EAST EUROPEAN JOURNAL OF PHYSICS. 2. 63-68 (2023)
DOI:10.26565/2312-4334-2023-2-03 ISSN 2312-4334

COMPOSITE FERMIONS QED LAGRANGIAN DENSITY IN FRACTIONAL FORMULATION†

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Received February 13, 2023; revised March 10, 2023; accepted March 12, 2023

Quantum electrodynamics (QED) is a highly precise and successful theory that describes the interaction between electrically charged particles and electromagnetic radiation. It is an integral part of the Standard Model of particle physics and provides a theoretical basis for explaining a wide range of physical phenomena, including the behavior of atoms, molecules, and materials. In this work, the Lagrangian density of Composite Fermions in QED has been expressed in a fractional form using the Riemann-Liouville fractional derivative. The fractional Euler-Lagrange and fractional Hamiltonian equations, derived from the fractional form of the Lagrangian density, were also obtained. When α is set to 1, the conventional mathematical equations are restored.
Keywords: *Quantum Electrodynamics; Composite Fermions; Fractional derivative; Lagrangian density; Euler-Lagrange equations*
PACS: 31.30.J, 71.10.Pm, 45.10.Hj, 45.20.Jj, 11.10.Ef

1. INTRODUCTION

Composite Fermions are a theoretical concept in condensed matter physics that explains the behavior of electrons when subjected to a strong magnetic field [1-4]. These electrons can form composite particles with unique physical properties, such as those seen in fractional quantum Hall states [5-7]. Understanding composite fermions provides insights into the behavior of electrons in high magnetic fields and has applications in various fields, from technology development to the discovery of basic physical principles.

Fractional derivatives [8-12] have become a valuable tool in various fields because they provide the ability to accurately model physical phenomena that cannot be captured by ordinary derivatives. There has been a surge of research in fractional calculus, leading to its application in physics, engineering, and related areas [13-16]. The Maxwell equations have been expressed in fractional form [17-19], as have those in quantum mechanics, including the fractional Schrödinger equation [20, 21] and the fractional Dirac equation [22]. These advancements demonstrate the versatility of fractional calculus in describing a wide range of physical systems.

The main goal of this work is to examine the composite Fermions QED Lagrangian density and transform it into a fractional form using the Riemann-Liouville (RL) fractional derivative. The ultimate purpose is to derive the fractional Hamilton's equations and fractional Euler-Lagrange (EL) equations from this reformulation, thereby providing a fresh perspective on the dynamics of composite Fermions within a QED framework.

The structure of the paper is as follows: In Sec. 2, a brief explanation of RL fractional derivative is provided. The topic of the QED Lagrangian density is discussed in Sec. 3. In Sec. 4, the fractional form of the Lagrangian density and the fractional Euler-Lagrange equations are presented. The focus of Sec. 5 is on the Hamiltonian equations derived from the Lagrangian density. The paper concludes with a concise summary of the key points in Sec. 6.

2. PRELIMINARIES

This section provides essential definitions used in this study. For a more comprehensive understanding, readers can refer to reference [23]. The following are the definitions of the left and right RL fractional derivative.
The Left RL fractional derivative

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-\tau)^{n-\alpha-1} f(\tau)\, d\tau. \tag{1}$$

The right RL fractional derivative

$$ {}_xD_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^n \int_x^b (\tau-x)^{n-\alpha-1} f(\tau)\, d\tau. \tag{2}$$

The value of α signifies the order of differentiation, where n-1≤α<n, with Γ symbolizing the gamma function. In cases where α is an integer, the derivative is calculated using the conventional definition.

$$\begin{cases} {}_aD_x^\alpha f(x) = \left(\frac{d}{dx}\right)^n f(x) \\ {}_aD_t^\alpha f(x) = \left(\frac{d}{dx}\right)^n f(t) \end{cases}. \tag{3}$$
$$\alpha = 1,2,\ldots$$

† *Cite as:* A.D. Al-Oqali, East Eur. J. Phys. **2**, 63 (2023), https://doi.org/10.26565/2312-4334-2023-2-03

3. COMPOSITE FERMIONS QED LAGRANGIAN DENSITY

The Lagrangian density for composite fermions in QED, with the speed of light set to 1, has the following mathematical form [1]:

$$\mathcal{L} = \overline{\Psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{\mu\nu\rho}A_{\mu}\ \partial_{\nu}A_{\rho} - \left(1 + v_{eff}\right)e\overline{\Psi}\,\gamma^{\mu}\psi\, A_{\mu}\,. \tag{4}$$

