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ANALYTICAL SOLUTIONS TO THE SCHRÖDINGER EQUATION WITH COLLECTIVE POTENTIAL MODELS: APPLICATION TO QUANTUM INFORMATION THEORY†

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In this study, the energy equation and normalized wave function were obtained by solving the Schrödinger equation analytically utilizing the Eckart-Hellmann potential and the Nikiforov-Uvarov method. Fisher information and Shannon entropy were investigated. Our results showed higher-order characteristic behavior for position and momentum space. Our numerical results showed an increase in the accuracy of the location of the predicted particles occurring in the position space. Also, our results show that the sum of the position and momentum entropies satisfies the lower-bound Berkner, Bialynicki-Birula, and Mycieslki inequality and Fisher information was also satisfied for the different eigenstates. This study's findings have applications in quantum chemistry, atomic and molecular physics, and quantum physics.
Keywords: Schrödinger equation; Eckart-Hellmann potential; Fisher information; Shannon entropy; Nikiforov-Uvarov method
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1. Introduction

With various analytical techniques, such as the Nikiforov-Uvarov (NU) method [1-10], the asymptotic iterative method (AIM) [11], the supersymmetric quantum mechanics method (SUSQM) [12], the Nikiforov-Uvarov functional analysis (NUFA) method [13-16], the series expansion method [17-21], the WKB approximation [22-24], and so on [25], the Schrödinger equation (SE) can be solved for a variety of potentials. Our knowledge of the underlying cause of a quantum system is significantly influenced by the analytical solutions to this equation with a physical potential. This is due to the fact that the eigenvalues and eigenfunctions convey essential information about the quantum system under investigation [26, 27]. However, the exact bound state solutions of the SE are possible in some cases [28]. One can solve the SE using appropriate approximation approaches, such as the Pekeris, Greene and Aldrich, and others [29–31], to obtain the approximate solutions when the arbitrary angular momentum quantum number is not equal to zero. The eigenvalues and eigenfunctions are of great importance in the study of mass spectra of heavy mesons [32], thermodynamic properties of the system [33], and the quantum theoretic information entropies [34] among others. According to the fundamental principle of information theory put forward by Claude Shannon, the global measures of Shannon entropy and Fisher information are crucial to quantum information-theoretic measures [36]. As a result of its numerous applications in physics and chemistry, scientists have actively investigated Shannon and Fisher entropies in various fields in recent years. The theory of communication is one field in which Shannon and Fisher entropies are applied [37,38]. The theoretical foundation of Fisher information was obtained much earlier [39], but the application was unknown until Sear et al.,[40], found a link between Fisher information and the kinetic energy of a quantum system. The significance of the global measure is to investigate the uncertainty associated with the probability distribution [41,42]. The position and momentum spaces of the Shannon entropy have an entropic relation derived by Berkner, Bialynicki-Birula and Mycieslki [43] and expressed as $s_r + s_p \geq D(1 + In\pi)$, where D is the spatial dimension. In view of this, many scholars have studied the Shannon and Fisher entropies [44-46], for instance, Edet et al., [47] used a class of Yukawa potential to study the global quantum information-theoretic measurements in the presence of magnetic and Aharanov-Bohm (AB) fields. Also, Olendski [48] used the quadratic and inverse quadratic dependencies on the radius to study the Shannon quantum information entropies, Fisher information, and Onicescu energies and complexities in the position and momentum spaces for the azimuthally symmetric two-dimensional nano-ring that is placed into the combination of the transverse uniform magnetic field and the AB flux. For time-dependent harmonic vector potential. Onate et al.[49] found the exact solution to the Feinberg-Horodecki equation. Explicitly, the quantized momentum and its corresponding un-normalized wave functions were obtained. Using the Hellman-Feynman theory, expectation values of time and momentum were used to determine the Fisher information (for time and momentum) and variance (for time and momentum). Also, Under the influence of an improved expression for the Wei potential energy function, Onate et al., [50] obtained an approximate solution of the one-dimensional Klein-Gordon equation. By using specific mappings, it was possible to derive the solution of the SE from that of the Klein-Gordon equation. The computation of expectation values was used to study the Fisher information for position space and momentum space. Furthermore, Onate et al., [51] obtained an approximate bound state solution of the three-dimensional SE for a potential family together with the corresponding

