East European Journal of Physics. **4**. 200-212 (2022)
DOI:10.26565/2312-4334-2022-4-21 ISSN 2312-4334

RELATIVISTIC SYMMETRIES OF BOSONIC PARTICLES AND ANTIPARTICLES IN THE BACKGROUND OF THE POSITION-DEPENDENT MASS FOR THE IMPROVED DEFORMED HULTHÉN PLUS DEFORMED TYPE-HYPERBOLIC POTENTIAL IN 3D-EQM SYMMETRIES[†]

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Received October 10, 2022; revised, November 12, 2022; accepted November 16, 2022

The bound state solutions of the deformed Klien-Gordon equation have been determined in the three-dimensional extended relativistic quantum mechanics 3D-ERQM symmetries using position-dependent mass (PDM) with unequal scalar and vector potential for the improved Hulthén plus improved deformed type-hyperbolic potential (PDM-SVID(H-TP)) models. PDM with unequal scalar and vector potential for the Hulthén plus deformed type-hyperbolic potential (PDM-(SVH-DTP)) models, as well as a combination of radial terms, which are coupled with the coupling $\mathbf{L\Theta}$, which explains the interaction of the physical features of the system with the topological deformations of space-space. The new relativistic energy eigenvalues have been derived using the parametric Bopp shift method and standard perturbation theory which is sensitive to the atomic quantum numbers (j,l,s,m), mixed potential depths (V_0,S_0,V_1,S_1), the rest, and perturbed mass $\left(m_{0,}m_1\right)$, the screening parameter's inverse α, and noncommutativity parameters $\left(\Theta,\tau,\chi\right)$. Within the framework of 3D-ERQM symmetries, we have treated certain significant particular instances that we hope will be valuable to the specialized researcher. We have also treated the nonrelativistic limit and applied our obtained results to generate the mass spectra of heavy-light mesons (HLM) such as $c\bar{c}$ and $b\bar{b}$ under PDM-SE with improved deformed Hulthén plus improved hyperbolic potential (PDM-ID(H-TP)) models. When the three simultaneous limits $\left(\Theta,\tau,\chi\right)$ were applied, we recovered the normal results of relativistic in the literature ($0,0,0$) for the PDM-ID(H-TP)) models.

Keywords: Klien-Gordon equation, deformed Hulthén plus deformed type-hyperbolic potential, heavy-light mesons, Noncommutative quantum mechanics and Bopp's shift method, Canonical noncommutativity
PACS: 03.65.−w;03.65.Ge;05.30.Jp

1. INTRODUCTION

One of the significant issues in quantum mechanics (QM) and noncommutative quantum mechanics (NCQM) or extended quantum mechanics (EQM) is the investigation of solutions to the nonrelativistic Schrödinger equation (SE) or relativistic Klien-Gordon (KG), Dirac and Duffin-Kemmer-Petiau equations for a particle with spin 0, 1/2 or (1,2...) under the real physical potentials. The hyperbolic and Hulthén potentials are considered to be one of the most important interactions that have received great attention. It has been the subject of an in-depth study by many researchers within the framework of fundamental equations [1-5] whether it is a single treatment or a combination of both. In their study of the bound and scattering states of the KGE with deformed Hulthén plus deformed hyperbolical potential for arbitrary states, Ikot *et al.* used supersymmetry quantum mechanics and factorization techniques [6]. The variable mass formalism provides relevant and practical theoretical predictions of a variety of experimental properties for many-body quantum systems for this goal [7,8]. The effective mass notion has been applied to numerous important issues in the literature, including nuclei, metallic clusters, ^{3}He clusters, quantum liquids, and nuclei [9-13]. In the present work, we aim to investigate the solution of KG and SE with deformed Hulthén plus deformed-type hyperbolic potential in 3D-ERQM and 3D-ENRQM symmetries to develop the physical concepts in ref. [6]. We aspire through this work to reveal more new applications within the framework of extended postulates that include more comprehensive axioms than we know about relativistic quantum mechanics (see below). These new postulates were connected to the deformation space-space and phase-phase. The divergence problem of the standard model, gravity quantization, the problem of unifying it with the rest of the fundamental interactions, and other significant physical problems have emerged despite the brilliant successes of quantum mechanics in treating physical and chemical systems in various research fields [14-21]. It should be mentioned that before the renormalization approach was created and gained popularity, Heisenberg proposed the idea of extended noncommutativity to the coordinates as a possible treatment for eliminating the limitless number of field theories in 1930. Snyder published the first work on QFT's history in 1947 [22], and Connes introduced its geometric analysis in 1991 and 1994 [23,24] to standardize QFT. I believe that this research will contribute to further subatomic scale investigations and scientific knowledge of elementary particles. The position-dependent mass with unequal scalar and vector potential for the improved deformed Hulthén plus improved type-hyperbolic potential (PDM-SVID(H-TP)) models in the 3D-ERQM symmetries was motivated by the fact that it had not been reported in the literature for bosonic particles and antiparticles.

[†] ***Cite as:*** A. Maireche, East Eur. J. Phys. **4**, 200 (2022), https://doi.org/10.26565/2312-4334-2022-4-21

The following are the vector and scalar that will be used in this study $F_{ht}(r) \equiv (V_{ht}(r_{nc}), S_{ht}(r_{nc})$ and $m_{ht}(r_{nc})$ which are unified in the following form:

$$F(r_{nc}) = F_{ht}(r) - \frac{1}{2r}\frac{\partial F_{ht}(r)}{\partial r}\mathbf{L}\Theta + O(\Theta^2), \tag{1}$$

where $V_{ht}(r), S_{ht}(r)$ and $m_{ht}(r)$ are the (vector, scalar) potentials and PDM, in 3D-RQM known in the literature [6]:

$$\begin{cases} V_{ht}/S_{ht} = -\dfrac{V_0/S_0 e^{(-2\alpha r)}}{1-qe^{(-2\alpha r)}} + \dfrac{V_1/S_1\left(1+qe^{(-2\alpha r)}\right)}{1-qe^{(-2\alpha r)}} \\ m_{ht}(r_{nc}) = m_0 + \dfrac{m_1}{1-qe^{(-2\alpha r)}} \end{cases}, \tag{2}$$

where V_0/V_1 stand for the potential wells' depths, q for deformation, and α is the screening parameter's inverse, m_0 is the integration constant (rest mass of the bosonic particles and antiparticles), m_1 is the perturbed mass, $(r_{nc}$ and $r)$ are the distances in the EQM and usual QM symmetries, respectively.

