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COMPARATIVE STUDY OF THE MASS SPECTRA OF HEAVY QUARKONIUM SYSTEM WITH AN INTERACTING POTENTIAL MODEL[†]

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In this work, the comparison of the mass spectra of heavy quarkonium system with an interacting potential (class of Yukawa potential) was studied. The Schrodinger equation is analytically solved using the Nikiforov-Uvarov (NU) method and the series expansion method (SEM). The approximate solutions of the eigen energy equation and corresponding eigenfunction in terms of Laguerre polynomials were obtained using the NU method and the solutions of the eigen energy equation were also obtained with the SEM. The mass spectra for heavy quarkonium system (HQS) for the potential under study were obtained for bottomonium $b\bar{b}$ and charmonium $c\bar{c}$. We compared the results obtained between the NU and the SEM. It was noticed that SEM solutions yield mass spectra very close to experimental data compared to solutions with the NU method. The obtained results were also compared with works by some other authors and were found to be improved. This study can be extended by using other exponential-type potential models with other analytical approach and a different approximation scheme to obtain the mass spectra of heavy quarkonium system. The relativistic properties using Klein-Gordon or Dirac equations can be explored to obtain the mass spectra of light quarkonia. Finally, the information entailed in the normalized wave-functions can also be studied.
Keywords: *Schrödinger equation; Nikiforov-Uvarov method; Class of Yukawa potential; Mass Spectra; Series expansion method*
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1. INTRODUCTION

The study of the fundamental or constituent blocks of matter has been for long time a fascinating field in Physics. In the nineteenth century, the atom was considered to be the fundamental particles from which all matters were composed. This idea was used to explain the basic structure of all elements. Experiments performed at the end of the nineteenth century and beginning of the twentieth century provided evidence for the structure of an atom [1].

The conclusions were that all atoms have a nucleus containing protons which is surrounded by elements and that the nucleus was very small compared with the size of the atom. The neutron was introduced to explain the discrepancy between the mass of the atom and the mass from the number of protons. In 1932, Chadwick discovered the neutron and the fundamental particles were considered to be proton, the neutron and the electron. The discovery of antimatter in cosmic radiation supported the theory developed from the special theory of relativity and quantum theory that all fundamental particles have corresponding antimatter particles. The matter and antimatter particles have the same mass but opposite charge. The problem of what were considered to be fundamental particles was resolved by the quarks. Quarks are the basic building blocks of hadrons, particles interacting with each other through strong interaction [2,3]. In nuclear physics, we are mostly concerned with the lightest members of the hadron's family; nucleons, which make up all the nuclei and pions which constitute the main carriers of nuclear force. Since their discoveries, investigation of heavy quarkonium system (HQS) provides us with great tools for quantitative tests of quantum chromodynamics (QCD) [4]. Because of the heavy masses of the constituent quarks, a good description of many features of these systems can be obtained using non-relativistic models, where one assumes that the motion of constituent quarks is non-relativistic, so that the quark-antiquark strong interaction is described by a phenomenological potential [5,6]. Heavy quarkonium system have turned out to provide extremely useful probes for the deconfined state of matter because the force between a heavy quark and anti-quark is weakened due to the presence of gluons which lead to the dissociation of quarkonium bound states [7]. The quarkonia with heavy quark and antiquark and their interaction are well described by the Schrodinger equation (SE). The solution of the Schrodinger equation with spherically symmetric potential is of major concern in describing the mass spectra (MS) of quarkonium system [8,9]. In simulating the interaction potentials for these systems, confining-type potentials are generally used. The holding potentials can be of any form. For instance, a variety of this type of potential is the Cornell potential (CP) with two terms one of which is responsible for the Coulomb interaction of the quarks and the other correspond to a confining term [10,11]. Researchers have studied the MS of heavy and heavy-light quarkonia using the CP and its extended form [12-14]. For studying the behavior of several physical problems in Physics, we require to solve the Schrodinger equation. The solutions to the Schrodinger equation can be established with analytical methods such as the Nikiforov-Uvarov (NU) method [15-24], the asymptotic iterative method (AIM) [25], the extended NU method [26], the Nikiforov-Uvarov functional analysis (NUFA) method [27-30], the series expansion

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method [30-34], the WKB approximation [35-37], and so on [38]. Recently, the study of MS of HQS with exponential-type potentials has attracted the attention of most researchers. For example, Inyang et al. [39] studied the MS of HQS with Yukawa potential using the NU method. Also, Akpan et al. [40], presented the mass spectra of HQS using Hulthen and Hellmann potential model through the solutions of the Schrodinger equation. Furthermore, Ibekwe et al. [41] studied the mass spectra of HQS using the NU with the combination of screened Coulomb and Kratzer potential. Abu-Shady and Inyang [42], suggested trigonometric Rosen-Morse potential as the quark-antiquark interaction potential for studying the masses of heavy and heavy-light mesons. In the present research, our interest is to compare the mass spectra of HQS with the class of Yukawa potential (CYP) using the Nikiforov-Uvarov and the series expansion methods. The CYP is a combination of Yukawa potential [43], Hellmann potential [44] and inverse quadratic Yukawa potential [45]. The CYP applications cut across other fields of physics such has atomic, nuclear and condensed matter physics, among others. The CYP takes the form [46],

$$V(r)=-\frac{a}{r}+\frac{be^{-\alpha_I r}}{r}-\frac{ce^{-2\alpha_I r}}{r^2}, \tag{1}$$

where a,b and c are potential strengths, α_I is the screening parameter.

