





THEORETICAL INVESTIGATION OF MESON SPECTRUM USING EXACT QUANTIZATION RULE TECHNIQUE[†]

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The energy eigenvalues with the Extended Cornell potential were obtained by analytically solving the radial Schrödinger equation using the Exact Quantization Rule technique. It was then used for computing the mass spectra of the heavy mesons like charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$) as well as heavy-light mesons such as bottom-charm ($b\bar{c}$) and charm-Strange ($c\bar{s}$) for various quantum states. Two exceptional cases such as the Coulomb and Cornell potentials, were taken into consideration when some of the potential parameters were set to zero. The current potential offers good outcomes when compared to experimental data and the work of other researchers with a maximum error of 0.0065 GeV .

Keywords: *Cornell potential; Schrödinger equation; Exact Quantization Rule; Mesons*

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1. INTRODUCTION

The development of the radial Schrödinger equation (SE) in quantum mechanics and its solutions plays a fundamental role in many fields of modern physics. The study of the behavior of quite a lot of physical problems in physics requires the solving of the SE. The solutions can be well-known only if we know the confining potential for a particular physical system [1-4]. The theory of quantum chromodynamics (QCD) which is described by the meson system is mediated by strong interactions [5]. The heavy mesons are the constituents of quark and antiquark such as charmonium and bottomonium that are considered as the non-relativistic system described by the SE [6]. In recent times, researchers have obtained the solutions of the SE and Klein-Gordon equation (KGE) with the quarkonium interaction potential model such as the Cornell or the Killingbeck potentials [7-9]. The Cornell potential is the sum of the Coulomb plus linear potentials. The Cornell potential and its extended form have been solved with SE with different analytical methods [10,11]. The exact solutions of the SE with some potentials are solvable for $l = 0$, but insolvable for any arbitrary angular momentum quantum number $l \neq 0$. In this case, several approximate techniques are employed in obtaining the solutions. Example of such techniques include, the asymptotic iteration method (AIM) [12], the Nikiforov-Uvarov functional analysis (NUFA) method [13-17] the Laplace transformation method [18], the Nikiforov-Uvarov (NU) method [19-24], the series expansion method (SEM) [25-27], analytical exact iterative method (AEIM) [28], WKB approximation method [29-31] and others [32,33].

Recently, the mass spectrum of the quarkonium system has been studied by researchers [34,35]. For instance, Vega and Flores [36] obtained the solution of the SE with the Cornell potential via the variational method and super symmetric quantum mechanics (SUSYQM). Ciftci and Kisoglu [37] addressed non-relativistic arbitrary l -states of quark-antiquark through the Asymptotic Iteration Method (AIM). An analytic solution of the N-dimensional radial SE with the mixture of vector and scalar potentials via the Laplace transformation method (LTM) was studied by [18]. Their results were employed to analyze the different properties of the heavy-light mesons. Also, Al-Jamel and Widyan [38] studied heavy quarkonium mass spectra in a Coulomb field plus quadratic potential by employing the Nikiforov-Uvarov (NU) method. In their work, the spin-averaged mass spectra of heavy quarkonia in a Coulomb plus quadratic potential is analyzed within the non-relativistic SE. In addition, Al-Oun et al. [39] examined heavy quarkonia characteristics in the general framework of a non-relativistic potential model consisting of a Coulomb plus quadratic potential. Furthermore, Omugbe et al. [29] solved the SE with Killingbeck potential plus an inversely quadratic potential model via the WKB method. They obtained the energy eigenvalues and the mass spectra of the heavy and heavy-light meson systems. In addition, Inyang et al. [40] obtained the KGE solutions for the Yukawa potential via the NU method. They obtained energy eigenvalues both in the relativistic and non-relativistic regimes, and the results were then applied to calculate heavy-meson masses of charmonium $c\bar{c}$ and bottomonium $b\bar{b}$. Ibekwe et al. [41] solved the radial SE with $b\bar{b}$ an exponential, generalized, harmonic

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Cornell potential via the series expansion method. They applied the bound state eigenvalues to study the energy spectra for CO, NO, CH and N₂ diatomic molecules and the mass spectra of heavy quarkonium systems.