The coupling $\mathbf{L\Theta}$ is the scalar product of the usual components of the angular momentum operator $\mathbf{L}(L_x, L_y, L_z)$ and the modified noncommutativity vector $\Theta(\theta_{12}, \theta_{23}, \theta_{13})/2$ which present the noncommutativity elements parameter. In the case of G_{NC}, the noncentral generators can be suitably realized as self-adjoint differential operators (x_μ^{nc}, p_ν^{nc}) in 3D-EQM symmetries. NC canonical commutations in a variety of canonical structures satisfying a deformed algebra of the form (we have used the natural units $\hbar = c = 1$) [25-30]:

$$\begin{cases} \left[x_\mu^{nc}\overset{*}{,} p_\nu^{nc}\right] = i\hbar_{eff}\delta_{\mu\nu} \\ \left[x_\mu^{nc}\overset{*}{,} x_\nu^{nc}\right] = i\varepsilon_{\mu\nu}\theta \end{cases}. \tag{3}$$

The corresponding generalizing momentums (x_μ and p_μ) in the usual QM symmetries, respectively. Here $\delta_{\mu\nu}$ is the Kronecker symbol, $(\mu, \nu = 1,2,3)$, $\theta_{\mu\nu}$ is antisymmetric real constant (3×3) matrices with the dimensionality (length)^2 parameterizing the deformation of space-space, $\varepsilon_{\mu\nu}$ is the Levi-Civita symbol ($\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu} = 1$ for $\mu \neq \nu$ and $\varepsilon_{\varepsilon\varepsilon} = 0$), and $\theta \in \mathrm{R}$ is the noncommutative parameter which measures the non-commutativity of coordinates, $\hbar_{eff} \cong \hbar$ is the effective Planck constant. In the first order of the noncommutativity parameter $\varepsilon^{\mu\nu}\theta$, the scalar product in 3D-EQM symmetries is expressed in terms $(h*g)(x)$ as follows [31-34]:

$$(h*g)(x) \approx (hg)(x) - i\varepsilon^{\mu\nu}\theta/2\partial_\mu^x h \partial_\nu^x g\Big|_{x^\mu = x^\nu}. \tag{4}$$

The outline of the paper is as follows: Sect. 2 presents an overview of the 3D-KGE under the PDM-SVID(H-TP) model. Sect. 3 is devoted to investigating the 3D-DKGE using the well-known Bopp's shift method to obtain the effective potential of the PDM-SVID(H-TP) model. Furthermore, using standard perturbation theory, we find the expectation values of some radial terms to calculate the corrected relativistic energy generated by the effect of the perturbed effective potential $W_{pert}^{ht}(r)$, and we derive the global corrected energies for *bosonic particles* and *bosonic antiparticles* whose spin quantum number has an integer value ($0,1,2...$). Sect. 4 is reserved for the study of important relativistic particular cases in 3D-ERQM symmetries. The next section is reserved for the nonrelativistic limits for PDM-SVID(H-TP) model*s* in 3D-ENRQM symmetries and we apply these results to generate mass spectra of HLM systems. Finally, we present our conclusion in Sec. 7.

2. AN OVERVIEW OF KGE UNDER THE PDM-SVD(H-TP) MODEL IN RQM SYMMETRY

The radial component $u_{nl}(r)$ of the wave function solution $\Psi_{nl}(r,\theta,\phi)$ satisfies the differential equation below [6]:

$$\left(\frac{d^2}{dr^2} + E_{nl}^2 - m_0^2 - W(r)\right)u_{nl}(r) = 0. \tag{5}$$

The effective potential $W_{eff}^{ht}(r)$ is determined from:

$$W_{eff}^{ht}(r)=\frac{\omega_1\exp(-4\alpha r)+\omega_2\exp(-2\alpha r)+\omega_3}{\left(1-q\exp(-2\alpha r)\right)^2}. \tag{6}$$

The parameters ω_1, ω_2 and ω_3 are determined in Ref. [6] as a function of ($V_0, S_0, V_1, S_1, m_0, m_1$). The author of this Ref. used the SUSYQM and factorization methods to obtain the expression $u_{nl}(r)$ as a function of generalized Jacobi polynomial $P^{(u,v)}(x)$ in RQM symmetries. We reformulate the relativistic wave function $\Psi_{nl}(r,\theta,\phi)$ in terms of the hypergeometric polynomials ${}_2F_1\,(-n, n+2\sqrt{\chi_{nl}^3}+2\beta_{nl}+1; 1+2\sqrt{\chi_{nl}^3}, z)$ as,

$$\Psi_{nl}=N_{nl}^{n}Y_{m}^{l}(\theta,\phi)\frac{z^{\sqrt{\chi_{nl}^3}}}{r}(1-z)^{\beta_{nl}}\;{}_2F_1\,(-n, n+2\sqrt{\chi_{nl}^3}+2\beta_{nl}+1; 1+2\sqrt{\chi_{nl}^3}, z), \tag{7}$$

where z equal $q\exp(-2\alpha r)$ while N_{nl} and (χ_{nl}^1, χ_{nl}^2 and χ_{nl}^3) are given by:

$$\begin{cases}\chi_{nl}^1=\dfrac{\omega_1}{4\alpha^2q^2}-\dfrac{E_{nl}^2-m_0^2}{4\alpha^2}, \chi_{nl}^2=-\dfrac{\omega_2}{4\alpha^2q^2}-\dfrac{E_{nl}^2-m_0^2}{4\alpha^2}\\ \chi_{nl}^3=\dfrac{\omega_3}{4\alpha^2q^2}-\dfrac{E_{nl}^2-m_0^2}{4\alpha^2}, \text{ and } \dfrac{N_{nl}\Gamma\left(n+2\sqrt{\chi_{nl}^3}+1\right)}{n!\Gamma\left(2\sqrt{\chi_{nl}^3}+1\right)},\end{cases} \tag{8}$$

with β_{nl} equal $\sqrt{\frac{1}{4}+\chi_{nl}^1-\chi_{nl}^2+\chi_{nl}^3}$. We obtained the energy for bosonic particles E_{nl}^{+} and bosonic antiparticles E_{nl}^{-}, from the square root of the equation of energy [6]:

$$E_{nl}^2-m_0^2=-\frac{1}{4q^2}\left(\frac{\omega_3q^2-\omega_1}{2\alpha\sigma}+2\alpha(n+\sigma)\right)^2+\omega_3, \tag{9}$$

where $\sigma=\frac{q}{2}\left(1+\sqrt{1+\frac{1}{\alpha^2q^2}\left(\omega_1+q\omega_2-q^2\omega_3\right)}\right)$

