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DUAL SOLUTIONS OF HYBRID NANOFLUID FLOW OVER A CONE WITH THE INFLUENCE OF THERMAL RADIATION AND CHEMICAL REACTION AND ITS STABILITY ANALYSIS[†]

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The main intention of this study is to differentiate the stable and realisable solutions between the dual solutions of the water-based hybrid nanofluid flow driven by a solid cone along with energy transfer in the form of heat and mass by employing a new approach called stability analysis. The deviation of thermal radiation, chemical responses and heat absorption/generation are reserved into account. The leading equations which support the mathematical representation of this study are renovated by utilizing a set of similarity variables and solved by the MATLAB built-in bvp4c solver scheme. The outcomes of this study are presented both graphically and numerically. From this study, two kind of flow solutions have been achieved where one of them is related to the time-independent solutions and stable in nature. Also, the speed of the hybrid nanofluid can be controlled by applying magnetic field, but we should keep in mind that excessive amount of magnetic parameter may damage the system by burning.
Keywords: *Hybrid nanofluid; Solid cone; Thermal radiation; Chemical reaction; Dual solutions; Stability Analysis*
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INTRODUCTION

In recent times, physics of nanofluid flow has achieved important significance due to their various significance in diverse areas such as medicine, electronics and heat transfer devices etc. Nanofluids have better performance in different thermo-physical properties compared to the base fluids like water and oil etc. The mixture of nanoparticles with base fluids are termed as "Nanofluid" which was first coined by Choi [1].

The nanoparticles with different oxidation stages such as metallic or non-metallic particles like Cu, Al, Ag, Fe, Al_2O_3 and CuO etc that are typically used in a base fluid like water, ethylene glycol, Kerosene and different bio-fluids to form nanofluids. Due to the vast applications of nanofluids, researchers have motivated to study the importance of nanofluids and interpreted different kind of results associated with nanofluids and their properties. Recently, Mishra *et al.* [2] have investigated the water based nanofluid containing Ag-nanoparticle under different slip effects. Chanie *et al.* [3] have explored the flow behaviour of water-based nanofluid containing Cu and Ag particles and found that the motion of the fluid containing Cu is more effective than the $Ag - H_2O$.

Hybrid nanofluids are new kind of fluids that are made up of two or more different nanoparticles with traditional base fluids. This kind of fluid has advance features than the general nanofluids. An individual matter can never has all the required characteristic that is the material may be omitted or deficient some properties. The hybrid nanoparticles can be customized in such a way that it can process better significant than the other nanofluids. $Al_2O_3 + Cu$, $Al_2O_3 + Ni$, $MgO + Fe$&$Al_2O_3 + SiO_2$ etc are some examples of hybrid nanomaterials. Turcu *et al.* [4] and Jana *et al.* [5] were the foremost authors who have studied hybrid nanofluid experimentally. They have examined that the rate of heat transfer of the hybrid nanofluids is noticeably superior than the general nanofluids. This kind of fluids have better significant thermo-physical properties than the nanofluids.

In this study, we have investigated the water-based hybrid nanofluid containing Cu and Al_2O_3 nanoparticles by considering a solid cone with the influence of both thermal and mass transmissions. This type of hybrid nanoparticles is used for oxygen storage, production and many other industrial applications [6]. There are many applications of fluid flow due to a solid cone in different industrial and engineering sciences such as the solder tip, the conical heater and the continuous variable transmission (CVT) in modern car [7]. Recently, many authors [2,7,8,9,10,11] have investigated the nanofluid and hybrid nanofluid flow caused due to a solid cone and given different importance outcomes and characteristics of cone.

Different physical areas recognise the importance of the synchronised effects of heat and mass transmissions on the flow of magnetised fluid under various flow geometries. Many industrial and engineering processes such as annealing and thickening of copper wire, paper production, MHD pump and MHD generators etc need both heat and mass transfers phenomenon with magnetic field effects. Dey and Borah [12] and Dey *et al.* [13,14 15] have investigated the boundary-layer fluid flow under the influence of both heat and mass transfers by considering different geometries. Alzahrani *et al.* [16] have examined the flow behaviour of hybrid nanofluid over a flat plate with the effects of both heat and mass transfers. Devi and Anjali Devi [17] and Khashi'ie *et al.* [18] have looked towards improving thermal transmission of $Cu - Al_2O_3$ hybrid nanofluid flow over a extending surface. Researchers [2,7,19] have investigated the hybrid nanofluid flow due to a solid and

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rotating cone under different flow factors such as heat and mass transfers etc. Abdullah *et al.* [20] have analysed the convective heat transfer characteristics of hybrid nanofluid driven by square enclosure.

Also, the thermal radiation and chemical reaction effects on the fluid flow problems have played an important role in different physical fields. From the several decades, many researchers have given attention on these two flow factors as because of their multifarious application in different industrial, engineering and medical applications. Chamkha *et al.* [21] have examined the effects of radiation on mixed convection fluid flow by suspending nanoparticles. Sulochana *et al.* [22] have explored the influence of both thermal radiation and chemical reaction on magnetohydrodynamics nanofluid flow due to moving surface. Recently, Sharma *et al.* [23] have explored the effects of radiation parameter on the hybrid nanofluid flow over an extending surface with Joule heating. Dey *et al.* [24] have analyzed the stability analysis of magnetized fluid flow with the effects of chemical reaction. Saleem *et al*. [25] have investigated the MHD nanofluid flow over a rotating cone under the influence thermal radiation.