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wave function, after which the Fisher information for a potential family was explicitly obtained via the methodology of expectation value and the radial expectation value. Also, some uncertainty relations that are closely related to Heisenberg-like uncertainty were obtained and numerical results were generated to justify the relations and inequalities. Onate et al., [52], obtained the solutions of the SE with Tietz-Hua potential using the Parametric NU method. The Shannon entropy and information energy were computed. Idiodi and Onate [53] studied the Shannon and Renyi information entropy for both position and momentum space and the Fisher information for the position-dependent mass SE with the Frost-Musulin potential. Onate et al [54], solved the approximate analytical solution of the SE in the framework of the parametric NU method with a hyperbolical exponential-type potential. Using the integral method, the Shannon entropy, information energy, Fisher information, and complexity measures were calculated. Edet et al., [55], investigated quantum information by a theoretical measurement approach of an Aharanov-Bohm (AB) ring with Yukawa interaction in curved space with disclination. They obtained the Shannon entropy through the eigenfunctions of the system. Ikot et al., [56] solved the SE for the Mobius square potential using the NUFA method. The Shannon entropy, Fisher information, Fisher-Shannon product and the expectation values for the Mobius square were investigated in position and momentum space. Amadi et al. [57] solved the SE with screened Kratzer potential to study the Shannon entropy and Fisher information. Their results showed that the sum of the position and momentum entropies satisfies the lower-bound Berkner, Bialynicki-Birula and Mycieslki inequality. Ikot et al., [58], solved the approximate solutions of the SE with the generalized Hulthen and Yukawa potential within the framework of the functional method. The obtained wave function and the energy levels are used to study the Shannon entropy, Renyl entropy, Fisher information, Shannon-Fisher complexity, Shannon power and Fisher-Shannon product in both position and momentum spaces for the ground and first excited states. Ikot et al. [59], studied the Shannon entropy and the Fisher Information entropies were investigated for a generalized hyperbolic potential in position and momentum spaces through the solutions of SE using the NUFA method. The position and momentum spaces for Shannon entropy and the Fisher Information entropies were calculated numerically.

Eckart potential [60], was proposed in 1930, and its application cut across molecular physics and other related areas. Numerous authors in the references [61–63] took into consideration the bound state solutions for this potential. Numerous scholars have extensively used the Hellmann potential [64] to obtain bound-state solutions in studying the condensed matter, atomic, nuclear, and particle physics [65-69]. Recently, a lot of scholars have expressed interest in the pairing of at least two potentials. The main goal of combining at least two physical potential models is to include additional physical applications and comparative analysis with previous studies [58,70,71]. Inyang et al.,[70] proposed Eckart and Hellmann potential (EHP) to study selected diatomic molecules. Furthermore, they adopted the potential to study the mass spectra and thermal properties of heavy mesons [71]. In this research, we aim at obtaining the approximate bound state analytical solutions to the SE with the Eckart plus Hellmann potential (EHP) using the NU method. The obtained energy eigenvalues and eigenfunctions will be applied to study the Shannon entropy and Fisher information. The combined potential takes the form [60,64].

$$V(r) = -\frac{A_0 e^{-\alpha r}}{1-e^{-\alpha r}} + \frac{A_1 e^{-\alpha r}}{\left(1-e^{-\alpha r}\right)^2} - \frac{A_2}{r} + \frac{A_3 e^{-\alpha r}}{r}, \tag{1}$$

where A_0, A_1, A_2, and A_3 are the strength of the potential, α is the screening parameter and r is inter-particle distance.

This paper is organized as follows: In Sect. 2 we solve the Schrödinger equation with the Eckart plus Hellmann potential to obtain the energy equation and normalized wave function. In Sect. 3, the derived eigenfunctions will be used to obtain the numerical computation of the Shannon entropy and Fisher information. In Sect. 3, we present the results and discussion. Conclusions are given in Sect. 4.

2. Analytical solutions of the Schrödinger equation with Eckart plus Hellmann potential

In this study, we adopt the NU method [1] which is based on solving the second-order differential equation of the hypergeometric type. The details can be found in Appendix A. The Schrödinger equation of a quantum physical system is characterized by a given potential $V(r)$ takes the form [72,73]

$$\frac{d^2W(r)}{dr^2} + \left[\frac{2\mu}{\hbar^2}\left(E_{nl} - V(r)\right) - \frac{l(l+1)}{r^2}\right]W(r) = 0 \tag{2}$$

E_{nl} is the energy eigenvalues of the quantum system, l is the angular momentum quantum number, μ is the reduced mass of the system, $\hbar$ is the reduced Planck's constant and r is the radial distance from the origin.