3. SOLUTIONS OF PDM-SVID(H-TP) MODELS IN 3D-ERQM SYMMETRIES

By applying the new principles which we have seen in the introduction, Eqs. (3) and (4), summarized in new relationships MASCCCRs and the notion of the Weyl-Moyal star product. These data allow us to rewrite the usual radial KG equations in Eq. (5) in 3D-ERQM symmetries as follows:

$$\left(\frac{d^2}{dr^2}+E_{nl}^2-m_0^2-W_{eff}^{ht}(r)\right)*u_{nl}(r)=0. \tag{10}$$

There are two approaches to including non-commutativity in the quantum field theory: either through the Moyal product on the space of ordinary functions or by redefining the field theory on a coordinate operator space that is inherently noncommutative [35-37]. It is known to specialists that the star product can be translated into the ordinary product known in the literature using what is called Bopp's shift method. F. Bopp was the first to consider pseudo-differential operators obtained from a symbol by the quantization rules $(x,p)\;\rightarrow\;\left(\hat{x}=x-\frac{i}{2}\partial_p, \hat{p}=x+\frac{i}{2}\partial_x\right)$ instead of ordinary correspondence $(x,p)\;\rightarrow\;\left(\hat{x}=x, \hat{p}=x+\frac{i}{2}\partial_x\right)$, respectively. This procedure is known as Bopp's shifts (BS) method, and this quantization procedure is known as Bopp quantization [38-45]. It is worth motioning that the BS method permutes us to reduce Eq. (10) in the simplest form:

$$\left(\frac{d^2}{dr^2}+E_{nl}^2-m_0^2-W_{eff}^{ht}(r_{nc})\right)u_{nl}(r)=0. \tag{11}$$

The Taylor expansion of $W\left(r_{nc}\right)$ can be expressed as in the 3D-ERQM symmetries, as [42-52]:

$$W_{eff}^{ht}\left(r_{nc}\right)=W_{eff}^{ht}\left(r\right)-\frac{\partial W_{eff}^{ht}\left(r\right)}{2r\partial r}\mathbf{L}\Theta+O\left(\Theta^{2}\right). \tag{12}$$

Substituting Eq. (12) into Eq. (11), we obtain the following, as in the Schrödinger equation:

$$\left(\frac{d^{2}}{dr^{2}}+E_{nl}^{2}-m_{0}^{2}-W_{eff}^{ht}\left(r\right)-W_{ht}^{pert}\left(r\right)\right)u_{nl}\left(r\right)=0, \tag{13}$$

with

$$W_{ht}^{pert}\left(r\right)=-\frac{1}{2r}\frac{\partial W_{eff}^{ht}\left(r\right)}{\partial r}\mathbf{L}\Theta+O\left(\Theta^{2}\right). \tag{14}$$

By comparing Eqs. (5) and (11), we observe an additive potential $W_{ht}^{pert}\left(r\right)$ dependent on new radial terms, which are coupled with the coupling $\mathbf{L\Theta}$ that explains the interaction of the physical features of the system with the topological deformations of space-space:

$$W_{ht}^{pert}\left(r\right)=\left(\begin{array}{l}\frac{2\alpha\omega_{1}\exp\left(-4\alpha r\right)}{r\left(1-qe^{\left(-2\alpha r\right)}\right)^{2}}+\frac{\alpha\omega_{2}\exp\left(-2\alpha r\right)}{r\left(1-q\exp\left(-2\alpha r\right)\right)^{2}}+\frac{2\alpha q\omega_{1}\exp\left(-6\alpha r\right)}{r\left(1-q\exp\left(-2\alpha r\right)\right)^{3}}+\\+\frac{2\alpha q\omega_{2}\exp\left(-4\alpha r\right)}{r\left(1-q\exp\left(-2\alpha r\right)\right)^{3}}+\frac{2\alpha q\omega_{3}\exp\left(-2\alpha r\right)}{r\left(1-q\exp\left(-2\alpha r\right)\right)^{3}}\end{array}\right)\mathbf{L}\Theta+O\left(\Theta^{2}\right), \tag{15}$$

Eq. (13) cannot be solved analytically for any state $l \neq 0$ because of the centrifugal term and the studied potential itself. The effective perturbative potential $W_{ht}^{pert}\left(r\right)$ in Eq. (15) has a strong singularity $r \to 0$, we need to use the suitable approximation of the centrifugal term proposed by Kurniawan *et al.* [46] and applied by Ikot *et al.* [47]. The radial part of the 3D-DKGE with the PDM-SVID(H-TP) model*s* contains the centrifugal term $l(l+1)/r^{2}$ and $l(l+1)/r^{4}$ since we assume $l \neq 0$. However, the PDM-SVID(H-TP) model is a kind of potential that cannot be solved exactly when the centrifugal term is taken into account unless $l=0$ is assumed. The conventional approximation used in this paper is as follows:

$$1/r^{2}\approx\alpha^{2}/\sinh_{q}^{2}\left(\alpha r\right)=4\alpha^{2}/\left(1-z\right)^{2}. \tag{16}$$

This gives the perturbative effective potential as follows:

$$W_{ht}^{pert}\left(r\right)=\left(\frac{\beta_{1}z^{2}}{\left(1-z\right)^{3}}+\frac{\beta_{1}z}{\left(1-z\right)^{3}}+\frac{\beta_{3}z^{3}}{\left(1-z\right)^{4}}+\frac{\beta_{4}z^{2}}{\left(1-z\right)^{4}}+\frac{\beta_{5}z}{\left(1-z\right)^{4}}\right)\mathbf{L\Theta}+O\left(\Theta^{2}\right), \tag{17}$$

with $\beta_{1}=\frac{4\alpha^{2}\omega_{1}}{q^{2}}$, $\beta_{2}=\frac{2\alpha^{2}\omega_{2}}{q}$, $\beta_{3}=\frac{4\alpha^{2}\omega_{1}}{q^{2}}$, $\beta_{4}=\frac{4\alpha^{2}\omega_{2}}{q}$ and $\beta_{5}=4\alpha^{2}\omega_{3}$.