Due to the lack of information about the smoothness of the surface of the considering geometry and mathematical tools with assumptions, some initial complexity in the flow is observed which may develop non-uniqueness flow solutions. This initial complexity in the flow classifies the flow solutions into two categories, one solution is stable and physically tractable. Markin [26] was the first author who has explored the idea of dual solutions and their stability. Recently, Ghosh and Mukhopadhyay [27], Dey and Borah [12], Dey *et al.* [14], Dey *et al.* [24], Waini *et al.* [28] and Dey *et al.* [29] have explored the nature of non-uniqueness solutions and their stability behaviour by considering different fluids model.

The present work is all inspired by the above literatures and its immense relevance in different physical fields. We were able to ascertain via the literature review that this study has novelty like effects of thermal radiation, considering nanoparticles to form hybrid nanofluid, dual solutions and its stability analysis and will have a significant influence on other experts in the field. To the authors' awareness, this type of fluid model obtained by inserting the nanoparticles $Cu + Al_2O_3$ with water which is driven by a solid cone along with energy transfers has not yet been taken into account while analysing dual solutions and their stability.

The intention of this work is to analyze the dual solutions and their stability of the water-based hybrid nanofluid driven by a solid cone which is sited in a porous medium under the influence of thermal radiation and chemical reaction with different slip flow effects such as velocity, thermal and concentration slips. Here, we have initially suspended the Cu solid nanoparticle of volume fraction $\phi_2 = 0.06$ into the water base fluid to form $Cu/water$ nanofluid. Again, Al_2O_3 solid nanoparticle of volume fraction $\phi_1 = 0.1$ is added into $Cu/water$ nanofluid and achieved the $Al_2O_3 - Cu/water$ hybrid nanofluid. Also, a harmonized magnetic field is applied in the normal direction of the conical surface which plays an important role to enhance the thermal properties of the fluid. The leading equations which support the mathematical model of this problem are renovated by utilizing a set of similarity variables and solved by the MATLAB built-in bvp4c solver technique. Stability analysis is executed between the flow solutions to characterise the stable and physically achievable solution. Also, for the verification of our numerical codes, we established a reasonable uniformity when we compared our findings to the previously published article Mishra *et al.* [2].

FORMULATION OF THE PROBLEM

We have constructed a mathematical model of this study by considering a permeable cone of radius $r(x)$ which is immersed in a steady, incompressible and two-dimensional hybrid nanofluid. The flow diagram and its coordinate system are shown in Fig. 1. Where, the x- axis is measured along the surface of the cone and y- axis is taken in the normal direction of the conical surface such that vertex of the cone is taken as the origin of the system. In this study, we have considered the influence of both heat and mass transfers such that T_w and C_w prescribe the constant wall temperature and concentration respectively and $T_\infty \& C_\infty$ the temperature and concentration at free stream region. A uniform magnetic field of strength B_0 is applied in the normal direction of the conical surface. Here, we have considered the hybrid nanofluid which is formed by adding Al_2O_3 nanoparticles into the Cu/H_2O nanofluid and hence $Al_2O_3 - Cu/H_2O$ hybrid nanofluid is found. Table 1 discusses the thermo-physical characteristics of the base fluid and solid particles.

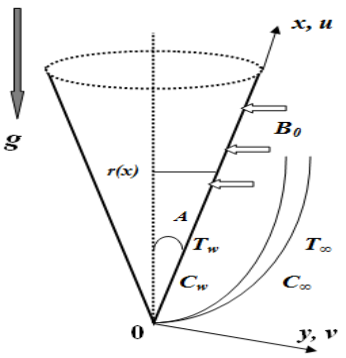

Figure 1. Flow diagram and its coordinate system.

Table 1. Thermo-physical properties of base fluid and solid particles (Mohmed *et al*. [7]).

Property	H_2O	Cu	Al_2O_3
$\rho\ (Kg\cdot m^{-3})$	997.1	8933	3970
$C_P\ [J(Kg\cdot K)^{-1}]$	4179	385	765
$k\ [W(mK)^{-1}]$	0.613	400	40
$\sigma\ (Sm^{-1})$	5.5×10^{-6}	59.6×10^{6}	35×10^{6}

Under the aforesaid assumptions and boundary-layer approximations, we have considered the following leading equations which govern the present problem ([7] and [2]).