This equation involves the Levi-Civita symbol $\varepsilon^{\mu\nu\rho}$, which is an antisymmetric tensor, as well as ψ which is a Dirac's spinor and made up of four complex parts, $\overline{\Psi} = \psi^{+}\gamma^{0}$, γ^{μ} is Dirac matrix, e is the charge of an electron, ϕ_0 is a unit of magnetic flux, it also includes a gauge parameter, θ, for the Chern-Simons fields. $F_{\mu\nu}$ is the electromagnetic field tensor and the A_{μ} are electromagnetic fields. The first term represents the fields that are associated with spinors, while the second term represents fields related to electromagnetism. The third term involves gauge Chern-Simons fields, and the final two terms describe the coupling of the spinor fields to both the electromagnetic fields and the Chern-Simons fields.

4. FRACTIONAL FORM OF COMPOSITE FERMIONS QED LAGRANGIAN DENSITY

The fractional Lagrangian density of (4) can be written as:

$$\mathcal{L} = \overline{\Psi}\left(i\gamma^{\mu}\,{}^{c}_{a}D^{\alpha}_{x_{\mu}} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{\mu\nu\rho}A_{\mu}\ {}_{a}D^{\alpha}_{x_{\nu}}A_{\rho} - \left(1 + v_{eff}\right)e\overline{\Psi}\,\gamma^{\mu}\psi\, A_{\mu}\,. \tag{5}$$

Begin by expanding the second term in the Lagrangian:

$$F_{\mu\nu}F^{\mu\nu} = g^{\mu\rho}g^{\nu\lambda}\left({}_{a}D^{\alpha}_{x_{\mu}}A_{\nu} - {}_{a}D^{\alpha}_{x_{\nu}}A_{\mu}\right)\left({}_{a}D^{\alpha}_{x_{\rho}}A_{\lambda} - {}_{a}D^{\alpha}_{x_{\lambda}}A_{\rho}\right)$$

Hence, we have

$$\mathcal{L} = \overline{\Psi}\left(i\gamma^{\mu}\,{}^{c}_{a}D^{\alpha}_{x_{\mu}} - m\right)\psi - \frac{1}{4}\,g^{\mu\rho}g^{\nu\lambda}\left({}_{a}D^{\alpha}_{x_{\mu}}A_{\nu} - {}_{a}D^{\alpha}_{x_{\nu}}A_{\mu}\right)\left({}_{a}D^{\alpha}_{x_{\rho}}A_{\lambda} - {}_{a}D^{\alpha}_{x_{\lambda}}A_{\rho}\right) +$$
$$+\frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{\mu\nu\rho}A_{\mu}\ {}_{a}D^{\alpha}_{x_{\nu}}A_{\rho} - \left(1 + v_{eff}\right)e\overline{\Psi}\,\gamma^{\mu}\psi\, A_{\mu} \tag{6}$$

In the case of a Lagrangian involving multiple fields, there will be a separate equation for each field. The EL equation for the field A_{β} is expressed as follows:

$$\frac{\partial\mathcal{L}}{\partial A_{\beta}} - {}_{a}D^{\alpha}_{x_{\sigma}}\left[\frac{\partial\mathcal{L}}{\partial\left({}_{a}D^{\alpha}_{x_{\sigma}}A_{\beta}\right)}\right] = 0. \tag{7}$$

Thus, the equation of motion is expressed as

$$\frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{\mu\nu\rho}\delta^{\beta}_{\mu}\ {}_{a}D^{\alpha}_{x_{\nu}}A_{\rho} - \left(1 + v_{eff}\right)e\overline{\Psi}\,\gamma^{\mu}\psi\, A_{\mu}\,\delta^{\beta}_{\mu} - {}_{a}D^{\alpha}_{x_{\sigma}}\left[-\frac{1}{4}\,g^{\mu\rho}g^{\nu\lambda}\left(\delta^{\sigma}_{\mu}\delta^{\beta}_{\nu} - \delta^{\sigma}_{\nu}\delta^{\beta}_{\mu}\right)\left({}_{a}D^{\alpha}_{x_{\rho}}A_{\lambda} -\right.\right.$$
$$\left.\left. {}_{a}D^{\alpha}_{x_{\lambda}}A_{\rho}\right) - -\frac{1}{4}\,g^{\mu\rho}g^{\nu\lambda}\left(\delta^{\sigma}_{\rho}\delta^{\beta}_{\lambda} - \delta^{\sigma}_{\lambda}\delta^{\beta}_{\rho}\right)\left({}_{a}D^{\alpha}_{x_{\mu}}A_{\nu} - {}_{a}D^{\alpha}_{x_{\nu}}A_{\mu}\right) + \frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{\mu\nu\rho}A_{\mu}\,\delta^{\sigma}_{\nu}\ \delta^{\beta}_{\rho}\right] = 0$$

