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INVESTIGATING THE EFFECT OF GRAVITY MODULATION ON WEAKLY NONLINEAR MAGNETOCONVECTION IN A NONUNIFORMLY ROTATING NANOFLUID LAYER[†]

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This paper investigates the impact of gravity modulation on weakly nonlinear magnetoconvection in a nanofluid layer that is nonuniformly rotating. The fundamental equations are obtained for the Cartesian approximation of the Couette flow using the Boussinesq approximation and gravitational modulation. The weakly nonlinear regime is analyzed using the method of perturbations with respect to the small supercritical parameter of the Rayleigh number, considering the effects of Brownian motion and thermophoresis in the nanofluid layer. Heat and mass transfer are evaluated in terms of finite amplitudes and calculated from the Nusselt numbers for the fluid and the volume concentration of nanoparticles. The findings demonstrate that gravitational modulation, nonuniform rotation, and differences in the volume concentration of nanoparticles at the layer boundaries can effectively control heat and mass transfer. Additionally, the negative rotation profile has a destabilizing effect. The study shows that the modulated system conveys more heat and mass than the unmodulated system.
Keywords: *Nanofluid; Nonuniformly rotating layer; Weakly nonlinear theory; Gravity modulation; Non-autonomous Ginzburg-Landau equation*
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1. INTRODUCTION

Nanofluids have received significant interest in recent years due to their unique thermal properties, which make them attractive for various applications. Convection in nanofluids is important for analyzing their behavior. The sudden enhancement in thermal conductivity and variety of behavior make it essential to investigate these models. Choi [1] introduced the study of nanofluids as fluids containing a scatter of solid particles with characteristic dimensions of 10 or 100 nm scaled. The use of nanofluids, which have smaller-sized particles providing a larger relative surface area than microsized particles, can improve heat and mass transfer properties and overcome problems such as clogging channels, drastic pressure drops, settling, and premature wear on channels and components. By adding nanoparticles to base fluids, their thermal conductivity can be enhanced by 15-40 %, which is crucial in modern engineering and research. These heat-exchanging situations are found in various fields, including biomechanics, spinning machines like nuclear reactors, food, geophysical problems, chemical processing, and the petroleum industry. Nanofluids have a variety of applications due to their ability to enhance heat and mass transfer using mixed nano-sized particles. These fluids can also control transport processes, making them useful in drug delivery systems.

In recent years, theoretical studies of the Rayleigh-Bénard convection (RBC) in nanofluids have attracted attention due to the potential for improved heat transfer and energy efficiency in various engineering applications. Several studies have investigated the effects of nanoparticle size, concentration, and shape on the onset of convection in Rayleigh-Benard cells. Buongiorno [2] conducted a thorough investigation of convective transport in nanofluids, with a particular focus on elucidating the enhanced heat transfer observed under convective flows. Tzo[3] utilized the transport equations proposed by Buongiorno to explore the onset of convection in a horizontally heated layer using nanofluids. The study revealed that the presence of Brownian motion and thermophoresis in the nanoparticles significantly reduces the critical Rayleigh number, by one to two orders of magnitude, compared to that of a conventional fluid.

Buongiorno and Hu [4], Kuznetsov and Niel [5] studied the effects of nanoparticle concentration on the onset of convection in nanofluids. The presence of nanoparticles in nanofluids can either enhance or suppress the onset of convection, leading to changes in heat and mass transport characteristics. This is attributed to the presence of concentration gradients of the nanoparticles within the fluid. Previous research has consistently shown that the instability of the flow is primarily driven by buoyancy forces, which are independent of the specific effects of Brownian motion and thermophoresis. Therefore, in order to control instability in the medium, it becomes necessary to modify the gravity field in nanofluids.

One of the possible ways to modify the gravitational field in a nanofluid is through its modulation. In general, modulation refers to the deliberate periodic change in certain parameters or conditions that affect convective processes. The study of the effect of modulation on convection is necessary to understand how external disturbances or changes in some parameters can affect the flow and transport phenomena within the system. In the study of RBC, several types of modulation techniques are commonly employed to investigate the influence of external perturbations on the convective flow. Some of the commonly used modulation techniques in RBC include:

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1. Temperature Modulation: This involves periodically varying the temperature difference across the fluid layer, either by applying a time-dependent boundary temperature or by modulating the heat input or extraction from the system. Temperature modulation can be achieved through various methods, such as sinusoidal variations \cite{6s}, square wave modulation, or modulating the temperature gradient [7].

2. Gravity Modulation: In this technique, the gravitational field acting on the fluid layer is periodically varied. It can be achieved by physically oscillating the entire system or by using time-dependent external forces or fields to induce periodic changes in the effective gravity [8].

3. Rotation Modulation: Rotation modulation involves periodically changing the rotation rate or angular velocity of the system. This can be achieved by modulating the speed of rotation, introducing intermittent rotation, or applying time-dependent torque or angular momentum variations [9].

4. Magnetic field modulation: The modulation of an external magnetic field on RBC refers to the time-dependent variation of the applied magnetic field in a system experiencing convective heat transfer [10].

Currently, researchers extensively utilize these types of modulation to investigate the characteristics of heat transfer in RBC across different media. Due to the vast amount of literature available on this topic, we will provide a concise overview focusing specifically on the impact of gravity field modulation on convective processes.

At present, studies of nanofluids in nonlinear modes of convection under modulation are rapidly developing. Bhadauria and Kiran[11], pioneered the investigation of gravity modulation effects on nanofluid convection in nonlinear modes. Bhadauria et al. [12] investigated the impact of gravity modulation on nanoconvection and observed that modulation plays a role in regulating transport phenomena at finite amplitudes. In addition, Kiran [13] studied the nonlinear thermal instability in a viscoelastic, nanofluid-saturated porous medium under gravitational modulation. Kiran et al. \cite{14s} also investigated the problem of internal heating in a similar way to the work by Bhadauria et al. \cite{12s}. Kiran and Narasimhulu [15]-[16] introduced the concept of out-of-phase modulation and lower boundary modulation in nanoconvection, and their findings revealed that modulation not only influences transport phenomena but also chaotic convection. Furthermore, Kiran et al. [17] explored the effect of throughflow on nanofluid convection and discovered that throughflow, whether inflow or outflow, can either enhance or diminish energy transfer in the medium. More recently, Kiran \cite{18s}-\cite{20s} conducted studies on the impact of g-jitter on RBC and Darcy convection. Kiran et al. [21] investigated the effect of g-jitter on the RBC of nanofluids with the Ginzburg-Landau (GL) model. Using nonlinear analysis, the thermal and concentration Nusselt numbers are calculated depending on other physical parameters. The effect of gravity modulation and rotation on thermal instability in a horizontal layer of a nanofluid was investigated by Manjula et al. [22].

The previous studies discussed concentrated mainly on the problem's plane geometry, using a Cartesian coordinate system. In laboratory experiments, however, the presence of Couette flows between two rotating cylinders with different angular speeds is extremely important. The angular velocity of fluid rotation in this configuration can be described by the relation:

$$\Omega(R)=\frac{\Omega_2 R_2^2-\Omega_1 R_1^2}{R_2^2-R_1^2}+\frac{(\Omega_1-\Omega_2)R_1^2R_2^2}{R^2(R_2^2-R_1^2)}=a+\frac{b}{R^2}\,,$$

where $R_{1,2}$ and $\Omega_{1,2}$ represent the inner and outer cylinders' radius and angular velocity, respectively. Chandrasekhar [23] and Velikhov [24] were the first to analyze the stability of such a flow in the presence of a magnetic field in a perfectly conducting medium. Research on magnetic convection in nonuniformly rotating media is currently scarce. This topic was partially explored in our previous articles [25]-[27], which focused on astrophysical applications, and in our nanofluid physics studies [28]. In our research, we suggested using electrically conducting nanofluids to simulate the magnetorotational instability (MRI) in laboratory settings. Theoretical studies conducted in [28] showed that standard MRI, azimuthal MRI, and helical MRI can be implemented in a nonuniformly rotating layer of an electrically conducting nanofluid. We also examined stationary regimes of nonuniformly rotating convection in axial and helical magnetic fields, taking into account temperature and nanoparticle concentration gradients. Our studies aimed to investigate the conditions for stabilizing and destabilizing stationary convection in axial and helical magnetic fields. The absence of a Ginzburg-Landau (GL) model in the existing literature for nonlinear magnetic convection in a nonuniformly rotating nanofluid layer subjected to gravity field modulation has motivated the present study.

The article is structured as follows. Section 1 (Introduction) provides an overview of recent research in this area and also indicates the need for a GL model for nonlinear magnetic convection in a non-uniformly rotating nanofluid layer under the influence of gravitational field modulation. Section 2 describes the problem in detail and derives evolution equations for small perturbations using the Boussinesq approximation. The study is focused on a rotating layer of an incompressible electrically conductive nanofluid in a modulated gravitational field, taking into account a constant temperature gradient and nanoparticle concentration. In Section 3, we study the weakly nonlinear stage of stationary convection in a nonuniformly rotating layer of an electrically conductive liquid in a modulated gravitational field. Using the method of perturbation theory with respect to the small supercritical parameter of the Rayleigh number $\varepsilon=\sqrt{(Ra-Ra_c)/Ra_c}$, we obtained a nonlinear GL equation with a periodic coefficient. In Section 4, we studied linear magnetic convection in a nonuniformly rotating layer of a nanofluid depending on variations in the system parameters.