$$\frac{\partial(ru)}{\partial x}+\frac{\partial(rv)}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu_{hnf}\frac{\partial^2 u}{\partial y^2}+g\beta(T-T_\infty)CosA-g\beta^*(C-C_\infty)CosA-\frac{\mu_{hnf}}{\rho_{hnf}}\frac{1}{k_p}u-\frac{\sigma_{hnf}}{\rho_{hnf}}B_0^2u=0, \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k_{hnf}}{(\rho C_P)_{hnf}}\frac{\partial^2 T}{\partial y^2}+\frac{\mu_{hnf}}{(\rho C_P)_{hnf}}\left(\frac{\partial u}{\partial y}\right)^2+\frac{Q_0}{(\rho C_P)_{hnf}}(T-T_\infty)+\frac{\sigma_{hnf}}{(\rho C_P)_{hnf}}B_0^2u^2-\frac{1}{(\rho C_P)_{hnf}}\frac{\partial q_r}{\partial y}, \tag{3}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_m\frac{\partial^2 C}{\partial y^2}-Kr(C-C_\infty). \tag{4}$$

Here, we have initially suspended the Cu solid nanoparticle of volume fraction $\phi_2=0.06$ into the water base fluid to form Cu/H_2O nanofluid. Again, Al_2O_3 solid nanoparticle of volume fraction $\phi_1=0.1$ is added into Cu/H_2O nanofluid and achieved the Al_2O_3-Cu/H_2O hybrid nanofluid. The properties of hybrid nanofluidis given below such that the base fluid and the nanoparticles are denoted with the subscript $f, s_1\&s_2$ [7].

$$\mu_{hnf}=\frac{\mu_f}{(1-\phi_1)^{2.5}(1-\phi_2)^{2.5}},\rho_{hnf}=(1-\phi_2)\left[(1-\phi_1)\rho_f+\phi_1\rho_{s_1}\right]+\phi_2\rho_{s_2},$$

$$(\rho C_P)_{hnf}=(1-\phi_2)\left[(1-\phi_1)(\rho C_P)_f+\phi_1(\rho C_P)_{s_1}\right]+\phi_2(\rho C_P)_{s_2},$$

$$\frac{k_{hnf}}{k_{bf}}=\frac{\left(k_{s_2}+2k_{bf}-2\phi_2(k_{bf}-k_{s_2})\right)}{\left(k_{s_2}+2k_{bf}+\phi_2(k_{bf}-k_{s_2})\right)},\frac{k_{bf}}{k_f}=\frac{\left(k_{s_1}+2k_f-2\phi_1(k_f-k_{s_1})\right)}{\left(k_{s_1}+2k_f+\phi_1(k_f-k_{s_1})\right)},$$

$$\frac{\sigma_{hnf}}{\sigma_{bf}}=1+\frac{3\left(\frac{\sigma_{s_2}}{\sigma_{bf}}-1\right)\phi_2}{\left(\frac{\sigma_{s_2}}{\sigma_{bf}}+2\right)-\left(\frac{\sigma_{s_2}}{\sigma_{bf}}-1\right)\phi_2},\frac{\sigma_{bf}}{\sigma_f}=1+\frac{3\left(\frac{\sigma_{s_1}}{\sigma_f}-1\right)\phi_1}{\left(\frac{\sigma_{s_1}}{\sigma_f}+2\right)-\left(\frac{\sigma_{s_1}}{\sigma_f}-1\right)\phi_1}.$$

The related boundary conditions are:

$$y=0: u=u_w+l_1\frac{\partial u}{\partial x}, v=v_w, T=T_w+l_2\frac{\partial T}{\partial y}, C=C_w+l_3\frac{\partial C}{\partial y}, \tag{5}$$
$$y\to\infty: u\to 0, T\to T_\infty, C\to C_\infty.$$

Where, $l_1, l_2\ \&\ l_3$ are the slip factors and vanishing of $l_1, l_2\ \&\ l_3$ implies the no slip flow in the system. To alter the nature of equations (1)-(4), we have adopted the following set of variables [2].

$$\eta=\sqrt{a/\upsilon_f}\,y,\psi=\sqrt{a\upsilon_f}\,xrf(\eta),r=x\sin A,\theta(\eta)=\frac{T-T_\infty}{T_w-T_\infty},\phi(\eta)=\frac{C-C_\infty}{C_w-C_\infty},$$
$$u=\frac{1}{r}\frac{\partial\psi}{\partial y}=axf'(\eta),v=-\frac{1}{r}\frac{\partial\psi}{\partial x}=-2\sqrt{a\upsilon_f}\,f(\eta). \tag{6}$$

From the Roseland approximation (following Prameela *et al.* [30]), the local radiative heat flux term for optically thick gray fluid is given by

$$q_r=-\frac{4\sigma_0}{3k_2}\frac{\partial T^4}{\partial y}\approx-\frac{16\sigma_0T_\infty^3}{3k_2}\frac{\partial T}{\partial y}, \tag{7}$$

where $T^4\approx 4T_\infty^3T-3T_\infty^4$ (by Taylor series approximation). As a result,

$$\frac{\partial q_r}{\partial y}\approx-\frac{16\sigma_0T_\infty^3}{3k_2}\frac{\partial^2 T}{\partial y^2}. \tag{8}$$

After utilizing the equations (6) and (8) into the equations (1)-(4), we have achieved the following set of equations and the equation (1) which represents the conservation of mass identically satisfies the similarity variables (6).