$$\frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{\beta\nu\rho}\ {}_{a}D^{\alpha}_{x_{\nu}}A_{\rho} - \left(1 + v_{eff}\right)e\overline{\Psi}\,\gamma^{\beta}\psi\, A_{\mu} - {}_{a}D^{\alpha}_{x_{\sigma}}\left[-\frac{1}{4}\left(g^{\sigma\rho}g^{\beta\lambda} - g^{\beta\rho}g^{\sigma\lambda}\right)\left({}_{a}D^{\alpha}_{x_{\rho}}A_{\lambda} - {}_{a}D^{\alpha}_{x_{\lambda}}A_{\rho}\right) -\right.$$
$$\left.-\frac{1}{4}\left(g^{\mu\sigma}g^{\nu\beta} - g^{\mu\beta}g^{\nu\sigma}\right)\left({}_{a}D^{\alpha}_{x_{\mu}}A_{\nu} - {}_{a}D^{\alpha}_{x_{\nu}}A_{\mu}\right) + \frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{\mu\sigma\beta}\ {}_{a}D^{\alpha}_{x_{\sigma}}A_{\mu}\right] = 0$$

$$\frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{\beta\nu\rho}\ {}_{a}D^{\alpha}_{x_{\nu}}A_{\rho} - \left(1 + v_{eff}\right)e\overline{\Psi}\,\gamma^{\beta}\psi - {}_{a}D^{\alpha}_{x_{\sigma}}\left[-\frac{1}{4}\left({}_{a}D^{\alpha}_{x^{\sigma}}A^{\beta} - {}_{a}D^{\alpha}_{x^{\beta}}A^{\sigma} - {}_{a}D^{\alpha}_{x^{\beta}}A^{\sigma} + {}_{a}D^{\alpha}_{x^{\sigma}}A^{\beta}\right) -\right.$$
$$\left.-\frac{1}{4}\left({}_{a}D^{\alpha}_{x^{\sigma}}A^{\beta} - {}_{a}D^{\alpha}_{x^{\beta}}A^{\sigma} - {}_{a}D^{\alpha}_{x^{\beta}}A^{\sigma} + {}_{a}D^{\alpha}_{x^{\sigma}}A^{\beta}\right) + \frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{\mu\sigma\beta}\ {}_{a}D^{\alpha}_{x_{\sigma}}A_{\mu}\right] = 0$$

or alternatively,

$$\frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\left(\varepsilon^{\beta\nu\rho}\ {}_{a}D^{\alpha}_{x_{\nu}}A_{\rho} - \varepsilon^{\mu\sigma\beta}\ {}_{a}D^{\alpha}_{x_{\sigma}}A_{\mu}\right) - \left(1 + v_{eff}\right)e\overline{\Psi}\,\gamma^{\beta}\psi - {}_{a}D^{\alpha}_{x_{\sigma}}\left({}_{a}D^{\alpha}_{x^{\sigma}}A^{\beta} - {}_{a}D^{\alpha}_{x^{\beta}}A^{\sigma}\right) = 0$$

using $F^{\sigma\beta} = \left({}_{a}D^{\alpha}_{x^{\sigma}}A^{\beta} - {}_{a}D^{\alpha}_{x^{\beta}}A^{\sigma}\right)$, as a result of this

$$\frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\left(\varepsilon^{\beta\nu\rho}\ {}_{a}D^{\alpha}_{x_{\nu}}A_{\rho} - \varepsilon^{\beta\sigma\mu}\ {}_{a}D^{\alpha}_{x_{\sigma}}A_{\mu}\right) - \left(1 + v_{eff}\right)e\overline{\Psi}\,\gamma^{\beta}\psi - {}_{a}D^{\alpha}_{x_{\sigma}}\,F^{\sigma\beta} = 0$$