Equation (2) cannot be exactly solved using the adopted potential. To deal with the centrifugal barrier, we thus employ an approximation approach suggested by Greene-Aldrich [29]. This approximation is a good approximation to the centrifugal term which is valid for $\alpha << 1$, and it becomes

$$\frac{1}{r^2} \approx \frac{\alpha^2}{\left(1-e^{-\alpha r}\right)^2}. \tag{3}$$

Substituting Eqs. (1) and (3) into Eq. (2), Eq. (4) is obtained as

$$\frac{d^2W(r)}{dr^2}+\left[\frac{2\mu}{\hbar^2}\left(E_{nl}-\frac{A_0e^{-\alpha r}}{1-e^{-\alpha r}}-\frac{A_1e^{-\alpha r}}{\left(1-e^{-\alpha r}\right)^2}+\frac{A_2\alpha}{1-e^{-\alpha r}}-\frac{A_3e^{-\alpha r}}{1-e^{-\alpha r}}\right)+\frac{l(l+1)\alpha^2}{\left(1-e^{-\alpha r}\right)^2}\right]W(r)=0, \tag{4}$$

By using the change of variable from r to s, our new coordinate becomes

$$s=e^{-\alpha r}. \tag{5}$$

We substitute Eq. (5) into Eq. (4) and after some simplifications; Eq. (6) is gotten as

$$\frac{d^2W(s)}{ds^2}+\frac{1-s}{s(1-s)}\frac{dW(s)}{ds}+\frac{1}{s^2(1-s)^2}\left[-(\varepsilon+\beta_3)s^2+(2\varepsilon-\beta_0-\beta_1-\beta_2-\beta_3)s-(\varepsilon-\beta_0-\beta_2+\gamma)\right]W(s)=0, \tag{6}$$

where

$$\left.-\varepsilon=\frac{2\mu E_{nl}}{\alpha^2\hbar^2},\quad \beta_0=\frac{2\mu A_0}{\alpha^2\hbar^2},\quad \beta_1=\frac{2\mu A_1}{\alpha^2\hbar^2},\quad \beta_2=\frac{2\mu A_2}{\alpha\hbar^2},\quad \beta_3=\frac{2\mu A_3}{\alpha\hbar^2},\quad \gamma=l(l+1)\right\}. \tag{7}$$

Comparing Eq. (6) with Eq. (A1) we obtain the following parameters

$$\left.\begin{array}{l}\tilde{\tau}(s)=1-s,\ \sigma(\mathrm{s})=s(1-s),\ \sigma'(s)=1-2s\\ \tilde{\sigma}(\mathrm{s})=-(\varepsilon+\beta_3)s^2+(2\varepsilon-\beta_0-\beta_1-\beta_2-\beta_3)s-(\varepsilon-\beta_0-\beta_2+\gamma)\end{array}\right\}. \tag{8}$$

Substituting Eq. (8) into Eq.(A9) we have

$$\pi(s)=-\frac{s}{2}\pm\sqrt{(P_0-k)s^2+(R_0+k)s+Q_0}, \tag{9}$$

where

$$\left.\begin{array}{l}P_0=\frac{1}{4}+\varepsilon+\beta_3,\ \mathrm{R}_0=-(2\varepsilon-\beta_0-\beta_1-\beta_2-\beta_3)\\ Q_0=\varepsilon-\beta_0-\beta_2+\gamma\end{array}\right\}. \tag{10}$$

To find the constant k, the discriminant of the expression under the square root of Eq. (9) must be equal to zero. As such we have that

$$k_-=-(\beta_1-\beta_0-\beta_2+\beta_3+2\gamma)-2\sqrt{\varepsilon-\beta_0-\beta_2+\gamma}\sqrt{\frac{1}{4}+\gamma+\beta_1} \tag{11}$$

Substituting Eqs. (10) and (11) in Eq. (9) we have

$$\pi_-(\mathrm{s})=-\frac{s}{2}-\left[\left(\sqrt{\varepsilon-\beta_0-\beta_2+\gamma}+\sqrt{\frac{1}{4}+\gamma+\beta_1}\right)s-\sqrt{\varepsilon-\beta_0-\beta_2+\gamma}\right]. \tag{12}$$

Differentiating Eq. (12) gives

$$\pi_-'(\mathrm{s})=-\frac{1}{2}-\left(\sqrt{\varepsilon-\beta_0-\beta_2+\gamma}+\sqrt{\frac{1}{4}+\gamma+\beta_1}\right). \tag{13}$$

Substituting Eqs. (11) and (13) into Eq.(A10) gives

$$\lambda=\beta_0-\beta_1+\beta_2-\beta_3-2\gamma-2\sqrt{\varepsilon-\beta_0-\beta_2+\gamma}\sqrt{\frac{1}{4}+\gamma+\beta_1}-\frac{1}{2}-\left(\sqrt{\varepsilon-\beta_0-\beta_2+\gamma}+\sqrt{\frac{1}{4}+\gamma+\beta_1}\right). \tag{14}$$

With $\tau(s)$ being obtained from Eq.(A7) as

$$\tau(s)=1-2s-2\sqrt{\varepsilon-\beta_0-\beta_2+\gamma}\,s-2\sqrt{\frac{1}{4}+\gamma+\beta_1}\,s+2\sqrt{\varepsilon-\beta_0-\beta_2+\gamma}. \tag{15}$$

