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THE INFLUENCE OF DEFORMATION PHASE-SPACE ON SPECTRA OF HEAVY QUARKONIA IN IMPROVED ENERGY POTENTIAL AT FINITE TEMPERATURE MODEL OF SHRODINGER EQUATION VIA THE GENERALIZED BOPP'S SHIFT METHOD AND STANDARD PERTURBATION THEORY†

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In this work, we obtain solutions of the deformed Schrödinger equation (DSE) with improved internal energy potential at a finite temperature model in a 3-dimensional nonrelativistic noncommutative phase-space (3D-NRNCPS) symmetries framework, using the generalized Bopp's shift method in the case of perturbed nonrelativistic quantum chromodynamics (pNRQCD). The modified bound state energy spectra are obtained for the heavy quarkonium system such as charmonium $c\overline{c}$ and bottomonium $b\overline{b}$ at finite temperature. It is found that the perturbative solutions of the discrete spectrum are sensible to the discreet atomic quantum numbers (j, l, s, m) of the $Q\overline{Q}$ $(Q = c, b)$ state, the parameters of internal energy potential $(T, \alpha_s(T), m_D(T), \beta, c)$, which are the Debye screening mass $m_D(T)$, the running coupling constant $\alpha_s(T)$, the critical temperature β, the free parameter c in addition to noncommutativity parameters $(\Theta,\overline{\theta})$. The new Hamiltonian operator in 3D-NRNCPS symmetries is composed of the corresponding operator in commutative phase-space and three additive parts for spin-orbit interaction, the new magnetic interaction, and the rotational Fermi-term. The obtained energy eigenvalues are applied to obtain the mass spectra of heavy quarkonium systems ($c\overline{c}$ and $b\overline{b}$). The total complete degeneracy of the new energy levels of the improved internal energy potential changed to become equal to the new value $3n^2$ in 3D-NRNCPS symmetries instead of the value n^2 in the symmetries of 3D-NRQM. Our non-relativistic results obtained from DSE will possibly be compared with the Dirac equation in high-energy physics.
Keywords: *Schrödinger equation; noncommutative phase-space; internal energy potential at finite temperature; Bopp shift method; heavy quarkonium systems*
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1. INTRODUCTION

It is well known that the ordinary Schrödinger equation (SE) describes the dynamics of quantum systems at low energy without considering the temperature effect. Recently, the finite temperature SE allows us to study quantum systems such as superconductivity mechanisms and Bose-Einstein condensates at an arbitrary temperature, and when the temperature is equal to zero, it becomes identical to the SE [1]. Very recently, many authors have studied the finite-temperature SE for hot quark-gluon plasma, heavy quarkonia in quark-gluon plasma, (electron and proton systems), and so on [2-5]. The problem of calculating the energy spectra of the SE with various types of potentials such as the internal energy potential and the Cornell potential at finite temperature has been attracting interest in recent years [2-8]. Abu-Shady has studied heavy-quarkonium mesons (HLM) using an internal energy potential and obtained wave function and energy spectra by solving SE using AEIM when the finite temperature is included [7]. The main objective is to develop the research article [7] and expand it to the large symmetry known by nonrelativistic noncommutative phase-space (NRNCPS) to achieve a more accurate physical vision so that this study becomes valid in the field of nanotechnology. Noncommutative quantum mechanics is an old idea that has been extensively discussed in the literature. It should be noted that noncommutativity (NC) was first introduced by Heisenberg in 1930 [9] and then by Snyder in 1947 [10]. It has appeared since the beginning of ordinary quantum mechanics. There has been a growing interest in this field since the discovery of string theory and the modified uncertainty principle. In addition, the NC idea is suggested as a result of the production of quantum gravity. It would provide a natural background for finding a suitable solution for a possible regularization of QFT [11-23]. During the past three decades, the NC theory has been the focus of extensive investigation and has produced a very interesting new class of quantum field theories with intriguing and occasionally unexpected properties [24]. Thus, the topographical properties of the NC space-space and phase-phase have a clear effect on the various physical properties of quantum systems and this has been a very interesting field in many fields of physics. The idea of noncommutativity has been studied in many articles, such as [24-36]. On the other hand, we explore the possibility of creating new applications and more profound interpretations in the sub-atomics and nanoscales using a new version of the improved internal energy potential, which has the following form:

$$V_{ip}(r) = F_1(r,T) - T\frac{\partial F_1(r,T)}{\partial T} \rightarrow \; V_{ip}(\hat{r}) = V_{ip}(r) + \left(\frac{A_1}{2r} - \frac{A_2}{2r^2} + \frac{D_3}{2r^3} - \frac{A_4}{2}r - \frac{A_3}{2}\right) exp(-m_D(T)r)\,\vec{\mathbf{L}}\vec{\mathbf{\Theta}} \quad (1)$$

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We refer to this term $\overrightarrow{L}\overrightarrow{\Theta}$ and the parameters $(\boldsymbol{A_1}, \boldsymbol{A_1}, \boldsymbol{A_1}, \boldsymbol{A_1}, \boldsymbol{D_3}, \boldsymbol{m_D}(\boldsymbol{T}))$ in the materials and methods section**.** The new structure of 3D-NRNCPS is based on new canonical commutation relations in both Schrödinger SP and Heisenberg HP, respectively, as follows (Throughout this paper, the natural units $\boldsymbol{c} = \boldsymbol{\hbar} = \boldsymbol{1}$ will be used) [36-46]:

$$[x_\mu, p_\nu] = i\delta_{\mu\nu}\hbar \rightarrow \; [\hat{x}_\mu \overset{*}{,} \hat{p}_\nu] = [\hat{x}_\mu(t) \overset{*}{,} \hat{p}_\nu(t)] = i\delta_{\mu\nu}\hbar_{eff} \Rightarrow \; |\Delta\hat{x}_\mu \Delta\hat{p}_\nu| \geq \frac{\hbar_{eff}\delta_{\mu\nu}}{2}, \tag{2.1}$$

and

$$[x_\mu, x_\nu] = 0 \rightarrow \; [\hat{x}_\mu \overset{*}{,} \hat{x}_\nu] = [\hat{x}_\mu(t) \overset{*}{,} \hat{x}_\nu(t)] = i\theta_{\mu\nu} \Rightarrow \; |\Delta\hat{x}_\mu \Delta\hat{x}_\nu| \geq \left|\frac{\theta_{\mu\nu}}{2}\right|, \tag{2.2}$$

and

$$[p_\mu, p_\nu] = 0 \rightarrow \; [\hat{p}_\mu \overset{*}{,} \hat{p}_\nu] = [\hat{p}_\mu(t) \overset{*}{,} \hat{p}_\nu(t)] = i\overline{\theta}_{\mu\nu} \; \Rightarrow \; |\Delta\hat{p}_\mu \Delta\hat{p}_\nu| \geq \left|\frac{\overline{\theta}_{\mu\nu}}{2}\right|, \tag{2.3}$$

the indices $\mu, \nu \equiv \overline{1,3}$, $\hbar_{eff}$ equal $\hbar\left(1 + \frac{\theta\overline{\theta}}{4}\right)$ denote the effective Planck constant. This means that the principle of uncertainty of Heisenberg is generalized to include another two new uncertainties related to the positions $(\hat{x}_\mu, \hat{x}_\nu)$ and the momenta $(\hat{p}_\mu, \hat{p}_\nu)$, in addition to the ordinary uncertainty $(\hat{x}_\mu, \hat{p}_\nu)$. The non-commutativity of the phase-space is based on the deformed Heisenberg–Weyl algebra, which is represented by the above commutation relations. here $\theta_{\mu\nu}$ and $\overline{\theta}_{\mu\nu}$ are invertible antisymmetric real constant (3 × 3) matrices which satisfied $\theta_{\mu\nu} = \varepsilon_{\mu\nu}\theta$ and $\overline{\theta}_{\mu\nu} = \varepsilon_{\mu\nu}\overline{\theta}$, with $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$ and $\varepsilon_{\mu\mu} = 0$, here $(\theta, \overline{\theta})$ are interpreted as being new constants in the quantum theory. The very small two parameters ($\theta^{\mu\nu}$ and $\overline{\theta}^{\mu\nu}$) (compared to the energy) are elements of two antisymmetric real matrixes, parameters of non-commutativity, and $(*)$ denote the Weyl Moyal star product, which is generalized between two arbitrary functions $(f, g)\,(x, p)$ to the new form $\hat{f}(\hat{x}, \hat{p})\hat{g}(\hat{x}, \hat{p}) \equiv (f * g)(x, p)$ in the 3D-NRNCPS symmetries as follows [47-55]:

$$(fg)\,(x, p) \rightarrow (f * g)(x, p) = \left(fg - \frac{i}{2}\theta^{\mu\nu}\partial_\mu^x f \partial_\nu^x g - \frac{i}{2}\overline{\theta}^{\mu\nu}\partial_\mu^p f \partial_\nu^p g\right)(x, p). \tag{3}$$

The second and third terms in the above equation are the present effects of (space-space) and (phase-phase) noncommutativity properties. However, the new operators $\hat{\xi}(t) = (\hat{x}_\mu \vee \hat{p}_\mu)(t)$ in HP are depending on the corresponding new operators $\hat{\xi} = \hat{x}_\mu \vee \hat{p}_\nu$ in SP from the following projection relations:

$$\chi(t) = exp\left(\frac{i}{\hbar}\hat{H}_{ip}(t - t_0)\right)\chi\, exp\left(-\frac{i}{\hbar}\hat{H}_{ip}(t - t_0)\right) \Rightarrow \; \hat{\chi}(t) = exp\left(\frac{i}{\hbar_{eff}}\hat{H}_{nc}^{ip}(t - t_0)\right) * \hat{\chi} * exp\left(-\frac{i}{\hbar_{eff}}\hat{H}_{nc}^{ip}(t - t_0)\right), \tag{4}$$

Here $\chi = x_\mu \vee p_\nu$ and$\chi(t) = (x_\mu \vee p_\nu)(t)$. The dynamics of the new systems $\frac{d\hat{\chi}(t)}{\mathrm{dt}}$ are described by the following motion equations in 3D-NRNCPS symmetries:

$$\frac{d\chi(t)}{\mathrm{dt}} = \frac{i}{\hbar}[\chi(t), \hat{H}_{ip}] + \frac{\partial\xi(t)}{\partial t} \Rightarrow \; \frac{d\hat{\chi}(t)}{\mathrm{dt}} = \frac{i}{\hbar_{eff}}[\hat{\xi}(t) \overset{*}{,} \hat{H}_{nc}^{ip}] + \frac{\partial\hat{\chi}(t)}{\partial t}. \tag{5}$$

The two operators ($\hat{H}_{nc}^{ip}$and $\hat{H}_{ip}$) are present as the quantum Hamiltonian operators for the internal energy potential and the improved internal energy potential in the 3D-NRNCPS symmetries and their extension. This paper consists of five sections and the organization scheme is given as follows: In the next section, the theory part, we briefly review the SE with internal energy potential at finite temperature based on refs. [7-8]. Section 3 is devoted to studying the DSE by applying the generalized Bopp's shift method and obtaining the improved internal energy potential and the modified spin-orbit operator at finite temperature. Then, we applied the standard perturbation theory to find the quantum spectrum of the ground state, the first excited state, and the $(n, l, m)^{th}$excited state produced by the effects of modified spin-orbit and newly modified Zeeman interactions. In the fourth section, a discussion of the main results is presented in addition to determining the new formula for determining the mass spectra of the quarkonium system in the 3D-NRNCPS symmetries framework. Finally, in the last section, a summary and conclusions are presented.

2. THEORY

2.1. Overview of the eigenfunctions and energy eigenvalues for the internal energy potential at finite temperature in the 3D-NRNCPS symmetries framework

As already mentioned, our objective was to obtain the spectrum of the improved internal energy potential at finite temperatures. To achieve this goal, it is useful to summarize the time-independent Schrödinger equation for the internal energy potential at a finite temperature [7-8]:

$$V_{ip}(r) = F_1(r, T) - T\frac{\partial F_1(r, T)}{\partial T}, \tag{6}$$

where $F_1(r, T)$ is determined from:

$$F_1(r, T) = \left(cr - \frac{4}{3}\frac{\alpha_s(T)}{r}\right) exp(-m_D(T)r),$$

and $m_D(T)$ is the Debye screening mass, c is a free parameter, the running coupling constant

$$\alpha_s(T) = \frac{2\prod}{11-2/3n_f\, ln(T/\beta T_c)},$$

Here n_f, T_c and β are the number of quark flavors, the critical temperature, and $(0{,}104 \pm 0.009)$, respectively, the relative spatial coordinate between the two quarks is r. By substituting $F_1(r,T)$ into Eq. (6), we obtain the internal energy potential that satisfies the following equation at a finite temperature [7-8]:

$$V_{ip}(r) = \left(D_2 + \frac{D_3}{r} + D_4 r + D_5 r^2\right) exp(-m_D(T)r), \tag{7}$$

where

$$d_2 = -2\mu D_2 = \frac{8\mu T}{3}\alpha_s(T)\frac{dm_D(T)}{dT},$$

$$d_3 = -2\mu D_3 = \frac{8\mu}{3}\alpha_s(T) + \frac{16\mu\prod(11-2/3n_f)}{3[11-2/3n_f\, ln(T/\beta T_c)]^2},$$

$$d_4 = -2\mu D_4 = -2\mu c\ ,$$

and

$$d_5 = -2\mu D_5 = 2\mu c T\frac{dm_D(T)}{dT}.$$

If we insert this potential into the Schrödinger equation, the radial part function $U_{nl}(r) = \frac{R_{nl}(r)}{r}$ is given as:

$$\frac{d^2U_{nl}(r)}{dr^2} + \frac{2}{r}\frac{dU_{nl}(r)}{dr} + 2\mu\left\{E - \left(D_2 + \frac{D_3}{r} + D_4 r + D_5 r^2\right) exp(-m_D(T)r) - \frac{l(l+1)}{2\mu r^2}\right\} U_{nl}(r) = 0, \tag{8}$$

and

$$\frac{d^2R_{nl}(r)}{dr^2} + 2\mu\left[E - \left(D_2 + \frac{D_3}{r} + D_4 r + D_5 r^2\right) e\,xp(-m_D(T)r) - \frac{l(l+1)}{2\mu r^2}\right] R_{nl}(r) = 0. \tag{9}$$