By replacing the dummy indices ν with μ and ρ with σ in the first term on the left-hand side, we obtain

$$\frac{e\pi\, v^2{}_{eff}}{\theta\phi_0}\varepsilon^{\beta\mu\sigma}\ {}_{a}D^{\alpha}_{x_{\sigma}}A_{\mu} - \left(1 + v_{eff}\right)e\overline{\Psi}\,\gamma^{\beta}\psi - {}_{a}D^{\alpha}_{x_{\sigma}}\,F^{\sigma\beta} = 0, \tag{8}$$

on the other hand, the EL equation for $\overline{\psi}$ reads as

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} - {}_aD^{\alpha}_{x_\sigma}\left[\frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{x_\sigma}\overline{\psi}\right)}\right] = 0, \tag{9}$$

which becomes

$$\left(i\gamma^{\mu}\, {}_aD^{\alpha}_{x_\mu} - m\right)\psi - \left(1 + v_{eff}\right)e\,\gamma^{\mu}\psi\, A_\mu = 0, \tag{10}$$

finally, the EL equation of the field ψ is presented as follows:

$$\frac{\partial \mathcal{L}}{\partial \psi} - {}_aD^{\alpha}_{x_\sigma}\left[\frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{x_\sigma}\psi\right)}\right] = 0, \tag{11}$$

which can be written as

$$-m\overline{\psi} - \left(1 + v_{eff}\right)e\,\gamma^{\mu}\overline{\psi}\, A_\mu - {}_aD^{\alpha}_{x_\sigma}\left(\overline{\psi}\, i\gamma^{\mu}\,\delta^{\sigma}_{\mu}\right) = 0$$

$$m\overline{\psi} + \left(1 + v_{eff}\right)e\,\gamma^{\mu}\overline{\psi}\, A_\mu = -i\gamma^{\sigma}\, {}_aD^{\alpha}_{x_\sigma}\overline{\psi}\,. \tag{12}$$

5. FRACTIONAL HAMILTONIAN FORMULATION

In the following section, we will derive the fractional Hamiltonian equations using the RL fractional derivative approach, based on the fractional Lagrangian density. Let us consider the fractional Hamiltonian density as

$$\mathcal{H} = \left(\overline{\psi}, \psi, {}_aD^{\alpha}_{x_j}\psi, A_\mu, {}_aD^{\alpha}_{x_j}A_\mu, \bar{\pi}, \pi, \pi_\mu\right). \tag{13}$$

Now, taking the total differential of $\mathcal{H}$, we get:

$$d\mathcal{H} = \frac{\partial \mathcal{H}}{\partial \overline{\psi}} d\overline{\psi} + \frac{\partial \mathcal{H}}{\partial \psi} d\psi + \frac{\partial \mathcal{H}}{\partial A_\mu} dA_\mu + \frac{\partial \mathcal{H}}{\partial\left({}_aD^{\alpha}_{x_j}\psi\right)} d\left({}_aD^{\alpha}_{x_j}\psi\right) + \frac{\partial \mathcal{H}}{\partial\left({}_aD^{\alpha}_{x_j}A_\mu\right)} d\left({}_aD^{\alpha}_{x_j}A_\mu\right) + \frac{\partial \mathcal{H}}{\partial \bar{\pi}} d\bar{\pi} + \frac{\partial \mathcal{H}}{\partial \pi} d\pi + \frac{\partial \mathcal{H}}{\partial \pi_\mu} d\,\pi_\mu \tag{14}$$

The canonical momenta $\bar{\pi}, \pi, \pi_\mu$ are given as follows:

$$\pi = \frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{t}\psi\right)}; \qquad \bar{\pi} = \frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{t}\overline{\psi}\right)}; \qquad \pi_\mu = \frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{t}A_\mu\right)}; \tag{15}$$

In order to construct $\mathcal{H}$, we start by defining it in its general form as follows:

$$\mathcal{H} = \bar{\pi}\, {}_aD^{\alpha}_{t}\overline{\psi} + \pi\, {}_aD^{\alpha}_{t}\psi \pm \pi_\mu\, {}_aD^{\alpha}_{t}A_\mu - \mathcal{L}\left(\psi, \overline{\psi}, {}_aD^{\alpha}_{x_\sigma}\psi, A_\beta, {}_aD^{\alpha}_{x_\sigma}A_\beta\right). \tag{16}$$

