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# EFFECTS OF GRAVITATIONAL FIELD OF A TOPOLOGICAL DEFECT ON STATISTICAL PROPERTIES OF HEAVY QUARK-ANTIQUARK SYSTEMS†

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In this paper, we determine eigen energies, eigenfunctions and statistical properties of non-relativistic heavy quarkonia interacting with the extended Cornel potential within a space-time generated by a cosmic-string. We extend the Cornel potential by adding the inverse square potential plus the quadratic potential. We have calculated the energy eigenvalues and the corresponding eigenstates using the Extended Nikiforov-Uvarov (ENU) method. Then, based on the equation of energy spectra, the thermodynamic properties like partition function, entropy, free energy, mean energy and specific heat capacity are calculated within the space-time of a cosmic-string. In the next step, we investigate the influence of the cosmic-string parameter on quantum states of heavy quarkonia and their statistical properties.
**Keywords:** thermodynamic properties, quark, Schrödinger wave equation, topological defect, space-time, cosmic string, extended Cornel potential, meson, extended Nikiforov-Uvarov method.
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## 1. INTRODUCTION

The study of quantum particles under the influence of a potential in a curved space-time with Topological defect has inspired a large number of research in recent years [1, 2, 3, 4, 5, 6] with different motivations. Kibble [4, 5] was the first to state the mechanism which would have led to the appearance of topological defects in the universe. According to grand unification theories (GUT) [4, 5, 6], the conditions which reigned in the very early moments of the universe were such that the four fundamental interactions were one [4, 5, 7]. A few moments after the Big-Bang, the universe began to expand and became less and less hot and its density decreased [7, 8]. From the initial phase where the grand unification symmetries would be realized, a spontaneous breaking of these symmetries would then have taken place and at this moment, stable topological defects were formed.

Thus, a large variety of topological defects have been studied in previous works, namely: point defects [4, 5, 6, 7, 8] called monopoles, linear defects or cosmic-strings [4, 5, 7], surface defects or "domain walls" [9], and also combinations between these different defects [10, 11, 12]. These faults are often stable; so, it is absolutely possible that some of them would have survived, until now [9]. However, cosmic strings are compatible with cosmological models [6] and current observations. Their cosmological effects could be confirmed with current and future work. It is therefore quite logical to give particular interest to the study of quantum systems on curved space-time with cosmic-string background. In [6], the author presents cosmic-strings as the most important topological defect in our universe.

Their effects on the environment are mainly gravitational in nature [6, 10, 11, 12]. To this end, a quantum particle placed in a gravitational field will be influenced by the topology of the space and by its interaction with the local curvature. From this interaction, it follows that an observer at rest with respect to the particle will see a shift in its spectrum. This would be of considerable interest both from a theoretical and an observational point of view. Apart from the effects due to the gravitational field, other effects can be induced by a cosmic-string, like for example bremsstrahlung process [13], the creation of the $(e^+, e^-)$ pair [14], gravitational leasing [15] and the gravitational Aharonov-Bohm effect [16].

The study of thermodynamic properties of quantum systems plays a major role in theoretical High Energy Physics (HEP) and related fields [17]. Usually, quarks and gluons remain confined in hadrons and in particular in the protons and neutrons which form the atomic nucleus. Indeed, Quantum Chromodynamics (QCD), a theory that accounts for the strong interaction predicts that by compressing or heating nuclei, it is possible to create a plasma of quarks and gluons [18, 19], a matter with poorly understood thermodynamic properties. In past few decades, some experimental works have been carry out to identify the existence of deconfinement transitions [19] and determine its signatures. Then, it was predicted

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that the suppression of charmonium (*a bound state of charm and anti-charm quarks*) meson is a possible signature of the QCD phase transition [18, 19, 20]. This phenomenon of deconfining matter is in principle observable through the study of the $\mathbf{J}/\psi$ particle [20], a meson as heavy as three surrounding protons. The suppression of the $c\overline{c}$ meson has been announced as being the signature of deconfinement transition, it would be interesting to study in depth the thermodynamic properties of heavy mesons, in this case heavy quarkonia.

In [21], the author obtained thermodynamic properties of the quark-gluon plasma from the constituent quasi particle model of quark-gluon plasma. Then in [22, 23, 24], thermodynamic properties are investigated using chiral quark models. In this work, we obtain the solutions of the Schrödinger equation produced by the gravitational field of a topological defect, for all values of the orbital angular momentum quantum number $l$. The Extended Nikiforov-Uvarov method [25, 26] is used to obtain exact analytical solutions of the radial Schrödinger equation in cosmic-string background. Next, we apply the results to obtain the thermodynamic properties of heavy quarkonia from cosmic-string geometry, which is not consider in other works.