Therefore, in this present work, we aim at studying the SE with the extended Cornell potential via the Exact quantization rule (EQR) to obtain the mass spectra of heavy mesons such as charmonium ($c\bar{c}$), bottomonium ($b\bar{b}$) and the heavy–light mesons such as the charm-Strange ($c\bar{s}$) and bottom-charm ($b\bar{c}$). The extended Cornell potential (ECP) takes the form [29].

$$V(r) = \eta_0 r^2 + \eta_1 r - \frac{\eta_2}{r} + \frac{\eta_3}{r^2} \tag{1}$$

where η_0 , η_1 , η_2 and η_3 are potential strength parameters. The second term in Eq. (1) is a linear term for confinement feature and the third term is the Coulomb potential that describes the short distance between quarks. While the first and the last terms are quadratic and the inverse quadratic potentials.

It is important to note that if we set $\eta_0 = \eta_1 = \eta_3 = 0$, the (ECP) reduces to the Coulomb potential also, if we set $\eta_0 = \eta_3 = 0$ the (ECP) reduces to the standard Cornell potential. The paper is organized as follows: in section 2, the brief EQR formalism is presented. Section 3, the analytical solution of the bound states of the SE is solved via the EQR. In section 4, we present the results of the mass spectrum of the mesons. Finally, in section 5, the study is concluded.

2. EXACT QUANTIZATION RULE FORMALISM

In this section, we give a brief review of exact quantization rule. The details can be found in [42,43]. It is a well known fact that, in one dimension, the SE is given as:

$$\frac{d^2\psi(x)}{dx^2} + \frac{2\mu}{\hbar^2} [E_{nl} - V(x)]\psi(x) = 0 \tag{2}$$

Equation (2) can be written in the following form:

$$\phi'(x) + \phi(x)^2 + k(x)^2 = 0 \tag{3}$$

with

$$k(x) = \sqrt{\frac{2\mu}{\hbar^2} [E_{nl} - V(x)]} \tag{4}$$

where $\phi(x) = \psi'(x) / \psi(x)$ is the logarithmic derivative of the wave function, μ is the reduced mass of the quarkonium particles, $k(x)$ is the momentum, and $V(x)$ is a piecewise continuous real potential function of x . The phase angle of the SE is the logarithmic derivative $\phi(x)$. From Eq. (3), as x increases across a node of wave function $\psi(x)$, $\phi(x)$ decreases to $-\infty$, jumps to $+\infty$, and then decreases again. We can generalize EQR to the three – dimensional radial SE with spherically symmetric potential by simply making the replacement $x \rightarrow r$ and $V(x) \rightarrow V_{eff}(r)$ [42,43].

$$\int_{r_a}^{r_b} k(r) dr = N\pi + \int_{r_a}^{r_b} \phi(r) \left[\frac{dk(r)}{dr} \right] \left[\frac{d\phi(r)}{dr} \right]^{-1} \tag{5}$$

$$k(r) = \sqrt{\frac{2\mu}{\hbar^2} [E_{nl} - V_{eff}(r)]} \tag{6}$$

where r_a and r_b are two turning points determined by $E = V_{eff}(r)$. $N = n + 1$ is the number of the nodes of $\phi(r)$ in the region $E_{nl} = V_{eff}(r)$ and is larger by 1 than the n of the nodes of the wave function $\psi(r)$. The first term $N\pi$ is the contribution from the nodes of the logarithmic derivatives of the wave function, and the second one is called the quantum correction. It was found that for all well-known exactly solvable quantum systems, this quantum correction is independent of the number of nodes of the wave function. This means that it is enough to consider the ground state in calculating the quantum correction (Q_c), i.e.

$$Q_c = \int_{r_a}^{r_b} k'_o(r) \frac{\phi'_o}{\phi_o} dr \tag{7}$$

To determine the energy eigenvalues, we equate Eqs. (5) and (7).

3. APPROXIMATE SOLUTIONS OF THE SCHRÖDINGER EQUATION WITH EXTENDED CORNELL POTENTIAL

The SE for two particles interacting via potential $V(r)$ is given by [44].

$$\frac{d^2 R(r)}{dr^2} + \frac{2\mu}{\hbar^2} \left[E_{nl} - V(r) - \frac{l(l+1)\hbar^2}{2\mu r^2} \right] R(r) = 0, \quad (8)$$

where l, μ, r and \hbar are the angular momentum quantum number, the reduced mass for the quarkonium particle, inter-particle distance and reduced plank constant respectively.