The reduced mass μ for the quarkonium particle for example ($c\bar{c}$and$b\bar{b}$) equal $\frac{m_q m_{\bar{q}}}{m_q + m_{\bar{q}}}$. The complete wave function $\Psi_{nlm}(r,\theta,\phi) = \frac{R_{nl}(r)}{r} Y_l^m(\theta,\phi)$ is given by [7]:

$$\Psi_{nlm}(r,\theta,\phi) = N_{nl}\prod_{i=1}^{n}\ (r-\alpha_i)\, r^{\delta-1}\, exp(-1/2\alpha r^2 - \beta r)\, Y_l^m(\theta,\phi). \tag{10}$$

Also, the energy E_{nl} of the potential in Eq. (7) is determined from the following equation:

$$E_{nl} = \frac{1}{2\mu}[\alpha(1 + 2(\delta + n) - \beta^2 - d_2 + m_D(T)d_3)], \tag{11}$$

where N_{nl} is a normalizing constant, n is a natural number accounting for the radial excitation, while l is a non-negative integer number that represents the orbital angular momentum,

$$\left.\begin{aligned} \alpha &= \sqrt{d_4 m_D - d_5 - \frac{d_2}{2}{m_D}^2} \\ \beta &= \frac{d_2 m_D - d_4 - d_3 m^2{}_D}{2\sqrt{d4 m_D - d_5 - \frac{d_2}{2}{m_D}^2}} \\ \delta &= \frac{1}{2}\left(1 \pm \sqrt{1 + 4(l+1/2)^2 - 1/4}\right) \end{aligned}\right\}$$

3. MATERIALS AND METHODS

3.1. DSE solution for an improved new internal energy potential at finite temperature in pNRQCD

In this subsection, we shall give an overview of a brief preliminary investigation of the improved internal energy potential in 3D-NRNCPS symmetries. To perform this task, the physical form of DSE, it is necessary to replace the ordinary three-dimensional Hamiltonian operators $\hat{H}_{ip}(x_\mu, p_\mu)$, the complex wave function $\Psi\left(\vec{r}\right)$, and energy E_{nl} with the new three Hamiltonian operators$\hat{H}_{nc}^{ip}(\hat{x}_\mu, \hat{p}_\mu)$, the new complex wave function$\Psi\left(\vec{\hat{r}}\right)$, and new values$E_{nc}^{ip}$, respectively. In addition to replacing the ordinary product with the Weyl-Moyal star product, which allows us to construct the DSE in the 3D-NRNCPS symmetries framework as [55-60]:

$$\frac{d^2R_{nl}(r)}{dr^2} + 2\mu\left[E_{nl} - \left(D_2 + \frac{D_3}{r} + D_4 r + D_5 r^2\right) e\,xp(-m_D(T)r) - \frac{l(l+1)}{2\mu r^2}\right] * R_{nl}(r) = 0. \tag{12}$$

Bopp's shift method [70-72] has been successfully applied to relativistic and nonrelativistic noncommutative quantum mechanical problems using the modified Dirac equation (MDE) [73-81], the modified Klein-Gordon equation (MKGE) [36-38, 48-61] and DSE [46, 64-69]. This method has produced very promising results for several situations of physical and chemical interest. The method reduces MDE, MKGE, and DSE to the Dirac equation, Klein-Gordon equation, and Schrödinger equation, respectively, under two simultaneous translations in space and phase. It is based on the following new commutators [46, 64-72]:

$$\begin{cases}[\hat{x}_\mu(t),\hat{p}_\nu(t)]=i\delta_{\mu\nu}\hbar_{eff}\\ [\hat{x}_\mu,\hat{x}_\nu]=[\hat{x}_\mu(t),\hat{x}_\nu(t)]=i\theta_{\mu\nu}\\ [\hat{p}_\mu,\hat{p}_\nu]=[\hat{p}_\mu(t),\hat{p}_\nu(t)]=i\overline{\theta}_{\mu\nu}\end{cases}. \tag{13}$$

The new generalized positions and momentum coordinates $(\hat{x}_\mu,\hat{p}_\nu)$ in 3D-NRNCPS are defined in terms of the commutative counterparts (x_μ,p_ν) in ordinary quantum mechanics via, respectively [46, 55-60]:

$$(x_\mu,p_\nu)\Rightarrow(\hat{x}_\mu,\hat{p}_\nu)=\left(x_\mu-\frac{\theta_{\mu\nu}}{2}p_\nu,p_\mu+\frac{\overline{\theta}_{\mu\nu}}{2}x_\nu\right). \tag{14}$$

The above equation allows us to obtain the two operators $(\hat{r}^2,\hat{p}^2)$ in the 3D-NRNCPS symmetries framework [28-31]:

$$(r^2,p^2)\Rightarrow(\hat{r}^2,\hat{p}^2)=\left(r^2-\vec{\mathbf{L}}\vec{\mathbf{\Theta}},p^2+\vec{\mathbf{L}}\vec{\overline{\mathbf{\theta}}}\right). \tag{15}$$

The two couplings $\vec{\mathbf{L}}\vec{\mathbf{\Theta}}$ and $\vec{\mathbf{L}}\vec{\overline{\mathbf{\theta}}}$ are $(L_x\Theta_{12}+L_y\Theta_{23}+L_z\Theta_{13})$ and $(L_x\overline{\theta}_{12}+L_y\overline{\theta}_{23}+L_z\overline{\theta}_{13})$, respectively, and ($L_x$, L_y andL_z) are the three components of the angular momentum operator$\vec{L}$ while $\Theta_{\mu\nu}$ equal $\theta_{\mu\nu}/2$. Thus, the reduced Schrödinger equation (without star product) can be written as:

$$\frac{d^2R_{nl}(r)}{dr^2}+2\mu\left(E-V_{ip}^{eff}(\hat{r})\right)R_{nl}(r)=0. \tag{16}$$

The new operator of Hamiltonian$H_{nc}^{ip}(\hat{x}_\mu,\hat{p}_\nu)$ can be expressed as:

$$H_{nc}^{ip}(\hat{x}_\mu,\hat{p}_\mu)=\frac{\hat{p}^2}{2\mu}+V_{ip}(\hat{r}), \tag{17}$$

here $\hat{r}$ equal $\sqrt{\left(x_\mu-\frac{\theta_{\mu\nu}}{2}p_\nu\right)\left(x^\mu-\frac{\theta^{\mu\alpha}}{2}p_\alpha\right)}$. The effectively improved internal energy potential $V_{ip}^{eff}(\hat{r})$ can be expressed in 3D-NRNCPS symmetries:

$$V_{ip}^{eff}(\hat{r})=\left(D_2+\frac{D_3}{\hat{r}}+D_4\hat{r}+D_5\hat{r}^2\right)exp(-m_D(T)\hat{r})+\frac{l(l+1)}{2\mu\hat{r}^2}. \tag{18}$$

Again, apply Eq. (15) to find the three terms ($\frac{D_3}{\hat{r}}$,$D_4\hat{r}$, $D_5\hat{r}^2$and$exp(-m_D(T)\hat{r})$), which will be used to determine the effective improved internal energy potential $V_{ip}^{eff}(\hat{r})$, as follows:

$$\frac{D_3}{r}\rightarrow\frac{D_3}{\hat{r}}=\frac{D_3}{r}+\frac{D_3}{2r^3}\vec{\mathbf{L}}\vec{\mathbf{\Theta}}+O(\Theta^2), \tag{19.1}$$

$$D_4r\rightarrow D_4\hat{r}=D_4r-\frac{D_4}{2r}\vec{\mathbf{L}}\vec{\mathbf{\Theta}}+O(\Theta^2), \tag{19.2}$$

$$D_5r^2\rightarrow D_5\hat{r}^2=D_5r^2-D_5\vec{\mathbf{L}}\vec{\mathbf{\Theta}}+O(\Theta^2), \tag{19.3}$$

$$\frac{l(l+1)}{r^2}\rightarrow\frac{l(l+1)}{\hat{r}^2}=\frac{l(l+1)}{r^2}+\frac{l(l+1)}{r^4}\vec{\mathbf{L}}\vec{\mathbf{\Theta}}+O(\Theta^2), \tag{19.4}$$

and

$$exp(-m_D(T)r)\rightarrow exp(-m_D(T)\hat{r})=exp(-m_D(T)r)-\frac{m_D(T)}{2r}\vec{\mathbf{L}}\vec{\mathbf{\Theta}}\,exp(-m_D(T)r). \tag{19.5}$$

Thus, we have the following.

$$\frac{D_3}{r}exp(-m_D(T)r)\rightarrow\frac{D_3}{\hat{r}}exp(-m_D(T)\hat{r})=\left(\frac{D_3}{r}+\frac{D_3}{2r^3}\vec{L}\vec{\Theta}\right)\left(1-\frac{m_D(T)}{2r}\vec{\mathbf{L}}\vec{\mathbf{\Theta}}\right)exp(-m_D(T)r), \tag{20.1}$$

$$D_4r\,exp(-m_D(T)r)\rightarrow D_4\hat{r}\,exp(-m_D(T)\hat{r})=\left(D_4r-\frac{D_4}{2r}\vec{L}\vec{\Theta}\right)\left(1-\frac{m_D(T)}{2r}\vec{\mathbf{L}}\vec{\mathbf{\Theta}}\right)exp(-m_D(T)r), \tag{20.2}$$

$$D_5r^2\,exp(-m_D(T)r)\rightarrow D_5\hat{r}^2\,exp(-m_D(T)\hat{r})=\left(D_5r^2-D_5\vec{L}\vec{\Theta}\right)\left(1-\frac{m_D(T)}{2r}\vec{\mathbf{L}}\vec{\mathbf{\Theta}}\right)exp(-m_D(T)r), \tag{20.3}$$

and

$$D_1\,exp(-m_D(T)r)\rightarrow D_1\,exp(-m_D(T)\hat{r})=D_1\,exp(-m_D(T)r)-D_1\frac{m_D(T)\,exp(-m_D(T)r)}{2r}\vec{\mathbf{L}}\vec{\mathbf{\Theta}}. \tag{20.4}$$

which gives immediately at the first order of the infinitesimal vector parameter $\vec{\mathbf{\Theta}}$ as follows:

$$\frac{D_3}{\hat{r}}exp(-m_D(T)\hat{r})=\frac{D_3}{r}exp(-m_D(T)r)-\left(\frac{m_D(T)D_3}{2r^2}exp(-m_D(T)r)+\frac{D_3\,exp(-m_D(T)r)}{2r^3}\right)\vec{\mathbf{L}}\vec{\mathbf{\Theta}}+O(\Theta^2), \tag{21.1}$$

$$D_4\hat{r}\,exp(-m_D(T)\hat{r})=D_4r\,exp(-m_D(T)r)-\left(\frac{m_D(T)D_4}{2}exp(-m_D(T)r)-\frac{D_4\,exp(-m_D(T)r)}{2r}\right)\vec{\mathbf{L}}\vec{\mathbf{\Theta}}+O(\Theta^2), \tag{21.2}$$

$$D_4\hat{r}\,exp(-m_D(T)\hat{r}) = D_4 r\,exp(-m_D(T)r) - \left(\frac{m_D(T)D_4}{2}exp(-m_D(T)r) - \frac{D_4\,exp(-m_D(T)r)}{2r}\right)\vec{\mathbf{L}}\vec{\mathbf{\Theta}} + O(\Theta^2), \quad (213)$$

and

$$D_1\,exp(-m_D(T)\hat{r}) = D_1\,exp(-m_D(T)r) - D_1\frac{m_D(T)\,exp(-m_D(T)r)}{2r}\vec{\mathbf{L}}\vec{\mathbf{\Theta}} + O(\Theta^2). \quad (21.4)$$

Substituting, Eq. (21) in Eq. (18), gives the new improved internal energy potential, we obtain the effective improved internal energy potential $V_{ip}^{eff}(\hat{r})$ in 3D-NRNCPS symmetries as follows:

$$V_{ip}^{eff}(\hat{r}) = V_{ip}(r) + \frac{l(l+1)}{2\mu r^2} + \left[\left(\frac{A_1}{2r} - \frac{A_2}{2r^2} + \frac{D_3}{2r^3} - \frac{A_4}{2}r - \frac{A_3}{2}\right)exp(-m_D(T)r) + \frac{l(l+1)}{r^4}\right]\vec{\mathbf{L}}\vec{\mathbf{\Theta}} + O(\Theta^2), \quad (22)$$

with

$$\left.\begin{aligned} A_1 &= m_D(T)D_5 - D_1 m_D(T) - D_4,\\ A_2 &= m_D(T)D_3,\\ A_3 &= m_D(T)D_4,\\ A_4 &= m_D(T)D_5. \end{aligned}\right\}.$$

By making the above substitution equation in Eq. (17), we find the global our working new modified Hamiltonian operator $H_{nc}^{ip}(\hat{r})$ in 3D-NRNCPS symmetries:

$$H_{nc}^{ip}(\hat{r}) = H_{ip}(x_\mu, p_\nu) + H_{pert}^{ip}(r,\Theta,\bar{\theta}), \quad (23)$$

here$H_{ip}(x_\mu, p_\nu)$ is just the ordinary Hamiltonian operator with internal energy potential in 3D-NR quantum mechanics symmetries:

$$H_{ip}(x_\mu, p_\nu) = \frac{p^2}{2\mu} + \left(D_2 + \frac{D_3}{r} + D_4 r + D_5 r^2\right)exp(-m_D(T)r), \quad (24)$$

while the rest part $H_{pert}^{ip}(r,\Theta,\bar{\theta})$, which we called the perturbative Hamiltonian operator, is proportional to two infinitesimals couplings $\vec{\mathbf{L}}\vec{\mathbf{\Theta}}$ and $\vec{\mathbf{L}}\vec{\bar{\mathbf{\theta}}}$:

$$H_{pert}^{ip}(r,\Theta,\bar{\theta}) = \left[f(r,A_i,D_3) + \frac{l(l+1)}{r^4}\right]\vec{\mathbf{L}}\vec{\mathbf{\Theta}} + \frac{\vec{\mathbf{L}}\vec{\bar{\mathbf{\theta}}}}{2\mu}, \quad (25)$$

here $f(r,A_i,D_3)$ is determined by:

$$f(r,A_i,D_3) = \left(\frac{A_1}{2r} - \frac{A_2}{2r^2} + \frac{D_3}{2r^3} - \frac{A_4}{2}r - \frac{A_3}{2}\right)exp(-m_D(T)r).$$

Thus, we can consider $\boldsymbol{H_{pert}^{ip}(r)}$ it as a perturbation term compared with the principal Hamiltonian operator $\boldsymbol{H_{ip}(x_\mu, p_\mu)}$ in 3D-NRPS symmetries.