$$A_1f'''+Gr\theta-Gm\phi-A_1k_1f'-A_2A_3Mf'-f'^2+2ff''=0, \tag{9}$$

$$A_4\frac{1}{\Pr}(1+Nr)\theta''+A_5Ecf''^2+A_6Q\theta+A_7MEcf'^2+2f\theta'=0, \quad (10)$$

$$\phi''+Scf\phi'-ScCr\phi=0. \quad (11)$$

The boundary restrictions become in the following form:

$$f(0)=S, f'(0)=1+L_v f''(0), \theta(0)=1+L_T\theta'(0), \phi(0)=1+L_m\phi'(0), \quad (12)$$
$$f'(\infty)\to 0, \theta(\infty)\to 0, \phi(\infty)\to 0.$$

Where, $A_i(i=1,2,3,4,5,6,7)$'s are defined in the following way:

$$A_1=\frac{\upsilon_{hnf}}{\upsilon_f}, A_2=\frac{\sigma_{hnf}}{\sigma_f}, A_3=\frac{\rho_f}{\rho_{hnf}}, A_4=\frac{k_{hnf}/k_f}{(\rho C_P)_{hnf}/(\rho C_P)_f}, A_5=\frac{\mu_{hnf}/\mu_f}{(\rho C_P)_{hnf}/(\rho C_P)_f},$$
$$A_6=(\rho C_P)_f/(\rho C_P)_{hnf}, A_7=\frac{\sigma_{hnf}/\sigma_f}{(\rho C_P)_{hnf}/(\rho C_P)_f}.$$

The dimensionless parameters are:

$$Gr=\frac{g\beta(T_w-T_\infty)}{a^2x}, Gm=\frac{g\beta^*(C_w-C_\infty)}{a^2x}, k_1=\frac{\upsilon_f}{ka}, M=\frac{\sigma_f B_0^2}{\rho_f a}, \Pr=\frac{(\rho C_P)_f\upsilon_f}{k_f}, Q=\frac{Q_0}{a(\rho C_P)_f},$$
$$Ec=\frac{(ax)^2\rho_f}{(\rho C_P)_f(T_w-T_\infty)}, Nr=\frac{16\sigma_0 T_\infty^3}{3k_2k_f}, Sc=\frac{\upsilon_f}{D_m}, Cr=\frac{k_r}{a}, L_v=\frac{l_1}{\sqrt{\upsilon_f/a}}, L_T=\frac{l_2}{\sqrt{\upsilon_f/a}}, L_m=\frac{l_3}{\sqrt{\upsilon_f/a}}.$$

During this study, we have observed three types of physical quantities which play an important role in different physical fields by computing shear stress, rate of heat transfer and rate of mass development at the surface. These physical quantities are defined in the following way [2], [7]:

$$C_f=2\frac{\mu_{hnf}}{\rho_f U_w^2}\left(\frac{\partial u}{\partial y}\right)_{y=0}, Nu_x=-\frac{xk_{hnf}}{k_f(T_w-T_\infty)}\left(\frac{\partial T}{\partial y}\right)_{y=0}, Sh_x=-\frac{x}{(C_w-C_\infty)}\left(\frac{\partial C}{\partial y}\right)_{y=0}. \quad (13)$$

Now, using the equation (6) into the equation (13), we have achieved the following normalize form of the above quantities as:

$$C_f=2\frac{\mu_{hnf}}{\mu_f}(\mathrm{Re}_x)^{-\frac{1}{2}}f''(0), Nu_x=-\frac{k_{hnf}}{k_f}(\mathrm{Re}_x)^{\frac{1}{2}}\theta'(0), Sh_x=-(\mathrm{Re}_x)^{\frac{1}{2}}\phi'(0), \quad (14)$$

where, the local Reynolds number, $Re_x=\frac{xU_w}{\nu_f}$.

FLOW STABILITY

Due to the considering geometry and lack of mathematical tools and assumptions, some initial disturbances in the flow have been occurred which are decay or growth with time. These disturbances classify the flow solutions into two categories, one of them converges to its time-independent flow solutions as the initial complexity in the flow decay with time. The flow stability is needed to characterize the stable and unstable solutions between the dual solutions. To implement the stability analysis of this problem, the unsteady governing equations are essential which are obtained by adding the terms $\frac{\partial u}{\partial t}, \frac{\partial T}{\partial t} \& \frac{\partial C}{\partial t}$ into the equations (2), (3) & (4) respectively. Due to the presence of time variable, equation (6) takes a modified form as given below:

$$\eta=\sqrt{\frac{a}{\nu_f}}y, \psi=\sqrt{a\nu_f}xrf(\eta,\tau), r=x\sin A, \theta(\eta,\tau)=\frac{T-T_\infty}{T_w-T_\infty}, \phi(\eta,\tau)=\frac{C-C_\infty}{C_w-C_\infty}, \tau=at \quad (15)$$