We substitute Eq. (1) into Eq. (8) and obtain

$$\frac{d^2 R(r)}{dr^2} + \frac{2\mu}{\hbar^2} \left[E_{nl} - V_{eff}(r) \right] R(r) = 0, \quad (9)$$

where

$$V_{eff}(r) = \eta_0 r^2 + \eta_1 r - \frac{\eta_2}{r} + \frac{\eta_3}{r^2} + \frac{l(l+1)\hbar^2}{2\mu r^2}. \quad (10)$$

We transform the coordinate of Eq. (9) by setting

$$x = \frac{1}{r}. \quad (11)$$

Upon substituting Eq. (11) into Eq. (10) we have

$$V_{eff}(x) = \frac{\eta_0}{x^2} + \frac{\eta_1}{x} - \eta_2 x + \eta_3 x^2 + \frac{l(l+1)\hbar^2 x^2}{2\mu}. \quad (12)$$

To deal with the first and second terms of Eq. (12), we propose the following approximation scheme. We assume that there is a characteristic radius r_0 of the meson. Then the scheme is based on the expansion of $\frac{\eta_1}{x}$ and $\frac{\eta_0}{x^2}$ in a power series around r_0 ; i.e. around $\delta \equiv \frac{1}{r_0}$, up to the second order. This is similar to Pekeris approximation, which helps to deform the centrifugal term such that the modified potential can be solved by NU method [45]. Setting $y = x - \delta$ and around $y = 0$ it can be expanded into a series of powers as:

$$\frac{\eta_1}{x} = \frac{\eta_1}{y + \delta} = \frac{\eta_1}{\delta \left(1 + \frac{y}{\delta} \right)} = \frac{\eta_1}{\delta} \left(1 + \frac{y}{\delta} \right)^{-1}, \quad (13)$$

which yields

$$\frac{\eta_1}{x} = \eta_1 \left(\frac{3}{\delta} - \frac{3x}{\delta^2} + \frac{x^2}{\delta^3} \right). \quad (14)$$

Similarly,

$$\frac{\eta_0}{x^2} = \eta_0 \left(\frac{6}{\delta^2} - \frac{8x}{\delta^3} + \frac{3x^2}{\delta^4} \right). \quad (15)$$

We then substitute Eqs. (14) and (15) into Eq. (12) and obtain

$$V_{eff}(x) = \xi_1 + \xi_2 x + \xi_3 x^2, \quad (16)$$

where

$$\left. \begin{aligned} \xi_1 &= \frac{6\eta_0}{\delta^2} + \frac{3\eta_1}{\delta}, & \xi_2 &= \frac{3\eta_1}{\delta^2} - \frac{8}{\delta^3} - \eta_2 \\ \xi_3 &= \frac{l(l+1)\hbar^2}{2\mu} + \frac{3\eta_0}{\delta^4} + \frac{\eta_1}{\delta^3} + \eta_3 \end{aligned} \right\}. \quad (17)$$

The non-linear Riccati equation for ground state is written in terms of the new variable x as,

$$-x^2 \phi'(x) + \phi^2(x) = k(x), \tag{18}$$

where

$$k(x) = \sqrt{\frac{2\mu}{\hbar^2} [\xi_3 x^2 + \xi_2 x + \xi_1 - E]}. \tag{19}$$

We now apply the quantization rule to study the potential. To this end we first calculate the turning points x_a and x_b , which is determined by solving the content of the square bracket of Eq. (19), these yields:

$$\left. \begin{aligned} x_a &= \frac{-\xi_2 - \sqrt{\xi_2^2 - 4\xi_3(\xi_1 - E)}}{2\xi_3} \\ x_b &= \frac{-\xi_2 + \sqrt{\xi_2^2 - 4\xi_3(\xi_1 - E)}}{2\xi_3} \end{aligned} \right\} \tag{20}$$

From Eq. (20) we have

$$\left. \begin{aligned} x_a x_b &= \frac{\xi_1 - E}{\xi_3} \\ x_a + x_b &= -\frac{\xi_2}{\xi_3} \end{aligned} \right\} \tag{21}$$

Also, from Eq. (19) we have

$$k(x) = \sqrt{\frac{2\mu}{\hbar^2} \xi_3 \left(x^2 + \frac{\xi_2}{\xi_3} x + \frac{\xi_1 - E}{\xi_3} \right)} \tag{22}$$

Substituting Eq.(21) into Eq.(22) we obtain

$$k(x) = \sqrt{\frac{2\mu\xi_3}{\hbar^2} (x - x_a)(x - x_b)} \tag{23}$$

where $k(x)$ is the momentum between the two turning points x_a and x_b .