Further, the results of numerical solutions of the non-autonomous GL equation are presented, showing the dependence of the heat transfer value (Nusselt number Nu) and mass transfer (nano-Nusselt number Nu_ϕ) on the parameters of the nanofluid $(\mathrm{Pr}, \mathrm{Rn}, \mathrm{L}e)$, the profile of the inhomogeneous rotation (Rossby numbers Ro), frequency Ω , and modulation amplitude δ . Conclusions (Section 5) represent the main findings of this paper.

2. GOVERNING EQUATIONS

Let us consider convective flows in a nonuniformly rotating layer of an electrically conductive nanofluid in an externally constant magnetic field and under the influence of gravity field modulation. In order to describe the nonlinear convective flows in the layer, a Cartesian approximation of the Couette flow is used instead of the cylindrical coordinate system (R, φ, z) (see Fig. 1)

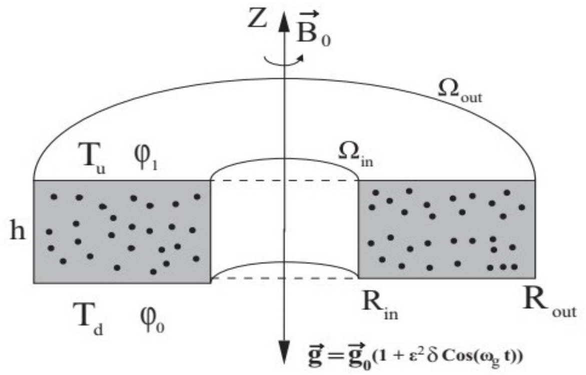

Figure. 1 Electrically conducting nanofluid fills the layer between two rotating cylinders with angular velocities of Ω_{in} and Ω_{out} and is located in an external uniform magnetic field: $\vec{B}_0 = B_0 \vec{e}_z$.

A fixed area of the fluid layer with radius R_0 and angular velocity $\vec{\Omega}$ is considered, and the coordinates $X = R - R_0$, $Y = R_0(\varphi - \varphi_0)$, and $Z = z$ are used to represent the radial, azimuthal, and vertical directions, respectively. The nonuniform rotation of the fluid layer can be locally represented as a rotation with an angular velocity Ω_0 and an azimuthal shear (see Fig. 2), whose velocity profile is locally linear:

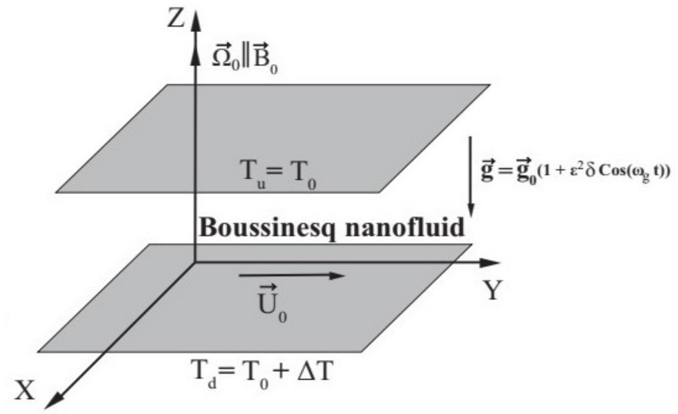

Figure. 2 Cartesian approximation for nonuniformly rotating magnetoconvection under the parametric influence of gravity modulation.

$\vec{U}_0 = -q\Omega_0 X \vec{e}_Y$, where $q = -d \ln \Omega / d \ln R$ is the dimensionless shear parameter determined from the angular velocity profile $\Omega(R) = \Omega_0 (R / R_0)^{-q}$. The shear parameter q is related to the hydrodynamic Rossby number Ro as follows: $q = -2Ro$. Examples of the values of the Rossby parameter include $Ro = 0$ for solid rotation, $Ro = -3/4$ for Keplerian rotation, and $Ro = -1$ for a Rayleigh angular velocity profile $\Omega(R) \sim R^{-2}$.

An electrically conductive nanofluid layer is located between two horizontal planes, $Z = 0$ and $Z = h$, and is heated from below and cooled from above, where $T_d = T_0 + \Delta T$ is the temperature at the lower boundary of the horizontal layer, and $T_u = T_0$ is the temperature at the upper boundary of the horizontal layer (see Fig. 2). We assume that the volume fraction of nanoparticles is also constant at the boundaries and has the corresponding values: ϕ_0 at

$Z=0$ and ϕ_1 at $Z=h$. If $\phi_1>\phi_0$, we have a top-heavy configuration of nanoparticles in nanofluid, and otherwise, for $\phi_1<\phi_0$, we have a bottom-heavy configuration of nanoparticles in nanofluid. Let us consider that the direction of the external constant magnetic field $\vec{B}_0$ coincides with the fluid rotation axis $\vec{\Omega}_0 \parallel OZ$.

The effect of gravity field modulation will be considered based on the equations of magnetohydrodynamics in a rotating coordinate system for an electrically conductive nanofluid in the Boussinesq approximation (see, for example, Kopp et al. [28]):

$$\rho\left(\frac{\partial\vec{v}}{\partial t}-q\Omega_0 X\frac{\partial\vec{v}}{\partial Y}+((\vec{v}+\vec{U}_0)\nabla)\vec{U}_0+(\vec{v}\nabla)\vec{v}\right)=-\nabla\left(P+\frac{\mu_e\vec{B}^2}{8\pi}\right)-2\rho(\vec{\Omega}_0\times(\vec{v}+\vec{U}_0))+\frac{\mu_e}{4\pi}(\vec{B}\nabla)\vec{B}+\mu\nabla^2\vec{v}+$$
$$+[\phi\rho_p+(1-\phi)(\rho_f(1-\beta(T-T_u)))]\vec{g}(t) \tag{1}$$

$$\frac{\partial\vec{B}}{\partial t}-q\Omega_0 X\frac{\partial\vec{B}}{\partial Y}+(\vec{v}\nabla)\vec{B}-(\vec{B}\nabla)\vec{U}_0-(\vec{B}\nabla)\vec{v}=\eta\nabla^2\vec{B} \tag{2}$$

$$(\rho c)_f\left(\frac{\partial T}{\partial t}-q\Omega_0 X\frac{\partial T}{\partial Y}+(\vec{v}\nabla)T\right)=k_f\nabla^2 T+(\rho c)_p\left(D_B\nabla\phi\nabla T+D_T\frac{(\nabla T)^2}{T_u}\right) \tag{3}$$

$$\frac{\partial\phi}{\partial t}-q\Omega_0 X\frac{\partial\phi}{\partial Y}+(\vec{v}\nabla)\phi=D_B\nabla^2\phi+\frac{D_T}{T_u}\nabla^2 T \tag{4}$$

$$div\vec{v}=0 \quad div\vec{B}=0 \tag{5}$$

Here $\vec{v}=(u,v,w)$ is the nanofluid velocity, ρ is the nanofluid density, ρ_p is the density of nanoparticles, ρ_f is the density of base fluid at temperature T_u, ϕ is the volume fraction of nanoparticles, $(\rho c)_f,(\rho c)_p$ are the effective heat capacities of the fluid and particle phases, respectively. μ, η and μ_e are the viscosity, magnetic viscosity, and magnetic permeability of nanofluid, respectively. D_B and D_T denote the Brownian diffusion coefficient and thermophoretic diffusion, respectively. The signs of the coefficients D_B and D_T are positive and they are

$$D_B=\frac{k_B T}{3\pi\mu d_p},\ D_T=\left(\frac{\mu}{\rho_f}\right)\left(\frac{0.26k_f}{2k_f+k_p}\right)\phi,$$

where d_p is the diameter of the nanoparticles, k_B is the Boltzmann constant, k_f,k_p are the thermal conductivity coefficients of the base fluid and nanoparticles. The coefficients of magnetic permeability μ_e, magnetic viscosity η and electrical conductivity are

$$\mu_e=\phi\mu_{ep}+(1-\phi)\mu_{ef},\ \eta=\frac{1}{4\pi\mu_e\sigma}=\phi\eta_p+(1-\phi)\eta_f,\ \sigma=\phi\sigma_p+(1-\phi)\sigma_f,$$

where μ_{ep},μ_{ef} are the magnetic permeability of the nanoparticles and the base fluid, η_p,η_f are the magnetic viscosity of the nanoparticles and the base fluid, σ_p,σ_f are the electrical conductivity coefficients nanoparticles and the base fluid.