3.2. The exact modified spin-orbit operator for heavy-quarkonium systems with improved internal energy potential in the pNRQCD system:

In this subsection, we apply the same strategy that we have seen exclusively in some of our published scientific works [46, 55-60, 73-76]. Under such a particular choice, one can easily reproduce both couplings ($\vec{\mathbf{L}}\vec{\mathbf{\Theta}}$ and $\vec{\mathbf{L}}\vec{\bar{\mathbf{\theta}}}$) to the new physical forms ($g_s\Theta\vec{\mathbf{L}}\vec{\mathbf{S}}$ and $g_s\bar{\theta}\vec{\mathbf{L}}\vec{\mathbf{S}}$), respectively. Thus, the perturbative Hamiltonian operator $H_{pert}^{ip}(r,\Theta,\bar{\theta})$ for the heavy quarkonium systems will be transformed into a modified spin-orbit operator $H_{so}^{ip}(r,\Theta,\bar{\theta})$, under the improved internal energy potential at a finite temperature as follows:

$$H_{so}^{ip}(r,\Theta,\bar{\theta}) \equiv g_s\left\{f(r,A_i,D_3)\Theta + \frac{l(l+1)}{r^4}\Theta + \frac{\bar{\theta}}{2\mu}\right\}\vec{\mathbf{L}}\vec{\mathbf{S}}, \quad (26)$$

here Θ and $\bar{\theta}$ are equals $\sqrt{{\Theta_{12}}^2 + {\Theta_{23}}^2 + {\Theta_{13}}^2}$ and $\sqrt{{\bar{\theta}_{12}}^2 + {\bar{\theta}_{23}}^2 + {\bar{\theta}_{13}}^2}$, respectively, and$g_s$ is a new constant, which plays the role of strong coupling in quantum chromodynamics or QCD theory, we have chosen two vectors ($\vec{\mathbf{\Theta}}$ and $\vec{\bar{\mathbf{\theta}}}$) parallel to the spin-s of the heavy quarkonium system. Furthermore, the above perturbative terms $H_{pert}^{ip}(r)$ can be rewritten to the following new form:

$$H_{so}^{ip}(r,\Theta,\bar{\theta}) = \frac{g_s}{2}\left[f(r,A_i,D_3)\Theta + \frac{l(l+1)}{r^4}\Theta + \frac{\bar{\theta}}{2\mu}\right]G^2, \quad (27)$$

where $G^2 \equiv \vec{\mathbf{J}}^2 - \vec{\mathbf{L}}^2 - \vec{\mathbf{S}}^2$ while $\vec{\mathbf{J}}$ and $\vec{\mathbf{S}}$ are the defined operators of the total angular momentum and spin of quarkonium systems. The operator $H_{so}^{iq}(r,\Theta,\bar{\theta})$ traduces the coupling between spin-orbit interaction $\vec{\mathbf{L}}\vec{\mathbf{S}}$. The set ($H_{so}^{ip}(r,\Theta,\bar{\theta})$, J^2, L^2,

$S^2$and J_z) forms a complete set of conserved physics quantities. For spin-1, the eigenvalues of the spin-orbit coupling operator are

$$k(l) \equiv \frac{1}{2}(j(j+1) - l(l+1) - 2)$$

corresponding $j = l + 1$ (spin great), $j = l$ (spin middle), and $j = l - 1$ (spin little), respectively, then, one can form a diagonal (3×3) matrix for the improved internal energy potential in 3D-NRNCPS symmetries, with diagonal elements$\left(H_{so}^{ip}\right)_{11}$, $\left(H_{so}^{ip}\right)_{22}$ and $\left(H_{so}^{ip}\right)_{33}$ are given by:

$$\left(H_{so}^{ip}\right)_{11} = g_s k_1(l)\left\{f(r, A_i, D_3)\Theta + \frac{l(l+1)}{r^4}\Theta + \frac{\bar{\theta}}{2\mu}\right\} \text{ if } j = l + 1, \tag{28.1}$$

$$\left(H_{so}^{ip}\right)_{22} = g_s k_2(l)\left\{f(r, A_i, D_3)\Theta + \frac{l(l+1)}{r^4}\Theta + \frac{\bar{\theta}}{2\mu}\right\} \text{ if } j = l, \tag{28.2}$$

and

$$\left(H_{so}^{ip}\right)_{33} = g_s k_3(l)\left\{f(r, A_i, D_3)\Theta + \frac{l(l+1)}{r^4}\Theta + \frac{\bar{\theta}}{2\mu}\right\} \text{ if } j = l - 1. \tag{28.3}$$

Here $\left(k_1(l), k_2(l), k_3(l)\right)$ are equals $\frac{1}{2}(l, -2, -2l-2)$, respectively, and j is the total quantum number. The non-null diagonal elements ($\left(H_{so}^{ip}\right)_{11}$, $\left(H_{so}^{ip}\right)_{22}$and $\left(H_{so}^{ip}\right)_{33}$) for the modified Hamiltonian operator $H_{nc}^{ip}(\hat{r})$ will change the energy values $E_{nl}$by creating three new values:

$$E_g^{ip} = \langle\Psi(r,\theta,\phi)\left|\left(H_{so}^{ip}\right)_{11}\right|\Psi(r,\theta,\phi)\rangle, \tag{29.1}$$

$$E_m^{ip} = \langle\Psi(r,\theta,\phi)\left|\left(H_{so}^{ip}\right)_{22}\right|\Psi(r,\theta,\phi)\rangle, \tag{29.2}$$

and

$$E_l^{ip} = \langle\Psi(r,\theta,\phi)\left|\left(H_{so}^{ip}\right)_{33}\right|\Psi(r,\theta,\phi)\rangle. \tag{29.3}$$

We will see them in detail in the next subsection. After profound calculation, one can show that the new radial function$R_{nl}(r)$satisfies the following differential equation for the improved internal energy potential:

$$\frac{d^2R_{nl}(r)}{dr^2} + 2\mu\left[E_{nl} - V_{ip}(r) - \frac{l(l+1)}{2\mu r^2} - [f(r, A_i, D_3) + l(l+1)r^{-4}]\vec{\mathbf{L}}\vec{\mathbf{\Theta}} - \frac{l(l+1)}{r^4}\vec{\mathbf{L}}\vec{\mathbf{\Theta}} - \frac{\vec{\mathbf{L}}\vec{\bar{\mathbf{\theta}}}}{2\mu}\right]R_{nl}(r) = 0. \tag{30}$$

Through our observation of the expression of $H_{pert}^{ip}(r, \Theta, \bar{\theta})$, which appears in equation (25), we see it as proportionate to the two infinitesimals parameters (Θ and$\bar{\theta}$), thus, in what follows, we proceed to solve the modified radial part of the DSE that is, equation (30) by applying standard perturbation theory to find an acceptable solution at the first order of two parameters Θ and $\bar{\theta}$. The proposed solutions for DSE under improved internal energy potential include energy corrections, which are produced automatically from two principal physical phonemes', the first one is the effect of modified spin-orbit interaction and the second is the modified Zeeman effect while the stark effect can appear in the linear part of improved internal energy potential at finite temperature model.

3.3. The exact modified spin-orbit spectrum for a heavy-quarkonium system under improved internal energy potential in pNRQCD

The purpose here is to give a complete prescription for determining the energy levels of the ground state, the first excited state, and $(n, l, m)^{th}$ the excited state, of heavy quarkonium systems. We first find the corrections ($E_{so}^{gip}(k_1(l), j, l, n)$, $E_{so}^{mip}(k_2(l), j, l, n)$ and $E_{so}^{lip}(k_3(l), j, l, n)$) for heavy quarkonium systems such as (charmonium and bottomonium) mesons that have the quark and antiquark flavor under a new improved internal energy potential at finite temperature, which have three polarities up and down $j = l + 1$ (spin great), $j = l$ (spin middle) and $j = l - 1$ (spin little), respectively, at the first order of two parameters (Θ and $\bar{\theta}$). Moreover, by applying the perturbative theory, in the case of perturbed non-relativistic quantum chromodynamics pNRQCD framework, we obtained the following results:

$$E_{so}^{gip} = g_s N_{nl}^2 k_1(l)\int_0^{+\infty}\left\{\prod_{i=1}^{n}(r-\alpha_i)\, r^{\delta-1}\right\}^2 exp(-\alpha r^2 - 2\beta r)\left(f(r, A_i, D_3)\Theta + \frac{l(l+1)}{r^4}\Theta + \frac{\bar{\theta}}{2\mu}\right)r^2dr, \tag{31.1}$$

$$E_{so}^{mip} = g_s N_{nl}^2 k_2(l)\int_0^{+\infty}\left\{\prod_{i=1}^{n}(r-\alpha_i)\, r^{\delta-1}\right\}^2 exp(-\alpha r^2 - 2\beta r)\left(f(r, A_i, D_3)\Theta + \frac{l(l+1)}{r^4}\Theta + \frac{\bar{\theta}}{2\mu}\right)r^2dr, \tag{31.2}$$

$$E_{so}^{lip} = g_s N_{nl}^2 k_3(l)\int_0^{+\infty}\left\{\prod_{i=1}^{n}(r-\alpha_i)\, r^{\delta-1}\right\}^2 exp(-\alpha r^2 - 2\beta r)\left(f(r, A_i, D_3)\Theta + \frac{l(l+1)}{r^4}\Theta + \frac{\bar{\theta}}{2\mu}\right)r^2dr. \tag{31.3}$$

We have used the standard identity:

$$\int Y_l^m(\theta,\phi)Y_{l'}^{m'}(\theta,\phi)\,sin(\theta)\,d\theta d\phi = \delta_{ll'}\delta_{mm'}$$

Now, we can rewrite the above equations to a simplified new form:

$$E_{so}^{mip}(k_1,j,n,l)=g_s N_{nl}^2 k_1(l)\left(\Theta\sum_{i=1}^{6}T_i^n+\frac{\bar{\theta}}{2\mu}\right),\tag{32.1}$$

$$E_{so}^{mip}(k_2,j,n,l)=g_s N_{nl}^2 k_2(l)\left(\Theta\sum_{i=1}^{6}T_i^n+\frac{\bar{\theta}}{2\mu}\right),\tag{32.2}$$

and

$$E_{so}^{lip}(k_3,j,n,l)=g_s N_{nl}^2 k_3(l)\left(\Theta\sum_{i=1}^{6}T_i^n+\frac{\bar{\theta}}{2\mu}\right).\tag{32.3}$$

Moreover, the expressions of the 6-factors$T_i^n\left(i=\overline{1,6}\right)$ are given by:

$$T_1^n=\frac{A_1}{2}\int_0^{+\infty}\left(\prod_{i=1}^{n}\;(r-\alpha_i)\,r^{\delta-1}\right)^2 exp\left(-\alpha r^2-\left(2\beta+m_D(T)\right)r\right)rdr,\tag{33.1}$$

$$T_2^n=-\frac{A_2}{2}\int_0^{+\infty}\left(\prod_{i=1}^{n}\;(r-\alpha_i)\,r^{\delta-1}\right)^2 exp\left(-\alpha r^2-\left(2\beta+m_D(T)\right)r\right)dr,\tag{33.2}$$

$$T_3^n=\frac{D_3}{2}\int_0^{+\infty}\left(\prod_{i=1}^{n}\;(r-\alpha_i)\,r^{\delta-1}\right)^2 exp\left(-\alpha r^2-\left(2\beta+m_D(T)\right)r\right)r^{-1}dr,\tag{33.3}$$

$$T_4^n=-\frac{A_4}{2}\int_0^{+\infty}\left(\prod_{i=1}^{n}\;(r-\alpha_i)\,r^{\delta-1}\right)^2 exp\left(-\alpha r^2-\left(2\beta+m_D(T)\right)r\right)r^3dr,\tag{33.4}$$

$$T_5^n=-\frac{A_3}{2}\int_0^{+\infty}\left(\prod_{i=1}^{n}\;(r-\alpha_i)\,r^{\delta-1}\right)^2 exp\left(-\alpha r^2-\left(2\beta+m_D(T)\right)r\right)r^2dr,\tag{33.5}$$

and

$$T_6^n=\int_0^{+\infty}\left(\prod_{i=1}^{n}\;(r-\alpha_i)\,r^{\delta-1}\right)^2 exp(-\alpha r^2-2\beta r)\frac{l(l+1)}{r^4}r^2dr.\tag{33.6}$$