From Eq.(18), since the logarithmic derivative $\phi_0(x)$ for the ground state has one zero and no pole, therefore we assume the trial solution for the ground states

$$\phi_0(x) = A + Bx \tag{24}$$

Substituting Eq. (24) into Eq. (18) and then solving the non-linear Riccati equation, we obtain the ground state energy as

$$E_0 = \xi_1 - \frac{\hbar^2 A^2}{2\mu} \tag{25}$$

Also, we obtain A and B as follows

$$\left. \begin{aligned} A &= \frac{\mu\xi_2}{B\hbar^2} \\ B &= \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{2\mu\xi_3}{\hbar^2}} \end{aligned} \right\} \tag{26}$$

Here we choose the positive sign in front of the square root for B . This is as a result of the logarithmic derivatives $\phi_0(x)$ which decreases exponentially, which is required. We now calculate the quantum correction and obtain

$$\int_{r_a}^{r_b} \phi(r) \left[\frac{dk(r)}{dr} \right] \left[\frac{d\phi(r)}{dr} \right]^{-1} dr = - \int_{x_a}^{x_b} \frac{k_0'(x) \phi_0(x)}{x^2 \phi_0'(x)} dx \tag{27}$$

From Eq. (27) we have

$$\sqrt{\frac{2\mu\xi_3}{\hbar^2}} \int_{x_a}^{x_b} \left(\frac{\frac{A}{B} - \frac{x_a + x_b}{2}}{x\sqrt{(x-x_a)(x-x_b)}} + \frac{\frac{A}{B} \left(\frac{x_a + x_b}{2} \right)}{x^2 \sqrt{(x-x_a)(x-x_b)}} \right) dx \tag{27a}$$

We utilized the integrals given by Appendix A, and obtain

$$\pi \sqrt{\frac{2\mu\xi_3}{\hbar^2}} \left[\frac{\frac{A}{B} - \frac{x_a + x_b}{2}}{\sqrt{1+(x_a+x_b)+x_ax_b}} + \frac{\frac{A}{B} \left(\frac{x_a + x_b}{2} \right) \left(\sqrt{x_ax_b} - \frac{1}{2(x_a+x_b)} \right)}{\sqrt{x_ax_b}} \right] \tag{27b}$$

We substitute Eq. (21) into Eq.(27b) and obtain

$$\pi \sqrt{\frac{2\mu\xi_3}{\hbar^2}} \left[\frac{\frac{A}{B} + \frac{\xi_2}{2\xi_3}}{\sqrt{\frac{\hbar^2}{2\mu\xi_3} (A+B)}} - \frac{A\xi_2 \left(\sqrt{\frac{\hbar^2}{2\mu\xi_3} A + \frac{\xi_3}{2\xi_2}} \right)}{\sqrt{\frac{\hbar^2}{2\mu\xi_3} A}} \right] \tag{27c}$$

Inserting Eq. (27c) into Eq. (5) we obtain

$$\pi \sqrt{\frac{2\mu\xi_3}{\hbar^2}} \left[\frac{\frac{A}{B} + \frac{\xi_2}{2\xi_3}}{\sqrt{\frac{\hbar^2}{2\mu\xi_3} (A+B)}} - \frac{A\xi_2 \left(\sqrt{\frac{\hbar^2}{2\mu\xi_3} A + \frac{\xi_3}{2\xi_2}} \right)}{\sqrt{\frac{\hbar^2}{2\mu\xi_3} A}} \right] + N\pi \tag{27d}$$

Furthermore, the integral of Eq.(7) is obtain as

$$\int_{r_a}^{r_b} k(r)dr = -\int_{x_a}^{x_b} \frac{k(x)}{x^2} dx \tag{28}$$

From Eq. (28) we have

$$-\sqrt{\frac{2\mu\xi_3}{\hbar^2}} \int_{x_a}^{x_b} \frac{\sqrt{(x-x_a)(x-x_b)}}{x^2} dx \tag{28a}$$

Using Eq. (29), Eq. (28a) becomes

$$-\sqrt{\frac{2\mu\xi_3}{\hbar^2}} \left[\frac{(x_a+x_b) - \sqrt{x_ax_b}}{2\sqrt{x_ax_b}} \right] \tag{28b}$$