In the momentum equation (1), the buoyancy force consists of two distinct components: the temperature variation within the fluid and the distribution of nanoparticles (which are generally heavier than the base fluid). Assuming that the fluid layer undergoes vertical harmonic oscillations with a frequency of ω_g and a small displacement amplitude of $\varepsilon^2\xi$, we can modify the equations of motion in the reference frame associated with the fluid layer. In the equation (1), the acceleration of gravity $\vec{g}(t)$ should be replaced by $\vec{g}_0(1+\varepsilon^2\delta\cos(\omega_g t))$, where δ is a small amplitude of gravitational modulation. Additionally, the influence of an external magnetic field is incorporated by introducing an additional force known as the Lorentz force, which affects the motion of the electrically conductive nanofluid. Equation (2) represents the description of induced magnetic fields that arise due to convective flows in electrically conductive nanofluids. The equations (3)-(4) are conservation laws for the thermal energy and volume fraction of nanoparticles. Equations (5) describe the solenoidality conditions for the fields $\vec{v}$ and $\vec{B}$. We impose the initial condition on temperature (T) and volumetric fraction of nanoparticles (ϕ) by assuming them to be constant at stress-free boundaries:

$$\vec{v}=0,\ T=T_d,\ \phi=\phi_0,\ \text{at } Z=0$$
$$\vec{v}=0,\ T=T_u,\ \phi=\phi_1,\ \text{at } Z=h \tag{6}$$

Equations (1)-(5), along with boundary conditions (6), provide a description of non-uniformly rotating magnetic convection in a nanofluid layer subjected to gravity field modulation. To facilitate the study of this phenomenon, it is advantageous to transform equations (1)-(5) into a dimensionless form by introducing dimensionless variables of the following nature:

$$(x^*,y^*,z^*)=\frac{(X,Y,Z)}{h},\ t^*=t\frac{\alpha_f}{h^2},\ T^*=\frac{T-T_u}{T_d-T_u}\ \phi^*=\frac{\phi-\phi_0}{\phi_1-\phi_0},\ \vec{v}^*(u^*,v^*,w^*)=\vec{v}(u,v,w)\frac{h}{\alpha_f},\ P^*=\frac{h^2P}{\alpha_f\mu},$$

$$\vec{B}^*(B_x^*,B_y^*,B_z^*)=\frac{\vec{B}(B_x,B_y,B_z)}{B_0},\ \omega_g^*=\omega_g\frac{h^2}{\alpha_f},$$

where $\alpha_f=k_f/(\rho c)_f$ is the coefficient of thermal diffusivity. Upon applying the aforementioned transformations to Eqs. (1)-(6), we obtain the following dimensionless governing system (after omitting the asterisk notation):

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial\vec{v}}{\partial t}+(\vec{v}\nabla)\vec{v}\right)+\mathrm{Ro}\sqrt{\mathrm{Ta}}\,x\frac{\partial\vec{v}}{\partial y}+\vec{e}_y\mathrm{Ro}\sqrt{\mathrm{Ta}}\,v_x+\sqrt{\mathrm{Ta}}(\vec{e}_z\times(\vec{v}+\vec{U}_0))=-\nabla\left(P+\mathrm{QPrPm}^{-1}\frac{\vec{B}^2}{2}\right)+\nabla^2\vec{v}+$$
$$+\mathrm{QPrPm}^{-1}(\vec{B}\nabla)\vec{B}-\vec{e}_zF_m\mathrm{Rm}-\vec{e}_zF_m\mathrm{Rn}\phi+\vec{e}_zF_m\mathrm{Ra}T \tag{7}$$

$$\frac{\partial\vec{B}}{\partial t}+\mathrm{Pr}Ro\sqrt{\mathrm{Ta}}\,x\frac{\partial\vec{B}}{\partial y}+(\vec{v}\nabla)\vec{B}-\mathrm{Pr}\,Ro\sqrt{\mathrm{Ta}}B_x\vec{e}_y-(\vec{B}\nabla)\vec{v}=\mathrm{PrPm}^{-1}\nabla^2\vec{B} \tag{8}$$

$$\frac{\partial T}{\partial t}+\mathrm{PrRo}\sqrt{\mathrm{Ta}}\,x\frac{\partial T}{\partial y}+(\vec{v}\nabla)T=\nabla^2T+\frac{N_B}{Le}\nabla\phi\nabla T+\frac{N_AN_B}{Le}(\nabla T)^2 \tag{9}$$

$$\frac{\partial\phi}{\partial t}+\mathrm{PrRo}\sqrt{\mathrm{Ta}}\,x\frac{\partial\phi}{\partial y}+(\vec{v}\nabla)\phi=\frac{1}{Le}\nabla^2\phi+\frac{N_A}{Le}\nabla^2T \tag{10}$$

where e_y,e_z are unit vectors along the Y, Z axes, respectively; $F_m=1+\varepsilon^2\delta\cos(\omega_g t)$. Then we write the boundary conditions (6) in dimensionless form as

$$\vec{v}=0,\ T=1,\ \phi=0,\ \text{at } z=0$$

$$\vec{v}=0,\ T=0,\ \phi=1,\ \text{at } z=1 \tag{11}$$

In equations (7)-(10), the following dimensionless parameters are used:

$Pr=\dfrac{\nu}{\alpha_f}$ is the Prandtl number, $Ta=\dfrac{4\Omega_0^2h^4}{\nu^2}$ is the Taylor number, $Pm=\dfrac{\nu}{\eta}$ is the magnetic Prandtl number, $Q=\dfrac{\mu_eB_0^2h^2}{4\pi\mu\eta}$ is the Chadrasekhar number, $\mathrm{Rm}=\dfrac{g_0h^3(\rho_p\phi_0+\rho_f(1-\phi_0))}{\mu\alpha_f}$ is the basic density Rayleigh number, $Ra=\dfrac{\rho_fg_0\beta h^3\Delta T}{\mu\alpha_f}$ is the Rayleigh number, $Rn=\dfrac{(\rho_p-\rho_f)(\phi_1-\phi_0)g_0h^3}{\mu\alpha_f}$ is the concentration Rayleigh number, $L_e=\dfrac{\alpha_f}{D_B}$ is the Lewis number, $N_B=(\phi_1-\phi_0)\cdot\dfrac{(\rho c)_p}{(\rho c)_f}$ is a coefficient characterizing the increment of the density of nanoparticles, $N_A=\dfrac{D_T(\Delta T)}{D_BT_0(\phi_1-\phi_0)}$ is a modified diffusion coefficient.

2.1. Basic state

Assume that the basic state is time independent and is given by

$$v_b=0,\ P=P_b(x,z),\ T=T_b(z),\ \phi=\phi_b(z) \tag{12}$$

Then, by representing all quantities in Eqs. (7)-(10) as the sum of the ground state and perturbed state, we can express them as follows:

$$\vec{v}=\vec{u}(u,v,w),\ \vec{B}=\vec{B}_0(0,0,B_0)+\vec{b}(\tilde{u},\tilde{v},\tilde{w}), \tag{13}$$

$$P = P_b(x,z) + P^{'} ,\ \phi = \phi_b(z) + \phi^{'} ,\ T = T_b(z) + T^{'}$$

The equations governing the ground state are given by:

$$\sqrt{\mathrm{T}a}(e_z \times U_0) = -\nabla P_b , \tag{14}$$

$$\frac{dP_b}{dz} = -\mathrm{Rm} - \mathrm{Rn}\phi_b + \mathrm{Ra}T_b , \tag{15}$$

$$\frac{d^2 T_b}{dz^2} + \frac{N_B}{Le}\frac{d\phi_b}{dz}\frac{dT_b}{dz} + \frac{N_A N_B}{Le}\left(\frac{dT_b}{dz}\right)^2 = 0 , \tag{16}$$

$$\frac{d^2\phi_b}{dz^2} + N_A \frac{d^2 T_b}{dz^2} = 0 . \tag{17}$$

The geostrophic equilibrium is described by Eq. (14), which balances the Coriolis force with the pressure gradient. The hydrostatic equation (15) ensures that the vertical component of the ground state is satisfied. Equatios (16)-(17) represent the mathematical expressions describing the stationary temperature profile $T_b = T_b(z)$, and the volume fraction of nanoparticles $\phi_b = \phi_b(z)$. Kuznetsov and Nield (2009) further suggested that the magnitude of the second and third terms in Eq. (16) is negligible and can be disregarded. Taking these facts into consideration, Eqs. (16) and (17) can be simplified to:

$$\frac{d^2 T_b}{dz^2} = 0 ,\ \frac{d^2\phi_b}{dz^2} = 0 \tag{18}$$

Equations (13) are solved under the boundary conditions (6), and the resulting solution is as follows:

$$T_b(z) = 1 - z ,\ \phi_b(z) = z \tag{19}$$

As will be shown below, we do not need an explicit form of the pressure $P_b(x,z)$.

2.2 Perturbated State

In this section, we obtain non-linear perturbed state equations used to describe three-dimensional nonuniformly rotating magnetic convection of a nanofluid under the influence of gravity field modulation. By subtracting the equations for the ground state (14)-(15) from Eqs. (7)-(10), we get the equations governing the evolution of perturbations:

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\nabla)\vec{u}\right) + \mathrm{Ro}\sqrt{\mathrm{Ta}}\,x\frac{\partial \vec{u}}{\partial y} + \vec{e}_y \mathrm{Ro}\sqrt{\mathrm{Ta}}\,u + \sqrt{\mathrm{Ta}}(\vec{e}_z \times \vec{u}) = -\nabla\tilde{P} + \nabla^2\vec{u} + \mathrm{QPrPm}^{-1}((\vec{e}_z\nabla)\vec{b} + (\vec{b}\nabla)\vec{b}) - \\ -\vec{e}_z F_m \mathrm{Rn}\phi^{'} + \vec{e}_z F_m \mathrm{Ra}T^{'} \tag{20}$$

$$\frac{\partial \vec{b}}{\partial t} + \mathrm{Pr}Ro\sqrt{\mathrm{Ta}}\,x\frac{\partial \vec{b}}{\partial y} + (\vec{u}\nabla)\vec{b} - \mathrm{Pr}\,Ro\sqrt{\mathrm{Ta}}\tilde{u}\vec{e}_y - (\vec{e}_z\nabla)\vec{u} - (\vec{b}\nabla)\vec{u} = \mathrm{PrPm}^{-1}\nabla^2\vec{b} \tag{21}$$

$$\frac{\partial T^{'}}{\partial t} + \mathrm{PrRo}\sqrt{\mathrm{Ta}}\,x\frac{\partial T^{'}}{\partial y} + (\vec{u}\nabla)T^{'} - w = \nabla^2 T^{'} + \frac{N_B}{Le}\left(\frac{\partial T^{'}}{\partial z} - \frac{\partial \phi^{'}}{\partial z}\right) + \frac{2N_A N_B}{Le}\frac{\partial T^{'}}{\partial z} \tag{22}$$

$$\frac{\partial \phi^{'}}{\partial t} + \mathrm{PrRo}\sqrt{\mathrm{Ta}}\,x\frac{\partial \phi^{'}}{\partial y} + (\vec{u}\nabla)\phi^{'} + w = \frac{1}{Le}\nabla^2\phi^{'} + \frac{N_A}{Le}\nabla^2 T^{'} \tag{23}$$

where $\tilde{P} = P^{'} + \mathrm{Q}PrPm^{-1}(\vec{e}_z\vec{b} + (\vec{b}^2/2))$ is the total perturbed pressure.