The PDM-SVID(H-TP) model is extended by including new radial terms $\frac{z^{2}}{\left(1-z\right)^{3}}$, $\frac{z}{\left(1-z\right)^{3}}$, $\frac{z^{3}}{\left(1-z\right)^{4}}$, $\frac{z^{2}}{\left(1-z\right)^{4}}$ and $\frac{z}{\left(1-z\right)^{4}}$ to become PDM-SVID(H-TP) model*s* in 3D-ERQM symmetries. The new additive part $W_{ht}^{pert}\left(r\right)$ is also proportional to the infinitesimal coupling $\mathbf{L\Theta}$, this is logical from a physical point of view because it explains the interaction between the physical properties of the studied potential $\mathbf{L}$ and the topological properties resulting from the deformation of space-space $\mathbf{\Theta}$. This allows us to consider the additive effective potential as a perturbation potential compared with the main potential $W_{ht}\left(r\right)$ (parent potential operator) in the symmetries of 3D-ERQM symmetries, that is, the inequality $W_{ht}^{pert}\left(r\right)\prec\prec W_{ht}\left(r\right)$ has become achieved. That is all the physical justifications for applying the time-independent perturbation theory become satisfied. This allows us to give a complete prescription for determining the energy level of the generalized $\left(n,l,m\right)^{th}$ excited states.

3.1. The expectation values $W_{ht}^{pert}(r)$ in the 3D-ERQM symmetries

In this subsection, we want to apply perturbative theory in the first order to find the expectation values $A_{(nlm)}^{1ht}$, $A_{(nlm)}^{2ht}$, $A_{(nlm)}^{3ht}$, $A_{(nlm)}^{4ht}$ and $A_{(nlm)}^{5ht}$ for bosonic particles and bosonic antiparticles taking into account the unperturbed $\Psi_{nl}(r,\theta,\phi)$ which we have seen previously in Eq. (7):

$$\begin{cases} A_{(nlm)}^{1ht} = N\int_0^{+q} z^{2\sqrt{\chi_{nl}^3}+2-1}(1-z)^{2\beta_{nl}-2-1} F dz \\ A_{(nlm)}^{2ht} = N\int_0^{+q} z^{2\sqrt{\chi_{nl}^3}+1-1}(1-z)^{2\beta_{nl}-2-1} F dz \\ A_{(nlm)}^{3ht} = N\int_0^{+q} z^{2\sqrt{\chi_{nl}^3}+3-1}(1-z)^{2\beta_{nl}-3-1} F dz \\ A_{(nlm)}^{4ht} = N\int_0^{+q} z^{2\sqrt{\chi_{nl}^3}+2-1}(1-z)^{2\beta_{nl}-3-1} F dz \\ A_{(nlm)}^{5ht} = N\int_0^{+q} z^{2\sqrt{\chi_{nl}^3}+1-1}(1-z)^{2\beta_{nl}-3-1} F dz \end{cases}, \tag{18}$$

with

$$F \equiv \left[{}_2F_1\left(-n, n+2\sqrt{\chi_{nl}^3}+2\beta_{nl}+1; 1+2\sqrt{\chi_{nl}^3}, z\right)\right]^2,$$

and $N = \frac{N_{nl}^{n2}}{2\alpha}$.

We have used useful abbreviations $\langle A\rangle_{(nlm)}^{iht} = \langle n,l,m\|A\|n,l,m\rangle$ to avoid the extra burden of writing. Furthermore, we have introduced the change of variable $z = q\exp(-2\alpha r)$. This maps the region $0 \le r \prec \infty$to $0 \le z \le q$ and allows us to obtain $dr = -1/2\frac{dz}{\alpha z}$. We can evaluate the above integrals either in a recurrence way through the physical values of the principal quantum number ($n = 0,1,...$) and then generalize the result to the general $(n,l,m)^{th}$ excited state or we use the method proposed by Dong *et al.* [48] and applied by Zhang [49], to obtain the general excited state directly. We calculate the integrals in Eqs. (20) with help of the special integral formula:

$$\int_0^1 z^{\xi-1}(1-s)^{\tau-1}\left[{}_2F_1(c_1,c_2;c_3;z)\right]^2 dz = \frac{\Gamma(\xi)\Gamma(\tau)}{\Gamma(\xi+\tau)}\ {}_3F_2(c_1, c_2, \beta; c_3, \beta+\alpha; 1), \tag{19}$$

here ${}_3F_2(c_1, c_2, \beta; c_3, \beta+\alpha; 1)$ equal $\sum_{n=0}^{+\infty}\frac{(c_1)_n(c_2)_n(\tau)_n}{(c_3)_n(\tau+\xi)n!}$, the symbol $(c_1)_n$ denotes the rising factorial or Pochhammer symbol while $\Gamma(\xi)0$ denoting the usual Gamma function. For the case $q = 1$ and by identifying Eqs. (18) with the integrals in Eq. (21), we obtain the following results:

$$\begin{cases} A_{(nlm)}^{1ht} = \frac{\Gamma\left(2\sqrt{\chi_{nl}^3}+2\right)\Gamma(2\beta_{nl}-2)}{N^{-1}\Gamma(B_{nl})}\ {}_3F_2(Y_{nl}, B_{nl}; 1) \\ A_{(nlm)}^{2ht} = \frac{\Gamma\left(2\sqrt{\chi_{nl}^3}+1\right)\Gamma(2\beta_{nl}-2)}{N^{-1}\Gamma(B_{nl}-1)}\ {}_3F_2(Y_{nl}, B_{nl}-1; 1) \\ A_{(nlm)}^{3ht} = \frac{\Gamma\left(2\sqrt{\chi_{nl}^3}+3\right)\Gamma(2\beta_{nl}-3)}{N^{-1}\Gamma(B_{nl})}\ {}_3F_2(Y_{nl}', B_{nl}; 1) \\ A_{(nlm)}^{4ht} = \frac{\Gamma\left(2\sqrt{\chi_{nl}^3}+2\right)\Gamma(2\beta_{nl}-3)}{N^{-1}\Gamma(B_{nl}-1)}\ {}_3F_2(Y_{nl}', B_{nl}-1; 1) \\ A_{(nlm)}^{5ht} = \frac{\Gamma\left(2\sqrt{\chi_{nl}^3}+1\right)\Gamma(2\beta_{nl}-3)}{N^{-1}\Gamma(B_{nl}-2)}\ {}_3F_2(Y_{nl}', B_{nl}-2; 1) \end{cases}, \tag{20}$$

with

$$B_{nl} = 2\sqrt{\chi_{nl}^3} + 2\beta_{nl},$$

$$Y_{nl} = \left(-n, n + 2\sqrt{\chi_{nl}^3} + 2\beta_{nl} + 1, X_{nl}; 1 + 2\sqrt{\chi_{nl}^3}\right),$$