The time-dependent governing equations after using equation (15) become:

$$A_1\frac{\partial^3 f}{\partial\eta^3}+Gr\theta-Gm\phi-A_1k_1\frac{\partial f}{\partial\eta}-A_2A_3M\frac{\partial f}{\partial\eta}-\left(\frac{\partial f}{\partial\eta}\right)^2+2f\frac{\partial^2 f}{\partial\eta^2}-\frac{\partial^2 f}{\partial\eta\partial\tau}=0, \quad (16)$$

$$\frac{A_4}{Pr}(1+Nr)\frac{\partial^2\theta}{\partial\eta^2}+A_5Ec\left(\frac{\partial^2 f}{\partial\eta^2}\right)^2+A_6Q\theta+A_7MEc\left(\frac{\partial f}{\partial\eta}\right)^2+2f\frac{\partial\theta}{\partial\eta}-\frac{\partial\theta}{\partial\tau}=0, \quad (17)$$

$$\frac{\partial^2\phi}{\partial\eta^2}+Scf\frac{\partial\phi}{\partial\eta}-ScCr\phi-Sc\frac{\partial\phi}{\partial\tau}=0. \quad (18)$$

The surface restrictions become:

$$f(0,\tau)=S, \frac{\partial f(0,\tau)}{\partial\eta}=1+L_v\frac{\partial^2 f(0,\tau)}{\partial\eta^2}, \frac{\partial f(\infty,\tau)}{\partial\eta}\to 0,$$

$$\theta(0,\tau) = 1 + L_T \frac{\partial\theta(0,\tau)}{\partial\eta}, \theta(\infty,\tau) \to 0, \tag{19}$$

$$\phi(0,\tau) = 1 + L_m \frac{\partial\phi(0,\tau)}{\partial\eta}, \phi(\infty,\tau) \to 0.$$

To test the stability of the steady flow solution $f(\eta) = f_0, \theta(\eta) = \theta_0$ and $\phi(\eta) = \phi_0$ fulfilling the equations (9)-(11), a set of perturb equations is employed for differentiating the variables ([26], [12] and [14, 24]):

$$f(\eta) = f_0(\eta) + e^{-\omega\tau}F(\eta),$$
$$\theta(\eta) = \theta_0(\eta) + e^{-\omega\tau}G(\eta), \tag{20}$$
$$\phi(\eta) = \phi_0(\eta) + e^{-\omega\tau}H(\eta).$$

Where, ω is an unknown eigenvalue parameter, $F, G\&H$ the small associated to time-independent solutions. Inserting equation (20) into the equations (16)-(18), fixing $\tau \to 0$ and simplifying the equations, we have achieved the following linearized eigenvalue problems.

$$A_1F''' + GrG - GmH - A_1k_1F' - A_2A_3MF' - 2f_0'F' + 2(f_0F'' + Ff_0'') + \omega F' = 0, \tag{21}$$

$$\frac{A_4}{Pr}(1 + Nr)G'' + 2A_5Ecf_0''F'' + A_6QG + 2A_7MEcf_0'F' + 2(f_0G' + F\theta_0') + \omega G = 0, \tag{22}$$

$$H'' + Sc(f_0H' + F\phi_0') - ScCrH + Sc\omega H = 0. \tag{23}$$

Relevant boundary conditions are:

$$F(0) = 0, F'(0) = 0, G(0) = 0, H(0) = 0,$$
$$F'(\infty) \to 0, G(\infty) \to 0, H(\infty) \to 0. \tag{24}$$

Following Mishra *et al.* [31] and Dey and Borah [12], the equations (21)-(23) are solved together with adjusted boundary conditions (referring Wahid *et al.* [28] and Dey and Borah [32]). It is noticed that this eigenvalue problem gives an infinite number of eigenvalues $\omega_1 < \omega_2 < \omega_3 < \cdots$, where ω_1 signifies the smallest eigenvalue. The flow stability can be examined with the help of this smallest eigenvalue. If $\omega_1 > 0$, then the initial complexity in the flow decay with time and the flow solution becomes stable. Otherwise, the flow solution to be unstable due to the escalation of complexity in the flow with time.

RESULT AND DISCUSSION

The set of equations (9)-(11) and (21)-(23) along with their two-point boundary conditions have been solved numerically by utilizing the MATLAB built-in bvp4c solver scheme. This technique performs the three-stage Lobatto IIIa formula and executes the finite difference method. It controls and adjusts the error upto 10^{-6} by its residuals.

The numerical explanations of this study have been achieved for the velocity $f'(\eta)$, thermal fraction$\theta(\eta)$and mass fraction $\phi(\eta)$ profiles of the hybrid nanofluid as an outcomes of different novel flow parameters and have been displayed in figures (2)-(9). In this problem, we have achieved two types of solutions, first solution is represented by the solid line and it is related to the time-independent solution. The dashed line signifies the second solution which is converged slowly to its free stream region due to the presence of flow disturbance.

Before conferring the numerical results, we afford confirmation of our numerical code by solving the model presented in [2] and comparing the present numerical results (first solution) with the results reported in [2]. Mishra *et al.* [2] have analysed the water based nanofluid flow containing the Ag nanoparticle over a solid cone with the influence of heat and mass transfers. In the non-appearance of thermal radiation and hybrid nanofluid, our leading equations during steady case of this study are matched with Mishra *et al.* [2] works. Table 2 reflects the comparison of the numerical code in terms of local Nusselt number and gives a good conformity for our results.