We will focus on the dynamics of axisymmetric perturbations, which means that all perturbed quantities in Eqs. (20)-(23) will depend solely on two variables, namely, x and z:

$$\vec{u} = (u(x,z), v(x,z), w(x,z)) ,\ \vec{b} = (\tilde{u}(x,z), \tilde{v}(x,z), \tilde{w}(x,z))$$

$$\tilde{P} = \tilde{P}(x,z) ,\ T^{'} = T^{'}(x,z) ,\ \phi^{'} = \phi^{'}(x,z) .$$

The solenoidal Eqs. (5) governing the axisymmetric velocity and magnetic field perturbations can be expressed in the following form:

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0\,,\ \frac{\partial \tilde{u}}{\partial x}+\frac{\partial \tilde{w}}{\partial z}=0\,. \tag{24}$$

To simplify the expressions of the perturbed velocity components, we can introduce the stream function, denoted as ψ . This allows us to express the components of the perturbed velocity as follows:

$$u=-\frac{\partial \psi}{\partial z}\,,\ w=\frac{\partial \psi}{\partial x}$$

Similarly, for the perturbations of the magnetic field, we can introduce the current function, denoted as φ . This allows us to express the components of the perturbed magnetic field as:

$$\tilde{u}=-\frac{\partial \varphi_m}{\partial z}\,,\qquad \tilde{w}=\frac{\partial \varphi_m}{\partial x}\,.$$

The Eqs. (20)-(23), expressed in terms of the introduced stream function and current function, will have the following form in the coordinate representation:

$$\begin{aligned}&\left(\frac{1}{\Pr}\frac{\partial}{\partial t}-\nabla^2\right)\nabla^2\psi+\sqrt{Ta}\frac{\partial v}{\partial z}-\mathrm{Q}\Pr Pm^{-1}\frac{\partial}{\partial z}\nabla^2\varphi_m+F_m\mathrm{Rn}\frac{\partial \phi'}{\partial x}-F_mRa\frac{\partial T'}{\partial x}=\\&=\mathrm{Q}\Pr Pm^{-1}J(\varphi_m,\nabla^2\varphi_m)-\Pr^{-1}J(\psi,\nabla^2\psi)\end{aligned} \tag{25}$$

$$\left(\frac{1}{\Pr}\frac{\partial}{\partial t}-\nabla^2\right)v-\sqrt{Ta}(1+Ro)\frac{\partial \psi}{\partial z}-\mathrm{Q}PrPm^{-1}\frac{\partial \tilde{v}}{\partial z}=\mathrm{Q}PrPm^{-1}J(\varphi_m,\tilde{v})-\Pr^{-1}J(\psi,v)\,, \tag{26}$$

$$\left(\frac{\partial}{\partial t}-PrPm^{-1}\nabla^2\right)\varphi_m-\frac{\partial \psi}{\partial z}=-J(\psi,\varphi_m)\,, \tag{27}$$

$$\left(\frac{\partial}{\partial t}-PrPm^{-1}\nabla^2\right)\tilde{v}-\frac{\partial v}{\partial z}+\Pr Ro\sqrt{Ta}\frac{\partial \varphi_m}{\partial z}=J(\varphi_m,v)-J(\psi,\tilde{v})\,, \tag{28}$$

$$\left(\frac{\partial}{\partial t}-\nabla^2\right)T'-\frac{\partial \psi}{\partial x}-\frac{N_B}{Le}\left(\frac{\partial T'}{\partial z}-\frac{\partial \phi'}{\partial z}\right)+\frac{2N_AN_B}{Le}\frac{\partial T'}{\partial z}=-J(\psi,T')\,, \tag{29}$$

$$\left(\frac{\partial}{\partial t}-\frac{1}{Le}\nabla^2\right)\phi'+\frac{\partial \psi}{\partial x}-\frac{N_A}{Le}\nabla^2T'=-J(\psi,\phi')\,, \tag{30}$$

where

$J(a,b)=\dfrac{\partial a}{\partial x}\dfrac{\partial b}{\partial z}-\dfrac{\partial a}{\partial z}\dfrac{\partial b}{\partial x}$, $\nabla^2\equiv\dfrac{\partial}{\partial x^2}+\dfrac{\partial}{\partial z^2}$. $J(a,b)$ is the Jacobian operator.

In the absence of gravity field modulation, Eqs. (25)-(29) were used to study the weakly nonlinear and chaotic regimes of stationary magnetic convection in a nonuniformly rotating "pure" (without nanoparticles) fluid, which were carried out by Kopp *et al.* [25], [29]-[30]. Magnetic convection in a nonuniformly rotating "pure" (without nanoparticles) fluid under gravity modulation was studied by Kopp *et al.* [31].

3. WEAK NONLINEAR ANALYSIS

In the weakly nonlinear theory of convective instability, the interaction between small amplitude convective cells can be described as follows. Assuming that the amplitude of the convective cells is of the order $O(\varepsilon^1)$, their interaction with each other leads to second harmonic and nonlinear effects of the order $O(\varepsilon^2)$, and subsequently to nonlinear effects of the order $O(\varepsilon^3)$, and so on. In this case, the nonlinear terms in equations (25)-(30) are considered perturbations to the linear convection problem. The Rayleigh parameter Ra , which governs convection, is assumed to be close to the critical value Ra_c . We further assume that the amplitude of the oscillating gravitational field, $\varepsilon^2\delta g_0$, is of the second order of smallness, $O(\varepsilon^2)$. Therefore, its influence on the nonlinear interaction of convective cells is expected to occur at the third order, $O(\varepsilon^3)$. Given the small influence of unstable modes, our objective is to derive equations that describe the interactions of these modes. The general procedure for constructing a weakly nonlinear theory is as follows. In the weakly nonlinear theory of convective instability, we introduce a small expansion parameter ε^2 , which represents the relative deviation of the Rayleigh number Ra from its critical value Ra_c :

$$\varepsilon^2 = \frac{Ra - Ra_c}{Ra_c} \ll 1 .$$

The perturbed quantities in the equations of the form $\mathcal{L}U = -N(U\,|\,U)$, where $N(\ldots)$ represents the nonlinear terms, can be expressed as a series expansion in powers of ε:

$$U \to \varepsilon U^{(1)} + \varepsilon^2 U^{(2)} + \varepsilon^3 U^{(3)} + \ldots$$

The equations for perturbations in different orders of ε take the following form:

$\varepsilon^1 : \mathcal{L}^{(0)} U^{(1)} = 0$,

$\varepsilon^2 : \mathcal{L}^{(0)} U^{(2)} = -N(U^{(1)}\,|\,U^{(1)})$,

$\varepsilon^3 : \mathcal{L}^{(0)} U^{(3)} = -\mathcal{L}^{(2)} U^{(0)} - N(U^{(1)}\,|\,U^{(2)}) - N(U^{(2)}\,|\,U^{(1)})$.

The solvability condition for this chain of nonlinear equations is known as Fredholm's alternative (see [32]). It can be expressed as:

$$\left\langle U^{\dagger}, R.H. \right\rangle = 0 , \tag{31}$$

where $U^{\dagger}$ is a non-trivial solution of the linear self-adjoint problem $\mathcal{L}^{\dagger} U^{\dagger} = 0$, and $\mathcal{L}^{\dagger}$ is a self-adjoint operator that is defined from the relation:

$$\left\langle U^{\dagger}, \mathcal{L}U \right\rangle \equiv \left\langle \mathcal{L}^{\dagger} U^{\dagger}, U \right\rangle , \tag{32}$$

where $\langle , \rangle$ denotes the inner product:

$$\left\langle f, g \right\rangle = \int_{z=0}^{1} \int_{x=0}^{2\pi/k_c} f \cdot g \, dx dz$$

$R.H.$ are the right parts of perturbed equations with non-linear terms.