$$Y'_{nl} = \left(-n, n + 2\sqrt{\chi_{nl}^3} + 2\beta_{nl} + 1 X'_{nl}; 1 + 2\sqrt{\chi_{nl}^3}\right),$$

and

$$X'_{nl} = 2\beta_{nl} - 3.$$

3.2. The corrected energy for the PDM-SVID(H-TP) models

The crucial goal of this sub-section is to identify the contribution under the PDM-SVID(H-TP) models, in 3D-ERQM symmetries, arising from deformation space-space using the method we have successfully applied in the past and are always working to develop. We can confirm that the PDM-SVD(H-TP) models are in place, which we provided through a summary of the *bosonic particles* and *bosonic antiparticles* in Eq. (9), produce a significant contribution to relativistic energy known in the literature under deformation KG theory, whereas the new contribution is generated from the topological properties under space-space deformation. The influence of the perturbed spin-orbit effective potential $W_{ht}^{pert}(r)$ corresponding to the *bosonic particles* and *bosonic antiparticles* with spin-s produces the first contribution. We obtain the perturbed spin-orbit effective potential by replacing the coupling of the angular momentum operator $\mathbf{L}$ and the NC vector Θ with the new equivalent coupling $\mathbf{L}\Theta \to \Theta\mathbf{LS}$ $\left(\Theta = \sqrt{\Theta_{12}^2 + \Theta_{23}^2 + \Theta_{13}^2}\right)$. This degree of freedom results from the arbitrary nature of the infinitesimal NC vector Θ. We have oriented the spin-s of the *bosonic particles* and *bosonic antiparticles* to become parallels to the vector Θ which interacted with the PDM-SVID(H-TP) model*s*. Additionally, we use the following transformation which is well known in QM symmetries:

$$\Theta\mathbf{LS} \to (\Theta/2)\mathbf{G}^2,$$

with

$$\mathbf{G}^2 = \mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2.$$

It is well known in QM symmetries, that the operators ($\hat{\mathbf{H}}_{\mathbf{nc}}^{\mathbf{ht}}$, $\mathbf{J}^2$, $\mathbf{L}^2$, $\mathbf{S}^2$ and $\mathbf{J}_z$) form a complete set of conserved physics quantities, and the eigenvalues $2F(j,l,s)$ of the operator $\mathbf{G}^2$ are equal to the values $j(j+1) - l(l+1) - s(s+1)$, $|l-s| \le j \le |l+s|$ in 3D-ERQM symmetry. As a direct consequence, the square partially corrected energies ΔE_{ht}^{so2} due to the perturbed effective potential $W_{ht}^{pert}(r)$ produced for the $(n,l,m)^{th}$ excited state, as follows:

$$\Delta E_{ht}^{so2} = \Theta F(j,l,s)\, \langle K \rangle_{(nlm)}^{ht}. \tag{21}$$

The global expectation values $\langle K \rangle_{(nlm)}^{ht}$ for the *bosonic particles* and *bosonic antiparticles*, which were created from the effect of the PDM-SVID(H-TP) model*s*, are determined from the following expression:

$$\langle K \rangle_{(nlm)}^{ht} = \sum_{\alpha=1}^{5} \beta_\alpha A_{(nlm)}^{\alpha ht}. \tag{22}$$

The second principal physical contribution for the perturbed potential $W_{ht}^{pert}(r)$ is proven when we substitute the coupling interaction $\mathbf{L}\Theta$ with physical coupling $\tau\vec{L}\vec{\aleph}$ and we chose $\aleph = \aleph e_z$ for simplification with physical condition $[\Theta] = [\tau][\aleph] \equiv$ (length)^2, here ($\aleph$ and τ) present the intensity of the magnetic field induced by the effect of the deformation of space-space geometry and a new infinitesimal noncommutativity parameter. This choice that the magnetic field is directed according to the (Oz) axis serves to simplify quantitative calculations without affecting the nature of the physical point of view; we also need to apply the identity $\langle n', l', m' | L_z | n, l, m \rangle$ which is equal $m\delta_{m'm}\delta_{l'l}\delta_{n'n}$ $(-|l| \le m \le +|l|)$. All of these data allow for the discovery of the new square improved energy shift ΔE_{ht}^{mg2} due to the perturbed Zeeman effect created by the influence of the PDM-SVID(H-TP) model*s* for the $(n,l,m)^{th}$ excited state in 3D-ERQM symmetries as follows:

$$\Delta E_{ht}^{mg2} = \tau\aleph\langle K\rangle_{(nlm)}^{ht} m \quad . \tag{23}$$

After we have completed the first and second stages of the self-production of energy, we are going to discover another very important case under the PDM-SVID(H-TP) model*s* in 3D-ERQM symmetries. This physical new phenomenon is produced automatically from the influence of perturbed effective potential $W_{ht}^{pert}(r)$. We consider the *bosonic particles* and *bosonic antiparticles* undergoing rotation with angular velocity Ω. The features of this subjective phenomenon are determined through the substitute of the arbitrary vector $\mathbf{\Theta}$ with the new physical quantity $\chi\Omega$. Allowing us to replace the coupling $\mathbf{L}\Theta$ with $\chi\mathbf{L}\Omega$, χ is just an infinitesimal real proportional constant. The effective potentials $W_{pert}^{ht-rot}(z)$ which induced the rotational movements can be expressed as follows:

$$W_{pert}^{ht-rot}(r) = \chi\langle K\rangle_{(nlm)}^{ht} \mathbf{L\Omega} . \tag{24}$$

We chose a rotational velocity $\mathbf{\Omega}$ parallel to the (Oz) axis ($\mathbf{\Omega} = \Omega\mathbf{e}_z$) to simplify the calculations. The perturbed generated spin-orbit coupling is then transformed into new physical phenomena as follows:

$$W_{pert}^{ht-rot}(z)\mathbf{L\Omega} = \chi\Omega W_{pert}^{ht-rot}(z) L_z . \tag{25}$$