For the ground state $n=0$, the expressions of the 6-factors$T_i^0\left(i=\overline{1,6}\right)$ will be simplified to the following form:

$$T_1^0=\frac{A_1}{2}\int_0^{+\infty}r^{2\delta-1}\,exp(-\alpha r^2-\varepsilon r)\,dr,\tag{34.1}$$

$$T_2^0=-\frac{A_2}{2}\int_0^{+\infty}r^{2\delta-1-1}\,exp(-\alpha r^2-\varepsilon r)\,dr,\tag{34.2}$$

$$T_3^0=\frac{D_3}{2}\int_0^{+\infty}r^{2\delta-2-1}\,exp(-\alpha r^2-\varepsilon r)\,dr,\tag{34.3}$$

$$T_4^0=-\frac{A_4}{2}\int_0^{+\infty}r^{2\delta+2-1}\,exp(-\alpha r^2-\varepsilon r)\,r^3dr,\tag{34.4}$$

$$T_5^0=-\frac{A_3}{2}\int_0^{+\infty}r^{2\delta+1-1}\,exp(-\alpha r^2-\varepsilon r)\,dr,\tag{34.5}$$

$$T_6^0=l(l+1)\int_0^{+\infty}r^{2\delta-3-1}\,exp(-\alpha r^2-2\beta r)\,dr,\tag{34.6}$$

where$\varepsilon=2\beta+m_D(T)$. It is convenient to apply the following special integral [82]:

$$\int_0^{\infty}x^{\nu-1}\exp\left(-\lambda x^2-\gamma x\right)dx=(2\lambda)^{-\frac{\nu}{2}}\Gamma(\nu)\exp\left(\frac{\gamma^2}{8\lambda}\right)D_{-\nu}\left(\frac{\gamma}{\sqrt{2\lambda}}\right),\tag{35}$$

where $D_{-\nu}\left(\frac{\gamma}{\sqrt{2\lambda}}\right)$ and $\Gamma(\nu)$denote the parabolic cylinder functions and the Gamma function. After straightforward calculations, we can obtain the explicit results:

$$T_1^0=\frac{A_1}{2}(2\alpha)^{-\frac{2\delta}{2}}\Gamma(2\delta)\,exp\left(\frac{\varepsilon^2}{8\alpha}\right)D_{-2\delta}\left(\frac{\varepsilon}{\sqrt{2\alpha}}\right),\tag{36.1}$$

$$T_2^0=-\frac{A_2}{2}(2\alpha)^{-\frac{2\delta-1}{2}}\Gamma(2\delta-1)\,exp\left(\frac{\varepsilon^2}{8\alpha}\right)D_{-(2\delta-1)}\left(\frac{\varepsilon}{\sqrt{2\alpha}}\right),\tag{36.2}$$

$$T_3^0=\frac{D_3}{2}(2\alpha)^{-\frac{2\delta-2}{2}}\Gamma(2\delta-2)\,exp\left(\frac{\varepsilon^2}{8\alpha}\right)D_{-(2\delta-2)}\left(\frac{\varepsilon}{\sqrt{2\alpha}}\right),\tag{36.3}$$

$$T_4^0=-\frac{A_4}{2}(2\alpha)^{-\frac{2\delta+2}{2}}\Gamma(2\delta+2)\,exp\left(\frac{\varepsilon^2}{8\alpha}\right)D_{-(2\delta+2)}\left(\frac{\varepsilon}{\sqrt{2\alpha}}\right),\tag{36.4}$$

$$T_5^0=-\frac{A_3}{2}(2\alpha)^{-\frac{2\delta+1}{2}}\Gamma(2\delta+1)\,exp\left(\frac{\varepsilon^2}{8\lambda}\right)D_{-(2\delta+1)}\left(\frac{\varepsilon}{\sqrt{2\lambda}}\right),\tag{36.5}$$

and

$$T_6^0=l(l+1)(2\alpha)^{-\frac{2\delta-3}{2}}\Gamma(2\delta-3)\,exp\left(\frac{\mathbf{4}\beta^2}{8\alpha}\right)D_{-(2\delta-3)}\left(\frac{2\beta}{\sqrt{2\alpha}}\right).\tag{36.6}$$

Let us second to obtain the exact modifications ($E_{so}^{gip}(k_1(l),j,l,n=0)$, $E_{so}^{mip}(k_2(l),j,l,n=0)$ and $E_{so}^{lip}(k_3(l),j,l,n=0)$) of the ground state as:

$$E_{so}^{gip}(k_1(l),j,l,n=0)=g_s\frac{(2\alpha)^{\frac{2\delta+1}{2}}exp(-\beta^2/2\alpha)}{\Gamma(2\delta+1)D_{-(2\delta+1)}\left(\beta\sqrt{2/\alpha}\right)}k_1(l)\left(\Theta T_{00}+\frac{\bar{\theta}}{2\mu}\right),\tag{37.1}$$

$$E_{so}^{mip}(k_2(l),j,l,n=0)=g_s\frac{(2\alpha)^{\frac{2\delta+1}{2}}exp(-\beta^2/2\alpha)}{\Gamma(2\delta+1)D_{-(2\delta+1)}\left(\beta\sqrt{2/\alpha}\right)}k_2(l)\left(\Theta T_{00}+\frac{\bar{\theta}}{2\mu}\right),\tag{37.2}$$

and

$$E_{so}^{lip}(k_3(l),j,l,n=0)=g_s\frac{(2\alpha)^{\frac{2\delta+1}{2}}exp(-\beta^2/2\alpha)}{\Gamma(2\delta+1)D_{-(2\delta+1)}\left(\beta\sqrt{2/\alpha}\right)}k_3(l)\left(\Theta T_{00}+\frac{\bar{\theta}}{2\mu}\right).\tag{37.3}$$

with $T_{00}=\sum_{i=1}^{6} T_i^0$. For the first excited state $n=1$, we replace $\left(\prod_{i=1}^{n}\ (r-\alpha_i)\, r^{\delta-1}\right)^2$ by $(r^{2\delta}-\alpha_1{}^2 r^{2\delta-2}-2\alpha_1 r^{2\delta-1})$, with $\alpha_1=\frac{d_3-2\beta(\delta+1)}{2\alpha}$, the expressions of the 6-factors$T_i^1\left(i=\overline{1{,}6}\right)$ will be simplified to the following form:

$$T_1^1=\frac{A_1}{2}\int_0^{+\infty}\left(r^{2\delta+2-1}-\alpha_1{}^2 r^{2\delta-1}-2\alpha_1 r^{2\delta+1-1}\right) exp(-\alpha r^2-\varepsilon r)\, dr, \tag{38.1}$$

$$T_2^1=-\frac{A_2}{2}\int_0^{+\infty} r^{2\delta+1-1}-\alpha_1{}^2 r^{2\delta-1-1}-2\alpha_1 r^{2\delta-1}\, exp(-\alpha r^2-\varepsilon r)\, dr, \tag{38.2}$$

$$T_3^1=\frac{D_3}{2}\int_0^{+\infty}\left(r^{2\delta-1}-\alpha_1{}^2 r^{2\delta-2-1}-2\alpha_1 r^{2\delta-1-1}\right) exp(-\alpha r^2-\varepsilon r)\, dr, \tag{38.3}$$

$$T_4^1=-\frac{A_4}{2}\int_0^{+\infty}\left(r^{2\delta+4-1}-\alpha_1{}^2 r^{2\delta+2-1}-2\alpha_1 r^{2\delta+3-1}\right) exp(-\alpha r^2-\varepsilon r)\, dr, \tag{38.4}$$

$$T_5^1=-\frac{A_3}{2}\int_0^{+\infty}\left(r^{2\delta+3-1}-\alpha_1{}^2 r^{2\delta+1-1}-2\alpha_1 r^{2\delta+2-1}\right) exp(-\alpha r^2-\varepsilon r)\, dr, \tag{38.5}$$

and

$$T_6^1=l(l+1)\int_0^{+\infty}\left(r^{2\delta-1-1}-\alpha_1{}^2 r^{2\delta-3-1}-2\alpha_1 r^{2\delta-2-1}\right) exp(-\alpha r^2-2\beta r)\, dr. \tag{38.6}$$

Evaluating the integral in Eq. (38) and applying the special integration, which is given by Eq. (33), we obtain the following results:

$$T_2^1=-\frac{A_2}{2}\Big\{(2\alpha)^{-\frac{2\delta+1}{2}}\Gamma(2\delta+1)\, exp\left(\frac{\varepsilon^2}{8\alpha}\right) D_{-(2\delta+1)}\left(\frac{\varepsilon}{\sqrt{2\alpha}}\right)+\alpha_1{}^2(2\alpha)^{-\frac{2\delta-1}{2}}\Gamma(2\delta-1)$$
$$exp\left(\frac{\varepsilon^2}{8\alpha}\right) D_{-(2\delta-1)}\left(\frac{\varepsilon}{\sqrt{2\alpha}}\right)-2\alpha_1(2\alpha)^{-\frac{2\delta}{2}}\Gamma(2\delta)\, exp\left(\frac{\varepsilon^2}{8\alpha}\right) D_{-(2\delta)}\left(\frac{\varepsilon}{\sqrt{2\alpha}}\right)\Big\}, \tag{39.1}$$

$$T_2^1=-\frac{A_2}{2}\Big\{(2\alpha)^{-\frac{2\delta+1}{2}}\Gamma(2\delta+1)\, exp\left(\frac{\varepsilon^2}{8\alpha}\right) D_{-(2\delta+1)}\left(\frac{\varepsilon}{\sqrt{2\alpha}}\right)+\alpha_1{}^2(2\alpha)^{-\frac{2\delta-1}{2}}\Gamma(2\delta-1)$$
$$exp\left(\frac{\varepsilon^2}{8\alpha}\right) D_{-(2\delta-1)}\left(\frac{\varepsilon}{\sqrt{2\alpha}}\right)-2\alpha_1(2\alpha)^{-\frac{2\delta}{2}}\Gamma(2\delta)\, exp\left(\frac{\varepsilon^2}{8\alpha}\right) D_{-(2\delta)}\left(\frac{\varepsilon}{\sqrt{2\alpha}}\right)\Big\}, \tag{39.2}$$

$$T_3^1=\frac{D_3}{2}\Big\{(2\alpha)^{-\frac{2\delta-1}{2}}\Gamma(2\delta-1)\, exp\left(\frac{\varepsilon^2}{8\alpha}\right) D_{-(2\delta-1)}\left(\frac{\varepsilon}{\sqrt{2\alpha}}\right)+\alpha_1{}^2(2\alpha)^{-\frac{2\delta-2}{2}}\Gamma(2\delta-2)$$
$$exp\left(\frac{\varepsilon^2}{8\alpha}\right) D_{-(2\delta-2)}\left(\frac{\varepsilon}{\sqrt{2\alpha}}\right)-2\alpha_1(2\alpha)^{-\frac{2\delta-1}{2}}\Gamma(2\delta-1)\, exp\left(\frac{\varepsilon^2}{8\alpha}\right) D_{-(2\delta-1)}\left(\frac{\varepsilon}{\sqrt{2\alpha}}\right)\Big\}, \tag{39.3}$$

$$T_4^1=-\frac{A_4}{2}\Big\{(2\alpha)^{-\frac{2\delta+4}{2}}\Gamma(2\delta+4)\, exp\left(\frac{\varepsilon^2}{8\alpha}\right) D_{-(2\delta+4)}\left(\frac{\varepsilon}{\sqrt{2\alpha}}\right)+\alpha_1{}^2(2\alpha)^{-\frac{2\delta+2}{2}}\Gamma(2\delta+2)$$
$$exp\left(\frac{\varepsilon^2}{8\alpha}\right) D_{-(2\delta+2)}\left(\frac{\varepsilon}{\sqrt{2\alpha}}\right)-2\alpha_1(2\alpha)^{-\frac{2\delta+3}{2}}\Gamma(2\delta+3)\, exp\left(\frac{\varepsilon^2}{8\alpha}\right) D_{-(2\delta+3)}\left(\frac{\varepsilon}{\sqrt{2\alpha}}\right)\Big\}, \tag{39.4}$$

$$T_5^1=-\frac{A_3}{2}\Big\{(2\alpha)^{-\frac{2\delta+3}{2}}\Gamma(2\delta+3)\, exp\left(\frac{\varepsilon^2}{8\alpha}\right) D_{-(2\delta+3)}\left(\frac{\varepsilon}{\sqrt{2\alpha}}\right)-\alpha_1{}^2(2\alpha)^{-\frac{2\delta+1}{2}}\Gamma(2\delta+1)$$
$$exp\left(\frac{\varepsilon^2}{8\alpha}\right) D_{-(2\delta+1)}\left(\frac{\varepsilon}{\sqrt{2\alpha}}\right)-2\alpha_1(2\alpha)^{-\frac{2\delta+2}{2}}\Gamma(2\delta+2)\, exp\left(\frac{\varepsilon^2}{8\alpha}\right) D_{-(2\delta+2)}\left(\frac{\varepsilon}{\sqrt{2\alpha}}\right)\Big\}, \tag{39.5}$$

and

$$T_6^1=l(l+1)(2\alpha)^{-\frac{2\delta+3}{2}}\Gamma(2\delta-1)\, exp\left(\frac{\mathbf{4}\beta^2}{8\alpha}\right) D_{-(2\delta-1)}\left(\frac{2\beta}{\sqrt{2\alpha}}\right)-l(l+1)\alpha_1{}^2(2\alpha)^{-\frac{2\delta+1}{2}}\Gamma(2\delta-3)$$
$$exp\left(\frac{\mathbf{4}\beta^2}{8\alpha}\right) D_{-(2\delta-3)}\left(\frac{2\beta}{\sqrt{2\alpha}}\right)-2l(l+1)\alpha_1(2\alpha)^{-\frac{2\delta+2}{2}}\Gamma(2\delta-2)\, exp\left(\frac{\mathbf{4}\beta^2}{8\alpha}\right) D_{-(2\delta-2)}\left(\frac{2\beta}{\sqrt{2\alpha}}\right). \tag{39.6}$$

Allow us to obtain the exact modifications ($E_{so}^{gip}(k_1(l),j,l,n=1)$, $E_{so}^{mip}(k_2(l),j,l,n=1)$ and $E_{so}^{lip}(k_3(l),j,l,n=1)$) of the first excited state as follows.

$$E_{so}^{gip}(k_1(l),j,l,n=1)=\frac{g_s k_1(l)}{I_1-2\alpha_1 I_2+\alpha_1{}^2 I_3}\left(\Theta T_{11}+\frac{\bar{\theta}}{2\mu}\right), \tag{40.1}$$

$$E_{so}^{mip}(k_2(l),j,l,n=1)=\frac{g_s k_2(l)}{I_1-2\alpha_1 I_2+\alpha_1{}^2 I_3}\left(\Theta T_{11}+\frac{\bar{\theta}}{2\mu}\right), \tag{40.2}$$

and

$$E_{so}^{lip}(k_3(l),j,l,n=1)=\frac{g_s k_3(l)}{I_1-2\alpha_1 I_2+\alpha_1{}^2 I_3}\left(\Theta T_{11}+\frac{\bar{\theta}}{2\mu}\right), \tag{40.3}$$

with

$$\left.\begin{array}{l} I_1=\frac{\Gamma(2\delta+3)D_{-(2\delta+3)}\left(\beta\sqrt{2/\alpha}\right)}{(2\alpha)^{(\delta+3/2)}\,exp(-\beta^2/2\alpha)} \\ I_2=\frac{\Gamma(2\delta+1)D_{-(2\delta+1)}\left(\beta\sqrt{2/\alpha}\right)}{(2\alpha)^{(\delta+1/2)}\,exp(-\beta^2/2\alpha)} \\ I_3=\frac{\Gamma(2\delta+3)D_{-(2\delta+3)}\left(\beta\sqrt{2/\alpha}\right)}{(2\alpha)^{\frac{2\delta+3}{2}}\,exp(-\beta^2/2\alpha)} \end{array}\right\},$$

and $T_{11}=\sum_{i=1}^{6}T_1^i$. In addition, in the same way, we find the exact modifications ($E_{so}^{gip}(k_1(l),j,l,n)$,$E_{so}^{mip}(k_2(l),j,l,n)$ and $E_{so}^{lip}(k_3(l),j,l,n)$) for the excited states $(n,l,m)^{th}$ of the heavy quarkonium system under the new improved internal energy potential in the global quantum group symmetry 3D-NRQM:

$$E_{so}^{gip}(k_1(l),j,l,n=1)=g_sk_1(l)N_{1l}^2\left(\Theta T_{1n}+\frac{\bar{\theta}}{2\mu}\right), \tag{41.1}$$

$$E_{so}^{mip}(k_2(l),j,l,n=1)=\gamma k2(l)N_{1l}^2\left(\Theta T_{1n}+\frac{\bar{\theta}}{2\mu}\right), \tag{41.2}$$

and

$$E_{so}^{lip}(k_3(l),j,l,n=1)=\gamma k_3(l)N_{1l}^2\left(\Theta T_{1n}+\frac{\bar{\theta}}{2\mu}\right), \tag{41.3}$$

with $T_{1n}=\sum_{i=1}^{6}T_n^i$.