Differentiating Eq. (15) yields

$$\tau'(s)=-2-2\left(\sqrt{\varepsilon-\beta_0-\beta_2+\gamma}+\sqrt{\frac{1}{4}+\gamma+\beta_1}\right). \tag{16}$$

And also taking the derivative of $\sigma'(s)$ with respect to s from Eq. (8), we have

$$\sigma''(s) = -2\,. \tag{17}$$

Substituting Eqs.(16) and (17) into Eq.(A11) and simplifying, yields

$$\lambda_n = n^2 + n + 2n\sqrt{\varepsilon - \beta_0 - \beta_2 + \gamma} + 2n\sqrt{\frac{1}{4} + \gamma + \beta_1}\,. \tag{18}$$

Equating Eqs. (14) and (18) and substituting Eq. (7) yields the energy eigenvalues equation of the Eckart plus Hellmann potential as

$$E_{nl} = \frac{\alpha^2\hbar^2\left(l+l^2\right)}{2\mu} - A_0 - A_2\alpha - \frac{\hbar^2\alpha^2}{8\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\left(\frac{1}{4}+l\right)^2+\frac{2A_1\mu}{\alpha^2\hbar^2}}\right)^2 - \frac{2A_0\mu}{\alpha^2\hbar^2} + \frac{2A_1\mu}{\alpha^2\hbar^2} - \frac{2A_2\mu}{\alpha\hbar^2} + \frac{2A_3\mu}{\alpha\hbar^2} + \left(l+l^2\right)}{n+\frac{1}{2}+\sqrt{\left(\frac{1}{4}+l\right)^2+\frac{2A_1\mu}{\alpha^2\hbar^2}}}\right]^2. \tag{19}$$

2.1. Special cases

1. We set $A_0 = A_1 = 0$ and obtain the energy eigenvalues for Hellmann potential

$$E_{nl} = \frac{\alpha^2\hbar^2\left(l+l^2\right)}{2\mu} - A_2\alpha - \frac{\hbar^2\alpha^2}{8\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\left(\frac{1}{4}+l\right)^2}\right)^2 - \frac{2A_2\mu}{\alpha\hbar^2} + \frac{2A_3\mu}{\alpha\hbar^2} + \left(l+l^2\right)}{n+\frac{1}{2}+\sqrt{\left(\frac{1}{4}+l\right)^2}}\right]^2. \tag{20}$$

2. We set $A_2 = A_3 = 0$ and obtain the energy eigenvalues for Eckart potential

$$E_{nl} = \frac{\alpha^2\hbar^2\left(l+l^2\right)}{2\mu} - A_0 - \frac{\hbar^2\alpha^2}{8\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\left(\frac{1}{4}+l\right)^2+\frac{2A_1\mu}{\alpha^2\hbar^2}}\right)^2 - \frac{2A_0\mu}{\alpha^2\hbar^2} + \frac{2A_1\mu}{\alpha^2\hbar^2} + \left(l+l^2\right)}{n+\frac{1}{2}+\sqrt{\left(\frac{1}{4}+l\right)^2+\frac{2A_1\mu}{\alpha^2\hbar^2}}}\right]^2. \tag{21}$$

3. We set $A_0 = A_1 = A_3 = \alpha = 0$ and obtain the energy eigenvalues for Coulomb potential

$$E_{nl} = -\frac{\mu A_2^2}{2\hbar^2(n+l+1)^2}\,. \tag{22}$$

4. We set $A_0 = A_1 = A_2 = 0$ and obtain the energy eigenvalues for Yukawa potential

$$E_{nl} = \frac{\alpha^2\hbar^2\left(l+l^2\right)}{2\mu} - \frac{\hbar^2\alpha^2}{8\mu}\left[\frac{\left(n+\frac{1}{2}+\sqrt{\left(\frac{1}{4}+l\right)^2}\right)^2 + \frac{2A_3\mu}{\alpha\hbar^2} + \left(l+l^2\right)}{n+\frac{1}{2}+\sqrt{\left(\frac{1}{4}+l\right)^2}}\right]^2. \tag{23}$$

2.2. Wave function

To obtain the corresponding wavefunction, we consider Eqs. (A4) and (A6) and upon substituting Eqs. (8) and (15) and integrating, we get

$$\phi(\mathrm{s}) = s^{\sqrt{\varepsilon-\beta_0-\beta_2+\gamma}}\left(1-s\right)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\gamma+\beta_1}}, \tag{24}$$

$$\rho(\mathrm{s}) = s^{2\sqrt{\varepsilon-\beta_0-\beta_2+\gamma}}\left(1-s\right)^{2\sqrt{\frac{1}{4}+\gamma+\beta_1}}. \tag{25}$$