The exponential terms in Eq. (1) are expanded with Taylor series up to order three, so that the potential can interact in the quark-antiquark system, and Eq. (2) is obtained.

$$V(r)=-\frac{\alpha_0}{r}+\alpha_1 r+\alpha_2 r^2+\frac{\alpha_3}{r^2}+\alpha_4 \tag{2}$$

where

$$\left.\begin{aligned} -\alpha_0 &= b-a+2c\alpha_I,\ \alpha_1=\frac{b\alpha_I^{\,2}}{2}-1.33\alpha_I^{\,3} \\ \alpha_2 &= -\frac{b\alpha_I^{\,3}}{6},\ \alpha_3=-c,\ \alpha_4=-b\alpha_I-2c\alpha_I^{\,2} \end{aligned}\right\} \tag{3}$$

2. REVIEW OF THE METHODS
2.1. The Nikiforov- Uvarov method

The Nikiforov-Uvarov (NU) method is based on solving the hypergeometric-type second-order differential equations by means of the special orthogonal functions [47]. For a given potential, the Schrodinger-like equations in spherical coordinates are reduced to a generalized equation of hypergeometric-type with an appropriate coordinate transformation $r \to x$ and then they can be solved systematically to find the exact solutions. The main equation which is closely associated with the method is given in the following form [48].

$$\psi''(x)+\frac{\tilde{\tau}(x)}{\sigma(x)}\psi'(s)+\frac{\tilde{\sigma}(x)}{\sigma^2(x)}\psi(x)=0 \tag{4}$$

where $\sigma(x)$ and $\tilde{\sigma}(x)$ are polynomials at most second- degree, $\tilde{\tau}(x)$ is a first- degree polynomial and $\psi(x)$ is a function of the hypergeometric-type.

By taking $\psi(x)=\phi(x)y(x)$ and choosing an appropriate function $\phi(x)$, Eq. (4) is reduced to a comprehensible form;

$$y''(x)+\left(2\frac{\phi'(x)}{\phi(x)}+\frac{\tilde{\tau}(x)}{\sigma(x)}\right)y'(x)+\left(\frac{\phi''(x)}{\phi(x)}+\frac{\phi'(x)}{\phi(x)}\frac{\tilde{\tau}(x)}{\sigma(x)}+\frac{\tilde{\sigma}(x)}{\sigma^2(x)}\right)y(x)=0 \tag{5}$$

The coefficient of $y'(x)$ is taken in the form $\tau(x)/\sigma(x)$, where $\tau(x)$ is a polynomial of degree at most one, i.e.,

$$2\frac{\phi'(x)}{\phi(x)}+\frac{\tilde{\tau}(x)}{\sigma(x)}=\frac{\tau(x)}{\sigma(x)}. \tag{6}$$

And hence the most regular form is obtained as follows,

$$\frac{\phi'(x)}{\phi(x)}=\frac{\pi(x)}{\sigma(x)} \tag{7}$$

where

$$\pi(x)=\frac{1}{2}\left[\tau(x)-\tilde{\tau}(x)\right] \tag{8}$$

The most useful demonstration of Eq. (8) is

$$\tau(x) = \tilde{\tau}(x) + 2\pi(x) \tag{9}$$

The new parameter π(x) is a polynomial of degree at most one. In addition, the term $\phi''(x)/\phi(x)$ which appears in the coefficient of $y(x)$ in Eq. (5) is arranged as follows,

$$\frac{\phi''(x)}{\phi(x)} = \left(\frac{\phi'(x)}{\phi(x)}\right)' + \left(\frac{\phi'(x)}{\phi(x)}\right)^2 = \left(\frac{\pi(x)}{\sigma(x)}\right)' + \left(\frac{\pi(x)}{\sigma(x)}\right)^2 \tag{10}$$

In this case, the coefficient of $y(x)$ is transformed into a more suitable form by taking the equality given in Eq.(31);

$$\frac{\phi''(x)}{\phi(x)} + \frac{\phi'(x)}{\phi(\mathrm{x})}\frac{\tilde{\tau}(x)}{\sigma(x)} + \frac{\tilde{\sigma}(x)}{\sigma^2(x)} = \frac{\bar{\sigma}(x)}{\sigma^2(x)} \tag{11}$$

where

$$\bar{\sigma}(x) = \tilde{\sigma}(x) + \pi^2(x) + \pi(x)\left[\tilde{\tau}(x) - \sigma'(x)\right] + \pi'(x)\sigma(x) \tag{12}$$

Substituting the right- hand sides of Eq. (6) and Eq. (11) into Eq. (5), an equation of hypergeometric-type is obtained as follows;

$$y''(x) + \frac{\tau(x)}{\sigma(x)}y'(x) + \frac{\bar{\sigma}(\mathrm{x})}{\sigma^2(x)}y(x) = 0 \tag{13}$$