3.4. The exact modified magnetic spectrum for heavy quarkonium systems under improved internal energy potential in pNRQCD

In addition to the important results obtained previously, now we consider another important physically meaningful phenomenon produced by the effect of the improved internal energy potential at finite temperature on the perturbative NRQCD related to the influence of an external uniform magnetic field $\vec{\mathbf{B}}$. To avoid repetition in the theoretical calculations, it is sufficient to apply the following replacements:

$$\vec{\mathbf{\Theta}}\to\quad\boldsymbol{\sigma}\vec{\mathbf{B}}\text{ and }\vec{\overline{\mathbf{\theta}}}\to\quad\overline{\boldsymbol{\sigma}}\vec{\mathbf{B}}. \tag{42}$$

Allows us to replace the physical quantities $f(r,A_i,D_3)\vec{\mathbf{L}}\vec{\mathbf{\Theta}}$, $\frac{l(l+1)}{r^4}\vec{\mathbf{L}}\vec{\mathbf{\Theta}}$ and $\frac{\vec{\mathbf{L}}\vec{\overline{\mathbf{\theta}}}}{2\mu}$ with corresponding new physical quantities $\sigma f(r)\vec{\mathbf{L}}\vec{\mathbf{B}}$, $\sigma\frac{l(l+1)}{r^4}\vec{\mathbf{L}}\vec{\mathbf{B}}$ and $\overline{\sigma}\frac{\vec{\mathbf{L}}\vec{\mathbf{B}}}{2\mu}$, respectively, here($\sigma$and$\overline{\sigma}$) are two infinitesimal real proportional constants, and we choose the arbitrary uniform external magnetic field $\vec{\mathbf{B}}$ parallel to the (Oz) axis, which allows us to introduce the new modified magnetic Hamiltonian $H_m^{ip}(r,\sigma,\overline{\sigma})$ in 3D-NRNCPS symmetries as:

$$H_m^{ip}(r,\chi,\overline{\sigma})=-\left(f(r,A_i,D_3)\chi+\frac{l(l+1)}{r^4}\chi-\frac{\overline{\sigma}}{2\mu}\right)\left\{\vec{\mathbf{B}}\vec{\mathbf{J}}-\aleph_z\right\}, \tag{43}$$

here $\aleph_z\equiv-\vec{\mathbf{S}}\vec{\mathbf{B}}$ denote to Zeeman effect in commutative quantum mechanics, while ($\aleph_{mod}^z=\vec{\mathbf{B}}\vec{\mathbf{J}}-\aleph_z$) is the new Zeeman effect. To obtain the exact NC magnetic modifications of energy for the ground state, the first excited state, and $(n,l,m)^{th}$ excited states of the heavy quarkonium system $E_{mag}^{ip}(m=0,l=0,n=0)$, $E_{mag}^{ip}\left(m=\overline{-l,+l},l,n=1\right)$ and $E_{mag}^{ip}\left(m=\overline{-l,+l},l,n\right)$ we just replace $k_1(l)$ and $\Theta\left(\overline{\theta}\right)$ in the Eqs. (37), (40), and (41) with the following parameters m and $\sigma(\overline{\sigma})$, respectively:

$$E_{mag}^{ip}(m=0,l=0,n=0)=g_s\frac{(2\alpha)^{\frac{2\delta+1}{2}}exp\left(-\beta^2/2\alpha\right)}{\Gamma(2\delta+1)D_{-(2\delta+1)}\left(\beta\sqrt{2/\alpha}\right)}B\left(\sigma T_{00}+\frac{\overline{\sigma}}{2\mu}\right)m, \tag{44.1}$$

$$E_{mag}^{ip}\left(m=\overline{-l,+l},l,n=1\right)=\frac{g_sB}{I_1-2\alpha_1I_2+\alpha_1{}^2I_3}\left(\sigma T_{11}+\frac{\overline{\sigma}}{2\mu}\right)m, \tag{44.2}$$

and

$$E_{mag}^{ip}\left(m=\overline{-l,+l},l,n\right)=g_sN_{nl}^2\boldsymbol{B}\left(\sigma T_{1n}+\frac{\overline{\sigma}}{2\mu}\right)m. \tag{44.3}$$

We have$-l\le m\le+l$, which allows us to fix $(2l+1)$ values for discreet numbers m. It should be noted that the results obtained in Eq. (44) we could find by direct calculation:

$$E_{mag}^{ip}=\langle\Psi(r,\theta,\phi)\left|H_m^{ip}(r,\sigma,\overline{\sigma})\right|\Psi(r,\theta,\phi)\rangle$$

that takes the following explicit relation:

$$E_{mag}^{ip}=N_{nl}^2mB\int_0^{+\infty}\left(\prod_{i=1}^{n}\ (r-\alpha_i)\,r^{\delta-1}\right)^2exp(-\alpha r^2-2\beta r)\left\{f(r,A_i,D_3)\sigma+\frac{l(l+1)}{r^4}\sigma+\frac{\overline{\sigma}}{2\mu}\right\}r^2dr. \tag{45}$$

Eq. (45) can be rewritten as follows:

$$E_{mag}^{ip}=g_s N_{nl}^2 B\left(\sigma\sum_{i=1}^{6}T_i^n+\frac{\overline{\sigma}}{2\mu}\right)m. \tag{46}$$

The 6-factors $T_i^n\left(i=\overline{1,6}\right)$ are given by Eq. (33). Then we find the magnetic specters of energy produced by the operator $H_m^{ip}(r,\chi,\overline{\sigma})$ for the ground state and first excited states repeating the same calculations in the previous subsection.

Having completed the first and second-induced perturbed both spin-orbit interaction and self-magnetic phenomena, now, for our purposes, we are interested in finding a new third automatically important symmetry for improved internal energy potential at zero temperature in RNCQM symmetries. This physical phenomenon is induced automatically by the influence of the perturbative Hamiltonian operator $H_{pert}^{ip-rot}(r,\chi,\overline{\chi})$, which we can obtain from the initial perturbed Hamiltonian operator in Eq. (25). We discover these important physical phenomena when our studied system the quarkonium particle such as ($c\overline{c}$ and$b\overline{b}$) undergoing rotation with angular velocity $\vec{\Omega}$ if we make the following two transformations to ensure that previous calculations are not repeated:

$$\vec{\mathbf{\Theta}}\rightarrow\boldsymbol{\chi}\vec{\mathbf{\Omega}}\ \text{ and }\ \vec{\overline{\mathbf{\theta}}}\rightarrow\overline{\boldsymbol{\chi}}\vec{\mathbf{\Omega}}. \tag{47}$$

Here $(\chi,\overline{\chi})$ are just two infinitesimal real proportional constants. We can express the perturbative Hamiltonian operator $H_{pert}^{ip-rot}(r,\chi,\overline{\chi})$ which induced the rotational movements of the quarkonium particle as follows:

$$H_{pert}^{ip-rot}(r,\chi,\overline{\chi})=\left[f(r,A_i,D_3)\,\chi+\frac{l(l+1)}{r^4}\chi+\frac{\overline{\chi}}{2\mu}\right]\vec{\mathbf{L}}\vec{\mathbf{\Omega}}. \tag{48}$$

To simplify the calculations without compromising physical content, we choose the rotational velocity $\vec{\Omega}=\Omega e_z$. Then we transform the spin-orbit coupling to the new physical phenomena as follows:

$$\left[f(r,A_i,D_3)\chi+\frac{l(l+1)}{r^4}\chi+\frac{\overline{\chi}}{2\mu}\right]\vec{\mathbf{L}}\vec{\mathbf{\Omega}}\rightarrow\ \left[f(r,A_i,D_3)\chi+\frac{l(l+1)}{r^4}\chi+\frac{\overline{\chi}}{2\mu}\right]\Omega\mathbf{L_z}. \tag{49}$$

To obtain the exact NC modifications of energy for the ground state, the first excited state, and $(n,l,m)^{th}$ excited states of the heavy quarkonium system $E_{rot}^{ip}(m=0,l=0,n=0)$, $E_{rot}^{ip}\left(m=\overline{-l,+l},l,n=1\right)$, and $E_{rot}^{ip}\left(m=\overline{-l,+l},l,n\right)$ we just replace$k_1(l)$ and $\Theta\left(\overline{\theta}\right)$ in Eqs. (37), (40), and (41) with the following parameters m and$\chi(\overline{\chi})$, respectively:

$$E_{rot}^{ip}(m=0,l=0,n=0)=g_s\frac{(2\alpha)^{\frac{2\delta+1}{2}}exp(-\beta^2/2\alpha)}{\Gamma(2\delta+1)D_{-(2\delta+1)}\left(\beta\sqrt{2/\alpha}\right)}\Omega\left(\chi T_{00}+\frac{\overline{\chi}}{2\mu}\right)m, \tag{50.1}$$

$$E_{rot}^{ip}\left(m=\overline{-l,+l},l,n=1\right)=\frac{g_s\Omega}{I_1-2\alpha_1 I_2+\alpha_1{}^2 I_3}\left(\chi T_{11}+\frac{\overline{\chi}}{2\mu}\right)m, \tag{50.2}$$

and

$$E_{rot}^{ip}\left(m=\overline{-l,+l},l,n\right)=g_s N_{nl}^2\Omega\left(\chi T_{1n}+\frac{\overline{\chi}}{2\mu}\right)m. \tag{50.3}$$

It is important to note that in Ref. [83], rotating isotropic and anisotropic harmonically confined ultra-cold Fermi gases were studied in two and three dimensions at absolute zero, but in that study, the rotational term had to be manually added to the Hamiltonian operator. In contrast, in our study, the rotation operator appears automatically because of the phase-space deformation caused by the improved internal energy potential models in the 3D-NRNCPS symmetries. It is crucial to note that perturbation theory cannot be utilized to find corrections of the second order ($\Theta^2$and $\overline{\theta}^2$) because we have only employed corrections of the first order of infinitesimal noncommutative parameters (Θ and $\overline{\theta}$).

4. MAIN RESULTS

In the previous subsections, we obtained the solution of the modified Schrödinger equation for new improved internal energy potential, which is given in Eq. (25) by using the generalized Bopp's shift method and standard perturbation theory in pNRQCD by the feature of 3D-NRNCPS symmetries. The modified eigenenergies $\left(E_{nc}^{gip},E_{nc}^{mip},E_{nc}^{lip}\right)$ $(T,c,n=0,m=0,l)$, $\left(E_{nc}^{gip},E_{nc}^{mip},E_{nc}^{lip}\right)\left(T,c,j,n=1,\left(m=\overline{-l,+l}\right),l\right)$ and $\left(E_{nc}^{gip},E_{nc}^{mip},E_{nc}^{lip}\right)$ $\left(T,c,j,n,\left(m=\overline{-l,+l}\right),l\right)$ with spin-1 for heavy quarkonium systems $Q\overline{Q}$ $(Q=c,b)$ with improved internal energy potential at finite temperature are obtained in this paper based on our original results presented in Eqs. (37), (40), (41), (44), and (50) in addition to the ordinary energy E_{nl} for the improved internal energy potential at a finite temperature which is presented in Eq. (11):

- For the ground state:

$$E_{nc}^{gip}(T,c,n=0,m=0,l)=E_{0l}+g_s\frac{(2\alpha)^{\frac{2\delta+1}{2}}exp(-\beta^2/2\alpha)}{\Gamma(2\delta+1)D_{-(2\delta+1)}\left(\beta\sqrt{2/\alpha}\right)}\left\{\begin{matrix}(\Theta k_1(l)+B\sigma m+\Omega\chi m)T_{00}\\+\frac{1}{2\mu}\left(k_1(l)\bar{\theta}+B\overline{\sigma}m+\Omega\overline{\chi}m\right)\end{matrix}\right\}, \tag{51.1}$$

$$E_{nc}^{mip}(T,c,n=0,m=0,l)=E_{0l}+g_s\frac{(2\alpha)^{\frac{2\delta+1}{2}}exp(-\beta^2/2\alpha)}{\Gamma(2\delta+1)D_{-(2\delta+1)}\left(\beta\sqrt{2/\alpha}\right)}\left\{\begin{matrix}(\Theta k_2(l)+B\sigma m+\Omega\chi m)T_{00}\\+\frac{1}{2\mu}\left(k_2(l)\bar{\theta}+B\overline{\sigma}m+\Omega\overline{\chi}m\right)\end{matrix}\right\}, \tag{51.2}$$

$$E_{nc}^{lip}(T,c,n=0,m=0,l)=E_{0l}+g_s\frac{(2\alpha)^{\frac{2\delta+1}{2}}exp(-\beta^2/2\alpha)}{\Gamma(2\delta+1)D_{-(2\delta+1)}(\beta\sqrt{2/\alpha})}\begin{Bmatrix}(\Theta k_3(l)+B\sigma m+\Omega\chi m)T_{00}\\+\frac{1}{2\mu}(k_3(l)\bar{\theta}+B\bar{\sigma}m+\Omega\bar{\chi}m)\end{Bmatrix}, \tag{51.3}$$