The total differential of $\mathcal{H}$ can also be defined as:

$$d\mathcal{H} = \bar{\pi}\, d\left({}_aD^{\alpha}_{t}\overline{\psi}\right) + {}_aD^{\alpha}_{t}\overline{\psi}\, d\bar{\pi} + \pi\, d\left({}_aD^{\alpha}_{t}\psi\right) + {}_aD^{\alpha}_{t}\psi\, d\pi + \pi_\mu\, d\left({}_aD^{\alpha}_{t}A_\mu\right) + {}_aD^{\alpha}_{t}A_\mu d\,\pi_\mu - \frac{\partial \mathcal{L}}{\partial \psi} d\psi -$$

$$- \frac{\partial \mathcal{L}}{\partial \overline{\psi}} d\overline{\psi} - \frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{x_\sigma}\psi\right)} d\left({}_aD^{\alpha}_{x_\sigma}\psi\right) - \frac{\partial \mathcal{L}}{\partial A_\beta} dA_\beta + \frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{x_\sigma}A_\beta\right)} d\left({}_aD^{\alpha}_{x_\sigma}A_\beta\right), \tag{17}$$

but

$$\frac{\partial \mathcal{L}}{\partial \psi} = {}_aD^{\alpha}_{t}\pi + {}_aD^{\alpha}_{x_j}\left(\frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{x_j}\psi\right)}\right), \tag{18}$$

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} = {}_aD^{\alpha}_{t}\bar{\pi} + {}_aD^{\alpha}_{x_j}\left(\frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{x_j}\overline{\psi}\right)}\right), \tag{19}$$

and

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = {}_aD^{\alpha}_{t}\,\pi_\mu + {}_aD^{\alpha}_{x_j}\left(\frac{\partial \mathcal{L}}{\partial\left({}_aD^{\alpha}_{x_j}A_\mu\right)}\right). \tag{20}$$

Using Eqs. (18), (19), and (20) Eq. (17) can be written as

$$d\mathcal{H} = \bar{\pi}\, d\left({}_aD_t^{\alpha}\overline{\Psi}\right) + {}_aD_t^{\alpha}\overline{\Psi}\ d\bar{\pi} + \pi\, d\left({}_aD_t^{\alpha}\psi\right) + {}_aD_t^{\alpha}\psi\ d\pi + \pi_\mu\, d\left({}_aD_t^{\alpha}A_\mu\right) + {}_aD_t^{\alpha}A_\mu d\,\pi_\mu - {}_aD_t^{\alpha}(\pi)d\psi - {}_aD_{x_j}^{\alpha}\left(\frac{\partial\mathcal{L}}{\partial\left({}_aD_{x_j}^{\alpha}\Psi\right)}\right)d\psi - {}_aD_t^{\alpha}(\bar{\pi})d\overline{\Psi} - {}_aD_{x_j}^{\alpha}\left(\frac{\partial\mathcal{L}}{\partial\left({}_aD_{x_j}^{\alpha}\overline{\Psi}\right)}\right)d\overline{\Psi} - \frac{\partial\mathcal{L}}{\partial\left({}_aD_{x_\sigma}^{\alpha}\Psi\right)}d\left({}_aD_{x_\sigma}^{\alpha}\Psi\right) - {}_aD_t^{\alpha}(\pi_\mu)dA_\beta - - {}_aD_{x_j}^{\alpha}\left(\frac{\partial\mathcal{L}}{\partial\left({}_aD_{x_j}^{\alpha}A_\beta\right)}\right)dA_\beta - \frac{\partial\mathcal{L}}{\partial\left({}_aD_{x_\sigma}^{\alpha}A_\beta\right)}d\left({}_aD_{x_\sigma}^{\alpha}A_\beta\right), \tag{21}$$

or

$$d\mathcal{H} = {}_aD_t^{\alpha}\overline{\Psi}\ d\bar{\pi} + {}_aD_t^{\alpha}\psi\ d\pi + {}_aD_t^{\alpha}A_\mu\, d\,\pi_\mu - {}_aD_t^{\alpha}(\pi)d\psi - {}_aD_{x_j}^{\alpha}\left(\frac{\partial\mathcal{L}}{\partial\left({}_aD_{x_j}^{\alpha}\Psi\right)}\right)d\psi - {}_aD_t^{\alpha}(\bar{\pi})d\overline{\Psi} - - {}_aD_{x_j}^{\alpha}\left(\frac{\partial\mathcal{L}}{\partial\left({}_aD_{x_j}^{\alpha}\overline{\Psi}\right)}\right)d\overline{\Psi} - \frac{\partial\mathcal{L}}{\partial\left({}_aD_{x_i}^{\alpha}\Psi\right)}d\left({}_aD_{x_i}^{\alpha}\Psi\right) - {}_aD_t^{\alpha}(\pi_\beta)dA_\beta - {}_aD_{x_j}^{\alpha}\left(\frac{\partial\mathcal{L}}{\partial\left({}_aD_{x_j}^{\alpha}A_\beta\right)}\right)dA_\beta - -\frac{\partial\mathcal{L}}{\partial\left({}_aD_{x_i}^{\alpha}A_\beta\right)}d\left({}_aD_{x_i}^{\alpha}A_\beta\right) \tag{22}$$