All of these data allow for the discovery of the new corrected square improved energy ΔE_{ht}^{rot2} due to the perturbed effective potential $W_{pert}^{ht-rot}(z)$ which is generated automatically by the influence of the PDM-SVID(H-TP) model*s* for the $(n,l,m)^{th}$ excited state in 3D-ERQM symmetries as follows:

$$\Delta E_{ht}^{rot2} = \chi\Omega\langle K\rangle_{(nlm)}^{ht} m . \tag{26}$$

It is worth noting that the authors of ref. [48] were studied rotating isotropic and anisotropic harmonically confined ultra-cold Fermi gases in two and 3D space at zero temperature, but in this case, the rotational term was manually added to the Hamiltonian operator, whereas, in our study, the rotation operator $W_{pert}^{ht-rot}(z)\mathbf{L\Omega}$ appears automatically due to the effect of the deformation of space-space under the PDM-SVID(H-TP) model*s*. The eigenvalues of the operations $\mathbf{G}^2$ for bosonic particles and antiparticles (negative energy) with spin $s = (1,2..)$ are equal to the following values $F(j,l,s)$. In the 3D-ERQM symmetries, the total relativistic improved energy E_{nc}^{ht} for the case of the bosonic particles and bosonic antiparticles with spin has an integer value ($0,1,2...$) and satisfies the Bose-Einstein statistics such as ($\pi^{\pm}$ and π^{0}) with PDM-SVID(H-TP) model*s*, corresponding to the generalized $(n,l,m)^{th}$ excited states are expressed as:

$$E_{nc}^{ht} = E_{nl}^{\pm} \pm [\langle K\rangle_{(nlm)}^{ht} (\Lambda(\aleph\Omega)m + \Theta F)]^{1/2} , \tag{27}$$

where

$$\Lambda(\aleph\Omega) \equiv \tau\aleph + \chi\Omega .$$

Here $E_{nl}^{\pm}$ are usual relativistic energies under the PDM-SVID(H-TP) model obtained from equations of energy in Eq. (9). It should be noted that the positive and negative sign denotes the improved energy of the bosonic particles which corresponds to the positive and negative energy of the bosonic antiparticles which corresponds to the negative energy. We can now generalize our obtained energies $E_{t-nc}^{ht-b/ap}$, in a unified formula, under the PDM-SVID(H-TP) model*s* that were produced with the global induced potential $W_{ht}^{pert}(r)$:

$$E_{t-nc}^{ht-b/ap} = E_{nc}^{ht}\theta\left(\left|E_{nc}^{ht}\right|\right) - E_{nc}^{ht-s}\theta\left(-\left|E_{nc}^{ht}\right|\right) , \tag{28}$$

by using the unit step function (also known as a viside step function $\theta(x)$ or simply the theta function). It is important to point out that because we have only used corrections of the first order of infinitesimal noncommutative parameters (Θ, τ, χ), perturbation theory cannot be used to find corrections of the second order $(\Theta^2, \tau^2, \chi^2)$.

4. STUDY OF IMPORTANT RELATIVISTIC PARTICULAR CASES IN 3D-ERQM SYMMETRIES

We will look at some specific examples involving the new bound state energy eigenvalues in Eq. (27) in this section. By adjusting relevant parameters of the PDM-SVID(H-TP) model*s* in the 3D-ERQM, we could derive some specific potentials useful for other physical systems for much concern the specialist reach.

(1). If we choose, $V_1 = 0$, $S_0 = S_1 = m_1 = 0$ and $\alpha \to \alpha/2$ in Eq. (1), we obtain the improved generalized Hulthén potential (GHP) and the global relativistic energy for the bosonic particles E_{nc}^{hp-p} (bosonic antiparticles E_{nc}^{hp-ap}) under the improved GHP in 3D-ERQM symmetries as:

$$\begin{cases} V_h(r_{nc}) = V_h(r) - \left(\dfrac{\frac{\alpha}{2} V_0 e^{(-\alpha r)}}{r\left(1-qe^{(-\alpha r)}\right)} + \dfrac{q \frac{\alpha}{2} V_0 e^{(-2\alpha r)}}{r\left(1-qe^{(-\alpha r)}\right)^2} \right) \mathbf{L}\Theta \\ E_{nc}^{hp} = E_{nl}^{hp\pm} \pm \left[\langle K \rangle_{(nlm)}^{hp} \left(\Lambda(\aleph\Omega) m + \Theta F \right) \right]^{1/2} \end{cases} . \tag{29}$$

Here $V_h(r)$ presents the GHP in 3D-RQM symmetries [51], while $\langle K \rangle_{(nlm)}^{hp}(n, V_0, \alpha, m)$ is determined from the limits:

$$\langle K \rangle_{(nlm)}^{hp}(n, V_0, \alpha, m) = \underset{(V_1=S_0=S_1=m_1,\alpha)\to(0,\alpha/2)}{Lim} \langle K \rangle_{(nlm)}^{ht} .$$

The first two parts $E_{nl}^{hp\pm}$ describe the relativistic energies of bosonic particles and bosonic antiparticles. In 3D-RQM symmetries, the rest of the terms present the topological effect of the deformation space-space (TDSS) on the thesis's main energies $E_{nl}^{hp\pm}$.

(2). If we choose, $\alpha = \dfrac{1}{R}, q = \exp\left(\dfrac{\theta}{R}\right)$ and $V_0 \to qV_0 = \exp\left(\dfrac{\theta}{R}\right)$ in Eq. (1), we obtain the improved Woods-Saxon potential (WSP), the global relativistic energy for the bosonic particles E_{nc}^{wp-p} (or bosonic antiparticles E_{nc}^{wp-ap}) under the improved WSP in 3D-ERQM symmetries as:

$$\begin{cases} V_{ws}(r_{nc}) = V_{ws}(r) + \dfrac{\frac{V_0}{2R}\exp\left(\frac{r-\theta}{R}\right)}{r\left(1-\exp\left(\frac{r-\theta}{R}\right)\right)^2} \mathbf{L}\Theta \\ E_{nc}^{wp} = E_{nl}^{wp\pm} \pm \left[\langle K \rangle_{(nlm)}^{wp} \left(\Lambda(\aleph\Omega) m + \Theta F \right) \right]^{1/2} \end{cases} , \tag{30}$$

where V_0, θ and R are the potential depth, the width of the potential, and the surface thickness whose values correspond to the ionization energies, respectively, $V_{ws}(r)$ present the standard WSP [52] while the rest terms give the influences of TDSS on the standard WSP, $\langle K \rangle_{(nlm)}^{wp}$ obtained from $\langle K \rangle_{(nlm)}^{hp}$ with $(V_0, \alpha) \to (qV_0, \frac{1}{R})$. The first two parts $E_{nl}^{wp\pm}$ in RHS of Eqs. (30) describe the relativistic energy of bosonic particles and bosonic antiparticles within the framework of 3D-RQM while the rest terms are present in the TDSS on the thesis's main energies $E_{nl}^{wp\pm}$ which are obtained from making these substitutes.