We substitute Eq. (21) into Eq. (28b) and obtain

$$-\sqrt{\frac{2\mu\xi_3}{\hbar^2}} \left[\frac{-\frac{\xi_2}{\xi_3} - \sqrt{\frac{\xi_1-E}{\xi_3}}}{2\sqrt{\frac{\xi_1-E}{\xi_3}}} \right] \tag{28c}$$

In order to obtain the integral of Eq.(28a), we used maple software to obtain the following useful integral, which is not available in integral table

$$\int_a^b \frac{\sqrt{(x-a)(b-x)}}{x^2} dx = \frac{(a+b) - \sqrt{ab}\pi}{2\sqrt{ab}} \tag{29}$$

By equating Eqs.(27d) and (28c) ,and substituting Eqs.(17) and (26) ,setting $\hbar = 1$ we obtain the energy equation for extended Cornell potential as

$$E_{nl} = \frac{3\eta_1}{\delta} + \frac{6\eta_0}{\delta^2} - \frac{2\mu \left(\frac{3\eta_1}{\delta^2} + \frac{8\eta_0}{\delta^3} + \eta_2 \right)^2}{\left[(2n+1) + \sqrt{1 + \frac{8\mu\eta_1}{\delta^3} + 4 \left(\left(l + \frac{1}{2} \right)^2 - \frac{1}{4} \right) - 8\mu\eta_3 + \frac{24\mu\eta_0}{\delta^4}} \right]^2} \tag{30}$$

3.1. Special case

In this subsection, we present some special cases of the energy eigenvalues of the ECP.

1. When we set $\eta_0 = \eta_1 = \eta_3 = 0$ we obtain energy eigenvalues expression for Coulomb potential

$$E_{nl} = -\frac{\mu\eta_2^2}{2(n+l+1)^2} \tag{31}$$

2. When we set $\eta_0 = \eta_3 = 0$ we obtain energy eigenvalues expression for Cornell potential

$$E_{nl} = \frac{3\eta_1}{\delta} - \frac{2\mu\left(\frac{3\eta_1}{\delta^2} + \eta_2\right)^2}{\left[(2n+1) + \sqrt{1 + \frac{8\mu\eta_1}{\delta^3} + 4\left(\left(l + \frac{1}{2}\right)^2 - \frac{1}{4}\right)}\right]^2} \tag{32}$$

The result of Eq. (32) is consistent with the result obtained in Eq. (30) in Ref. [9]

4. RESULTS

Using the relation in Ref. [46], we calculate the mass spectra of the heavy quarkonia such as charmonium and bottomonium, and heavy-light mesons.

$$M = 2m + E_{nl} \tag{33}$$

where m is quarkonium bare mass and E_{nl} is energy eigenvalues.

By substituting Eq. (30) into Eq. (33) we obtain the mass spectra for extended Cornell potential for heavy quarkonia as,

$$M = 2m + \frac{3\eta_1}{\delta} + \frac{6\eta_0}{\delta^2} - \frac{2\mu\left(\frac{3\eta_1}{\delta^2} + \frac{8\eta_0}{\delta^3} + \eta_2\right)^2}{\left[(2n+1) + \sqrt{1 + \frac{8\mu\eta_1}{\delta^3} + 4\left(\left(l + \frac{1}{2}\right)^2 - \frac{1}{4}\right)} - 8\mu\eta_3 + \frac{24\mu\eta_0}{\delta^4}\right]^2} \tag{34}$$

We also obtain the mass spectra of the special case of Eq. (32) for heavy-light mesons by substituting into Eq.(33)

$$M = m_s + m_c + \frac{3\eta_1}{\delta} - \frac{2\mu\left(\frac{3\eta_1}{\delta^2} + \eta_2\right)^2}{\left[(2n+1) + \sqrt{1 + \frac{8\mu\eta_1}{\delta^3} + 4\left(\left(l + \frac{1}{2}\right)^2 - \frac{1}{4}\right)}\right]^2} \tag{35}$$

We test the accuracy of the predicted results, by using the Chi square function defined by [47]

$$\chi^2 = \frac{1}{n} \sum_{i=1}^n \frac{(M_i^{\text{Exp.}} - M_i^{\text{Theo.}})^2}{\Delta_i} \tag{36}$$

where n runs over selected samples of mesons, $M_i^{\text{Exp.}}$ is the experimental mass of mesons, while $M_i^{\text{Theo.}}$ is the corresponding theoretical prediction. The Δ_i quantity is experimental uncertainty of the masses. Intuitively, Δ_i should be one. The tendency of overestimating Chi square value is that, it reflects some mean error.