Table 3 is developed to check the flow stability between the dual solutions with the help of evaluated smallest eigenvalues for different values of suction parameter (S). From this table, it is noticed that the smallest eigenvalues are positive and negative for first solution and second solution respectively. Due to the positive values of least eigenvalues, the initial disturbances in the flow lie down as time evolves and the flow solution converges quickly to its time-independent solution. Hence, the first solution becomes stable and physically achievable. But, negative values of least eigenvalues develop the initial disturbances in the flow and hence the flow solution (second solution) is being as unstable behaviour and slowly converges to its free stream region.

Table 4 is inserted to show the skin friction coefficient numerically during time-dependent and time-independent cases for developing values of the Prandtl, Schmidt and Eckert numbers. These quantities help to evaluate the effects of shear stress at the surface of the cone. It appears from this data that the skin friction coefficient has been experienced a reduction from the noble gas $(Pr = 0.015)$ to sea-water $(Pr = 13.5)$. Again, the effects of shear stress at the surface have been enhanced from the hydrogen $(Sc = 0.22)$ to water vapour $(Sc = 0.60)$. It is also seen from this table is that increasing values of the Eckert number (Ec) raises the influence of shear stress during both the solutions at the surface of the cone. It is also observed that the effects of shear stress of the fluid at the surface of the cone during time-dependent case (second solution) is fewer than the first solution. Table 5 is inserted to show the heat transfer rate of the fluid at the surface of the cone for developing values of Prandtl and Eckert numbers. From this table, it is achieved that the Prandtl number develops the rate of heat transfer of the fluid during both the cases. But the Eckert number reduces the rate of heat transfer at the surface of the cone.

Table 2. Numerical presentation of Nusselt number $\left(-\theta'(0)\right)$ of Ag/H_2O nanofluid for different values of M when $Pr = 0.7, Sc = 1,\ L_v = 0.1, L_T = 0.1, L_m = 0.1, Ec = 0.05, Q = 0.1, S = 0.4, \phi_1 = 0.1, Gr = Gm = 0.1, k_1 = 0.5, Nr = 0.$

	Mishra *et al.* [2] works	Present Results
$\boldsymbol{M}$	$-\boldsymbol{\theta}'(\mathbf{0})$	$-\boldsymbol{\theta}'(\mathbf{0})$
0.5	2.279557	2.265545
0.7	2.259399	2.251232
1	2.231499	2.221432

Table 3. Numerical presentation of smallest eigenvalue for different values of S when $M = 0.5, Gr = 1, Gm = 2, Q = 0.1, Nr = 0.2, k_1 = 0.5, L_v = 0.2, L_T = 0.1, L_m = 0.2, Sc = 0.22, Pr = 0.71, Ec = 0.3.$

	Smallest Eigenvalue ($\boldsymbol{\omega_1}$)	
$\boldsymbol{S}$	**First Solution**	**Second Solution**
0.25	0.20209420	-1.220506212
0.63	1.20955422	-3.02148542
0.82	2.26992567	-4.00797213

Table 4. Numerical presentation of skin friction coefficient for different values of $Pr, Sc \& Ec$ when $M = 0.5, Gr = 1, Gm = 2, Q = 0.1, Nr = 0.2, k_1 = 0.5, L_v = 0.2, L_T = 0.1, L_m = 0.2.$

$\boldsymbol{Pr}$	$\boldsymbol{Sc}$	$\boldsymbol{Ec}$	Skin friction coefficient ($\boldsymbol{C_f}$)	
			First Solution	**Second Solution**
0.015	0.22	0.005	-1.88491462	-2.35204790
7			-1.91532212	-2.41139078
13.5			-1.92660748	-2.40977806
0.71	0.22		-1.88406043	-2.35823329
	0.30		-1.88286201	-2.35613715
	0.60		-1.87858313	-2.34883267
	0.22	0.0	-1.88408938	-2.35828831
		0.3	-1.88235554	-2.35499662
		1	-1.87833480	-2.34738988

Table 5. Numerical presentation of local Nusselt number for different values of $Pr \& Ec$ when $M = 0.5, Gr = 1, Gm = 2, Q = 0.1, Nr = 0.2, k_1 = 0.5, L_v = 0.2, L_T = 0.1, L_m = 0.2, Sc = 0.22.$

$\boldsymbol{Pr}$	$\boldsymbol{Ec}$	Local Nusselt Number ($\boldsymbol{Nu_x}$)	
		First Solution	**Second Solution**
0.015	0.005	0.24259499	0.264506742
7		1.26955410	3.12148393
13.5		2.29992551	4.32797261
0.71	0.0	0.18628192	0.35992906
	0.3	0.07307090	0.11873327
	1.0	-0.18963456	-0.43981793

The effect of M on the hybrid nanofluid's velocity and thermal fraction is shown in Figs. (2) and (3).