Equation 25 is known as the weight function.
By substituting Eqs. (8) and (25) into Eq. (A5) we obtain the Rodrigue's equation as

$$y_n\left(s\right) = B_{nl} s^{-2\sqrt{\varepsilon-\beta_0-\beta_2+\gamma}}\left(1-s\right)^{-2\sqrt{\frac{1}{4}+\gamma+\beta_1}} \frac{d^n}{ds^n}\left[s^{n+2\sqrt{\varepsilon-\beta_0-\beta_2+\gamma}}\left(1-s\right)^{n+2\sqrt{\frac{1}{4}+\gamma+\beta_1}}\right], \tag{26}$$

where B_{nl} is normalization constant.
Equation (26) is equivalent to

$$P_n^{\left(2\sqrt{\varepsilon-\beta_0-\beta_2+\gamma},2\sqrt{\frac{1}{4}+\gamma+\beta_1}\right)}\left(1-2s\right), \tag{27}$$

where $p_n^{(\alpha,\beta)}$ is Jacobi Polynomial
The wave function is given by

$$\psi_{nl}\left(s\right) = B_{nl} s^{\sqrt{\varepsilon-\beta_0-\beta_2+\gamma}}\left(1-s\right)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\gamma+\beta_1}} P_n^{\left(2\sqrt{\varepsilon-\beta_0-\beta_2+\gamma},2\sqrt{\frac{1}{4}+\gamma+\beta_1}\right)}\left(1-2s\right). \tag{28}$$

Using the normalization condition, we obtain the normalization constant as follows

$$\int_0^\infty \left|\psi_{nl}(r)\right|^2 dr = 1. \tag{29}$$

From our coordinate transformation of Eq. (5), we have that

$$-\frac{1}{\alpha s}\int_1^0 \left|\psi_{nl}(\mathrm{s})\right|^2 ds = 1. \tag{30}$$

By letting, $y = 1-2s$, we have

$$\frac{B_{nl}^2}{\alpha}\int_{-1}^{1}\left(\frac{1-y}{2}\right)^{2\sqrt{\varepsilon+\beta_0-\beta_1+\gamma}}\left(\frac{1+y}{2}\right)^{1+2\sqrt{\frac{1}{4}+\gamma}}\left[P_n^{\left(2\sqrt{\varepsilon+\beta_0-\beta_1+\gamma},2\sqrt{\frac{1}{4}+\gamma}\right)} y\right]^2 dy = 1. \tag{31}$$

Let

$$\left.\begin{aligned} &\eta = 1+2\sqrt{\frac{1}{4}+\gamma},\ \eta-1 = 2\sqrt{\frac{1}{4}+\gamma} \\ &u = 2\sqrt{\varepsilon+\beta_0-\beta_1+\gamma} \end{aligned}\right\}. \tag{32}$$

Substituting Eq. (32) into Eq. (31) we have

$$\frac{B_{nl}^2}{\alpha}\int_{-1}^{1}\left(\frac{1-y}{2}\right)^{\varsigma}\left(\frac{1+y}{2}\right)^{\eta}\left[P_n^{(2\mathrm{u},\mu-1)} y\right]^2 dy = 1. \tag{33}$$

According to Ebomwonyi et al. [74], integral of the form in Eq,(33) can be expresses as

$$\int_{-1}^{1}\left(\frac{1-p}{2}\right)^{x}\left(\frac{1+p}{2}\right)^{y}\left[P_n^{(2\mathrm{x},2y-1)} p\right]^2 dp = \frac{2\Gamma(x+n+1)\Gamma(y+n+1)}{n!x\Gamma(x+y+n+1)}. \tag{34}$$

Hence, comparing Eq. (33) with the standard integral of Eq.(34), we obtain the normalization constant as

$$B_{nl} = \sqrt{\frac{n!u\alpha\Gamma(\mathrm{u}+\eta+\mathrm{n}+1)}{2\Gamma(u+n+1)\Gamma(\eta+n+1)}}. \tag{35}$$

2.3. Shannon Entropy for the Eckart plus Hellmann potential

Entropy is a thermodynamic quantity representing the unavailabity of a systems thermal energy for conversion into mechanical work [53]. The Shannon entropy is defined in position and momentum spaces as [75]

$$S_r = -\int_c^b \rho(r) In\rho(r) dr\,, \tag{36}$$

and

$$S_p = -\int_c^b \rho(p) In\rho(p) dp\,, \tag{37}$$

where S_r is the position space Shannon entropy, S_p is the momentum space Shannon entropy,

$$\rho(r) = |\psi(r)|^2\,, \tag{38}$$

and

$$\rho(p) = |\psi(p)|^2\,, \tag{39}$$

are the probability densities in the position and momentum spaces, respectively. $\psi(p)$ is the wave function in the momentum coordinate obtained by the Fourier transform of $\psi(r)$. The Shannon entropic uncertainty relation proposed by Beckner,Bialynicki-Birula and Mycielski(BBM) takes the form [43]

$$S_T = S_r + S_p \geq D(1+\text{In}\pi), \tag{40}$$

where D is the spatial dimension.