This paper is organized as follows: in Section 2, the Extended Nikiforov-Uvarov method is reviewed. The bound state solutions for the extended Cornel potential are obtained for the Schrödinger equation in cosmic-string background in Section 3. Then in section 4, thermodynamic properties of heavy quarkonia within cosmic-string geometry are presented. In Section 5 results are discussed, and conclusions are presented in Section 6.

## 2. BASIC CONCEPTS OF THE EXTENDED NIKIFOROV-UVAROV METHOD

In this section, we briey present the Extended Nikiforov-Uvarov (ENU) technique, for more details see [25, 26, 27]. The extended Nikiforov-Uvarov method is obtained by changing the boundary conditions of the standard Nikiforov-Uvarov (NU) method and is used for solving any second order differential equation which has at most four singular points. The equation to be solved is of the form:

$$\psi''(r)+\frac{\overline{\tau}(r)}{\sigma(r)}\psi'(r)+\frac{\overline{\sigma}(r)}{\sigma^2(r)}\psi(r)=0, \tag{1}$$

such that $\overline{\tau}(r)$, $\sigma(r)$ and $\overline{\sigma}(r)$ are polynomials of at most second, third and fourth degree, respectively. By choosing $\psi$ such that:

$$\psi(r)=\phi(r)Y(r), \tag{2}$$

where $\phi(r)$ and $Y(r)$ are functions to be determined later. Moreover, considering the above substitution Eq. (1) reduces to an equation of the form

$$\sigma(r)Y''(r)+\tau(r)Y'(r)+h(r)Y(r)=0\,, \tag{3}$$

where $\phi(r)$ solves the equation

$$\frac{\phi'(r)}{\phi(r)}=\frac{\pi(r)}{\sigma(r)}, \tag{4}$$

$$h(r)-\pi'(r)=G(r)\,. \tag{5}$$

And the polynomials of the function $Y(r)$ satisfy the Rodriguez formula [27]

$$Y_n(r)=\frac{B_n}{\rho(r)}\frac{d^n}{dr^n}\left[\sigma^n(r)\rho(r)\right]. \tag{6}$$

In Eq. (6), $B_n$ is the normalization factor and $\rho$ is the density function. The function $\rho$ solves the following equation:

$$\left(\sigma(r)\rho(r)\right)'=\tau(r)\rho(r)\,. \tag{7}$$

The function $\pi(r)$ required for this method are given by:

$$\pi(r)=\frac{\sigma'(r)-\overline{\tau}(r)}{2}\pm\sqrt{\left(\frac{\sigma'(r)-\overline{\tau}(r)}{2}\right)^2-\overline{\sigma}(r)+G(r)\sigma(r)}, \tag{8}$$

where $\pi(r)$ is a second-degree polynomial. The function $h_n(r)$ is determined from the equation

$$h_n(r) = -\frac{n}{2}\tau'(r) - \frac{n(n-1)}{6}\sigma''(r) + C_n, \tag{9}$$

where $\tau(r) = \overline{\tau}(r) + 2\pi(r)$ and $C_n$ is an integration constant. In the order to obtain the eigenvalue solution of the problem by ENU method [25, 26, 27], the polynomials $h(r)$ and $h_n(r)$ are equal and $Y(r) = Y_n(r)$ is a particular solution of degree $n$ of Eq. (3). The function $\phi(r)$ is defined as a logarithmic derivative given in Eq. (4), and then eigenfunction spectra can be obtained analytically. For more details one can see Refs. [26] and [27].

## 3. SCHRÖDINGER WAVE EQUATION IN THE SPACE-TIME OF A COSMIC STRING

In spherical coordinates, the line element describing the cosmic string space-time [6] is given by $\left(x^0 = ct,\ x^1 = r,\ x^2 = \theta,\ x^3 = \varphi\right)$

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + \left[\chi d\theta + \alpha r \sin\theta d\varphi\right]^2, \tag{10}$$

where $0 < r < \infty$, $0 < \theta < \pi$, and $0 < \varphi < 2\pi$, $0 < \alpha = 1 - 4J$ is the topological parameter of the cosmic string, $\chi = \frac{4GJ}{c^3}$, is the torsion [23] parameter and $J$ denotes the linear mass density [28, 29,30,31] of the cosmic string. From General Relativity (GR), we know that the values of $J$ varies in the interval $J \in \left]0,1\right[$ .