- For the first excited state:

$$E_{nc}^{gip}(T,c,j,n=1,(m=\overline{-l,+l}),l)=E_{1l}+\frac{g_s}{I_1-2\alpha_1I_2+\alpha_1{}^2I_3}\begin{Bmatrix}(\Theta k_1(l)+B\sigma m+\Omega\chi)T_{01}\\+\frac{1}{2\mu}(k_1(l)\bar{\theta}+B\bar{\sigma}m+\Omega\bar{\chi}m)\end{Bmatrix}, \tag{52.1}$$

$$E_{nc}^{mip}(T,c,j,n=1,(m=\overline{-l,+l}),l)=E_{1l}+\frac{g_sB}{I_1-2\alpha_1I_2+\alpha_1{}^2I_3}\begin{Bmatrix}(\Theta k_2(l)+B\sigma m+\Omega\chi)T_{01}\\+\frac{1}{2\mu}(k_2(l)\bar{\theta}+B\bar{\sigma}m+\Omega\bar{\chi}m)\end{Bmatrix}, \tag{52.2}$$

$$E_{nc}^{lip}(T,c,j,n=1,(m=\overline{-l,+l}),l)=E_{1l}+\frac{g_sB}{I_1-2\alpha_1I_2+\alpha_1{}^2I_3}\begin{Bmatrix}(\Theta k_3(l)+B\sigma m+\Omega\chi m)T_{01}\\+\frac{1}{2\mu}(k_3(l)\bar{\theta}+B\bar{\sigma}m+\Omega\bar{\chi}m)\end{Bmatrix}, \tag{52.3}$$

- For any $(n,l,m)^{th}$ excited state:

$$E_{nc}^{gip}(T,c,j,n,(m=\overline{-l,+l}),l)=E_{nl}+g_sN_{nl}^2\left\{(\Theta k_1(l)+B\sigma m+\Omega\chi m)T_{1n}+\frac{1}{2\mu}(k_1(l)\bar{\theta}+B\bar{\sigma}m+\Omega\bar{\chi}m)\right\}, \tag{53.1}$$

$$E_{nc}^{mip}(T,c,j,n,(m=\overline{-l,+l}),l)=E_{nl}+g_sN_{nl}^2\left\{(\Theta k_2(l)+B\sigma m+\Omega\chi m)T_{1n}+\frac{1}{2\mu}(k_2(l)\bar{\theta}+B\bar{\sigma}m+\Omega\bar{\chi}m)\right\}, \tag{53.2}$$

$$E_{nc}^{lip}(T,c,j,n,(m=\overline{-l,+l}),l)=E_{nl}+g_sN_{nl}^2\left\{(\Theta k_3(l)+B\sigma m+\Omega\chi)mT_{1n}+\frac{1}{2\mu}(k_3(l)\bar{\theta}+B\bar{\sigma}m+\Omega\bar{\chi}m)\right\}, \tag{53.3}$$

where E_{0l} and E_{1l} are the energy of the ground state and the first excited state of heavy quarkonium systems in the symmetries of quantum mechanics under internal energy potential at finite temperature:

$$E_{0l}=\frac{\alpha(1+2\delta-\beta^2-d_2+m_D(T)d_3)}{2\mu}, \tag{54.1}$$

and

$$E_{1l}=\frac{\alpha(1+2(\delta+1)-\beta^2-d_2+m_D(T)d_3)}{2\mu}. \tag{54.2}$$

This is one of the main objectives of our research and by noting that the obtained eigenvalues of energy are real's and then the NC diagonal Hamiltonian $H_{nc}^{ip}(x_\mu,p_\mu)$ is Hermitian. Furthermore, it's possible to write the three elements $(H_{nc}^{ip})_{11}$, $(H_{nc}^{ip})_{22}$ and $(H_{nc}^{ip})_{33}$ as follows:

$$H_{ip}(x_\mu,p_\mu)\to H_{nc}^{ip}(x_\mu,p_\mu)\equiv diag\left((H_{nc}^{ip})_{11},(H_{nc}^{ip})_{22},(H_{nc}^{ip})_{33}\right), \tag{55}$$

where

$$\left.\begin{matrix}(H_{nc}^{ip})_{11}=-\dfrac{\Delta_{nc}}{2\mu}+H_{int}^{gip}\\(H_{nc}^{ip})_{22}=-\dfrac{\Delta_{nc}}{2\mu}+H_{int}^{mip}\\(H_{nc}^{ip})_{33}=-\dfrac{\Delta_{nc}}{2\mu}+H_{int}^{lip}\}\end{matrix}\right\}$$

In the symmetries of 3D-NRNCPS, the new kinetic term $\frac{\Delta_{nc}}{2\mu}$ can be expressed as:

$$\frac{\Delta_{nc}}{2\mu}=\frac{\Delta-\vec{\mathbf{L}}\vec{\bar{\boldsymbol{\theta}}}-\vec{\mathbf{L}}\vec{\bar{\boldsymbol{\sigma}}}-\vec{\mathbf{L}}\vec{\bar{\boldsymbol{\chi}}}}{2\mu}$$

The three modified interaction elements $(H_{int}^{gip},H_{int}^{mip},H_{int}^{lip})$ are given by the following expressions:

$$H_{int}^{gip}=\left(D_2+\frac{D_3}{r}+D_4r+D_5r^2\right)exp(-m_D(T)r)\ {}_s(k_1(l)\Theta+\sigma\aleph_{mod}^{z_z})(r,A_i,D_3)\Big), \tag{56.1}$$

$$H_{int}^{mip}=\left(D_2+\frac{D_3}{r}+D_4r+D_5r^2\right)exp(-m_D(T)r)\ {}_s(k_2(l)\Theta+\sigma\aleph_{mod}^{z_z})(r,A_i,D_3)\Big), \tag{56.2}$$

and

$$H_{int}^{lip}=\left(D_2+\frac{D_3}{r}+D_4r+D_5r^2\right)exp(-m_D(T)r)\ {}_s(k_3(l)\Theta+\sigma\aleph_{mod}^{z_z})(r,A_i,D_3)\Big). \tag{56.3}$$

Thus, the ordinary kinetic term for the internal energy potential $(-\frac{\Delta}{2\mu})$ and ordinary interaction $\left(D_2+\dfrac{D_3}{r}+D_4r+D_5r^2\right)exp(-m_D(T)r)$ are replaced by a new modified form of the kinetic term $(\frac{\Delta_{nc}}{2\mu})$ and new modified interactions modified to the new form $(H_{int}^{gip},H_{int}^{mip},H_{int}^{lip})$ in 3D-NRNCPS symmetries. On the other hand, it is evident consider the quantum number m takes $(2l+1)$ values and we have also three values for$(j=l\pm 1,l)$, thus every state in the usually three-dimensional space of energy for a heavy quarkonium system under improved internal energy potential will be $3(2l+1)$sub-states. To obtain the complete total degeneracy of the energy level of the improved internal energy potential in 3D-NRNCPS symmetries, we need to sum all allowed values l. Total degeneracy is thus,

$$\underbrace{\sum_{l=0}^{n-1}(2l+1)=n^2}_{\text{3D-NRQM}} \to \underbrace{3\left(\sum_{l=0}^{n-1}(2l+1)\right) \equiv 3n^2}_{\text{3D-NRNCPS}}. \tag{57}$$

Note that the obtained new energy eigenvalues $\left(E_{nc}^{gip}, E_{nc}^{mip}, E_{nc}^{lip}\right)\left(T,c,j,n,\left(m=\overline{-l,+l}\right),l\right)$ now depend on new discrete atomic quantum numbers (n,j,l,s) and$m$ in addition to the parameters of the internal energy potential. It is pertinent to note that when the atoms have spin-0, the total operator can be obtained from the interval $|l-s| \leq j \leq |l+s|$, which allows us to obtain the eigenvalues of the operator G^2as $k(j,l,s) \equiv 0$ and then the nonrelativistic energy spectrum $\left(E_{nc}^{gip}, E_{nc}^{mip}, E_{nc}^{lip}\right)\left(T,j,n,\left(m=\overline{-l,+l}\right),l\right)$ reads [68, 69]:

$$\left(E_{nc}^{gip}, E_{nc}^{mip}, E_{nc}^{lip}\right)\left(T,c,j,n,\left(m=\overline{-l,+l}\right),l\right) = E_{nl} + g_s N_{nl}^2 \left\{\sigma T_{1n} + \frac{\bar{\sigma}}{2\mu}\right\} Bm. \tag{58}$$

It is important to apply the present results (53) and (58) to quarkonium mesons. One of the most important applications in the extended model of pNRQCD is to calculate the modified mass spectra of the heavy quarkonium systems (the mass of the quarkonium bound state), such as charmonium and bottomonium mesons, which have the quark and antiquark flavor in the symmetries of NCQM under improved internal energy potential at finite temperature. To achieve this goal, we generalize the traditional formula [84-91],

$$M = 2m_q + E_{nl}$$

which defines the total mass of the different quarkonium states (resonance masses), to the new form:

$$M = 2m_q + E_{nl} \to M_{nc}^{ip} = 2m_q + \frac{1}{3}\left(E_{nc}^{gip} + E_{nc}^{mip} + E_{nc}^{lip}\right)\left(T,c,j,n,\left(m=\overline{-l,+l}\right),l\right). \tag{59}$$

Here m_q is the bare mass of quarkonium or twice the reduced mass of the system. Moreover, $\frac{1}{3}\left(E_{nc}^{gip} + E_{nc}^{mip} + E_{nc}^{lip}\right)$ $\left(T,c,j,n,\left(m=\overline{-l,+l}\right),l\right)$ is the non-polarized energies, which can determine from Eqs. (49) and (54). Thus, at finite temperature, the modified mass of the quarkonium system M_{nc}^{ip} obtain:

$$M_{nc}^{ip} = M - g_s N_{nl}^2 \begin{cases} \left\{\left(\sigma Bm + \chi\Omega m - \frac{l+4}{6}\Theta + \right) T_{1n} + \frac{m}{2\mu}\left(B\bar{\sigma} + \Omega\bar{\chi}\right) - \frac{\bar{\theta}(l+4)}{12\mu}\right\} & \text{for spin-1} \\ \left\{(\sigma B + \chi\Omega) T_{1n} m + \frac{1}{2}\left(B\bar{\sigma} + \Omega\bar{\chi}\right)m\right\} & \text{for spin-0} \end{cases}. \tag{60}$$

Here$M$is the heavy quarkonium system at a finite temperature with improved internal energy potential in commutative quantum mechanics, which is defined in ref. [7]. If we consider$(\Theta,\sigma,\chi) \to (0,0,0)$, we recover the results of the commutative space of ref. [7] for the improved internal energy potential, which means that our calculations are correct. The novelty in this work is that the generalized Bopp shift method is successfully applied to find the solution of the 3-radial DSE at finite temperature in the symmetries of the 3D-NRNCPS framework. The automatic appearance of the spin in the term of improved energy as a quantum number clearly shows that the deformed Schrödinger equation under the influence of the improved energy potential model at finite temperature rises to the descriptor of the Dirac equation, meaning that this system can be valid in the field of high energies.

5. CONCLUSION

In the present work, the 3-dimensional deformed Schrodinger equation is analytically solved using the generalized Bopp's shift method and standard perturbation theory by the feature of 3D-NRNCPS symmetries. The improved internal energy potential at finite temperature is extended to include the effect of the non-commutativity space phase based on ref. [7]; we resume the main results:

- The ordinary Hamiltonian operator at finite temperature $H_{ip}(x_\mu, p_\mu)$ in 3D-NRNCPSsymmetries was replaced by a new modified operator $H_{nc}^{ip}(x_\mu, p_\mu)$ which equals $diag\left(\left(H_{nc}^{ip}\right)_{11}, \left(H_{nc}^{ip}\right)_{22}, \left(H_{nc}^{ip}\right)_{33}\right)$ in the 3D-NRNCPS symmetries framework for the heavy quarkonium system such as $Q\bar{Q}$ $(Q=c,b)$,
- The ordinary kinetic term $-\frac{\Delta}{2\mu}$ in 3D-NRNCPSsymmetries is modified to the new form $\frac{\Delta_{nc}}{2\mu}$ which is equal $\left(\frac{\Delta - \vec{\mathbf{L}}\vec{\boldsymbol{\theta}} - \vec{\mathbf{L}}\vec{\boldsymbol{\sigma}} - \vec{\mathbf{L}}\vec{\boldsymbol{\chi}}}{2\mu}\right)$ to a heavy quarkonium system under the influence of the improved internal energy potential at the finite-temperature model.
- We have obtained the perturbative corrections $\left(\left(E_{nc}^{gip}, E_{nc}^{mip}, E_{nc}^{lip}\right)\left(T,c,j,n,\left(m=\overline{-l,+l}\right),l\right), \left(E_{nc}^{gip}, E_{nc}^{mip}, E_{nc}^{lip}\right)\right.$ $\left(T,c,j,n=1,\left(m=\overline{-l,+l}\right),l\right)$ and $\left(E_{nc}^{gip}, E_{nc}^{mip}, E_{nc}^{lip}\right)\left(T,c,j,n,\left(m=\overline{-l,+l}\right),l\right)$ for the ground state, the first excited state,

and the $(n, l, m)^{th}$ excited state with (spin-1 and spin-0) for the heavy quarkonium system under the influence of the improved internal energy potential model at finite temperature are obtained.

➢ We have obtained, at finite temperature, the modified mass of the quarkonium system M_{nc}^{ip} which equals the sum of the corresponding values M in the 3D-NRNCPSsymmetries, and two perturbative terms proportional with two parameters (Θ and $\overline{\theta}$).