The probability density's logarithmic functional measure of randomness and uncertainty in a particle's spatial localization is called Shannon entropy. The lower this entropy, the more concentrated is the wave function, the smaller the uncertainty and the higher is the accuracy in predicting the localization of the particle [74].

Using Eqs. (28) and (35) we obtained the total wavefunction as

$$\psi_{nl}(s) = \sqrt{\frac{n!u\alpha\Gamma(\text{u}+\eta+\text{n}+1)}{2\Gamma(u+n+1)\Gamma(\eta+n+1)}} e^{-\alpha r\sqrt{\varepsilon-\beta_0-\beta_2+\gamma}} \left(1-e^{-\alpha r}\right)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\gamma+\beta_1}} P_n^{\left(2\sqrt{\varepsilon-\beta_0-\beta_2+\gamma},2\sqrt{\frac{1}{4}+\gamma+\beta_1}\right)}\left(1-2e^{-\alpha r}\right). \tag{41}$$

The normalized wave function for two low lying states $n = 0,1$ is given as

$$\psi_0(s) = \sqrt{\frac{u\alpha\Gamma(\text{u}+\eta+1)}{2\Gamma(u+1)\Gamma(\eta+1)}} e^{-\alpha r\sqrt{\varepsilon-\beta_0-\beta_2+\gamma}} \left(1-e^{-\alpha r}\right)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\gamma+\beta_1}}\,, \tag{42}$$

and

$$\psi_1(s) = \frac{1}{1+2(\varepsilon-\beta_0-\beta_2+\gamma)} \sqrt{\frac{u\alpha(1+2(\varepsilon-\beta_0-\beta_2+\gamma))\Gamma(\text{u}+\eta+2)}{(1+(\varepsilon-\beta_0-\beta_2+\gamma))2\Gamma(u+n+1)\Gamma(\eta+n+1)}}\,, \\ e^{-\alpha r\sqrt{\varepsilon-\beta_0-\beta_2+\gamma}} \left(1-e^{-\alpha r}\right)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\gamma+\beta}} \left(1-2e^{-\alpha r}\right) \tag{43}$$

where $\Gamma(x)$ is the gamma function given as $\Gamma(x) = \int_0^\infty t^{x-1}e^{-x}dx$ [76].

The corresponding normalized wave function in the momentum space for two low lying states $n = 0,1$ is obtained as [77]

$$\psi_o(p) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \psi_0(r) e^{-ipr} dr\,, \tag{44}$$

$$\psi_0(p) = \sqrt{\frac{u\alpha\Gamma(\text{u}+\eta+1)}{2\Gamma(u+1)\Gamma(\eta+1)}} \frac{\Gamma\left(\frac{ip}{\alpha}+\varepsilon-\beta_0-\beta_2+\gamma\right)\Gamma\left(1+\varepsilon-\beta_0-\beta_2+\gamma\right)}{\sqrt{2\pi}\alpha\Gamma\left(1+\frac{ip}{\alpha}+\varepsilon-\beta_0-\beta_2+\gamma\right)}\,, \tag{45}$$

$$\psi_1(p) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \psi_1(r) e^{-ipr} dr\,, \tag{46}$$

$$\psi_1(p)=\sqrt{\frac{u\alpha(1+2(\varepsilon-\beta_0-\beta_2+\gamma))\Gamma(\mathrm{u}+\eta+2)}{(2(\varepsilon-\beta_0-\beta_2+\gamma))2\Gamma(u+n+1)\Gamma(\eta+n+1)}}\left(\frac{\left(-2ip+\alpha(1+2(\varepsilon-\beta_0-\beta_2+\gamma))\right)\Gamma\left(\frac{ip}{\alpha}+\varepsilon-\beta_0-\beta_2+\gamma\right)}{\sqrt{2\pi}\alpha^2(1+2(\varepsilon-\beta_0-\beta_2+\gamma))\Gamma\left(2+\frac{ip}{\alpha}+\varepsilon-\beta_0-\beta_2+\gamma\right)}\right). \tag{47}$$