In the weakly nonlinear theory, we represent all the variables in the Eqs. (25)-(30) using an asymptotic expansion. The Rayleigh number Ra is expanded as

$$Ra = Ra_c + \varepsilon^2 Ra_2 + \varepsilon^4 Ra_4 + \ldots$$

and the perturbed quantities are expressed as follows:

$$\psi = \varepsilon\psi_1 + \varepsilon^2\psi_2 + \varepsilon^3\psi_3 + \ldots ,\ v = \varepsilon v_1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + \ldots ,\ \varphi_m = \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3 + \ldots ,\ \tilde{v} = \varepsilon\tilde{v}_1 + \varepsilon^2\tilde{v}_2 + \varepsilon^3\tilde{v}_3 + \ldots ,$$

$$T' = \varepsilon T_1 + \varepsilon^2 T_2 + \varepsilon^3 T_3 + \ldots ,\ \phi' = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + \ldots ,$$

where Ra_c represents the critical Rayleigh number for convection without modulation. The amplitudes of the perturbed quantities are solely dependent on the slow time variable $\tau = \varepsilon^2 t$. To simplify the analysis, we will consider nonlinear terms only in the equations governing the temperature and volume concentration of nanoparticles (29)-(30). These equations are solved using a perturbation approach by considering various powers of ε. At the lowest order, i.e., ε, the following solution is obtained

$$\hat{L} M_1 = 0 \tag{34}$$

where $M_1 = [\psi_1, v_1, \varphi_1, \tilde{v}_1, T_1, \phi_1]^{Tr}$, $\hat{L}$ is a matrix operator of the form

$$\hat{L} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{pmatrix} ,$$

where $a_{11} = -\nabla^4$, $a_{12} = \sqrt{Ta}\frac{\partial}{\partial z}$, $a_{13} = -\mathrm{Q}\,\mathrm{Pr}\,Pm^{-1}\frac{\partial}{\partial z}\nabla^2$, $a_{14} = 0$, $a_{15} = -Ra_c\frac{\partial}{\partial x}$, $a_{16} = \mathrm{Rn}\frac{\partial}{\partial x}$, $a_{21} = -\sqrt{Ta}(1+Ro)\frac{\partial}{\partial z}$, $a_{22} = -\nabla^2$, $a_{23} = 0$, $a_{24} = -\mathrm{Q}PrPm^{-1}\frac{\partial}{\partial z}$, $a_{25} = 0$, $a_{26} = 0$, $a_{31} = -\frac{\partial}{\partial z}$, $a_{32} = 0$, $a_{33} = -PrPm^{-1}\nabla^2$, $a_{34} = a_{35} = a_{36} = 0$,

$a_{41} = 0$, $a_{42} = -\frac{\partial}{\partial z}$, $a_{43} = \mathrm{Pr}\,Ro\sqrt{Ta}\frac{\partial}{\partial z}$, $a_{44} = -PrPm^{-1}\nabla^2$, $a_{45} = a_{46} = 0$, $a_{51} = -\frac{\partial}{\partial x}$, $a_{52} = a_{53} = a_{54} = 0$, $a_{55} = -\nabla^2 - \frac{N_B}{Le}\frac{\partial}{\partial z} + \frac{2N_A N_B}{Le}\frac{\partial}{\partial z}$, $a_{56} = \frac{N_B}{Le}\frac{\partial}{\partial z}$, $a_{61} = \frac{\partial}{\partial x}$, $a_{62} = a_{63} = a_{64} = 0$, $a_{65} = -\frac{N_A}{Le}\nabla^2$, $a_{66} = -\frac{1}{Le}\nabla^2$.

The solutions of the system of Eqs. (34) satisfying the boundary conditions (11) are as follows:

$\psi_1 = A(\tau)\sin k_c x \sin \pi z$,

$v_1 = \frac{A(\tau)\pi\sqrt{Ta}}{a^2}\cdot\frac{(1+Ro)a^4 + \pi^2 QPmRo}{a^4+\pi^2 Q}\sin k_c x\cos\pi z$, $\varphi_1 = \frac{A(\tau)\pi Pm}{a^2\,\mathrm{Pr}}\sin k_c x\cos\pi z$,

$\tilde{v}_1 = -\frac{A(\tau)\pi^2\sqrt{Ta}(1+Ro(1-Pm))Pm}{\mathrm{Pr}(a^4+\pi^2 Q)}\sin k_c x\sin\pi z$, $T_1 = \frac{A(\tau)k_c}{a^2}\cos k_c x\sin\pi z$,

$$\phi_1 = -\frac{A(\tau)k_c}{a^2}(Le+N_A)\cos k_c x\sin\pi z,\quad a^2 = k_c^2+\pi^2. \tag{35}$$

The critical value of the Rayleigh number Ra_c for steady magnetoconvection in a non-uniformly rotating nanofluid is obtained from the first equation of system (34) and can be expressed as follows:

$$Ra_c = \frac{a^6}{k_c^2} + \frac{\pi^2 a^2}{k_c^2}Q + \frac{\pi^2 a^4 Ta}{k_c^2(a^4+\pi^2 Q)} + \frac{\pi^2 TaRo(a^4+\pi^2 QPm)}{k_c^2(a^4+\pi^2 Q)} - \mathrm{R}n(Le+N_A). \tag{36}$$

Expression (36) was first obtained by Kopp *et al.* [28], which agrees very well with the known results. When there are no nanoparticles present $(\mathrm{Rn}=0)$, the expression (36) aligns with the findings reported by Kopp *et al.*[25], [29]. In the absence of rotation ($\mathrm{Ta}=0$) and a magnetic field ($\mathrm{Q}=0$), Nield and Kuznetsov [33] derived the expression for the thermal Rayleigh number in the context of stationary motions. Chand [34], on the other hand, investigated a uniformly rotating $\mathrm{Ro}=0$ nanofluid without a magnetic field $\mathrm{Q}=0$ and obtained an expression for the thermal Rayleigh number in the case of stationary convection, considering the top-heavy configuration of nanoparticles. Without rotation $\mathrm{Ta}=0$, expression (36) coincides with the results of Gupta *et al.* [35] and Yadav *et al.* [36]. In the case when there is no heating of the nanofluid $Ra=0$, then in system, the development of MRI is possible. As established by Kopp *et al.* [28], in the presence of a concentration of nanoparticles, the area of development of the MRI becomes larger compared to a “pure” fluid.

For the second order in ε we get the following equation:

$$\hat{L}M_2 = N_2, \tag{37}$$

where

$$M_2 = [\psi_2, v_2, \varphi_2, \tilde{v}_2, T_2, \phi_2]^{Tr},\ N_2 = [N_{21}, N_{22}, N_{23}, N_{24}, N_{25}, N_{26}]^{Tr},\ N_{21} = N_{22} = N_{23} = N_{24} = 0,$$

$$N_{25} = -\left(\frac{\partial\psi_1}{\partial x}\frac{\partial T_1}{\partial z} - \frac{\partial T_1}{\partial x}\frac{\partial\psi_1}{\partial z}\right),\ N_{26} = -\left(\frac{\partial\psi_1}{\partial x}\frac{\partial\phi_1}{\partial z} - \frac{\partial\phi_1}{\partial x}\frac{\partial\psi_1}{\partial z}\right).$$

The following second-order solutions are obtained using the first-order solutions provided in Eqs. (35):

$$\psi_2 = 0,\ v_2 = 0,\ \varphi_2 = 0,\ \tilde{v}_2 = 0, \tag{38}$$

$$T_2 = -\frac{A^2(\tau)k_c^2}{8\pi a^2}\sin(2\pi z),\ \phi_2 = \frac{A^2(\tau)k_c^2}{8\pi a^2}Le(Le+N_A)\sin(2\pi z).$$

To analyze the intensity of heat transfer, we introduce a horizontally-averaged heat flux at the boundary of the electrically conducting nanofluid layer, known as the Nusselt number:

$$Nu(\tau)=1+\frac{\left[\dfrac{k_c}{2\pi}\int\limits_0^{2\pi/k_c}\left(\dfrac{\partial T_2}{\partial z}\right)dx\right]_{z=0}}{\left[\dfrac{k_c}{2\pi}\int\limits_0^{2\pi/k_c}\left(\dfrac{\partial T_b}{\partial z}\right)dx\right]_{z=0}}=1+\frac{k_c^2}{4a^2}A^2(\tau) \tag{39}$$

The nanoparticle concentration Nusselt number, denoted as $Nu_\phi(\tau)$, is defined in the same way as the Nusselt number. It can be expressed as follows:

$$Nu_\phi(\tau)=1+\frac{k_c^2A^2(\tau)}{4a^2}Le(Le+N_A) \tag{40}$$

The heat/mass transfer quotients $Nu(\tau)$ and $Nu_\phi(\tau)$ will be analyzed once the expression for the amplitude $A(\tau)$ is obtained. It can be observed from the asymptotic expansion (33) that the effect of gravity modulation only contributes to the third order in ε. Therefore, in the third order ε, we obtain the following expression:

$$\hat{L}M_3=N_3, \tag{41}$$

where

$$M_3=[\psi_3,v_3,\varphi_3,\tilde{v}_3,T_3,\phi_3]^{Tr}\ ,\ N_3=[N_{31},N_{32},N_{33},N_{34},N_{35},N_{36}]^{Tr}.$$

The terms on the right side of Eq. (41) are defined by the following expression:

$$N_{31}=-\frac{1}{\Pr}\frac{\partial}{\partial\tau}\nabla^2\psi_1+Ra_2\frac{\partial T_1}{\partial x}+Ra_c\delta\cos(\Omega\tau)\frac{\partial T_1}{\partial x}-\mathrm{R}n\delta\cos(\Omega\tau)\frac{\partial\phi_1}{\partial x}\ ,\ \Omega=\frac{\omega_g}{\varepsilon^2}, \tag{42}$$

$$N_{32}=-\frac{1}{\Pr}\frac{\partial v_1}{\partial\tau},\ N_{33}=-\frac{\partial\phi_1}{\partial\tau},\ N_{34}=-\frac{\partial\tilde{v}_1}{\partial\tau}, \tag{43}$$

$$N_{35}=-\frac{\partial T_1}{\partial\tau}-\left(\frac{\partial\psi_1}{\partial x}\frac{\partial T_2}{\partial z}-\frac{\partial T_2}{\partial x}\frac{\partial\psi_1}{\partial z}+\frac{\partial\psi_2}{\partial x}\frac{\partial T_1}{\partial z}-\frac{\partial T_1}{\partial x}\frac{\partial\psi_2}{\partial z}\right), \tag{44}$$

$$N_{36}=-\frac{\partial\phi_1}{\partial\tau}-\left(\frac{\partial\psi_1}{\partial x}\frac{\partial\phi_2}{\partial z}-\frac{\partial\phi_2}{\partial x}\frac{\partial\psi_1}{\partial z}+\frac{\partial\psi_2}{\partial x}\frac{\partial\phi_1}{\partial z}-\frac{\partial\phi_1}{\partial x}\frac{\partial\psi_2}{\partial z}\right). \tag{45}$$