5. SE WITH PDM-ID(H-TP) MODES IN 3D-ENRQM SYMMETRIES

To realize a study of the nonrelativistic limit, in 3D extended nonrelativistic QM (3D-ENRQM) symmetries, for the PDM-ID(H-TP) models, two steps must be applied. The first corresponds to the NR limit, in 3D-NRQM symmetries. This is done by applying the following simultaneous replacements, ($E_{nl} + m_0$ and $E_{nl} - m_0$) by ($2m_0$ and E_{nl}^{nr}), respectively in addition to the setv $S_0 = S_1 = 0$. After straightforward calculation, we can obtain the NR-energy equation for PDM-D(H-TP) models as:

$$E_{nl}^{nr} = -\frac{\left(\frac{\omega_3^{nr} q^2 - \omega_1^{nr}}{2\alpha\sigma^{nr}} + 2\alpha\left(n + \sigma^{nr}\right)\right)^2}{8q^2\left(m_0 - V_1\right)} + V\,, \tag{31}$$

with σ_{nr} and V are equal $\frac{q}{2}\left(1+\sqrt{1-\frac{\rho^{nr}}{\alpha^2 q^2}}\right)$ and $\frac{2V_0V_1 + 2m_0m_1 + V_1^2 + m_1^2}{2\left(m_0 - V_1\right)}$, respectively, while:

$$\begin{cases} \omega_1^{nr} = 2qE_{nl}^{nr}V_0 - 2q^2E_{nl}^{nr}V_1 - V_0^2 + q^2V_1^2 \\ \omega_2^{nr} = -2E_{nl}^{nr}V_0 + 2qV_0V_1 - 2qm_0m_1 + 4l\left(l+1\right)\alpha^2 \\ \omega_3^{nr} = 2E_{nl}^{nr}V_1 + 2\left(m_0 - V_1\right)V \end{cases}, \tag{32}$$

with

$$\rho^{nr} = q^2\omega_3^{nr} - \omega_1^{nr} - q\omega_2^{nr}\,.$$

Now, under the conditions of NR-limit, the new NR-expectation values $B_{(nlm)}^{1ht}$, $B_{(nlm)}^{2ht}$, $B_{(nlm)}^{3ht}$, $B_{(nlm)}^{4ht}$ and $B_{(nlm)}^{5ht}$ are obtained from the expectation values $A_{(nlm)}^{1ht}$, $A_{(nlm)}^{2ht}$, $A_{(nlm)}^{3ht}$, $A_{(nlm)}^{4ht}$ and $A_{(nlm)}^{5ht}$, by setting $S_0 = S_1 = 0$ and $k \to l$ in Eq. (20). As a direct consequence, the new NR improved energy E_{nc-nl}^{nr-ht} of the excited state $\left(n,l,m\right)^{th}$ in 3D-ENRQM symmetries under the PDM-ID(H-TP) models equals the NR-energy E_{nl}^{nr} in Eq. (31) under PDM-D(H-TP) models plus the NR corrections which are generated with the effect of deformation space-space, as:

$$E_{nc-nl}^{nr-ht} = E_{nl}^{nr} + \left[\left\langle K\right\rangle_{(nlm)}^{nr-ht}\left(\Lambda\left(\aleph\Omega\right)m + \Theta F\right)\right]^{1/2}, \tag{33}$$

where

$$\left\langle K\right\rangle_{(nlm)}^{nr-ht} = \sum_{\alpha=1}^{5}\beta_\alpha^{nr}B_{(nlm)}^{\alpha ht}.$$

6. Spin-averaged mass spectra of HLM under PDM-ID(H-TP) modes

The quark-antiquark interaction potentials, are spherically symmetrical and provide a good description of HLM such as $c\bar{c}$ and $b\bar{b}$ under PDM-ID(H-TP) modes. This would give us a strong incentive to dedicate this section to the purpose to determine the modified spin-averaged mass spectra of HLM under the PDM-ID(H-TP) modes interaction by using the following formula:

$$+M_{nl}^{ht} = m_q + m_{\bar{q}} + E_{nl}^{nr} \Rightarrow M_{nc}^{ht} = m_q + m_{\bar{q}} + \begin{cases} \frac{1}{3}\sum_{\alpha=1}^{3}E_{nc}^{ht\alpha} & \text{for spin-1} \\ E_{nc}^{ht} & \text{for spin-0} \end{cases}. \tag{34}$$

The LHS of Eq. (34) describes spin-averaged mass spectra of *HLM* in usual QM symmetries [53-57], while the RHS is our self-generalization to this formula in 3D-ENRQM symmetries, m_q and $m_{\bar{q}}$ are the quark mass and the antiquark mass, M_{nl}^{ht} is the spin-averaged mass spectra of *HLM* under the mass-dependent SE with the vector quark-antiquark interaction in usual NRQM symmetries, E_{nl}^{nr} is the nonrelativistic energy under PDM-ID(H-TP) modes, which is determined by generalizing Eq. (33) while (E_{nc}^{ht1}, E_{nc}^{ht2} and E_{nc}^{ht3}) are the modified energies of *HLM* which have spin-1 while E_{nc}^{ht} is the modified energies of *HLM* that have spin-0. We need to replace the factor $F\left(j,l,s\right)$ with new generalized values as follows:

$$F = \begin{cases} \frac{l}{2} & \text{for } \left(j = l+1,\, s = 1\right) \\ -1 & \text{for } \left(j = l,\, s = 1\right) \\ -\left(l+1\right) & \text{for } \left(j = l-1,\, s = 1\right) \\ 0 & \text{for } \left(j = l,\, s = 0\right) \end{cases}. \tag{35}$$

The modified energies (E_{nc}^{ht1}, E_{nc}^{ht2}, E_{nc}^{ht3} and E_{nc}^{ht}) correspond to Eq. (35) and can be expressed by the following formula:

$$\begin{cases} E_{nc}^{ht1} = E_{nl}^{nr} + \left[\langle K \rangle_{(nlm)}^{nr-ht} \left(\Lambda(\aleph\Omega)m + \frac{\Theta l}{2}\right)\right]^{1/2} \\ E_{nl}^{ht2} = E_{nl}^{nr} + \left[\langle K \rangle_{(nlm)}^{nr-ht} \left(\Lambda(\aleph\Omega)m - \Theta\right)\right]^{1/2} \\ E_{nl}^{ht3} = E_{nl}^{nr} + \left[\langle K \rangle_{(nlm)}^{nr-ht} \left(\Lambda(\aleph\Omega)m - \Theta(l+1)\right)\right]^{1/2} \\ E_{nc}^{ht} = E_{nl}^{nr} + \left[\langle K \rangle_{(nlm)}^{nr-ht} \Lambda(\aleph\Omega)m)\right]^{1/2} \end{cases}. \tag{36}$$