➢ Since the main quantum number, spin, appears clearly and automatically in the expression of the global Hamiltonian and its eigenvalues, this is an indication that our results are valid in the field of high energies where the Dirac equation is applied.

Through high-value results, which we have achieved in the present work, we hope to extend our recent work for further investigations of particle physics and other characteristics of quarkonium at finite temperatures.

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REFERENCES

[1] X.Y. Wu, B.J. Zhang, X.J. Liu, Y.H. Wu, Q.C. Wang, and Y. Wang, “Finite temperature Schrödinger equation”, Int. J. Theor. Phys., **50**(8), 2546 (2017). https://doi.org/10.1007/s10773-011-0745-7

[2] C.Y. Wong, “Heavy quarkonia in quark-gluon plasma”, Phys. Rev. C., **72**(3), 034906 (2005). https://doi.org/10.1103/PhysRevC.72.034906

[3] T. Matsui, and H. Satz, “J/ψ suppression by quark-gluon plasma formation”, Phys. Lett. B, **178**(4), 416 (1986). https://doi.org/10.1016/0370-2693(86)91404-8

[4] V. Mateu, P. G. Ortega, D. R. Entem, and F. Fernández, “Calibrating the naïve Cornell model with NRQCD”, Eur. Phys. J. C, **79** (4), 323 (2019). https://doi.org/10.1140/epjc/s10052-019-6808-2

[5] J. Fingberg, “Heavy quarkonia at high temperature”, Physics Letters B, **424**(3-4), 343 (1998). https://doi.org/10.1016/s0370-2693(98)00205-6

[6] A.I. Ahmadov, C. Aydin, and O. Uzun, “Bound state solution of the Schrödinger equation at finite temperature”, Journal of Physics: Conference Series, **1194**, 012001 (2019). https://doi.org/10.1088/1742-6596/1194/1/012001

[7] M. Abu-Shady, “Multidimensional Schrödinger Equation and Spectral Properties of Heavy-Quarkonium Mesons at Finite Temperature”, Advances in Mathematical Physics, **2016**, 1 (2016). https://doi.org/10.1155/2016/4935940

[8] M.E. Naggar, N.I. Abou Salem, L.G. A. Shalaby, and A.M. Bourham, “The Equation of State for Non-ideal Quark Gluon Plasma”, Physical Science International Journal, **4**(7), 912 (2014). https://doi.org/10.9734/PSIJ/2014/10296

[9] W. Heisenberg, *Letter to R. Peierls*, in: *Wolfgang Pauli, Scientific Correspondence*, edited by K. von Meyenn – 1985. (Springer Verlag, 1930), Vol. III, p. 15.

[10] H.S. Snyder, “Quantized Space-Time”, Phys. Rev., **71**, 38 (1947). https://doi.org/10.1103/PhysRev.71.38,“The Electromagnetic Field in Quantized Space-Time”, Phys. Rev., **72**, 68 (1947). https://doi.org/10.1103/PhysRev.72.68

[11] A. Connes, M.R. Douglas, and A. Schwarz, Noncommutative geometry and Matrix theory”, Journal of High Energy Physics, **02**, 003 (1998). https://doi.org/10.1088/1126-6708/1998/02/003

[12] N. Seiberg, and E. Witten, String theory and noncommutative geometry”, Journal of High Energy Physics, **1999**(09) 032 (1999). https://doi.org/10.1088/1126-6708/1999/09/032

[13] S. Capozziello, G. Lambiase, and G. Scarpetta, “Generalized Uncertainty Principle from Quantum Geometry”, Int. J. Theor. Phys., **39**, 15 (2000). https://doi.org/10.1023/A:1003634814685

[14] S.K. Doplicher, and J.E. Fredenhagen, “Spacetime quantization induced by classical gravity, Roberts”, Phys. Lett. B, **331**(1- 2), 39 (1994). https://doi.org/10.1016/0370-2693(94)90940-7

[15] E. Witten, “Reflections on the Fate of Spacetime”, Phys. Today, **49**(4), 24 (1996). https://doi.org/10.1063/1.881493

[16] A. Kempf, G. Mangano, and R.B. Mann, “Hilbert space representation of the minimal length uncertainty relation”, Phys. Rev. D, **52**(2), 1108 (1995). https://doi.org/10.1103/physrevd.52.1108

[17] F. Scardigli, “Some Heuristic Semi-Classical Derivations of the Planck the Hawking effect and the Unruh effect. Il Nuovo Cimento B Series, 11 B (9), 1029 (1995). https://doi.org/10.1007/bf02726152

[18] R.J. Adler, and D.I. Santiago, On gravity and the uncertainty principal”, Mod. Phys. Lett. A, **14**(20), 1371 (1999). https://doi.org/10.1142/s0217732399001462

[19] H. Benzaira, M. Meradc, T. Boudjedaad, and A. Makhlouf “Relativistic Oscillators in a Noncommutative Space: a Path Integral Approach”, Z. Naturforsch. **67**a, 77 (2012). https://doi.org/10.5560/ZNA.2011-0060

[20] F. Scardigli, “Generalized uncertainty principle in quantum gravity from micro-black hole gedanken experiment”, Phys. Lett. B, **452**(1-2), 39 (1999). https://doi.org/10.1016/s0370-2693(99)00167-7

[21] P.M. Ho and H.C. Kao, Noncommutative Quantum Mechanics from Noncommutative Quantum Field Theory”, Phys. Rev. Lett., **88**(15), 151602-1(2002). https://doi.org/10.1103/physrevlett.88.151602

[22] O. Bertolami, G.J. Rosa, C.M.L. Dearagao, P. Castorina, and D. Zappala, “Scaling variables and the relation between noncommutative parameters in noncommutative quantum mechanics”, Mod. Phys. Lett. A, **21**(10), 795 (2006). https://doi.org/10.1142/s0217732306019840

[23] P. Gnatenko, "Parameters of noncommutativity in Lie-algebraic noncommutative space", Phys. Rev. D., **99**(2), 026009-1(2019). https://doi.org/10.1103/physrevd.99.026009
[24] E.E. N'Dolo, D.O. Samary, B. Ezinvi, and M.N. Hounkonnou, Noncommutative Dirac and Klein–Gordon oscillators in the background of cosmic string: Spectrum and dynamics. International Journal of Geometric Methods in Modern Physics, 17(05), 2050078 (2020). https://doi.org/10.1142/S0219887820500784
[25] O. Bertolami, and P. Leal, Aspects of phase-space noncommutative quantum mechanics", Phys. Lett. B., **750**, 6 (2015). https://doi.org/10.1016/j.physletb.2015.08.024
[26] M.A. De Andrade, and C. Neves, "Noncommutative mapping from the symplectic formalism", J. Math. Phys., **59**(1), 012105 (2018). https://doi.org/10.1063/1.4986964
[27] K.P. Gnatenko, and V.M.Tkachuk, "Upper bound on the momentum scale in noncommutative phase space of canonical type", EPL, **127**(2), 20008 (2019). https://doi.org/ 10.1209/0295-5075/127/20008
[28] K.P. Gnatenko, and V.M. Tkachuk, "Composite system in rotationally invariant noncommutative phase space", Int. J. Mod. Phys. A, 33(07), 1850037(2018). https://doi.org/10.1142/s0217751x18500379
[29] A. Maireche, "Bound-state solutions of the modified Klien-Gordon and Schrödinger for arbitrary noncommutative quantum mechanics", J. Phys. Stud. **25**(1), 1002 (2021). https://doi.org/10.30970/jps.25.1002
[30] A. Maireche, "A Theoretical Model of Deformed Klein-Gordon Equation with Generalized Modified Screened Coulomb Plus Inversely Quadratic Yukawa Potential in RNCQM Symmetries", Few-Body Syst. **62**, 12 (2021). https://doi.org/10.1007/s00601-021-01596-2
[31] A. Maireche, " The Relativistic and Nonrelativistic Solutions for the Modified Unequal Mixture of Scalar and Time-Like Vector Cornell Potentials in the Symmetries of Noncommutative Quantum Mechanics", Jordan Journal of Physics, **14**(1), 59 (2021). https://doi.org/10.47011/14.1.6
[32] J. Gamboa, M. Loewe, and J.C. Rojas, Noncommutative quantum mechanics Phys. Rev. D., **64**, 067901(2001). .
[33] E.F. Djemaï, and H. Smail, "On Quantum Mechanics on Noncommutative Quantum Phase Space", Commun. Theor. Phys. (Beijing, China), 41(6), 837 (2004). https://doi.org/10.1088/0253-6102/41/6/837
[34] Y. Yi, K. Kang, W. Jian-Hua, and C. Chi-Yi, Spin-1/2 relativistic particle in a magnetic field in NC phase space", Chin. Phys. C., **34**(5), 543 (2010). https://doi.org/10.1088/1674-1137/34/5/005G.
[35] G. Valencia-Ortega, and L.A. Arias-Hernandez, "Thermodynamic properties of diatomic molecule systems under SO (2,1)-anharmonic Eckart potential International", Journal of Quantum Chemistry, **118**(14), e25589(2018). https://doi.org/10.1002/qua.25589
[36] A. Maireche, "Solutions of Klein-Gordon equation for the modified central complex potential in the symmetries of noncommutative quantum mechanics", Sri Lankan Journal of Physics, **22**(1), 1 (2021). http://doi.org/10.4038/sljp.v22i1.8079
[37] A. Maireche, "Theoretical Investigation of the Modified Screened cosine Kratzer potential via Relativistic and nonrelativistic treatment in the NCQM symmetries", Lat. Am. J. Phys. Educ. **14** (3), 3310-1 (2020). https://dialnet.unirioja.es/descarga/articulo/7803865.pdf
[38] A. Maireche, "Modified Unequal Mixture Scalar Vector Hulthén–Yukawa Potentials Model as a Quark–Antiquark Interaction and Neutral Atoms via Relativistic Treatment Using the Improved Approximation of the Centrifugal Term and Bopp's Shift Method", Few-Body Syst., **61**, 30 (2020). https://doi.org/10.1007/s00601-020-01559-z
[39] O. Bertolami, J.G. Rosa, C.M.L. De Aragão, P. Castorina, and D. Zappalà, "Noncommutative gravitational quantum well Phys", Rev. D., **72**(2), 025010-1(2005). https://doi.org/10.1103/PhysRevD.72.025010
[40] J. Zhang, "Coherent states of the deformed Heisenberg–Weyl algebra in non-commutative space", Phys. Lett. B, **584**(1-2), 204 (2004). https://doi.org/10.1016/j.physletb.2005.03.040
[41] J.F.G. Santos, "Heat flow and noncommutative quantum mechanics in phase-space," J. Math. Phys., **61**, 122101, (2020). https://doi.org/10.1063/5.0010076.
[42] E.M.C. Abreu, C. Neves, and OW. liveira, "Noncommutativity from the symplectic point of view", Int. J. Mod. Phys. A., **21**, 5359 (2006). https://doi.org/10.1142/S0217751X06034094.
[43] E.M.C. Abreu, J.A. Neto, A.C.R Mendes, C. Neves, W. Oliveira, and M.V. Marcial, "Lagrangian formulation for noncommutative nonlinear systems Int. J. Mod. Phys. A, **27**, 1250053(2012). https://doi.org/10.1142/S0217751X12500534
[44] J. Wang, and K. Li, "The HMW effect in noncommutative quantum mechanics", J. Phys. A Math. Theor., **40**(9), 2197 (2007). https://doi.org/10.1088/1751-8113/40/9/021
[45] K. Lin, and J. Wang, "The topological AC effect on non-commutative phase space", Eur. Phys. J. C, **50**(4), 1007 (2007). https://doi.org/10.1140/epjc/s10052-007-0256-0
[46] A. Maireche, A theoretical investigation of nonrelativistic bound state solution at finite temperature using the sum of modified Cornell plus inverse quadratic potential", Sri Lankan Journal of Physics, 21, 11-35(2020). http://doi.org/10.4038/sljp.v21i1.8069
[47] A. Maireche, "Extended of the Schrödinger equation with new Coulomb potentials plus linear and Harmonic radial terms in the symmetries of noncommutative quantum mechanics", J. Nano-Electron. Phys., **10**(6), 06015-1 (2018). https://doi.org/10.21272/jnep.10(6).06015
[48] A. Maireche, "The Klein-Gordon equation with modified Coulomb plus inverse-square potential in the noncommutative three-dimensional space", Mod. Phys. Lett. A, **35**(5), 052050015 (2020). https://doi.org/10.1142/s0217732320500157
[49] A. Maireche,"The Klein–Gordon equation with modified Coulomb potential plus inverse-square–root potential in three-dimensional noncommutative space", To Physics Journal, **3**, 186 (2019). https://www.purkh.com/articles/the-kleingordon-equation-with-modified-coulomb-potential-plus-inversesquareroot-potential-in-threedimensional-non-commut.pdf
[50] A. Maireche, "Bound-state solutions of Klien-Gordon and Schrödinger equations for arbitrary *l*-state with combination of Hulthén and Kratzer potentials", Afr. Rev Phys. **15**(0003), 19 (2020). http://lamp.ictp.it/index.php/aphysrev/article/download/1779/620
[51] H. Motavalli, and A.R. Akbarieh, "Klien-Gordon equation for the Coulomb potential in noncommutative space", Phys. Lett. A, **25**(29), 2523 (2010). https://doi.org/10.1142/s0217732310033529
[52] M. Darroodi, H. Mehraban, and H. Hassanabadi, "The Klein–Gordon equation with the Kratzer potential in the noncommutative space", Mod. Phys. Lett. A, **33** (35), 1850203 (2018). https://doi.org/10.1142/s0217732318502036
[53] A. Maireche, "A New Theoretical Investigation of the Modified Equal Scalar and Vector Manning-Rosen Plus Quadratic Yukawa Potential Within the Deformed Klein-Gordon and Schrödinger Equations Using the Improved Approximation of the Centrifugal