For $\alpha \to 1$ and $\chi \to 0$, the metric given by Eq. (10) reduces to the usual Minkowski metric in spherical coordinates. The metric tensor for the space-time described by Eq. (10) is:

$$g_{\mu\nu}(x) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \chi^2 + r^2 & \chi\alpha r \sin\theta \\ 0 & 0 & \chi\alpha r\sin\theta & \alpha^2 r^2 \sin^2\theta \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & \left(g_{ij}\right) \end{bmatrix}, \tag{11}$$

with the inverse metric,

$$g^{\mu\nu}(x) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & -\frac{\chi}{\alpha r^3 \sin\theta} \\ 0 & 0 & -\frac{\chi}{\alpha r^3 \sin\theta} & \frac{\chi^2 + r^2}{\alpha^2 r^4 \sin^2\theta} \end{pmatrix}, \tag{12}$$

We adopt the signature $(-,+,+,+)$ for the metric tensor $g^{\mu\nu}$, and its determinant is given by $g = det\left(g^{\mu\nu}\right) = -\alpha^2 r^4 \sin^2\theta$, with $\mu,\nu = 0,1,2,3$. In the curvilinear coordinates system $ds^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij} dx^i \otimes dx^j$ such that $r \to x^1$, $\theta \to x^2$, $\varphi \to x^3$ the metric tensor of the internal 3-dimensional Euclidian space is:

$$g_{ij}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \chi^2 + r^2 & \chi\alpha r \sin\theta \\ 0 & \chi\alpha r \sin\theta & \alpha^2 r^2 \sin^2\theta \end{pmatrix}. \tag{13}$$

The Laplace-Beltrami operator of the local coordinates system can be expressed as:

$$\Delta_{LB} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left( g^{ij} \sqrt{g} \frac{\partial}{\partial x^i} \right) \qquad i,j = 1,2,3 \qquad \text{and } g = det\left(g_{ij}\right) = \alpha^2 r^4 \sin^2\theta. \tag{14}$$

Then, considering Eq. (14) and for small values of the torsion parameter $\chi \ll 1$, the Laplace-Beltrami operator becomes:

$$\Delta_{LB} = \frac{1}{r^2}\left\{ \frac{\partial}{\partial r}\left[ r^2 \left( \frac{\partial}{\partial r} \right) \right] + \cot\theta \frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2} + \frac{1}{\alpha^2 \sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right\}, \tag{15}$$

from this we can write the Hamiltonian operator in natural units ( $\hbar = c = 1$ ) as:

$$H = -\frac{1}{2M}\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\cot\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2}\frac{1}{\alpha^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right] + V(r,\theta,\varphi), \tag{16}$$

$M = m_q m_{\bar{q}} / (m_q + m_{\bar{q}})$ is the reduced mass of the quark-antiquark system, where $m_q$ and $m_{\bar{q}}$ are the mass of quark and antiquark respectively [8, 26].

The non-relativistic Schrödinger equation in curved background space-time is [16, 23]

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = -\frac{\hbar^2}{2M}\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\cot\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2}\frac{1}{\alpha^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\Psi(\vec{r},t) + V(\vec{r})\Psi(\vec{r},t), \tag{17}$$

here, we consider bound states of two heavy quarks interacting with the extended Cornel potential [26, 32]

$$V(r) = -\frac{K}{r} + \alpha r + \frac{g}{r^2} + br^2. \tag{18}$$

The first term is a Coulomb-like potential due to one-gluon exchange processes between quark-antiquark [32], the second term is a linear confinement term. The additional part was added to improve quark-antiquark properties [26, 32, 33]. As $V = V(r)$, the following commutation relations must be satisfied

$$\left[L_z, V(r)\right] = 0, \qquad \left[L^2, V(r)\right] = 0. \tag{19}$$

Consequently, $L_z$ and $L^2$ are good quantum numbers to describe the quantum states of the system under the influence of this potential. Inserting the new form of $\Psi$ as

$$\Psi(\vec{r},t) = \frac{\psi_{nl}(r)}{r}H_l^m(\theta)\Phi_m(\varphi)e^{-iE_{nl}t}, \tag{20}$$

and substituting in Eq. (17) gives the following set of second-order differential equations,

$$\frac{d^2\psi_{nl}(r)}{dr^2} + \left[-\frac{2M}{\hbar^2}V(r) + \frac{2M}{\hbar^2}E_{nl} - \frac{\delta}{r^2}\right]\psi_{nl}(r) = 0, \tag{21}$$

$$\frac{d^2H_l^m(\theta)}{d\theta^2} + \cot\theta\frac{dH_l^m(\theta)}{d\theta} + \left[\delta - \frac{m^2}{\alpha^2\sin^2\theta}\right]H_l^m(\theta) = 0, \tag{22}$$

$$\frac{d^2\Phi_m(\varphi)}{d\varphi^2} + m^2\Phi_m(\varphi) = 0, \tag{23}$$

where $\delta$ and $m^2$ are the separation constants. Using the following boundary condition: $\Phi_m(\varphi + 2\pi) = \Phi_m(\varphi)$, we can easily obtain the solution of Eq. (23) as

$$\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}}e^{im\varphi}, \qquad m = 0, \pm1, \pm2, \pm3, \dots. \tag{24}$$