By substituting Eqs. (36) and (35) into Eq. (34), the new mass spectrum of the HLM systems in 3D-ENRQM symmetries under the PDM-ID(H-TP) models for any arbitrary radial and angular momentum quantum numbers becomes:

$$M_{nc}^{ht} = M_{nl}^{ht} + \begin{cases} \langle E_{nc}^{ht} \rangle_{np} & \text{For spin} - 1 \\ \left(\langle K \rangle_{(nlm)}^{nr-ht}\right)^{1/2} \left[(\tau\aleph + \chi\Omega)m)\right]^{1/2} & \text{For spin} - 0 \end{cases}, \tag{37}$$

with $\langle E_{nc}^{ht} \rangle_{np}$ mean physically the value of the nonpolarized energy which takes into account different all spin values:

$$\langle E_{nc}^{ht} \rangle_{np} = \frac{\left(\langle K \rangle_{(nlm)}^{nr-ht}\right)^{1/2}}{3} h(\Theta, \tau, \chi, m, l), \tag{38}$$

and $h(\Theta, \tau, \chi, m, l)$ is given by:

$$h(\Theta,\tau,\chi,m,l) = \left[\left[((\tau\aleph + \chi\Omega)m + \Theta l/2)\right]^{1/2} + \left[((\tau\aleph + \chi\Omega)m - \Theta)\right]^{1/2} + \left[((\tau\aleph + \chi\Omega)m - \Theta(l+1))\right]^{1/2} \right]. \tag{39}$$

It is important to notice that the new function $h(\Theta,\tau,\chi,m,l)$ describe the topological defect de deformation space-space because it disappears in the absence of the non-commutativity parameters (Θ,τ,χ). The LHS of Eq. (37) is the spin-averaged mass spectra M_{nl}^{ht} of *HLM* under the PDM-D(H-TP) modes in 3D-NRQM symmetries and the RHS is produced with the effect of deformation space-space which is sensitive to the atomic quantum numbers (n,l,j,s,m) and potential depths (V_0, S_0, V_1, S_1).

7. CONCLUSIONS

This paper presents an approximate analytical solution of the 3D-ERQM and 3D-ENRQM symmetries with PDM-SVID(H-TP) and PDM-ID(H-TP) models using the parametric Bopp shift method and standard perturbation theory. Under the deformed features of space-space, we found new bound-state energies that appear sensitive to quantum numbers (n, j, l, s, m), the mixed potential depths (V_0, S_0, V_1, S_1), the rest and perturbed mass (m_0, m_1), the screening parameter's inverse α, and the noncommutativity parameter (Θ, τ, χ). Moreover, the nonrelativistic limit of the studied potential in 3D-ENRQM symmetries has been investigated. The modified spin-averaged mass spectra of *HLM* in both 3D-NRQM (commutative space CS) and 3D-ENRQM symmetries were determined by applying our results of the new nonrelativistic energies that represent the binding energy between the quark and antiquark. We have treated certain significant particular instances that we hope will be valuable to the specialized researcher such as the improved GHP and the improved WSP in the context of 3D-ERQM symmetries. It is shown that the PDM-SVID(H-TP) model in a 3D-ERQM has a behavior similar to the dynamics of bosonic particles and bosonic antiparticles under the PDM with PDM-SVD(H-TP) in a 3D-RQM symmetry (CS) influenced by the effect of constant magnetic field and a self-rotational which can be similar to the behavior of coupling to spin-orbit. As a result, the dynamics of PDM-SVID(H-TP) models in a 3D-ERQM symmetry under the DKGE are similar to the dynamics of a particle in a 3D-RQM symmetry under the Duffin-Kemmer equation which describes bosonic particles with spin-1.

Competing interests. The author declares that they have no competing interests.

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РЕЛЯТИВІСТСЬКІ СИМЕТРІЇ БОЗОННИХ ЧАСТИНОК І АНТИЧАСТИНОК НА ФОНІ ПОЗИЦІЙНО-ЗАЛЕЖНОЇ МАСИ ДЛЯ ВДОСКОНАЛЕНОГО ДЕФОРМОВАНОГО ХЮЛЬТЕН ПЛЮС ПОКРАЩЕНОГО ПОТЕНЦІАЛУ ГІПЕРБОЛІЧНОГО ТИПУ У СИМЕТРІЯХ 3D-EQM

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Розвязання зв'язаного стану деформованого рівняння Клієна-Гордона були визначені в симетріях тривимірної розширеної релятивістської квантової механіки 3D-ERQM з використанням позиційно-залежної маси (PDM) з нерівним скалярним і векторним потенціалом для вдосконаленого деформованого Hulthén плюс покращеного потенціалу гіперболічного типу (PDM-SVID(H-TP)). PDM з нерівним скалярним і векторним потенціалом для моделей Hulthén плюс гіперболічний потенціал деформованого типу (PDM-(SVH-DTP)), а також комбінація радіальних членів, які пов'язані з $\mathbf{L\Theta}$, що пояснює взаємодію фізичного особливості системи з топологічними деформаціями простір-простір. Нові релятивістські власні значення енергії були отримані за допомогою параметричного методу зсуву Боппа та стандартної теорії збурень, яка чутлива до атомних квантових чисел (j,l,s,m), змішаних потенціальних глибин (V_0,S_0,V_1,S_1), спокою та збуреної маси $\left(m_{0,}m_1\right)$, параметра екранування, зворотного параметру α, і параметрів некомутативності (Θ,τ,χ). У рамках симетрій 3D-ERQM ми розглянули певні важливі окремі випадки, які, як ми сподіваємося, будуть цінними для дослідників-спеціалістів. Ми також розглянули нерелятивістську межу та застосували наші отримані результати для створення мас-спектрів важких і легких мезонів (HLM), таких як $c\bar{c}$ та $c\bar{c}$ у рамках PDM-SE з покращеними деформованими моделями Hulthén плюс покращеного гіперболічного потенціалу (PDM ID(H-TP)). Коли було застосовано три одночасні обмеження (Θ,τ,χ), ми відновили нормальні релятивістські результати в літературі ($0,0,0$) для моделей PDM ID(H-TP).

Ключові слова: рівняння Клієна-Гордона, деформований Хультен плюс деформований гіперболічний потенціал, важкі-легкі мезони, Некомутативна квантова механіка та метод зсуву Боппа, Канонічна некомутативность