depicts the evolution of equation of state (EoS) over time (t). It illustrates a transition from an accelerating to a decelerating phase, eventually returning to an accelerating phase of the Universe over late time.

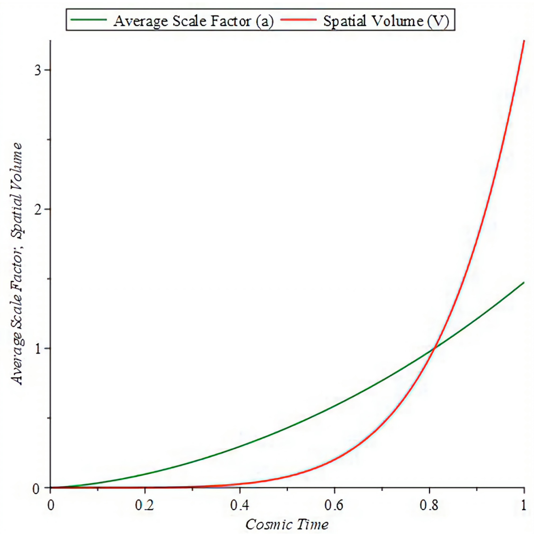

Figure 1. Variation of Average Scale Factor (a) & Volume (V) against Cosmic Time (t) for $\alpha = 0.5$ and $\eta = 1.5$

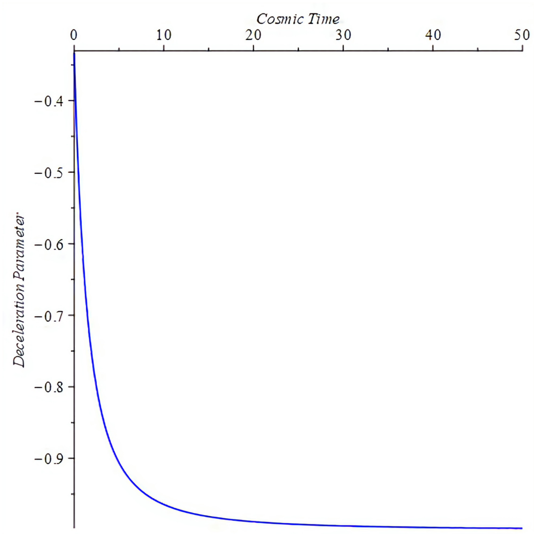

Figure 2. Variation of Deceleration Parameter (q) against Cosmic Time (t) for $\alpha = 0.5$ and $\eta = 1.5$

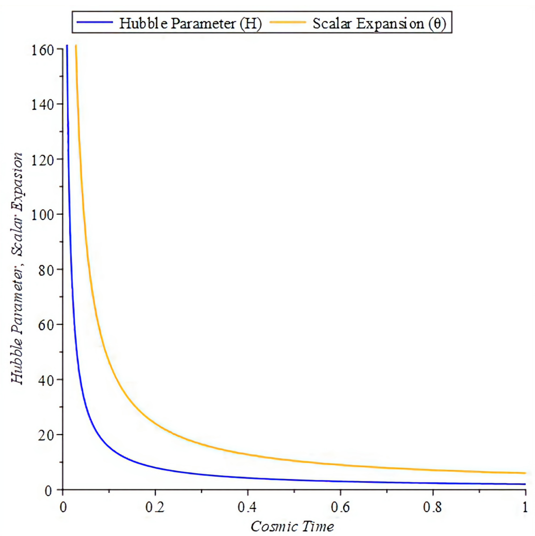

Figure 3. Variation of Hubble Parameter (H) & Scalar Expansion (θ) against Cosmic Time (t) for $\alpha = 0.5$ and $\eta = 1.5$