We calculate mass spectra of charmonium and bottomonium for states from 1S to 1F, as presented in Tables 1 and 2. Also calculated the mass spectra of heavy-light mesons for states from 1S to 1D as presented in Tables 3 and 4. The free parameters of Eq. (34) were obtained by solving two algebraic equations of mass spectra in Eq. (34) for charmonium.

We followed the same procedure for bottomonium and obtained the free parameters of mass spectra. Equation (35) was fitted with experimental data of mass spectra of 1S, 2S to obtain the free parameters for bottom-charm ($b\bar{c}$) and charm-strange ($c\bar{s}$) heavy-light mesons.

For bottomonium $b\bar{b}$, charmonium $c\bar{c}$ and strange systems we adopt the numerical values of $m_b = 4.823 \text{ GeV}$, $m_c = 1.209 \text{ GeV}$ and $m_s = 0.419 \text{ GeV}$, and the corresponding reduced mass are $\mu_b = 2.4115 \text{ GeV}$, $\mu_c = 0.6045 \text{ GeV}$

and $\mu_s = 0.2095 \text{ GeV}$, respectively [48-50]. We note that the theoretical prediction of the mass spectra of charmonium and bottomonium are in excellent agreement with experimental data and the work of other researchers like; Refs.[28,9,18,37,41] as shown in Tables 1 and 2. Also, the results for charm-strange meson presented in Table 4, are in excellent agreement with the work of other authors in Refs.[28,37] Furthermore, in Table 3, the mass spectra of the bottom-charm meson are very close to the ones obtained in Refs[28,18,37] with other methods and experimental data indicating an improvement compared to the other methods. The maximum error in comparison with experimental data is 0.0065 GeV .

Table 1. Mass spectra of charmonium in (GeV) ($m_c = 1.209 \text{ GeV}$, $\mu = 0.6045 \text{ GeV}$, $\eta_1 = 0.001 \text{ GeV}$, $\eta_2 = 14.94 \text{ GeV}$, $\eta_0 = 0.02 \text{ GeV}$, $\eta_3 = -15.04 \text{ GeV}$, $\delta = 1.7 \text{ GeV}$)

State	Present work	AEIM[28]	NU[9]	AIM[37]	LTM[18]	SEM[41]	Experiment[48,49]
1S	3.096	3.0954	3.095	3.096	3.0963	3.095922	3.097
2S	3.686	3.5673	3.685	3.686	3.5681	3.685893	3.686
1P	3.295	3.5677	3.258	3.214	3.5687	-	3.525
2P	3.802	4.0396	3.779	3.773	3.5687	3.756506	3.773
3S	4.040	4.0392	4.040	4.275	4.0400	4.322881	4.040
4S	4.269	4.5110	4.262	4.865	4.5119	4.989406	4.263
1D	3.583	4.0396	3.510	3.412	4.0407	-	3.770
2D	3.976	-	-	-	-	-	4.159
1F	3.862	-	-	-	-	-	-

Table 2. Mass spectra of bottomonium in (GeV) ($m_b = 4.823 \text{ GeV}$, $\mu = 2.4115 \text{ GeV}$, $\eta_1 = 0.798 \text{ GeV}$, $\eta_2 = 5.051 \text{ GeV}$, $\eta_0 = 0.02 \text{ GeV}$, $\eta_3 = -3.854 \text{ GeV}$, $\delta = 1.5 \text{ GeV}$)

State	Present work	AEIM[28]	NU[9]	AIM[37]	LTM[18]	SEM[41]	Experiment[48,49]
1S	9.460	9.74473	9.460	9.460	9.745	9.515194	9.460
2S	10.023	10.02315	10.022	10.023	10.023	10.01801	10.023
1P	9.661	10.02406	9.609	9.492	10.025	-	9.899
2P	10.138	10.30248	10.109	10.038	10.303	10.09446	10.260
3S	10.355	10.30158	10.360	10.585	10.302	10.44142	10.355
4S	10.567	10.58000	10.580	11.148	10.580	10.85777	10.580
1D	9.943	10.30248	9.846	9.551	10.303	-	10.164
2D	10.306	-	-	-	-	-	-
1F	10.209	-	-	-	-	-	-