As a consequence of the algebraic transformations mentioned above, the functional form of Eq. (4) is protected in a systematic way. If the polynomial $\bar{\sigma}(x)$ in Eq. (13) is divisible by $\sigma(x)$, i.e.,

$$\bar{\sigma}(x) = \lambda\sigma(x) \tag{14}$$

where λ is a constant, Eq. (13) is reduced to an equation of hypergeometric-type

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda y(x) = 0 \tag{15}$$

And so its solution is given as a function of hypergeometric-type. To determine the polynomial $\pi(x)$, Eq. (12) is compared with Eq. (14) and then a quadratic equation for $\pi(x)$ is obtained as follows,

$$\pi^2(x) + \pi(x)\left[\tilde{\tau}(x) - \sigma'(x)\right] + \tilde{\sigma}(x) - k\sigma(x) = 0 \tag{16}$$

where

$$k = \lambda - \pi'_{-}(x) \tag{17}$$

The solution of this quadratic equation for π(x) yields the following equality

$$\pi(x) = \frac{\sigma'(x) - \tilde{\tau}(x)}{2} \pm \sqrt{\left(\frac{\sigma'(x) - \tilde{\tau}(x)}{2}\right)^2 - \tilde{\sigma}(x) + k\sigma(x)} \tag{18}$$

In order to obtain the possible solutions according to plus and minus of Eq. (18), the parameter k within the square root sign must be known explicitly. To provide this requirement, the expression under the square root sign has to be the square of a polynomial, since $\pi(x)$ is a polynomial of degree at most one. In this case, an equation of the quadratic form is available for the constant k . Setting the discriminant of this quadratic equal to zero, the constant k is determined clearly. After determining k , the polynomial $\pi(x)$ is obtained from Eq. (18), and then $\tau(x)$ and λ are also obtained by using Eq.(8) and Eq.(17), respectively.

A common trend that has been followed to generalize the solutions of Eq. (15) is to show that all the derivatives of hypergeometric-type functions are also of the hypergeometric-type. Equation (15) is differentiated by using the representation $v_1(x) = y'(x)$.

$$\sigma(x)\,\mathrm{v}_1''(x) + \tau_1(x)v_1'(x) + \mu_1 v_1(x) = 0 \tag{19}$$

where $\tau_1(x) = \tau(x) + \sigma^{\prime}(x)$ and $\mu_1 = \lambda + \tau^{\prime}(x)$. $\tau_1(x)$ is a polynomial of degree at most one and μ_1 is a parameter that is independent of the variables. It is clear that Eq. (19) is an equation of hypergeometric- type. By taking $v_2(x) = y^{\prime\prime}(x)$ as a new representation, the second derivative of Eq. (15) becomes

$$\sigma(x)\, v_2^{\prime\prime}(x) + \tau_2(x) v_2^{\prime}(x) + \mu_2 v_2(x) = 0 \tag{20}$$

where

$$\tau_2(x) = \tau_1(x) + \sigma^{\prime}(x) = \tau(x) + 2\sigma^{\prime}(x) \tag{21}$$

$$\mu_2 = \mu_1 + \tau_1^{\prime}(x) = \lambda + 2\tau^{\prime}(x) + \sigma^{\prime\prime}(x) \tag{22}$$

In a similar way, an equation of hypergeometric–type can be constructed as a family of particular solutions of Eq. (15) by taking $v_n(x) = y^{\prime\prime}(x)$;

$$\sigma(x) v_n^{\prime\prime}(x) + \tau_n(x) v_n^{\prime}(x) + \mu_n v_n(x) = 0 \tag{23}$$

And here the general recurrence relations for $\tau_n(x)$ and μ_n are found as follows, respectively,

$$\tau_n(x) = \tau(x) + n\sigma^{\prime}(x) \tag{24}$$

$$\mu_n = \lambda + n\tau^{\prime}(x) + \frac{n(n-1)}{2}\sigma^{\prime\prime}(x). \tag{25}$$

When $\mu_n = 0$, Eq. (25) becomes as follows

$$\lambda_n = -n\tau^{\prime}(x) - \frac{n(n-1)}{2}\sigma^{\prime\prime}(x), (n = 0,1,2,3,...). \tag{26}$$

And then Eq. (23) has a particular solution of the form $y(x) = y_n(x)$ which is a polynomial of degree n. To obtain an eigenvalue solution through the NU method, the relationship between λ and λ_n must be set up by means of Eq.(17) and Eq.(26). $y_n(x)$ is the hypergeometric –type function whose polynomial solutions are given by the Rodrigues relation

$$y_n(x) = \frac{B_n}{\rho(x)} \frac{d^n}{dx^n}\left[\sigma^n(x)\rho(\mathrm{x})\right] \tag{27}$$

where B_n is a normalization constant and the weight function $\rho(x)$ must satisfy the condition below

$$\left(\sigma(x)\rho(x)\right)^{\prime} = \tau(x)\rho(x). \tag{28}$$

2.2 The series expansion method

The series expansion method is based on solving the hypergeometric-type second-order differential equations. For a given potential the wave function of SE is chosen in the form.

$$R(r) = e^{-\alpha r^2 - \beta r} F(\mathrm{r}) \tag{29}$$

where $\alpha\ and\ \beta\ are$ parameters whose values are to be determined in terms of potential strength parameters. The functional series for $F(r)$ is taken to be

$$F(\mathrm{r}) = \sum_{n=0}^{\infty} a_n r^{2n+L} \tag{30}$$

where a_n is an expansion coefficient [48].