By substituting the expressions for $\psi_1,v_1,\varphi_1,\tilde{v}_1,T_1,T_2,\phi_1$, and ϕ_2 from (35) and (38) into (42)-(45), we can easily obtain expressions for $N_{31},N_{32},N_{33},N_{34},N_{35}$, and N_{36} in terms of the amplitude $A(\tau)$. Applying the condition for the solvability of the existence of a third-order solution Eq. (32), we find the Ginzburg-Landau equation for stationary convection with time-periodic coefficients in the following form:

$$A_1\frac{\partial A}{\partial\tau}-A_2(\tau)A+A_3A^3=0, \tag{46}$$

Where

$$A_1=\frac{a^2}{\Pr}+\frac{k_c^2}{a^4}Ra_c+\frac{k_c^2}{a^4}RnLe(Le+N_A)-\frac{\pi^2}{a^2}\frac{QPm}{\Pr}-\frac{\pi^2Ta\left((1+Ro)a^4+\pi^2QPm(RoPm-1)\right)}{\Pr(a^4+\pi^2Q)^2}-$$

$$-\frac{\pi^4TaRoQPm^2}{a^4(a^4+\pi^2Q)\Pr},\ A_2(\tau)=\frac{k_c^2}{a^2}Ra_c\left(\frac{Ra_2}{Ra_c}+\delta\cos(\Omega\tau)\right)+\frac{k_c^2}{a^2}Rn(Le+N_A)\delta\cos(\Omega\tau),\ A_3=\frac{k_c^4}{8a^4}(Ra_c+RnLe^2(Le+N_A)).$$

Obtaining an analytical solution for the non-autonomous Ginzburg-Landau (GL) Eq. (46) with time-varying coefficients is a difficult task. Therefore, we solve this equation numerically using Mathematica software. The initial value $A(0)=A_0$ is set, where A_0 represents the value of the initial amplitude. We assume $Ra_2\approx Ra_c$, since the nonlinearity is considered near the critical state of convection, i.e., the Rayleigh number in this system is: $Ra\approx Ra_c(1+\varepsilon^2)$. In the absence of nonuniform rotation and an external magnetic field, the non-autonomous Ginzburg-Landau equation for a non-conductive nanofluid was obtained by Kiran *et al.* [21].

In concluding this section, we note that in the case of an unmodulated system, the Ginzburg-Landau equation given above (47) can be simplified to:

$$A_1 \frac{\partial \tilde{A}}{\partial \tau} - A_2 \tilde{A} + A_3 \tilde{A}^3 = 0\,, \tag{48}$$

where $\tilde{A}(\tau)$ represents the amplitude of convection for the unmodulated case, and A_1 and A_3 have the same expressions as given in (47), while $A_2 = k_c^2 Ra_2 / a^2 \Pr$. An analytical solution of the equation (48) with a known initial condition $A_0 = A(0)$ can be obtained using the Lagrange method (constant variation), which is expressed as:

$$\tilde{A}(\tau) = \frac{A_0}{\sqrt{\dfrac{A_3}{A_2} A_0^2 + \left(1 - A_0^2 \dfrac{A_3}{A_2}\right) \exp\left(-\dfrac{2\tau A_2}{A_1}\right)}} \tag{49}$$

The thermal and nanoparticle concentration Nusselt numbers in this case can be obtained from Eqs. (39) and (40) by substituting the amplitude of convection (49).

4. RESULTS AND DISCUSSION

The Ginzburg-Landau equation, derived from the perturbation analysis, provides insights into the behavior of the system at finite amplitudes. We utilized a numerical solver, implementing the equation with proper initial conditions to obtain numerical solutions. NDSolve in Mathematica efficiently handles the equation's nonlinear and time-dependent nature. Through setting the appropriate parameters, initial amplitude $A(0) = 0.3$, and gravity modulation strength (δ, Ω), we simulated the system's evolution over time. The external control of convection through the modulation effect is important in the study of thermal instability and therefore has implications for a variety of applications, including the enhancement of heat transfer in industrial processes and the design of thermal management systems.

The graphical representation of the results of our numerical calculations is illustrated in Figs. 3-6.

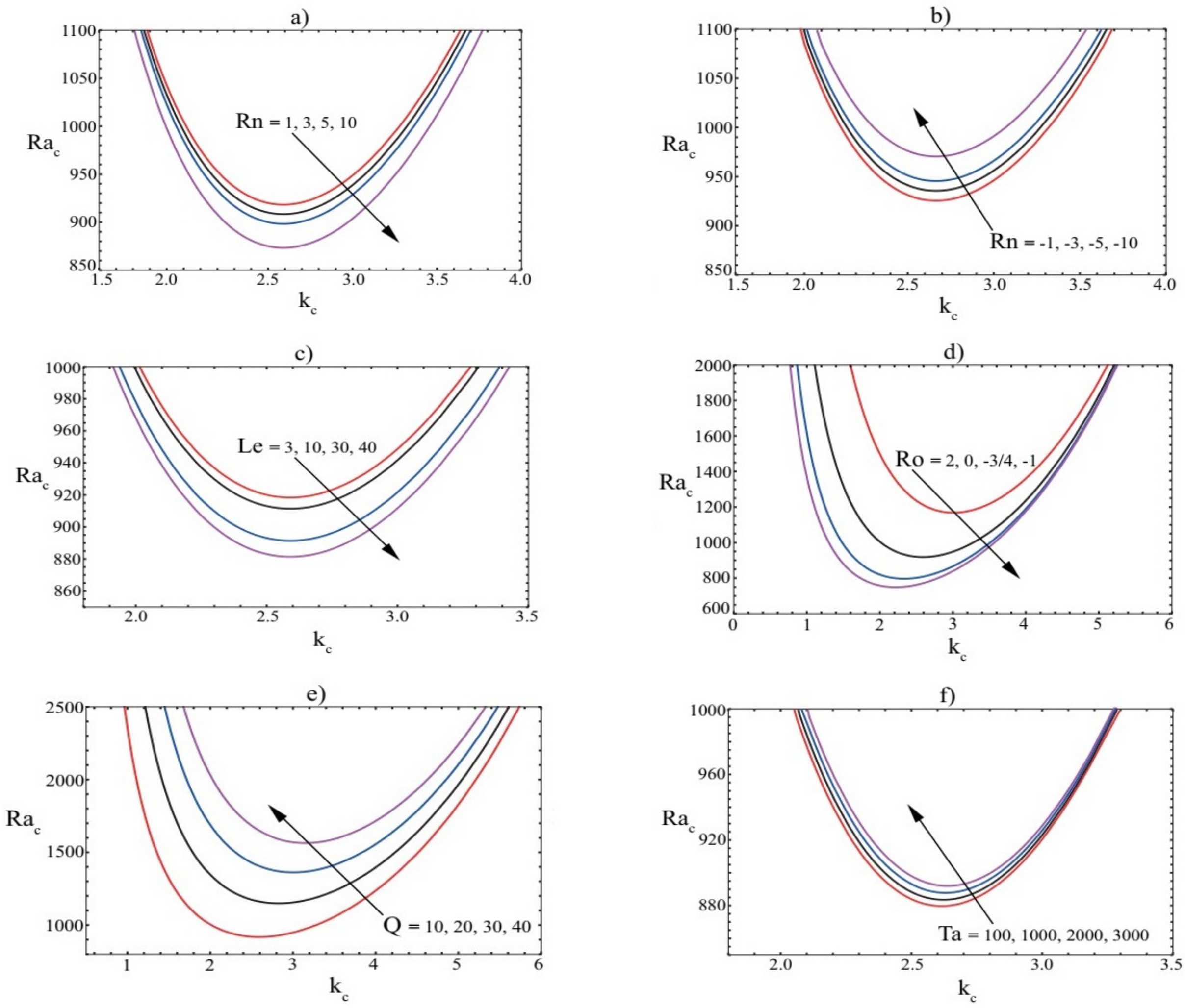

Figure. 3 Plots of the critical Rayleigh number Ra_c versus wave numbers k_c

We analyzed the dependencies of heat transfer Nu and mass transfer Nu_{ϕ} on the dimensionless time parameter τ. By varying the nanofluid parameters, such as Pr, Rn, Ro, and Le, as well as the modulation parameters (δ, Ω), we could investigate their impact on the heat and mass transfer characteristics. We assume that the fluid layer's viscosity is not high, so moderate Pr values are used in the calculations. δ values are also small due to low amplitude modulation. Additionally, gravity modulation is assumed to have a low frequency Ω, which maximizes its effect on the onset of convection and heat transport at lower frequencies. The stability curves for the stationary Rayleigh number (36) in the linear theory are shown in Fig. 3. From Figs. 3a and 3c, it can be seen that with an increase in parameters $Rn > 0$ (top-heavy configuration of nanofluid) and Le, the minimum values of the Rayleigh numbers decrease. Therefore, an increase in the parameters $Rn > 0$ and Le has a destabilizing effect on the onset of convection. From Fig. 3b, we see that in the case of a bottom-heavy configuration of nanofluid ($Rn < 0$, $N_A < 0$) an increase in the parameters Rn has a stabilizing effect on the onset of convection. Based on Fig. 3d, it can be observed that when the Rossby number has a positive profile $(Ro > 0)$, the minimum value of the critical Rayleigh number (Ra_{min}) is higher compared to negative rotation profiles. As a result, negative rotation profiles have a lower threshold for instability development compared to uniform $(Ro = 0)$ and nonuniform $(Ro = 2)$ rotations. In Fig. 3e, the Rayleigh number is displayed in relation to the dimensionless wavenumber for various vertical magnetic field values (Chandrasekhar number Q). Fig. 3e illustrates that with an increase in magnetic field (Chadrasekhar number Q) values, the Rayleigh number also increases, leading to the stabilization of stationary convection. Fig. 3f illustrates that with an increase in Taylor number Ta values, the Rayleigh number also increases, leading to the stabilization of stationary convection.