By comparing Eqs. (14) and (22), we obtain Hamilton's equation of motion

$$\frac{\partial\mathcal{H}}{\partial A_\mu} = -\,{}_aD_t^{\alpha}(\pi_\mu) - {}_aD_{x_j}^{\alpha}\left(\frac{\partial\mathcal{L}}{\partial\left({}_aD_{x_j}^{\alpha}A_\mu\right)}\right), \tag{23}$$

$$\frac{\partial\mathcal{H}}{\partial\psi} = -\,{}_aD_t^{\alpha}\pi - {}_aD_{x_j}^{\alpha}\left(\frac{\partial\mathcal{L}}{\partial\left({}_aD_{x_j}^{\alpha}\Psi\right)}\right), \tag{24}$$

$$\frac{\partial\mathcal{H}}{\partial\overline{\Psi}} = -\,{}_aD_t^{\alpha}(\bar{\pi}) - {}_aD_{x_j}^{\alpha}\left(\frac{\partial\mathcal{L}}{\partial\left({}_aD_{x_j}^{\alpha}\overline{\Psi}\right)}\right), \tag{25}$$

$${}_aD_t^{\alpha}A_\mu = \frac{\partial\mathcal{H}}{\partial\pi_\mu},\quad {}_aD_t^{\alpha}\psi = \frac{\partial\mathcal{H}}{\partial\pi},\quad {}_aD_t^{\alpha}\overline{\Psi} = \frac{\partial\mathcal{H}}{\partial\bar{\pi}}, \tag{26}$$

$$\frac{\partial\mathcal{L}}{\partial\left({}_aD_{x_j}^{\alpha}\Psi\right)} = -\frac{\partial\mathcal{H}}{\partial\left({}_aD_{x_j}^{\alpha}\Psi\right)}\ ,\ \frac{\partial\mathcal{L}}{\partial\left({}_aD_{x_j}^{\alpha}A_\beta\right)} = -\frac{\partial\mathcal{H}}{\partial\left({}_aD_{x_j}^{\alpha}A_\mu\right)}. \tag{27}$$

Consider the Lagrangian density given in Eq. (5)

$$\mathcal{L} = \overline{\Psi}\left(i\gamma^\mu\,{}_aD_{x_\mu}^{\alpha} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e\pi\, {v^2}_{eff}}{2\theta\phi_0}\varepsilon^{\mu\nu\rho}A_\mu\ {}_aD_{x_\nu}^{\alpha}A_\rho - \left(1+v_{eff}\right)e\overline{\Psi}\,\gamma^\mu\psi\, A_\mu$$

We can determine $\bar{\pi}$, π and π_i such that:

$$\bar{\pi} = \frac{\partial\mathcal{L}}{\partial\left({}_aD_t^{\alpha}\overline{\Psi}\right)} = 0, \tag{28a}$$

$$\pi == \frac{\partial\mathcal{L}}{\partial\left({}_aD_t^{\alpha}\Psi\right)} = i\gamma^0\overline{\Psi}, \tag{28b}$$

$$\begin{cases} \pi_i = \dfrac{\partial\mathcal{L}}{\partial\left({}_aD_t^{\alpha}A_i\right)} = -F^{oi} + \dfrac{e\pi\, {v^2}_{eff}}{2\theta\phi_0}\varepsilon^{ij}A_j \\ \pi_0 = \dfrac{\partial\mathcal{L}}{\partial\left({}_aD_t^{\alpha}A_0\right)} = 0 \end{cases}. \tag{28c}$$