By substituting $F(r), F^{\prime}(r)$ and $F^{\prime\prime}(r)$ into the SE, rearranging and equating coefficients of the corresponding powers of r to zero. The eigen-values are subsequently obtained.

3. APPROXIMATE SOLUTIONS OF THE SCHRODINGER EQUATION WITH CLASS OF YUKAWA POTENTIAL USING THE NU METHOD

The Schrodinger equation takes the form [49]

$$\frac{d^2\,\mathrm{U}(r)}{dr^2} + \left[\frac{2\mu}{\hbar^2}\left(E_{nl} - V(r)\right) - \frac{l(l+1)}{r^2}\right]\mathrm{U}(r) = 0 \tag{31}$$

where l, is the angular momentum quantum number, μ, is the reduced mass for the quark-antiquark particle, r is the inter-particle distance and $\hbar$ is reduced plank constant respectively.
Substituting Eq.(2) into Eq.(31) gives,

$$\frac{d^2R(r)}{dr^2}+\left[\frac{2\mu E}{\hbar^2}+\frac{2\mu\alpha_0}{\hbar^2 r}-\frac{2\mu\alpha_1 r}{\hbar^2}-\frac{2\mu\alpha_2 r^2}{\hbar^2}-\frac{2\mu\alpha_3}{\hbar^2 r^2}-\frac{2\mu\alpha_4}{\hbar^2}-\frac{l(l+1)}{r^2}\right]R(r)=0 \tag{32}$$

Transforming the coordinate of Eq.(32) we set

$$x=\frac{1}{r} \tag{33}$$

Using Eqs. (32) and (33) we have

$$\frac{d^2R(\mathrm{x})}{dx^2}+\frac{2}{x}\frac{dR}{dx}+\frac{1}{x^4}\left[\begin{matrix}\frac{2\mu E}{\hbar^2}+\frac{2\mu\alpha_0 x}{\hbar^2}-\frac{2\mu\alpha_1}{\hbar^2 x}\\ -\frac{2\mu\alpha_2}{\hbar^2 x^2}-\frac{2\mu\alpha_3 x^2}{\hbar^2}-\frac{2\mu\alpha_4}{\hbar^2}-l(l+1)x^2\end{matrix}\right]R(r)=0 \tag{34}$$

Next, we propose the following approximation scheme on the term $\frac{\alpha_1}{x}$ and $\frac{\alpha_2}{x^2}$.

Let us assume that there is a characteristic radius r_0 of the meson. Then the scheme is based on the expansion of $\frac{\alpha_1}{x}$ and $\frac{\alpha_2}{x^2}$. in a power series around r_0; i.e. around $\delta\equiv\frac{1}{r_0}$, in the x-space up to the second order. This is similar to Pekeris approximation, which helps to deform the centrifugal term such that the modified potential can be solved by the NU method [12].
Setting $y=x-\delta$ and around $y=0$ it can be expanded into a series of powers we obtain;

$$\frac{\alpha_1}{x}=\alpha_1\left(\frac{3}{\delta}-\frac{3x}{\delta^2}+\frac{x^2}{\delta^3}\right) \tag{35}$$

and

$$\frac{\alpha_2}{x^2}=\alpha_2\left(\frac{6}{\delta^2}-\frac{8x}{\delta^3}+\frac{3x^2}{\delta^4}\right) \tag{36}$$

Putting Eqs. (35) and (36) into Eq. (34) and simplifying gives

$$\frac{d^2R(x)}{dx^2}+\frac{2x}{x^2}\frac{dR(x)}{dx}+\frac{1}{x^4}\left[-\varepsilon+\alpha x-\beta x^2\right]R(x)=0 \tag{37}$$

where

$$\left.\begin{aligned}-\varepsilon&=\left(\frac{2\mu E}{\hbar^2}-\frac{6\mu\alpha_1}{\hbar^2\delta}-\frac{12\mu\alpha_2}{\hbar^2\delta^2}-\frac{2\mu\alpha_4}{\hbar^2}\right)\\ \alpha_{II}&=\left(\frac{2\mu\alpha_0}{\hbar^2}+\frac{6\mu\alpha_1}{\hbar^2\delta^2}+\frac{16\mu\alpha_2}{\hbar^2\delta^3}\right)\\ \beta_{II}&=\left(\frac{2\mu\alpha_1}{\hbar^2\delta^2}+\frac{6\mu\alpha_2}{\hbar^2\delta^4}+\frac{2\mu\alpha_3}{\hbar^2}+\gamma\right)\\ \gamma&=l(l+1)\end{aligned}\right\} \tag{38}$$

Comparing Eq. (37) and Eq. (4) we obtain

$$\left.\begin{aligned}&\tilde{\tau}(x)=2x,\sigma(x)=x^2,\tilde{\sigma}(x)=-\varepsilon+\alpha x-\beta x^2\\ &\sigma'(x)=2x,\sigma''(x)=2\end{aligned}\right\} \tag{39}$$