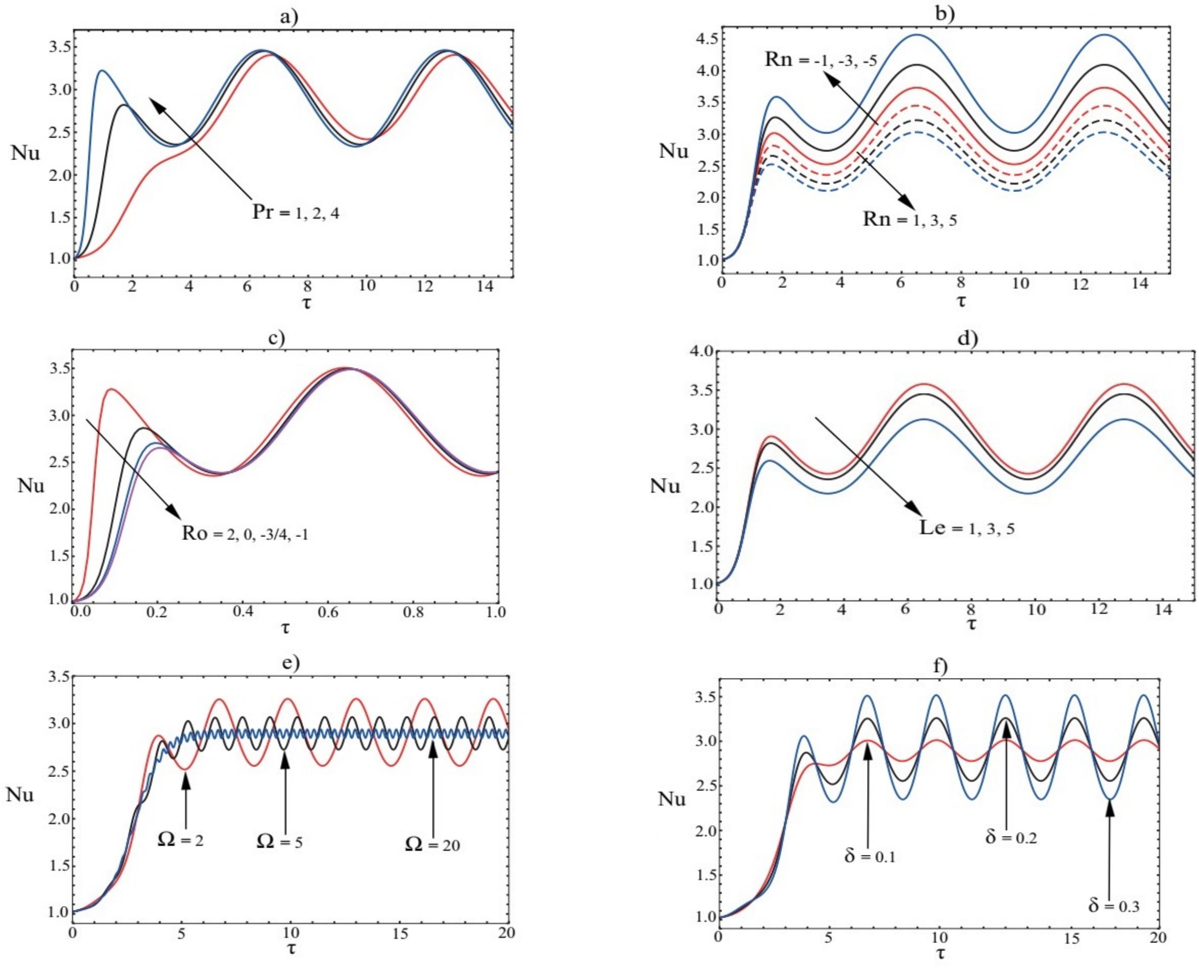

Figure 4. Dependence of the Nusselt number Nu on the time τ for a) Pr, b) Rn, c) Ro, d) Le, e) Ω, f) δ variations

Fig. 4a and Fig. 6a show the effect of Prandtl number Pr variations on heat and mass transfer, assuming the other parameters are fixed: $Rn = 1, Le = 3, Ro = 2, \Omega = 1, \delta = 0.3$. As can be seen from Fig. 4a and Fig. 5a, an increase in the Prandtl number Pr causes a rise in heat and concentration transfer over a brief period of time. As a result, systems with higher Prandtl numbers tend to exhibit more efficient heat transfer characteristics. Therefore, the Prandtl number (Pr) plays a significant role in enhancing heat and concentration transport, especially at low time values. Next, we consider the influence of concentration Rayleigh number Rn on both thermal and concentration Nusselt numbers at fixed parameters: $\Pr = 2, Le = 3, \mathrm{Ro} = 2, \Omega = 1, \delta = 0.3$. Similarly, the concentration Rayleigh number Rn has a significant impact on both thermal and concentration Nusselt numbers, leading to enhanced heat and concentration transport in the case of a bottom-heavy configuration of nanofluid ($\mathrm{Rn} < 0$, $N_A < 0$), as shown in Figures 4b and 5b. Increasing the concentration of nanoparticles at the lower hot boundary enhances the fluid's thermal conductivity. Nanoparticles typically have higher thermal conductivity than the base fluid, and this higher thermal conductivity aids in conducting heat from the lower hot boundary to the surrounding fluid. As a result, the convective heat transfer is improved, leading to an increase in both the thermal and concentration Nusselt numbers. It is noteworthy that Rn has a dual role in regulating the heat and mass transfer properties of the medium. The positive values of Rn can induce the reverse nature of heat and mass transport within the layer. When the concentration of nanoparticles increases at the upper cold boundary, they can enhance the overall thermal conductivity of the fluid layer. This increased thermal conductivity facilitates heat transfer through conduction, reducing the reliance on convective heat transfer. As a result, the thermal Nusselt number decreases. The presence of nanoparticles can disrupt the fluid flow and impede its motion. This hindered fluid motion reduces the convective heat and mass transfer rates. Consequently, the concentration Nusselt number, which characterizes mass transfer, decreases. It is important to note that these effects can be observed for all profiles of rotation, not just positive profiles.

Figures 4c and 5c exhibit the temporal variations of the Nusselt number Nu and nano-Nusselt number Nu_ϕ for various rotation profiles $\mathrm{Ro} = (2, 0, -3/4, -1)$ of the electrically conductive nanofluid subjected to an oscillating gravitational field with a frequency $\Omega = 10$ and an amplitude $\delta = 0.3$ with fixed parameters $\Pr = 1, Le = 3$. The results presented in these figures indicate that both heat and concentration transfer in the nanofluid are enhanced when the nanofluid experiences nonuniform rotation characterized by a positive Rossby number $(\mathrm{Ro} = 2)$. This demonstrates the useful impact of nonuniform rotation on the overall heat and concentration transport processes within the nanofluid system.