2.4. Fisher Information theory for the Eckart plus Hellmann potential

We examine the Fisher information in position and momentum spaces. Fisher information is the sole component of the local measure, and is mainly concerned with local changes that occur in probability density [57,78]. Density functional is important for the investigation of Fisher information [79]. It is stated as:

$$I_r=\int\frac{\left|\rho'(r)\right|^2}{\rho(r)}dr=4\int\left|\psi'(r)\right|^2dr=4\langle p^2\rangle-2(2l+1)|m|\langle r^{-2}\rangle, \tag{48}$$

$$I_p=\int\frac{\left|\rho'(p)\right|^2}{\rho(p)}dp=4\int\left|\psi'(p)\right|^2dp=4\langle r^2\rangle-2(2l+1)|m|\langle p^{-2}\rangle. \tag{49}$$

Fisher information inequality becomes [80]

$$I_rI_p\geq 9\left[2-\frac{2l+1}{l(l+1)}|m|\right]^2\geq 36. \tag{50}$$

We solve Eqs. (48) and (49) numerically, which are complicated to solve analytically due to the form of the integral.

3. RESULTS AND DISCUSSION

In this section, we will discuss our numerical results. For both cases, the screening criterion was set to $0.1\leq\alpha\leq0.9$. These parameters were selected in order to compare results [57].

Our results were obtained numerically. Fisher information and Shannon entropy give important details about the precision and degree of uncertainty in particle localization predictions. Lower Shannon entropy denotes greater stability, higher localization, and reduced uncertainty. The kinetic energy and Fisher information are related, and more Fisher information indicates greater localization and energy fluctuation. For the various values of α, the numerical results for Shannon entropy and Fisher information are shown in Tables 1 and 2, respectively.

Table 1. Numerical values of Shannon entropy for Eckart plus Hellmann potential

n	α	S_r	S_p	S_T
0	0.1	5.78107	0.692356	6.4734260
	0.2	5.97095	0.498294	6.4692440
	0.3	6.20766	0.267158	6.4748180
	0.4	6.49426	-0.017446	6.4768131
	0.5	6.85881	-0.378718	6.4800920
	0.6	7.34806	-0.856921	6.4911890
	0.7	8.03556	-1.529530	6.5060300
	0.8	9.09978	-2.571370	6.5284100
	0.9	11.0879	-4.524130	6.5637700
1	0.1	9.67144	-2.915280	6.7561600
	0.2	11.7253	-5.091890	6.6334100
	0.3	11.4772	-4.873510	6.6036900
	0.4	8.71251	-2.048560	6.6639500
	0.5	7.39803	-0.709438	6.6885920
	0.6	6.50485	-0.198122	6.3067280
	0.7	5.82308	0.887327	6.7104070
	0.8	5.26825	1.445710	6.7139600
	0.9	4.80376	1.914240	6.7180000

The Shannon entropy values show a deceasing order in the position space, which signifies a lower uncertainty and higher accuracy in predicting localization and the stability. This is complimented in the momentum space by an increasing Shannon entropy. For $n=1$, it increased and decreased afterward. This similar behavior is also observed in the momentum spaces. Negative values mean that the Shannon entropy is highly localized [57]. The numerical analysis of Fisher information for $n=0$ and $n=1$ is shown in Table 2. Similar phenomena are seen in momentum spaces as well and negative values indicate a strongly localized Shannon entropy. Table 2 displays the numerical analysis of Fisher information for $n=0$ and $n=1$ indicating ground and first excited states respectively. Here, there was a similar pattern of behavior, and the alternative increase and decrease for $n=1$ are noticed. The increasing Fisher information observed in these different states implies an increasing localization. In both cases, the Shannon entropy uncertainty relation condition is satisfied as seen in Eq. (40) and Fisher uncertainty relation is satisfied as seen in Eq. (50).

Table 2. Numerical values of Fisher information for Eckart plus Hellmann potential

n	α	I_r	I_p	$I_r I_p$
0	0.1	1.195670	33.4215	39.96108
	0.2	1.052470	38.0402	40.03616
	0.3	0.903119	44.4792	40.17001
	0.4	0.748496	54.0185	40.43263
	0.5	0.590304	69.2190	40.86025
	0.6	0.431695	96.4388	41.63214
	0.7	0.278445	154.136	42.91839
	0.8	0.141132	318.584	44.96239
	0.9	0.038628	1212.07	46.81983
1	0.1	0.137107	465.101	63.76860
	0.2	0.032212	1782.30	57.41144
	0.3	0.037720	1516.76	57.21218
	0.4	0.251035	241.175	60.54336
	0.5	0.606999	100.787	61.17761
	0.6	1.108780	55.6926	61.75084
	0.7	1.753940	35.4030	62.09332
	0.8	2.542350	24.5121	62.31833
	0.9	3.473940	17.9847	62.47776

4. CONCLUSION

In this research, the Schrödinger equation is solved with the Eckart plus Hellmann potential to obtain the energy equation and normalized wave function. We studied the charactertic properties of Shannon entropy and Fisher information for the position and momentum spaces for ground state and first excited state. Our results was presented numerically. We observed a similar behavior for Shannon entropy and Fisher information values. This behavior is related to the probability density distribution's concentration. Our findings showed that several eigenstates had negative values in the position space. This implies a higher localization for the collective potential models. The potential models also show increasing accuracy in predicting particle localization in the position space of Shannon entropy and Fisher information. This research can be extended to other global measures, such as Renyi entropy, Tsallis entropy etc.