Substituting Eq. (39) into Eq. (18) gives

$$\pi(x)=\pm\sqrt{\varepsilon-\alpha_{II}x+\left(\beta_{II}+k\right)x^2} \tag{40}$$

To determine k, we take the discriminant of the function under the square root.

$$k=\frac{\alpha_{II}{}^2-4\beta_{II}\varepsilon}{4\varepsilon} \tag{41}$$

We substitute Eq. (41) into Eq. (40) and have

$$\pi(x) = \pm\left(\frac{\alpha_{II}x}{2\sqrt{\varepsilon}} - \frac{\varepsilon}{\sqrt{\varepsilon}}\right) \tag{42}$$

Taking the negative part of Eq. (48), gives

$$\pi'_{-}(x) = -\frac{\alpha_{II}}{2\sqrt{\varepsilon}} \tag{43}$$

Substituting Eqs.(39) and (43) into Eq.(9) we have

$$\tau(x) = 2x - \frac{\alpha_{II}x}{\sqrt{\varepsilon}} + \frac{2\varepsilon}{\sqrt{\varepsilon}} \tag{44}$$

From Eq. (44) we have

$$\tau'(x) = 2 - \frac{\alpha_{II}}{\sqrt{\varepsilon}} \tag{45}$$

Substituting Eqs. (41) and (43) into Eq.(17) we have

$$\lambda = \frac{\alpha_{II}^{2} - 4\beta_{II}\varepsilon}{4\varepsilon} - \frac{\alpha_{II}}{2\sqrt{\varepsilon}} \tag{46}$$

We substitute Eqs. (39) and (45) into Eq.(26) and obtain

$$\lambda_n = \frac{n\alpha_{II}}{\sqrt{\varepsilon}} - n^2 - n \tag{47}$$

To obtain the energy equation, we equate Eqs. (46) and (47) and then substitute Eqs. (3) and (38)

$$E_{nl} = \frac{3}{\delta}\left(\frac{b\alpha_I^{2}}{2} - 1.33\alpha_I^{3}\right) - \frac{b\alpha_I^{3}}{\delta^2} - b\alpha_I - 2c\alpha_I^{2} - \frac{\hbar^2}{8\mu}\left[\frac{\frac{6\mu}{\hbar^2\delta^2}\left(\frac{b\alpha_I^{2}}{2} - 1.33\alpha_I^{3}\right) + \frac{2\mu}{\hbar^2}\left(a - b - 2c\alpha_I\right) - \frac{8\mu b\alpha_I^{3}}{3\hbar^2\delta^3}}{n + \frac{1}{2} + \sqrt{\left(l+\frac{1}{2}\right)^2 + \frac{2\mu}{\hbar^2\delta^3}\left(\frac{b\alpha_I^{2}}{2} - 1.33\alpha_I^{3}\right) - \frac{\mu b\alpha_I^{3}}{\hbar^2\delta^4} - \frac{2\mu c}{\hbar^2}}}\right]^2 \tag{48}$$

The wave function in terms of Laguerre polynomials is

$$\psi(x) = N_{nl}x^{-\frac{\alpha_I}{2\sqrt{\varepsilon}}}e^{-\frac{\varepsilon}{x\sqrt{\varepsilon}}}L_n^{\frac{\alpha_I}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{x\sqrt{\varepsilon}}\right) \tag{49}$$

where N_{nl} is normalization constant, which can be obtain from

$$\int_0^\infty |N_{nl}(r)|^2 dr = 1 \tag{50}$$

4. EXACT SOLUTIONS OF THE SCHRODINGER EQUATION WITH CLASS OF YUKAWA POTENTIAL USING THE SEM

We consider the radial Schrodinger equation of the form [50]

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} + \left[\frac{2\mu}{\hbar^2}\left(E - V(r)\right) - \frac{l(l+1)}{r^2}\right]R(r) = 0 \tag{51}$$

where l is angular quantum number taking the values 0,1,2,3,4…, μ is the reduced mass for the quarkonium particle, and r is the internuclear separation.

Putting Eq. (2) into Eq. (51) gives

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} + \left[\varepsilon + \frac{A}{r} - Br - Cr^2 - \frac{\mathrm{L}(\mathrm{L}+1)}{r^2}\right]R(r) = 0 \tag{52}$$

where

$$\left.\varepsilon=\frac{2\mu}{\hbar^2}\left(E-\alpha_4\right), A=\frac{2\mu\alpha_0}{\hbar^2}, B=\frac{2\mu\alpha_1}{\hbar^2}, C=\frac{2\mu\alpha_2}{\hbar^2}\right\} \tag{53}$$

$$L\left(L+1\right)=\frac{2\mu\alpha_3}{\hbar^2}+l\left(l+1\right) \tag{54}$$

From Eq. (54) we have

$$L=-\frac{1}{2}+\frac{1}{2}\sqrt{\left(2l+1\right)^2+\frac{8\mu\alpha_3}{\hbar^2}} \tag{55}$$