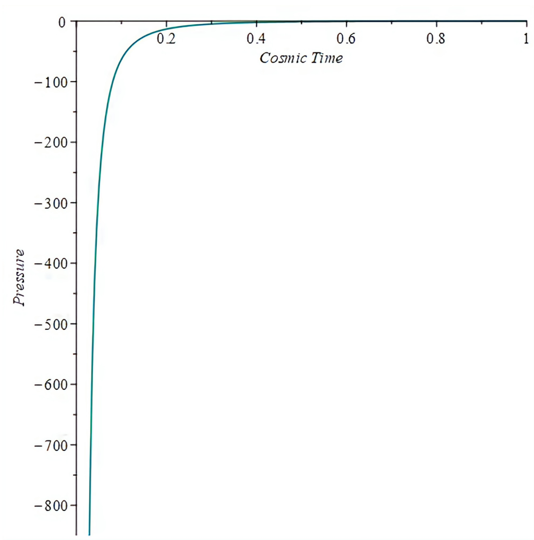

Figure 4. Variation of Pressure (P) against Cosmic Time (t) for $\alpha = 0.5$ and $\eta = 1.5$

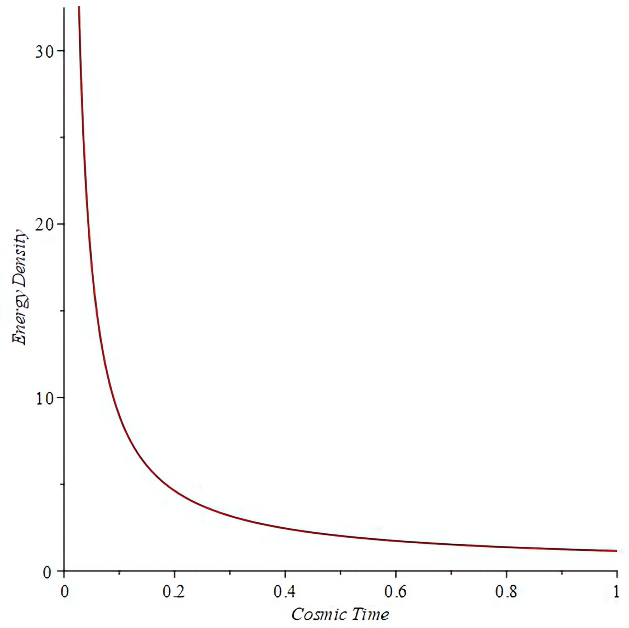

Figure 5. Variation of Energy Density (ρ) against Cosmic Time (t) for α=0.5 and $\eta = 1.5$

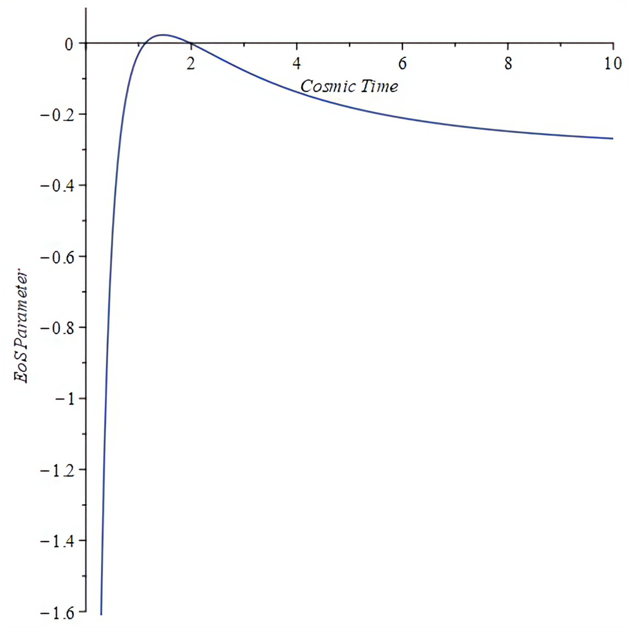

Figure 6. Variation of Equations of state Parameter (EoS) against Cosmic Time(t) for $\alpha = 0.5$ and $\eta = 1.5$