Then we can write the fractional Hamiltonian density of the system as follows:

$$\mathcal{H} = \mathcal{H} = \bar{\pi}\ {}_aD_t^{\alpha}\overline{\Psi} + \pi\,{}_aD_t^{\alpha}\psi + \pi_\mu\ {}_aD_t^{\alpha}A_\mu - \mathcal{L}\left(\psi, \overline{\Psi}, {}_aD_{x_\sigma}^{\alpha}\Psi, A_\beta, {}_aD_{x_\sigma}^{\alpha}A_\beta\right). \tag{29}$$

By inserting Eqs. (5) and (28) into Eq. (29), we obtain

$$\mathcal{H} = i\gamma^0\overline{\Psi}\,{}_aD_t^{\alpha}\psi + \pi_i\left(-\pi_i + \frac{e\pi\, {v^2}_{eff}}{2\theta\phi_0}\varepsilon^{ij}A_j + {}_aD_{x^i}^{\alpha}A_0\right) - i\gamma^0\overline{\Psi}\,{}_aD_t^{\alpha}\psi - \overline{\Psi}\left(i\gamma^\mu\,{}_a^cD_{x_j}^{\alpha} - m\right)\psi + \frac{1}{2}F_{0i}F^{0i} + \frac{1}{4}F_{ij}F^{ij} - \frac{e\pi\, {v^2}_{eff}}{2\theta\phi_0}\varepsilon^{\mu 0i}A_\mu\left(-\pi_i + \frac{e\pi\, {v^2}_{eff}}{2\theta\phi_0}\varepsilon^{ij}A_j + {}_aD_{x_i}^{\alpha}A_0\right) - \frac{e\pi\, {v^2}_{eff}}{2\theta\phi_0}\varepsilon^{\mu i\rho}A_\mu\ {}_aD_{x_i}^{\alpha}A_\rho + \left(1+v_{eff}\right)e\overline{\Psi}\,\gamma^\mu\psi\, A_\mu$$

Substituting for F^{0i} from equation (28c), we get

$$\mathcal{H} = -\frac{1}{2}\pi_i^{\ 2} + \pi_i\ {}_aD_{x^i}^{\alpha} A_0 - \overline{\psi}\left(i\gamma^j\ {}_aD_{x_j}^{\alpha} - m\right)\psi - \frac{1}{2}\left(\frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{ij}\right)^2 A_j^{\ 2} + \frac{1}{4}F_{ij}F^{ij} - \frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{\mu 0 i} A_\mu\left(-\pi_i - {}_aD_{x_i}^{\alpha} A_0\right) - \frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{\mu i\rho} A_\mu\ {}_aD_{x_i}^{\alpha} A_\rho + (1+v_{eff})e\overline{\psi}\,\gamma^\mu\psi\, A_\mu . \tag{30}$$

Now we will find the Hamiltonian equations of motion for the same system. Initially, the equation of motion for A_μ

$$\frac{\partial\mathcal{H}}{\partial A_\mu} = -\ {}_aD_t^{\alpha}(\pi_\mu) - {}_aD_{x_j}^{\alpha}\left(\frac{\partial\mathcal{L}}{\partial\left({}_aD_{x_j}^{\alpha}A_\mu\right)}\right) - \left(\frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{i\mu}\right)^2 A_\mu - \frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{k0\mu}\left(-\pi_\mu + {}_aD_{x_\mu}^{\alpha}A_0\right) - \frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{\mu i\rho}\ {}_aD_{x_j}^{\alpha}A_\rho + (1+v_{eff})e\overline{\psi}\,\gamma^\mu\psi = {}_aD_t^{\alpha}F^{0j} - \frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{i\mu 0}\ {}_aD_t^{\alpha}A_\mu - {}_aD_{x_j}^{\alpha}(-F^{jk}) + + {}_aD_{x_j}^{\alpha}(-F^{0j}) - \frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{\mu jk}\ {}_aD_{x_j}^{\alpha}A_\mu . \tag{31}$$

The substitution of ${}_aD_\sigma^{\alpha}F^{\sigma\beta} = {}_aD_t^{\alpha}F^{0\beta} + {}_aD_j^{\alpha}F^{j0} + {}_aD_j^{\alpha}F^{jk}$ into Eq. (31) gives

$$-\left(\frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{i\mu}\right)^2 A_\mu - \frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{k0\mu}\left(-\pi_\mu + {}_aD_{x_\mu}^{\alpha}A_0\right) - \frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{\mu i\rho}\ {}_aD_{x_j}^{\alpha}A_\rho + (1+v_{eff})e\overline{\psi}\,\gamma^\mu\psi = \frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{i\mu 0}\ {}_aD_t^{\alpha}A_\mu + -\frac{e\pi\, v^2{}_{eff}}{2\theta\phi_0}\varepsilon^{\mu jk}\ {}_aD_{x_j}^{\alpha}A_\mu + {}_aD_\sigma^{\alpha}F^{\sigma\beta}. \tag{32}$$