From Eq. (29), Eqs.(56) and (57) are obtained

$$R'(\mathrm{r})=F'(\mathrm{r})\mathrm{e}^{-\alpha r^2-\beta r}+F(r)\left(-2\alpha r-\beta\right)e^{-\alpha r^2-\beta r} \tag{56}$$

$$\begin{aligned}R''(\mathrm{r})&=\mathrm{F}''(\mathrm{r})\mathrm{e}^{-\alpha r^2-\beta r}+F'(r)\left(-2\alpha r-\beta\right)e^{-\alpha r^2-\beta r}\\&+\left[\left(-2\alpha\right)+\left(-2\alpha r-\beta\right)\left(-2\alpha r-\beta\right)\right]F(r)e^{-\alpha r^2-\beta r}\end{aligned} \tag{57}$$

Substituting Eqs. (29), (56) and (57) into Eq. (51) and divide through by $e^{-\alpha r^2-\beta r}$ we obtain

$$F''(\mathrm{r})+\left[-4\alpha r-2\beta+\frac{2}{r}\right]F'(\mathrm{r})+\left[\begin{matrix}\left(4\alpha^2-C\right)r^2+\left(4\alpha\beta-B\right)r\\+\left(A-2\beta\right)\frac{1}{r}-\frac{L(L+1)}{r^2}+\left(\varepsilon+\beta^2-6\alpha\right)\end{matrix}\right]F(r)=0 \tag{58}$$

Also, from Eq. (30), we obtain the following

$$F'(r)=\sum_{n=0}^{\infty}(2n+L)a_n r^{2n+L-1} \tag{59}$$

$$F''(r)=\sum_{n=0}^{\infty}(2n+L)\left(2n+L-1\right)a_n r^{2n+L-2} \tag{60}$$

We substitute Eqs. (30),(59) and (60) into Eq.(58) and obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}(2n+L)\left(2n+L-1\right)a_n r^{2n+L-2}+\left[-4\alpha r-2\beta+\frac{2}{r}\right]\sum_{n=0}^{\infty}(2n+L)a_n r^{2n+L-1}\\&+\left[\left(4\alpha^2-C\right)r^2+\left(4\alpha\beta-B\right)r+\frac{\left(A-2\beta\right)}{r}-\frac{L\left(L+1\right)}{r^2}+\left(\varepsilon+\beta^2-6\alpha\right)\right]\sum_{n=0}^{\infty}a_n r^{2n+L}=0\end{aligned} \tag{61}$$

By collecting powers of r in Eq. (61) we have

$$\sum_{n=0}^{\infty}a_n\left\{\begin{matrix}\left[\left(2n+L\right)\left(2n+L-1\right)+2\left(2n+L\right)-L\left(L+1\right)\right]r^{2n+L-2}\\+\left[-2\beta\left(2n+L\right)+\left(A-2\beta\right)\right]r^{2n+L-1}\\+\left[-4\alpha\left(2n+L\right)+\varepsilon+\beta^2-6\alpha\right]r^{2n+L}\\+\left[4\alpha\beta-B\right]r^{2n+L+1}+\left[4\alpha^2-C\right]r^{2n+L+2}\end{matrix}\right\}=0 \tag{62}$$

Equation (62) is linearly independent implying that each of the terms is separately equal to Zero, noting that r is a non-zero function; therefore, it is the coefficient of r that is zero. With this in mind, we obtain the relation for each of the terms.

$$\left(2n+L\right)\left(2n+L-1\right)+2\left(2n+L\right)-L\left(L+1\right)=0 \tag{63}$$

$$-2\beta\left(2n+L\right)+A-2\beta=0 \tag{64}$$

$$-4\alpha\left(2n+L\right)+\varepsilon+\beta^2-6\alpha=0 \tag{65}$$

$$4\alpha\beta-B=0 \tag{66}$$

$$4\alpha^2-C=0 \tag{67}$$

From Eq. (64) we have

$$\beta = \frac{A}{4n+2L+2} \tag{68}$$

From Eq. (67) we have

$$\alpha = \frac{\sqrt{C}}{2} \tag{69}$$

The energy equation of the CYP is obtain by substituting Eqs. (53), (55), (68) and (69) into Eq. (65) and simplifying we have

$$\begin{aligned} E_{nl} = {} & \sqrt{\frac{-\hbar^2 b\alpha_I^{\,3}}{12\mu}}\left(4n+2+\sqrt{(2l+1)^2-\frac{8\mu c}{\hbar^2}}\right) \\ & -\frac{2\mu}{\hbar^2}\left(a-b-2c\alpha_I\right)^2\left(4n+1+\sqrt{(2l+1)^2-\frac{8\mu c}{\hbar^2}}\right)^{-2}-b\alpha_I-2c\alpha_I^{\,2} \end{aligned} \tag{70}$$

4. RESULTS AND DISCUSSION

4.1 Results

We calculate mass spectra of the heavy quarkonium system such as charmonium and bottomonium that have the quark and antiquark flavor, and apply the following relation [51-53]

$$M = 2m + E_{nl} \tag{71}$$

where

$M =$ Mass spectra of the heavy quarkonium,
$m =$ Quarkonium bare mass,
$E_{nl} =$ Energy eigenvalue.