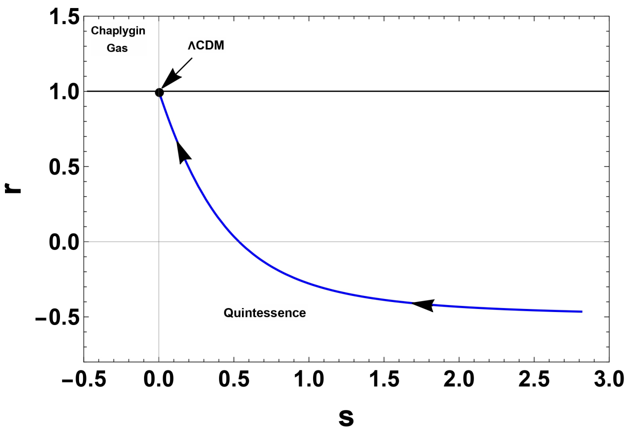

Figure 7. Evolution trajectory of $r - s$ Plane

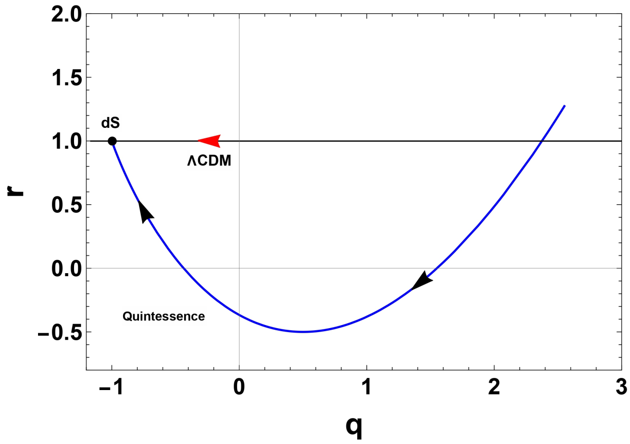

Figure 8. Evolution trajectory of $r - q$ Plane

- Yet it is clear from the aforementioned traits that the evolutionary trajectories split depending on different parameter choices and vary from one model to another. Particularly noteworthy is Figure 7, which shows how to plot the $r - s$ planes. Compared to looking at the r and s evolution separately, analyzing the $r - s$ plane is clearer. This clarity is very helpful when comparing various cosmological models. Given this situation, the $r - s$ planes' usefulness increases because of their clear evolutionary paths and obvious directional cues, which make differentiating between discrepancies easier.

- Figure 8 is a $r - q$ plane, which vividly demonstrate the consistency with the characteristics deliberated upon in this section. Moreover, it is noteworthy that the most suitable-fit model within this category showcases the capacity to transition from an initial phase of decelerating expansion to a subsequent phase of cosmic acceleration in the late stages of the Universe's evolution.

- We have observed from Figure 7 and 8 that, in the long run, they both have a tendency to evolve in a way that resembles a ΛCDM model, i.e., $\{r, s\} = \{1, 0\}$ and $\{q, r\} = \{-1, 1\}$ in future.

7. DISCUSSION AND CONCLUDING REMARKS

In this article, we investigates the late-time cosmic accelerated expansion of the Universe using a specific form of $f(R, L_m)$ gravity as non-linear model, $f(R, L_m) = \frac{R}{2} + L_m^n$, where n is free model parameter. During this study we derived the motion equations for the isotropic and homogeneous FLRW cosmological model with strange quark matter (SQM). We have obtained cosmological model using hybrid scale factor and exhibited a smooth transition of the Universe from accelerating phase to decelerated phase and retain to accelerating phase.

The results of this study are extremely persuasive, leading to the formulation of the subsequent conclusions:

- We have noted that the average scale factor and spatial volume (V) remain bounded around $t = 0$, progressively expanding as time (t) advances, ultimately approaching an infinitely large value as $t \to \infty$, as depicted in Figure 1. This signifies that the expansion of the Universe initiates with a finite volume, progressively extending as time unfolds and results matched with [55, 56, 57].