Table 3. Mass spectra of bottom-charm ($b\bar{c}$) in (GeV) ($m_b = 4.823 \text{ GeV}$, $m_c = 1.209 \text{ GeV}$, $\eta_1 = 0.202 \text{ GeV}$, $\eta_2 = 1.213 \text{ GeV}$, $\delta = 0.371 \text{ GeV}$)

State	Present work	AEIM[28]	LTM[18]	AIM[37]	Experiment[50]
1S	6.274	6.2774	6.2770	6.277	6.275
2S	6.845	7.0376	7.0372	6.814	6.842
3S	7.125	7.7978	7.7973	7.351	-
4S	7.283	7.0386	-	7.889	-
1P	6.519	7.7987	7.0381	6.340	-
2P	6.959	-	7.7983	6.851	-
1D	6.813	-	-	6.452	-

Table 4. Mass spectra of charm-strange ($c\bar{s}$) meson in (GeV) ($m_s = 0.419 \text{ GeV}$, $m_c = 1.209 \text{ GeV}$, $\eta_1 = 0.202 \text{ GeV}$, $\eta_2 = 2.046 \text{ GeV}$, $\delta = 0.561 \text{ GeV}$)

State	Present work	AEIM [18]	AIM [37]	Experiment [48, 51]
1S	1.969	1.968	2.512	1.968[48]
2S	2.709	2.709	2.709	2.709[51]
3S	2.913	2.932	2.906	-
4S	2.998	-	3.102	-
1P	2.601	2.565	2.649	-
2P	2.877	-	2.860	-
1D	2.863	2.857	2.859	2.859[51]

5. CONCLUSION

In this study, we used the EQR technique to derive the approximate Schrödinger equation solutions for energy eigenvalues with extended Cornell potential. Consideration was given to two particular instances that lead to Cornell and Coulomb potentials. We use the current findings to determine the masses of heavy mesons like charmonium and

bottomonium as well as the heavy-light mesons such as bottom-charm and charm-strange for various quantum states. We noticed that the mass spectra of the meson systems reported in this current work are also consistent with those of other researchers and are enhanced. This research could be expanded to study the thermodynamic properties of the mesons.

Appendix A: Some Useful Standard Integrals

$$\int_{r_a}^{r_b} \frac{1}{\sqrt{(r-r_a)(r_b-r)}} dr = \pi \quad (\text{A1})$$

$$\int_{r_a}^{r_b} \frac{1}{(a+br)\sqrt{(r-r_a)(r_b-r)}} dr = \frac{\pi}{\sqrt{(a+br_b)(a+br_a)}} \quad (\text{A2})$$

$$\int_{r_a}^{r_b} \frac{1}{r\sqrt{(r-r_a)(r_b-r)}} dr = \frac{\pi}{2}(r_a+r_b) - \pi\sqrt{r_a r_b} \quad (\text{A3})$$

$$\int_{r_a}^{r_b} \frac{1}{r\sqrt{(r-r_a)(r_b-r)}} dr = \frac{\pi}{\sqrt{r_a r_b}} \quad (\text{A4})$$

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ТЕОРЕТИЧНЕ ДОСЛІДЖЕННЯ СПЕКТРУ МЕЗОНІВ ЗА МЕТОДИКОЮ ПРАВИЛА ТОЧНОГО КВАНТУВАННЯ

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Шляхом аналітичного розв'язання радіального рівняння Шредінґера за допомогою техніки точного правила квантування були отримані власні значення енергії з розширеним потенціалом Корнелла. Потім його використовували для обчислення мас-спектрів важких мезонів, таких як чармоній ($c\bar{c}$) і боттоній ($b\bar{b}$), а також важких і легких мезонів, таких як bottom-charm ($b\bar{c}$) і charm-strange ($c\bar{s}$) для різних квантових станів. Були взяті до уваги, два виняткових випадки, такі як потенціали Кулона та Корнелла, коли деякі з параметрів потенціалу були встановлені на нуль. Поточний потенціал забезпечує хороші результати в порівнянні з експериментальними даними та роботою інших дослідників з максимальною похибкою у 0.0065 GeV .

Ключові слова: потенціал Корнелла; рівняння Шредінґера; Правило точного квантування; Мезони