By substituting Eq. (48) into Eq. (71) we obtain the mass spectra for class of Yukawa potential using the NU method as,

$$\begin{aligned} M = {} & 2m+\frac{3}{\delta}\left(\frac{b\alpha_I^{\,2}}{2}-1.33\alpha_I^{\,3}\right)-\frac{b\alpha_I^{\,3}}{\delta^2}-b\alpha_I-2c\alpha_I^{\,2} \\ & -\frac{\hbar^2}{8\mu}\left[\frac{\frac{6\mu}{\hbar^2\delta^2}\left(\frac{b\alpha_I^{\,2}}{2}-1.33\alpha_I^{\,3}\right)+\frac{2\mu}{\hbar^2}\left(a-b-2c\alpha_I\right)-\frac{8\mu b\alpha_I^{\,3}}{3\hbar^2\delta^3}}{n+\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{2\mu}{\hbar^2\delta^3}\left(\frac{b\alpha_I^{\,2}}{2}-1.33\alpha_I^{\,3}\right)-\frac{\mu b\alpha_I^{\,3}}{\hbar^2\delta^4}-\frac{2\mu c}{\hbar^2}}}\right]^2 \end{aligned} \tag{72}$$

By substituting Eq. (70) into Eq.(71) we obtain the MS for CYP using the SEM as,

$$\begin{aligned} M = {} & 2m+\sqrt{\frac{-\hbar^2 b\alpha_I^{\,3}}{12\mu}}\left(4n+2+\sqrt{(2l+1)^2-\frac{8\mu c}{\hbar^2}}\right) \\ & -\frac{2\mu}{\hbar^2}\left(a-b-2c\alpha_I\right)^2\left(4n+1+\sqrt{(2l+1)^2-\frac{8\mu c}{\hbar^2}}\right)^{-2}-b\alpha_I-2c\alpha_I^{\,2} \end{aligned} \tag{73}$$

4.2. Determination of the potential strength parameters

The reduced mass μ is defined in the standard way as $\mu = \frac{m}{2}$,where $m =$ mass of the constituent quarks and antiquarks. For bottomonium $b\bar{b}$ and charmonium $c\bar{c}$ systems we adopt the numerical values of these masses as $m_b =$ 4.823 GeV for bottomonium and $m_c =$ 1.209 GeV for charmonium [54]. Then, the corresponding reduced mass are $\mu_b =$ 2.4115 GeV and $\mu_c = 0.6045\,GeV$. The potential parameters of Eqs. (72) and (73) are fitted with experimental data. Experimental data are taken from [55]. The parameters for charmonium and bottomonium of the Eq. (72) are

$$\begin{pmatrix} m_c = 1.209\ GeV, \mu = \ 0.6045\ GeV,\ a\ =\ -19.045\ GeV,\ b\ =\ 5.885\ GeV, \\ c\ =\ -1.188\ GeV, \delta = \ 0.23\ GeV, \alpha_i =\ 1.52\ GeV, \hbar = \ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} m_b = 4.823\ GeV, \mu = 2.4115\ GeV,\ a = -1.591\ GeV,\ b = 8.875\ GeV, \\ c = -11.153\ GeV, \delta = 0.23\ GeV, \alpha_I = 1.52\ GeV, \hbar = 1 \end{pmatrix}$$

respectively. In the same vain, the parameters for charmonium and bottomonium of the Eq. (73) are

$$\begin{pmatrix} m_c = 1.209\ GeV, \mu = 0.6045\ GeV,\ \hbar = 1, \alpha_I = 1.52\ GeV \\ a = 0.489\ GeV,\ b = -0.695\ GeV,\ c = 5.679\ GeV \end{pmatrix}$$

and

$$\begin{pmatrix} m_b = 4.823\ GeV, \mu = 2.4115\ GeV,\ a = 1.192\ GeV, \\ c = -13.876\ GeV, \alpha_I = 1.52\ GeV, \hbar = 1,\ b = 0.998\ GeV \end{pmatrix}$$

respectively.

4.3. Discussion of results

The mass spectra of charmonium and bottomonium for class of Yukawa potential for the NU and the SEM were calculated as shown in Tables 1and 2 respectively using Eqs. (72) and (73).

Table 1. Comparison of mass spectra of charmonium in (GeV) for the class of Yukawa potential $M_{n,l}^{CYP}$, between the NU, SEM, some authors and experimental data

State	$M_{n,l}^{CYP}$ with the NU	$M_{n,l}^{CYP}$ with the SEM	[12]	[13]	Experiment [55]
1s	3.096	3.096	3.096	3.095	3.096
2s	3.686	3.686	3.686	3.685	3.686
1p	3.493	3.524	3.255	3.258	3.525
2p	3.772	3.773	3.779	3.779	3.773
3s	4.040	4.040	4.040	4.040	4.040
4s	4.267	4.263	4.269	4.262	4.263
1d	3.763	3.769	3.504	3.510	3.770
2d	4.146	4.156	-	3.928	4.159
1f	3.962	4.081	-	_-	_-