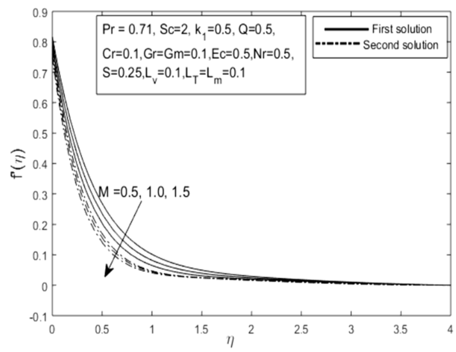

Figure 2. Velocity profile for incremental amount of M

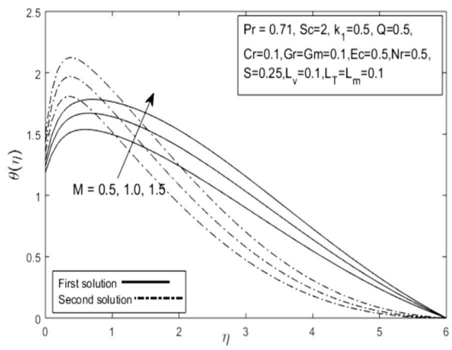

Figure 3. Sketch of thermal fraction for incremental amount of M

From these figures, it is believed that the hybrid nanofluid's velocity has been decelerated with the increasing amount of Mduring both the cases, whereas the thermal fraction of the fluid has been developed with M. A resistive type force known as the "Lorentz force" is produced as a result of the application of a magnetic field and has the power to control fluid motion. As a result, the hybrid nanofluid's velocity is decreased with increased values of M. Again, due to the effects of magnetic field, the velocity of the fluid has been decelerated and a frictional force is developed between the fluid and surface of the cone which generates additional energy in terms of heat at the surface of the cone and hence the thermal fraction of the hybrid nanofluid is enhanced with$M$. From Figure (2), it appears that the hybrid nanofluid moves more slowly in the time-dependent (second solution) situation than in the time-independent (first solution) case. Due to this reason, fluid's temperature during second solution superiors than the first solution in the vicinity of the surface of the cone. Figures (4)-(6) are depicted to show the effects of porosity of the porous medium on the velocity, thermal fraction and mass fraction of the hybrid nanofluid.

From Figure (4), it is noticed that the speed of the fluid has been enhanced with the increasing values of k_1 during both the cases. As a result, the overall pressure of the hybrid nanofluid reduces and the thermal fraction and mass fraction of the fluid are directly proportional to the pressure and hence the thermal fraction and mass fraction are experienced reduction with k_1 (see Fig. 5 and Fig. 6). Also, the second solution of the velocity distribution and mass fraction of the fluid are comparatively fewer than the first solution.

Figure (7) shows how Nr affects the thermal fraction of the hybrid nanofluid. It can be seen from this figure that when Nr increases, the thermal portion of the hybrid nanofluid decreases in both cases.