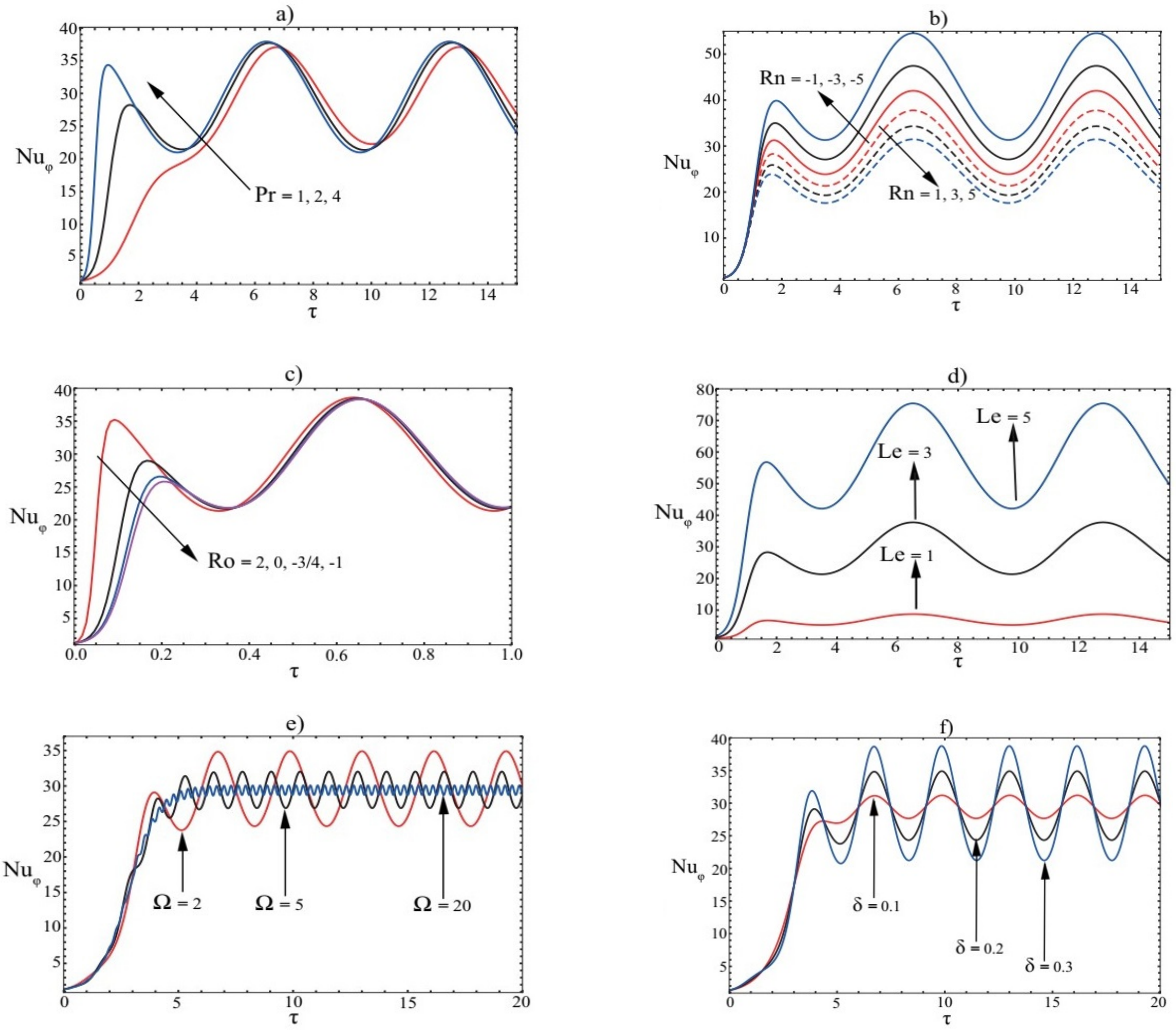

Figure 5. Dependence of the nano-Nusselt number Nu_ϕ on the time τ for a) $\Pr$, b) Rn, c) Ro, d) Le, e) Ω, f) δ variations

On Fig. 4d the effect of variations of the Lewis number Le on heat transfer is shown for fixed other parameters: $\mathrm{Rn}=1, \mathrm{Pr}=2, \mathrm{Ro}=2, \Omega=1, \delta=0.3$. It was observed that as the Lewis number increased, the heat transfer decreased. On the contrary, growing Le enhances concentration transport, as depicted in Fig. 5d. These results are in good agreement with the conclusions of the papers by Aleng *et al.* [38], Alam *et al.* [39]. The impact of the modulation frequency Ω is illustrated in Figures 4e and 5e. Specifically, at lower modulation rates, corresponding to low-frequency cases $(\Omega=2)$, higher heat and mass transfer are achieved compared to higher vibrational rates $(\Omega=20)$.

Fig. 4f and Fig. 5f depict the influence of the modulation amplitude δ on heat and mass transfer. The range of δ considered in the study is from 0.1 to 0.5, aimed at enhancing heat and mass transfer. It is worth mentioning that the frequency of modulation Ω has a diminishing effect on heat and mass transfer, which aligns with the findings reported by Gresho and Sani [8] and Kopp *et al.* [31] for the case of ordinary fluid. These results highlight the significance of considering low-frequency g-jitter to optimize the transport process. The outcomes obtained from our investigation on nanofluids can also be compared with the studies conducted by Kiran *et al.* [14]-[18] and Bhadauria and Agarwal [40]. Equation (48) provides an analytical expression for the amplitude of convection in the unmodulated case. By utilizing this amplitude, a comparison between the modulated system and the unmodulated system is depicted in Fig. 6. The graph demonstrates that there is a sudden increase in $\mathrm{N}u(\tau)$ and $\mathrm{N}u_\phi(\tau)$ for low values of the time parameter τ, and it stabilizes for higher values of τ. However, in the case of the modulated system, both $\mathrm{N}u(\tau)$ and $\mathrm{N}u_\phi(\tau)$ exhibit oscillatory behavior.

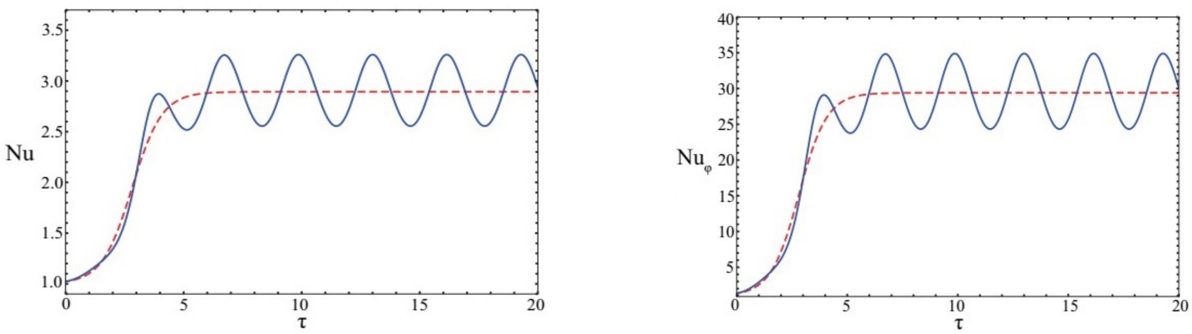

Figure 6. Variations of the Nusselt numbers Nu and Nu_ϕ in the absence of δ (dashed line) and the presence of $\delta=0.3$, $\Omega=2$ (solid line) modulation of the gravity field

5. CONCLUSIONS

We have developed a weakly nonlinear theory to investigate the effects of gravity modulation on stationary convection in a nonuniformly rotating electrically conductive nanofluid under a constant vertical magnetic field. Our analysis utilizes perturbation theory with respect to the small supercriticality parameter, $\varepsilon=\sqrt{(Ra-Ra_c)/Ra_c}$, where Ra is the Rayleigh number and Ra_c is the critical Rayleigh number. We consider the small amplitude of the modulated gravity field to be of second order in ε. In the first order of ε, the parametric modulation does not influence the development of convection, leading to results consistent with linear theory. However, in the third order of ε, we obtained a nonlinear Ginzburg-Landau equation with a time-periodic coefficient. Through numerical analysis, we draw the following conclusions based on the obtained results:

- The numbers $\mathrm{Nu}(\tau)$ and $\mathrm{Nu}_\phi(\tau)$ increase when the values of the parameters Pr and $\mathrm{Rn}<0$ are increased.
- With an increase in the Le number, a decrease in heat transfer and an increase in mass transfer were observed.
- Nonuniform rotation with a positive Rossby number $\mathrm{Ro}>0$ enhances heat and mass transfer in the nanofluid system.
- Increasing the modulation frequency Ω leads to a decrease in the variations of the Nusselt numbers $\mathrm{Nu}(\tau)$ and $\mathrm{N}u_\phi(\tau)$, resulting in suppressed heat and mass transfer for both positive $(\mathrm{Ro}>0)$ and negative $(\mathrm{Ro}<0)$ rotation profiles.
- Irrespective of the rotation profile, increasing the modulation amplitude δ enhances heat and mass transfer.

These findings provide valuable insights into the behavior of stationary magnetoconvection in a nonuniformly rotating electrically conductive nanofluid under the influence of gravity modulation. Understanding the role of the parameters $\mathrm{Pr}, \mathrm{Rn}, \mathrm{Ro}$, Le, Ω, and δ and their influence on convection is crucial for managing and controlling the behavior of the system to optimize heat and mass transfer.

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ДОСЛІДЖЕННЯ ВПЛИВУ ГРАВІТАЦІЙНОЇ МОДУЛЯЦІЇ НА СЛАБОНЕЛІНІЙНУ МАГНІТОКОНВЕКЦІЮ В ШАРІ НАНОРІДІНИ, ЩО НЕРІВНОМІРНО ОБЕРТАЄТЬСЯ

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В цій роботі досліджується вплив гравітаційної модуляції на слабонелінійну магнітоконвекцію в шарі нанорідіни, що нерівномірно обертається. Отримано основні рівняння для декартової апроксимації течії Куетта з використанням наближення Бусінеска та гравітаційної модуляції. Слабонелінійний режим аналізується методом збурень за малим параметром надкритичності числа Релея з урахуванням ефектів броунівського руху та термофорезу у шарі нанорідіни. Тепломасоперенос оцінюється в термінах кінцевих амплітуд і розраховується за числами Нуссельта для рідини та об'ємної концентрації наночастинок. Отримані дані показують, що гравітаційна модуляція, нерівномірне обертання та відмінності в об'ємній концентрації наночастинок на межах шарів можуть ефективно управляти тепломасопереносом. Крім того, негативний профіль обертання має ефект дестабілізації. Дослідження показує, що модульована система передає більше тепла та маси, ніж немодульована система.

Ключові слова: *нанорідина; шар, що нерівномірно обертається; слабонелінійна теорія; гравітаційна модуляція; неавтономне рівняння Гінзбурга-Ландау*