The free parameters are fitted with experimental data. In addition, quark masses are obtained from Ref. [55]. We note the spectra masses of charmonium from states 1s,2s, 3s and 2p from both the NU and the series expansion methods agree with experimental data and 1s,2s,3s and 4s states for bottomonium agree with experimental data for both methods as shown in Tables 1 and 2. Other states appear to be close with experimental data, but the SEM solutions appear to be very close to experimental data for charmonium and bottomonium compared to the NU method. It was noticed that in the 1f state for charmonium and 2d and 1f states for bottomonium the values of the experimental data are not available. The mass spectra obtained agree with Ref. [12]. Our results are improved in comparison with works of other researcher like Ref. [12] as shown in the Tables in which the author investigated the N-radial SE analytically. The Cornell potential was extended to finite temperature. The energy eigenvalue and the wave functions were calculated in the N-dimensional form using the NU method. Also, the mass spectra obtained using Eqs. (72) and (73) are improved in comparison with the works of Ref. [13] in which they studied the N-dimensional radial Schrodinger equation using the analytical exact iteration method, in which the Cornell potential is generalized to finite temperature and chemical potential.

Table 2. Comparison of mass spectra of bottomonium in (GeV) for the class of Yukawa potential $M_{n,l}^{CYP}$, between the NU, SEM, some authors and experimental data

State	$M_{n,l}^{CYP}$ with the NU	$M_{n,l}^{CYP}$ with the SEM	[12]	[13]	Experiment [55]
1s	9.460	9.460	9.460	9.460	9.460
2s	10.023	10.023	10.023	10.022	10.023
1p	9.761	9.889	9.619	9.609	9.899
2p	10.258	10.260	10.114	10.109	10.260
3s	10.355	10.355	10.355	10.360	10.355
4s	10.577	10.579	10.567	10.580	10.580
1d	9.989	10.164	9.864	9.846	10.164
2d	10.336	10.575	_	_	_
1f	10.279	10.299	_	_	_

CONCLUSION

In this work, the Schrodinger equation is analytically solved using the Nikiforov- Uvarov and series expansion methods with the class of Yukawa potential. The approximate solutions of the eigen energy equation and corresponding eigenfunction in terms of Laguerre polynomials were obtained using the NU method. The solutions of the eigen energy equation were also obtained with the SEM. The mass spectra for heavy quarkonium system for the potential under study were obtained for bottomonium $b\bar{b}$ and charmonium $c\bar{c}$. We adopted the numerical values of these masses as $m_b = 4.823\ GeV$ for bottomonium and $m_c = 1.209\ GeV$ for charmonium. We compared the results obtained between the Nikiforov- Uvarov and series expansion methods. It was noticed that SEM solutions yield mass spectra very close to experimental data compared to solutions with the NU method. The obtained results were also compared with works by some other authors [12,13] with different analytical methods. The values obtained are improved in comparison with their works. This work can be extended by using other exponential-type potential models with other analytical approach and a different approximation scheme to obtain the mass spectra of heavy quarkonium system. The relativistic properties using Klein-Gordon or Dirac equations can be explored to obtain the mass spectra of light quarkonia. Finally, the information entailed in the normalized wave-functions can also be studied.

Declarations

Availability of data and materials. All data generated during this study are included in the references in the paper.

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Authors contributions. EPI and JEO conceived and designed the study, acquired, analyzed and interpreted the data and handled the review; ESW handled the computational analysis, DEB and EPI handled writing-review and editing. All authors read and approved the final manuscript.

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ПОРІВНЯЛЬНЕ ДОСЛІДЖЕННЯ СПЕКТРІВ МАСИ ВАЖКОЇ КВАРКОНІЄВОЇ СИСТЕМИ З МОДЕЛЛЮ ПОТЕНЦІАЛУ ВЗАЄМОДІЇ

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У цій роботі досліджено порівняння мас-спектрів важкої кварконієвої системи з потенціалом взаємодії (клас потенціалу Юкави). Рівняння Шредінгера аналітично розв'язується за допомогою методу Нікіфорова-Уварова (NU) і методу розкладання в ряд (SEM). Наближені розв’язки рівняння власної енергії та відповідної власної функції через поліноми Лагерра були отримані за допомогою методу NU, також за допомогою SEM були отримані розв’язки рівняння власної енергії. Мас-спектри важкої кварконієвої системи (ВСК) для досліджуваного потенціалу отримано для боттонієвої та чармонієвої ВСК. Ми порівняли результати, отримані між NU та SEM. Було помічено, що SEM-рішення дають мас-спектри, дуже близькі до експериментальних даних, порівняно з розв’язанням методом NU. Отримані результати також порівнювали з роботами деяких інших авторів і визнали їх покращеними. Це дослідження можна розширити, використовуючи інші потенційні моделі експоненціального типу з іншим аналітичним підходом і іншою схемою наближення для отримання мас-спектрів важкої кварконієвої системи. Релятивістські властивості за допомогою рівнянь Клейна-Гордона або Дірака можна досліджувати для отримання мас-спектрів легких кварконіїв. Нарешті, також можна вивчати інформацію, що міститься в нормалізованих хвильових функціях.

Ключові слова: *рівняння Шредінгера; метод Нікіфорова-Уварова; клас Юкава-потенціалу; мас-спектри; метод серійного розширення*