By substituting Eq. (28c) into Eq. (32) and undergoing mathematical manipulation, the following equation is obtained:

$$\frac{e\pi\, v^2{}_{eff}}{\theta\phi_0}\varepsilon^{\beta\mu\sigma}\ {}_aD_{x_\sigma}^{\alpha}A_\mu - (1+v_{eff})e\overline{\psi}\,\gamma^\beta\psi - {}_aD_{x_\sigma}^{\alpha}F^{\sigma\beta} = 0. \tag{33}$$

While the equation of motion for ψ reads

$$\frac{\partial\mathcal{H}}{\partial\psi} = -\ {}_aD_t^{\alpha}\pi - {}_aD_{x_j}^{\alpha}\left(\frac{\partial\mathcal{L}}{\partial\left({}_aD_{x_j}^{\alpha}\psi\right)}\right)$$

$$m\,\overline{\psi} + (1+v_{eff})e\,\gamma^\mu\overline{\psi}\,A_\mu = -i\gamma^0\ {}_aD_t^{\alpha}\,\overline{\psi} - {}_aD_{x_j}^{\alpha}\left(\overline{\psi}\,i\gamma^j\right), \tag{34}$$

but

$$i\gamma^0\ {}_aD_t^{\alpha}\,\overline{\psi} + i\gamma^j\ {}_aD_{x_j}^{\alpha}\,\overline{\psi} = i\gamma^\sigma\ {}_aD_{x_\sigma}^{\alpha}\overline{\psi}$$

Thus, Eq. (34) becomes

$$m\,\overline{\psi} + (1+v_{eff})e\,\gamma^\mu\overline{\psi}\,A_\mu = -i\gamma^\sigma\ {}_aD_{x_\sigma}^{\alpha}\overline{\psi}. \tag{35}$$

Similarly, the equation of motion for $\overline{\psi}$ is

$$\frac{\partial\mathcal{H}}{\partial\overline{\psi}} = -\ {}_aD_t^{\alpha}(\bar{\pi}) - {}_aD_{x_j}^{\alpha}\left(\frac{\partial\mathcal{L}}{\partial\left({}_aD_{x_j}^{\alpha}\overline{\psi}\right)}\right)$$

or

$$\left(i\gamma^j\ {}_aD_{x_j}^{\alpha} - m\right)\psi - (1+v_{eff})e\,\gamma^\mu\psi\,A_\mu = 0. \tag{36}$$

The results from Eqs. (33), (35), and (36) are in full accordance with those derived from the fractional EL method.

6. CONCLUSION

The Riemann-Liouville fractional derivative were employed to reformulate the composite Fermions QED Lagrangian density. It was observed that the fractional Euler-Lagrange equations and the fractional Hamilton's equations of motion, both derived from the same Lagrangian density, produced the same outcomes. The fractional formulation was demonstrated to encompass the classical results as a specific case.

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ЩІЛЬНІСТЬ ЛАГРАНЖІАНА КОМПОЗИЦІЙНИХ ФЕРМІОНІВ QED У ДРОБОВОМУ ФОРМУЛЮВАННІ

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Квантова електродинаміка (КЕД) – точна й успішна теорія, яка описує взаємодію між електрично зарядженими частинками та електромагнітним випромінюванням. Вона є невід'ємною частиною Стандартної моделі фізики елементарних частинок і забезпечує теоретичну основу для пояснення широкого спектру фізичних явищ, у тому числі поведінки атомів, молекул і матеріалів. У цій роботі щільність Лагранжіана композитних ферміонів у КЕД була виражена у дробовій формі за допомогою дробової похідної Рімана-Ліувіля. Були також отримані дробове рівняння Ейлера-Лагранжа і дробове рівняння Гамільтона, виведені з дробової форми лагранжіана густини. Коли α встановлено на 1, відновлювались звичайні математичні рівняння.

Ключові слова: *квантова електродинаміка; композитні ферміони; дробова похідна; щільність лагранжіана; рівняння Ейлера-Лагранжа*