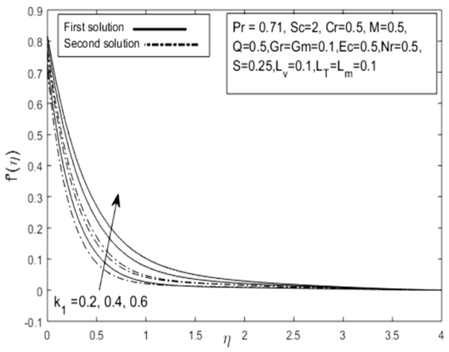

Figure 4. Velocity profile for incremental amount of k_1

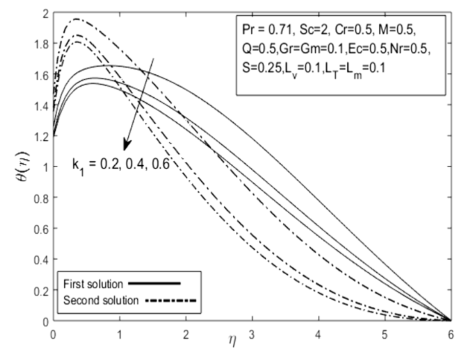

Figure 5. Sketch of thermal fraction for incremental amount of k_1

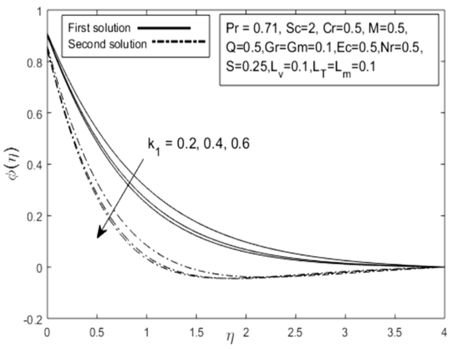

Figure 6. Sketch of mass fraction for incremental amount of k_1

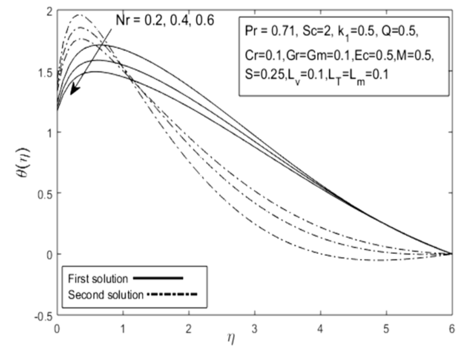

Figure 7. Sketch of thermal fraction for incremental amount of Nr

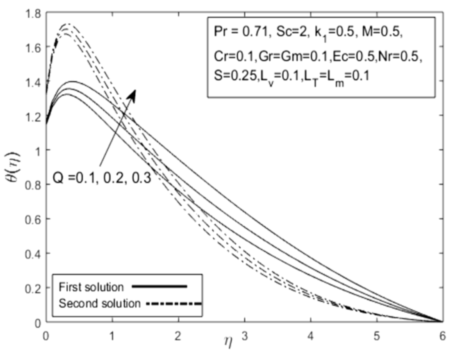

Figure 8. Sketch of thermal fraction for incremental amount of Q

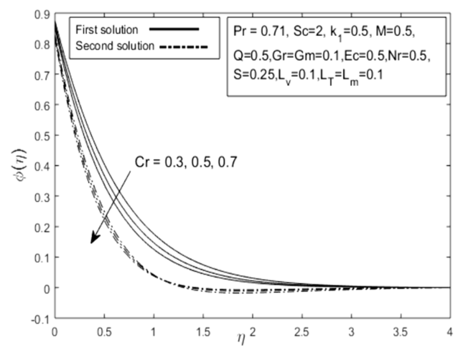

Figure 9. Sketch of mass fraction for incremental amount of Cr.

The cause of this occurrence is that the thermal radiation parameter (Nr) is directly proportional to the third power of the temperature of fluid at free stream region. Therefore, as Nr increases, the hybrid nanofluid's thermal fraction over the cone decreases. Also, it is seen that the thermal fraction of the fluid during the second solution is larger than the first solution near the surface of the cone. Figure (8) reflects the impact of heat generation parameter (Q) on the thermal fraction of the fluid. From this figure, it is perceived that the thermal fraction of the hybrid nanofluid has been increased with Q. Influence of Cr on the mass fraction of the hybrid nanofluid is presented in Fig. 9. The chemical reaction is occurred mainly due to the effects of suction/injection of the fluid flow and presence of flow slips effects. The chemical reaction parameter has the capacity to fall down the mass fraction of the considering fluid. It can also be observed that the concentration level of the hybrid nanofluid during the second solution is fewer and slowly conversed to its free stream region over the first solution.

We have found from this study that all flow profiles display dual solutions up to a specific area of the similarity variable (η) and asymptotically meet the far-field boundary conditions.

CONCLUSION

This paper analyzes $Cu - Al_2O_3$/water hybrid nanofluid flow driven by a solid cone which sited in a porous medium. The energy transfer in terms of heat and mass is encountered with the effects of magnetic field, thermal radiation and chemical reaction. The leading equations are solved through numerical method called “three-stage Lobatto IIIa formula” by using MATLAB built in bvp4c solver scheme. Major results have been obtained as follows:

1) The velocity of the hybrid nanofluid has been decelerated with the improving amount of M, but it has enhanced the thermal fraction of the fluid during both the cases.
2) Due to the increasing amount of k_1, the velocity of the fluid has accelerated. Whereas, thermal and mass fractions of the hybrid nanofluid have been controlled by employing k_1.
3) Thermal fraction of the hybrid nanofluid flow is a decreasing function of the thermal radiation (Nr) and an increasing function of the heat generation (Q) parameters.
4) Mass fraction of the hybrid nanofluid during the second solution is fewer than the first solution with the increasing amount of Cr.
5) Effects of shear stress is dropped down for incremental values of the Prandtl number. But we should keep in mind that the excessive amount of the Prandtl number may damage the system by developing the rate of heat transfer at the surface of the cone.
6) “According to the stability point of view, the first solution, which converges to the steady flow solution more quickly, is stable, whereas the second solution is unstable and impractical.

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ДУАЛЬНІ РІШЕННЯ ДЛЯ ГІБРИДНОЇ ТЕЧІЇ НАНОРІДИНИ ПО КОНУСУ З ВПЛИВОМ ТЕПЛОВОГО ВИПРОМІНЮВАННЯ І ХІМІЧНОЇ РЕАКЦІЇ ТА АНАЛІЗ ЇЇ СТАБІЛЬНОСТІ

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Основна мета цього дослідження полягає в тому, щоб розрізнити стабільні та реалізовані рішення між подвійними рішеннями потоку гібридної нанофлюїди на водній основі, що рухається по твердому конусу, разом із передачею енергії у формі тепла та маси, використовуючи новий підхід, який називається аналізом стабільності. Враховано відхилення теплового випромінювання, хімічних реакцій і поглинання/утворення тепла. Провідні рівняння, які підтримують математичне представлення цього дослідження, оновлені за допомогою набору змінних подібності та розв'язані за допомогою вбудованої схеми рішення рівнянь MATLAB bvp4c. Представлені графічні та чисельні результати цього дослідження. У результаті цього дослідження було отримано два типи потокових рішень, де один із них пов'язаний із незалежними від часу рішеннями та є стабільним за своєю природою. Крім того, швидкість течії гібридного нанофлюїду можна контролювати, застосовуючи магнітне поле, але треба мати на увазі, що надмірна кількість магнітного параметра може пошкодити систему шляхом окислення.

Ключові слова: *гібридний нанофлюїд; твердий конус; теплове випромінювання; хімічна реакція; подвійні рішення; аналіз стабільності*