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Authors Contributions.

FA and EPI conceived and designed the study, acquired, analyzed and interpreted the data and handled the review; EAI and KML handled the computational analysis. All authors read and approved the final manuscript.

Competing interests.

The authors declare that they have no competing interests.

APPENDIX A

Review of Nikiforov-Uvarov (NU) method

The NU method was proposed by Nikiforov and Uvarov [1] to transform Schrödinger-like equations into a second-order differential equation via a coordinate transformation $s=s(r)$, of the form

$$\psi''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s)=0, \tag{A1}$$

where $\tilde{\sigma}(s)$, and $\sigma(s)$ are polynomials, at most second degree and $\tilde{\tau}(s)$ is a first-degree polynomial. The exact solution of Eq.(A1) can be obtained by using the transformation.

$$\psi(s)=\phi(s)y(s). \tag{A2}$$

This transformation reduces Eq.(A1) into a hypergeometric-type equation of the form

$$\sigma(s)y''(s)+\tau(s)y'(s)+\lambda y(s)=0. \tag{A3}$$

The function $\phi(x)$ can be defined as the logarithm derivative

$$\frac{\phi'(s)}{\phi(s)}=\frac{\pi(s)}{\sigma(s)}. \tag{A4}$$

With $\pi(s)$ being at most a first-degree polynomial. The second part of $\psi(s)$ being $y(s)$ in Eq. (A2) is the hypergeometric function with its polynomial solution given by Rodrigues relation as

$$y(s)=\frac{B_{nl}}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\rho(s)\right], \tag{A5}$$

where B_{nl} is the normalization constant and $\rho(s)$ the weight function which satisfies the condition below;

$$(\sigma(s)\rho(s))'=\tau(s)\rho(s), \tag{A6}$$

where also

$$\tau(s)=\tilde{\tau}(s)+2\pi(s). \tag{A7}$$

For bound solutions, it is required that

$$\tau'(s)<0. \tag{A8}$$

The eigenfunctions and eigenvalues can be obtained using the definition of the following function $\pi(s)$ and parameter λ, respectively:

$$\pi(s)=\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\pm\sqrt{\left(\frac{\sigma'(s)-\tilde{\tau}(s)}{2}\right)^2-\tilde{\sigma}(s)+k\sigma(s)}, \tag{A9}$$

and

$$\lambda=k_-+\pi'_-(s). \tag{A10}$$

The value of k can be obtained by setting the discriminant in the square root in Eq. (A9) equal to zero. As such, the new eigenvalues equation can be given as

$$\lambda+n\tau'(s)+\frac{n(n-1)}{2}\sigma''(s)=0,(n=0,1,2,...). \tag{A11}$$

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АНАЛІТИЧНІ РІШЕННЯ РІВНЯННЯ ШРЕДІНГЕРА З МОДЕЛЯМИ КОЛЕКТИВНОГО ПОТЕНЦІАЛУ: ЗАСТОСУВАННЯ ДО КВАНТОВОЇ ТЕОРІЇ ІНФОРМАЦІЇ

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У цьому дослідженні шляхом аналітичного розв'язання рівняння Шредінгера з використанням потенціалу Екарта-Гельмана та методу Нікіфорова-Уварова були отримані рівняння енергії та нормалізована хвильова функція. Досліджено інформацію Фішера та ентропію Шеннона. Наші результати показали характерну поведінку вищого порядку для положення у просторі та імпульсу. Наші чисельні результати показали підвищення точності визначення місця розташування передбачуваних частинок, що зустрічаються у положенні в просторі. Крім того, наші результати показують, що сума ентропій позиції та імпульсу задовольняє нижню межу нерівності Беркнера, Бялиніцького-Бірулі та Міческі, а інформація Фішера також задовольняється для різних власних станів. Висновки цього дослідження знайдуть застосування в квантовій хімії, атомній і молекулярній фізиці та квантовій фізиці.

Ключові слова: рівняння Шредінгера; потенціал Екарта-Гельмана; Інформація Фішера; ентропія Шеннона; Метод Нікіфорова-Уварова