- The deceleration parameter (q) is a measure of the rate at which the expansion of Universe is changing over time (t). It is used to describe the acceleration or deceleration of the universe's expansion under the influence of gravitational forces and other factors. In our study, we observed the deceleration parameter (q) is decreasing function of cosmic time (t) and approaches -1, it suggests that the expansion of the universe is accelerating, and we got the ΛCDM model.

- We noticed that, the Hubble parameter (H) and scalar expansion (θ) both initially have large value and as time progresses i.e. $t \to \infty$, the values of H and θ approaches to zero. This reflects that, in the initial moments after the Big Bang, the Universe expanded rapidly and as time progresses the expansion rate begins to decreases which is good agreement with results [58, 59].

- Our study reveals that the Universe's pressure for SQM (p_{sq}) experiences growth with cosmic time (t). It starts at a highly negative value and gradually approaches zero at the current epoch. Recent cosmological findings attribute the Universe's accelerated expansion to dark energy, characterized by negative pressure. Consequently, the model aligns well with observations of the type Ia Supernovae [1].

- In the study, it was determined that the energy density for SQM (ρ_{sq}) consistently remains positive and decreases as cosmic time. Initially, the energy density remained constant during the early epoch. The Universe could potentially reach a steady state in the distant future, as the (ρ_{sq}) tends to diminish significantly over extended periods of time.

- Initially, as time (t) approaches zero, the cosmological model exhibits a Phantom phase with $\omega < -1$, signifying an accelerated expansion. After a finite period, the model converges towards $\omega = -1$, representing the cosmological constant (Λ) and aligning with the ΛCDM model, characterized by continued accelerated expansion. Subsequently, the model enters the quintessence region with $\omega > -1$, maintaining this state for a certain duration. Beyond $t = 1.1$, the model transitions from an accelerating to a decelerating phase (for transition phase of the universe one can refer [60]), but it later reverts to an accelerating phase. Ultimately, the model reenters the quintessence region, where it persists during late times.

- In the present paradigm, we have conducted an assessment of the statefinder parameters and subsequently depicted the graphical representations of the $r - s$ and $r - q$ planes. Our investigation has yielded results indicating that $\{r, s\} = \{1, 0\}$ and $\{q, r\} = \{-1, 1\}$ respectively, providing a striking manifestation that the current model aligns closely with the characteristics of the de Sitter point, a foundational feature of the ΛCDM cosmological framework.

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КОСМОЛОГІЯ FLRW ІЗ ГІБРИДНИМ МАСШТАБНИМ КОЕФІЦІЄНТОМ У $f(R, L_m)$ ГРАВІТАЦІЇ

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У цій статті ми маємо за мету описати пізнє космічне прискорення Всесвіту в гравітаційній системі $f(R, L_m)$, запропонованій Харко (2010) за допомогою рівняння стану дивної кваркової матерії. Щоб досягти цього, ми приймаємо певну форму сили тяжіння $f(R, L_m)$ як $f(R, L_m) = \frac{R}{2} + L_m^n$, де n є довільна константа. Тут ми використовуємо гібридний масштабний коефіцієнт для вирішення модифікованих рівнянь поля в контексті гравітації $f(R, L_m)$ для ізотропної та однорідної метрики Фрідмана–Леметра–Робертсона–Уокера (FLRW) у присутності дивної кваркової матерії (SQM). Крім того, ми аналізуємо динаміку щільності енергії, тиску і параметрів шукача стану та пояснюємо відмінності між нашою моделлю та поточними моделями темної енергії за наявності SQM. Ми спостерігали перехід від фази прискорення до фази уповільнення у Всесвіті, а потім повернення до фази прискорення у пізніший час Крім того, ми проаналізували діагностику визначника стану та рівняння параметрів стану та виявили, що модель продемонструвала квінтесенційну поведінку. Наше дослідження дійшло висновку, що запропонована космологічна модель $f(R, L_m)$ добре узгоджується з нещодавніми спостережними дослідженнями та ефективно описує космічне прискорення, яке спостерігалося в останні часи.

Ключові слова: *космологічна модель; $f(R, L_m)$ гравітація; дивна кваркова матерія; гібридний масштабний фактор*