Term and Bopp's Shift Method in RNCQM and NRNCQM Symmetries", Spin J. **11**, No. 4, 2150029 (2021). https://doi.org/10.1142/S2010324721500296
[54] A. Maireche, "Heavy quarkonium systems for the deformed unequal scalar and vector Coulomb–Hulthén potential within the deformed effective mass Klein–Gordon equation using the improved approximation of the centrifugal term and Bopp's shift method in RNCQM symmetries", International Journal of Geometric Methods in Modern Physics, **18**(13), 2150214 (2021). https://doi.org/10.1142/S0219887821502133
[55] A. Maireche, "A new theoretical study of the deformed unequal scalar and vector Hellmann plus modified Kratzer potentials within the deformed Klein–Gordon equation in RNCQM symmetries", Modern Physics Letters A, **36**(33), 2150232. (2021). https://doi.org/10.1142/S0217732321502321
[56] A. Maireche, "A theoretical study of the modified equal scalar and vector Manning-Rosen potential within the deformed Klein-Gordon and Schrödinger in RNCQM and NRNCQM symmetries", Rev. Mex. Fís., **67**, no. 5 050702 1–18, (2021). https://doi.org/10.31349/revmexfis.67.050702
[57] A. Maireche, "The Investigation of Approximate Solutions of Deformed Klein–Gordon and Schrödinger Equations Under Modified More General Exponential Screened Coulomb Potential Plus Yukawa Potential in NCQM Symmetries", Few-Body Syst 62, 66 (2021). https://doi.org/10.1007/s00601-021-01639-8
[58] A. Maireche, "The Relativistic and Nonrelativistic Solutions for the Modified Unequal Mixture of Scalar and Time-Like Vector Cornell Potentials in the Symmetries of Noncommutative Quantum Mechanics", Jordan Journal of Physics, **14**(1), 59 (2021). https://doi.org/10.47011/14.1.6
[59] A. Maireche, "A Theoretical Model of Deformed Klein–Gordon Equation with Generalized Modified Screened Coulomb Plus Inversely Quadratic Yukawa Potential in RNCQM Symmetries", Few-Body Syst, **62**, 12 (2021). https://doi.org/10.1007/s00601-021-01596-2
[60] A. Maireche, "New bound-state solutions of the deformed Kilien-Gordon and Schrödinger equations for arbitrary *l*-state with modified equal vector and scalar Manning-Rosen plus a class of Yukawa potential in RNCQM and NRNCQM symmetries", Journal of Physical Studies, **25**(4). 4301 (2021). https://doi.org/10.30970/jps.25.4301
[61] A. Maireche, "Bound-state solutions of the modified Klein-Gordon and Schrödinger equations for arbitrary *l*-state with the modified morse potential in the symmetries of noncommutative quantum mechanics", Journal of Physical Studies, **25**(1), 1002 (2021). https://doi.org/10.30970/jps.25.1002
[62] A. Saidi, and M.B. Sedra, "Spin-one (1 + 3)-dimensional DKP equation with modified Kratzer potential in the non-commutative space", Phys. Lett. A, **35**(5), 2050014 (2020). https://doi.org/10.1142/s0217732320500145
[63] A. Maireche, "A model of modified Klien-Gordon equation with modified scalar-vector Yukawa potential", Afr. Rev Phys. **15**(0001), 1 (2020). http://lamp.ictp.it/index.php/aphysrev/article/view/1777/618
[64] A. Maireche," The relativistic treatment of hydrogen-like and neutral atoms subjected to the generalized perturbed Yukawa potential with centrifugal barrier in the symmetries of noncommutative quantum mechanics", International Journal of Geometric Methods in Modern Physics, **17**(5), 2050067 (2020). https://doi.org/10.1142/S021988782050067X
[65] A. Maireche, "A Recent Study of Excited Energy Levels of Diatomics for Modified more General Exponential Screened Coulomb Potential: Extended Quantum Mechanics", J. Nano- Electron. Phys. **9**(3), 03021 (2017). https://doi.org/10.21272/jnep.9(3).03021
[66] A. Maireche, "A new study of energy levels of hydrogenic atoms and some molecules for new more general exponential screened coulomb potential", Open Acc J Math Theor Phy.,**1**(6), 232 (2018). https://doi.org/10.15406/oajmtp.2018.01.00040
[67] A. Maireche, "A New Approach to the Approximate Analytic Solution of the Three-Dimensional Schrödinger Equation for Hydrogenic and Neutral Atoms in the Generalized Hellmann Potential Model", Ukr. J. Phys. **65**(11), 987 (2020). https://doi.org/10.15407/ujpe65.11.987
[68] A. Maireche,"Effects of Two-Dimensional Noncommutative Theories on Bound States Schrödinger Diatomic Molecules under New Modified Kratzer-Type Interactions", International Letters of Chemistry, Physics and Astronomy, **76**, 1 (2017). https://doi.org/10.18052/www.scipress.com/ILCPA.76.1
[69] A. Maireche, "Any L-states solutions of the modified Schrödinger equation with generalized Hellmann–Kratzer potential model in the symmetries of NRNCQM", To Physics Journal, **4**, 16 (2019). https://www.purkh.com/articles/any-lstates-solutions-of-the-modified-schrdinger-equation-with-generalized-hellmannkratzer-potential-model-in-the-symmet.pdf
[70] L. Mezincescu, "Star Operation in Quantum Mechanics", (2000). https://arxiv.org/abs/hep-th/0007046
[71] L. Gouba, "A comparative review of four formulations of noncommutative quantum mechanics", Int. J. Mod. Phys. A, **31**(19), 1630025 (2016). https://doi.org/10.1142/s0217751x16300258
[72] F. Bopp, "La mécanique quantique est-elle une mécanique statistique classique particulière", Ann. Inst. Henri Poincaré, **15**(2), 81 (1956). http://www.numdam.org/item/AIHP_1956__15_2_81_0.pdf
[73] A. Maireche, "New Relativistic Bound States for Modified Pseudoharmonic Potential of Dirac Equation with Spin and Pseudo-Spin Symmetry in One-electron Atoms", Afr. Rev. Phys. **12**(0018), 130 (2017). http://lamp.ictp.it/index.php/aphysrev/article/view/1533/564
[74] A. Maireche, "New Relativistic Atomic Mass Spectra of Quark (u, d and s) for Extended Modified Cornell Potential in Nano and Plank's Scales J. Nano- Electron. Phys., **8** (1), 01020 (2016). https://doi.org/10.21272/jnep.8(1).01020
[75] A. Maireche, "A New Relativistic Study for Interactions in One-electron atoms (Spin ½ Particles) with Modified Mie-type Potential J. Nano-Electron. Phys., **8**(4), 04027 (2016). https://doi.org/10.21272/jnep.8(4(1)).04027
[76] A. Maireche, "Investigations on the Relativistic Interactions in One-Electron Atoms with Modified Yukawa Potential for Spin 1/2 Particles", International Frontier Science Letters, **11**, 29 (2017). https://doi.org/10.18052/www.scipress.com/IFSL.11.29
[77] A. Maireche, "On the interaction of an improved Schiöberg potential within the Yukawa tensor interaction under the background of deformed Dirac and Schrödinger equations", Indian J Phys, **96**(10), (2022). https://doi.org/10.1007/s12648-022-02433-w
[78] A. Maireche, "New Approximate Solutions to a Spatially-Dependent Mass Dirac Equation for Modified Hylleraas Plus Eckart Potential with Improved Yukawa Potential as a Tensor in the DQM Framework", Few-Body Syst, **63**, 63 (2022). https://doi.org/10.1007/s00601-022-01766-w
[79] A. Maireche, "Approximate Arbitrary k State Solutions of Dirac Equation with Improved Inversely Quadratic Yukawa Potential within Improved Coulomb-like Tensor Interaction in Deformation Quantum Mechanics Symmetries", Few-Body Syst, **63**, 54 (2022). https://doi.org/10.1007/s00601-022-01755-zS

[80] A. Maireche, “Approximate k-state solutions of the deformed Dirac equation in spatially dependent mass for the improved Eckart potential including the improved Yukawa tensor interaction in ERQM symmetries”, International Journal of Geometric Methods in Modern Physics **19**, No. 06, 2250085 (2022). https://doi.org/10.1142/S0219887822500852
[81] A. Maireche, “Relativistic symmetries of the deformed Dirac equation through the improved Hulthén plus a class Yukawa potential including a Coulomb-like in deformed quantum mechanics”, Journal of Physical Studies, **26**(2), 2001-1 (2022). https://doi.org/10.30970/jps.26.2001
[82] I.S. Gradshteyn, and I.M. Ryzhik, “Table of Integrals, Series and Products, 7th. ed.; Elsevier, edited by Alan Jeffrey (the University of Newcastle upon Tyne, England) and Daniel Zwillinger (Rensselaer Polytechnic Institute USA) 2007.
[83] K. Bencheikh, S. Medjedel, and G. Vignale, “Current reversals in rapidly rotating ultracold Fermi gases”, Phys. Lett. A, **89**(6), 063620 (2014). https://doi.org/10.1103/physreva.89.063620
[84] M. Abu-Shady, T.A. Abdel-Karim, and S.Y. Ezz-Alarab, “Masses and thermodynamic properties of heavy mesons in the non-relativistic quark model using the Nikiforov–Uvarov method”, J. Egypt Math. Soc. **27**, 14 (2019). https://doi.org/10.1186/s42787-019-0014-0
[85] R. Rani, S.B. Bhardwaj, and F. Chand, “Mass Spectra of Heavy and Light Mesons Using Asymptotic Iteration Method”, Communications in Theoretical Physics, **70**(2), 179 (2018). https://doi.org/10.1088/0253-6102/70/2/179
[86] A. Maireche, “A New Asymptotic Study to the 3-Dimensional Radial Schrodinger Equation under Modified Quark-antiquark Interaction Potential”, J. Nanosci. Curr. Res. **4**(1), 1000131 (2019). https://www.hilarispublisher.com/open-access/a-new-asymptotic-study-to-the-3dimensional-radial-schrdinger-equation-under-modified-quarkantiquark-interaction-potentia.pdf
[87] A. Maireche, “Heavy-light mesons in the symmetries of extended nonrelativistic quark model”, Yanbu Journal of Engineering and Science, **17**, 51 (2019). https://doi.org/10.53370/001c.23732
[88] A. Maireche, “Analytical Expressions to Energy Eigenvalues of the Hydrogenic Atoms and the Heavy Light Mesons in the Framework of 3D-NCPS Symmetries Using the Generalized Bopp's Shift Method”, Bulg. J. Phys. **49**(3), 239 (2022). https://doi.org/10.55318/bgjp.2022.49.3.239
[89] A. Maireche, “The Impact of Deformed Space-Phase Parameters into HAs and HLM Systems with the Improved Hulthén Plus Hellmann Potentials Model in the Presence of Temperature-Dependent Confined Coulomb Potential Within the Framework of DSE”. Rev. Mex. Fis., **68**, no. 5 050702 1 (2022), https://doi.org/10.31349/RevMexFis.68.050702.
[90] A. Maireche, “The investigation of approximate solutions of Deformed Schrödinger Equations for the Hydrogenic Atoms, Heavy Quarkonium Systems ($Q\overline{Q}$ = $b\overline{b}$, $c\overline{c}$) and Diatomic molecule Bound-State Problem under Improved Exponential, Generalized, Anharmonic Cornell potential Model in NCPS symmetries”, Lat. Am. J. Phys. Educ. **12**(2), 2304 (2022). http://www.lajpe.org/jun22/16_2_04.pdf
[91] A. Maireche, “Relativistic symmetries of bosonic particles and antiparticles in the background of the position-dependent mass for the improved deformed Hulthén plus deformed type-hyperbolic potential in 3D-EQM symmetries”, East Eur. J. Phys. **4**, 201 (2022). https://periodicals.karazin.ua/eejp/article/view/20865/19710

ВПЛИВ ДЕФОРМАЦІЇ ФАЗОВОГО ПРОСТОРУ НА СПЕКТРИ ВАЖКОГО КВАРКОНІЮ В ПОКРАЩЕНОМУ ЕНЕРГЕТИЧНОМУ ПОТЕНЦІАЛІ ЗА СКІНЧЕННОЇ ТЕМПЕРАТУРНОЇ МОДЕЛІ РІВНЯННЯ ШРЕДІНГЕРА ЧЕРЕЗ МЕТОД УЗАГАЛЬНЕНОГО ЗСУВУ БОППА ТА СТАНДАРТНУ ТЕОРІЮ ЗБУРЕНЬ

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У цій роботі ми отримуємо розв’язання деформованого рівняння Шредінгера (DSE) з покращеним внутрішнім енергетичним потенціалом при кінцевій температурній моделі в 3-вимірній нерелятивістській некомутаційній системі симетрії фазового простору (3D-NRNCPS), використовуючи узагальнений метод зсуву Боппа у випадку збуреної нерелятивістської квантової хромодинаміки (pNRQCD). Отримано модифіковані енергетичні спектри зв’язаного стану для важкої кварконієвої системи, такої як чармоній $c\overline{c}$ і боттононій $b\overline{b}$ при кінцевій температурі. Встановлено, що пертурбативні розв’язки дискретного спектру чутливі до дискретних атомних квантових чисел (*j,l,s,m*) стану $Q\overline{Q}$ $(Q = c, b)$, параметрів потенціалу внутрішньої енергії $(T, \alpha_s(T), m_D(T), \beta, c)$, які є екрануючою масою Дебая $m_D(T)$, поточною константою зв’язку $\alpha_s(T)$, критичною температурою β, вільним параметром c на додаток до параметрів некомутативності (Θ,$\overline{\theta}$). Новий оператор Гамільтона в симетріях 3D-NRNCPS складається з відповідного оператора в комутативному фазовому просторі та трьох адитивних частин для спін-орбітальної взаємодії, нової магнітної взаємодії та обертального терма Фермі. Отримані власні енергетичні значення використовуються для отримання мас-спектрів важких кварконієвих систем ($c\overline{c}$ and $b\overline{b}$). Загальна повна виродженість нових енергетичних рівнів покращеного потенціалу внутрішньої енергії змінилася і стала рівною новому значенню $3n^2$ у симетріях 3D-NRNCPS замість значення n^2 у симетріях 3D-NRQM. Наші нерелятивістські результати, отримані із DSE, , будуть за можливості зіставлені з рівнянням Дірака у фізиці високих енергій.

Ключові слова: *рівняння Шредінгера; некомутативний фазовий простір; потенціал внутрішньої енергії при кінцевій температурі; метод зсуву Боппа, важкі кварконієві системи*