To find the solutions of Eq. (22), we introduce the variable $\eta = \cos\theta$, which map Eq. (22) into

$$\frac{d^2H_l^m(\eta)}{d\eta^2} - \frac{2\eta}{(1-\eta^2)}\frac{dH_l^m(\eta)}{d\eta} + \left[\delta - \frac{m^2}{\alpha^2(1-\eta^2)}\right]H_l^m(\eta) = 0. \tag{25}$$

To obtain suitable solutions of Eq. (25), it is therefore necessary to analyze the way the solutions behave around singular points, namely $\pm\eta$. Then we assume the following form for the solution:

$$H_l^m(\eta) = (1-\eta^2)^{\frac{m}{2\alpha}}g_l(\eta), \tag{26}$$

where $g_l$ is analytic everywhere except at $\eta \to \pm\infty$. From Eq. (25) and Eq. (26), and introducing the following generalized quantum numbers, namely $l_{(\alpha)}$, $m_{(\alpha)} = \frac{m}{\alpha}$, from the cosmic-string geometry, we obtain the following equation:

$$\left(1-\eta^2\right)\frac{d^2 g_{l(\alpha)}(\eta)}{d\eta^2} - 2\left(m_{(\alpha)}+1\right)\eta\frac{dg_{l(\alpha)}(\eta)}{d\eta} - \left[m^2_{(\alpha)} + m_{(\alpha)} - \delta\right] g_{l(\alpha)}(\eta). \tag{27}$$

From the expansion of $g_{l(\alpha)}(\eta)$ into a power series,

$$g_{l(\alpha)}(\eta) = \sum_{-\infty}^{+\infty} a_n \eta^n , \tag{28}$$

and inserting in Eq. (27) gives us the following recurrence relation:

$$a_{n+2} = \frac{n(n-1) + 2\left(m_{(\alpha)}+1\right)n - \delta + m_{(\alpha)}\left(m_{(\alpha)}+1\right)}{(n+1)(n+2)} a_n , \tag{29}$$

where $n = 0,1,2,3,...$ . For physically acceptable solutions, the series must be truncated at a certain value of $n$ :

$$n(n-1) + m_{(\alpha)}\left(m_{(\alpha)}+1\right) + 2\left(m_{(\alpha)}+1\right)n - \delta = 0 . \tag{30}$$

Solving for $\delta$ gives:

$$\delta = l_{(\alpha)}\left(l_{(\alpha)}+1\right) \quad \text{with} \quad l_{(\alpha)} = m_{(\alpha)} + n , \tag{31}$$

$l_{(\alpha)}$ being the generalized angular orbital quantum number. The generalized quantum numbers $l_{(\alpha)}$ and $m_{(\alpha)}$ are not always integers. $l_{(\alpha)} = m_{(\alpha)} + n = \frac{m}{\alpha} + n = l - \left(l - \frac{1}{\alpha}\right) m$, where $l = 0,1,2,...$ Eq. (25) can therefore take the form:

$$\left(1-\eta^2\right)\frac{d^2 H^{m_{(\alpha)}}_{l(\alpha)}(\eta)}{d\eta^2} - 2\eta\frac{dH^{m_{(\alpha)}}_{l(\alpha)}(\eta)}{d\eta} + \left[l_{(\alpha)}\left(l_{(\alpha)}+1\right) - \frac{\left[m_{(\alpha)}\right]^2}{\left(1-\eta^2\right)}\right] H^{m_{(\alpha)}}_{l(\alpha)}(\eta), \tag{32}$$

where Eq. (32) is the generalized Legendre equation within the cosmic-string space-time and $H^{m_{(\alpha)}}_{l(\alpha)}(\eta)$ are the generalized Legendre polynomials given by the formula:

$$H^{m_{(\alpha)}}_{l(\alpha)}(\eta) = P^{m_{(\alpha)}}_{l(\alpha)}(\eta) = \frac{(-1)^{l(\alpha)}}{2^{l(\alpha)} l_{(\alpha)}!}(1-\eta)^{\frac{m_{(\alpha)}}{2}} \frac{d^{l_{(\alpha)}+m_{(\alpha)}}}{d\eta^{l_{(\alpha)}+m_{(\alpha)}}}\left[\left(1-\eta^2\right)^{l(\alpha)}\right]. \tag{33}$$

Let's now turn to the radial equation.

Substituting the proposed potential in Eq. (21) and choosing the separation constant being $\delta = l_{(\alpha)}\left(l_{(\alpha)}+1\right)$, the radial equation becomes:

$$\frac{d^2\psi_{nl}(r)}{dr^2} + \frac{1}{r^2}\left[-2Mg - l_{(\alpha)}\left(l_{(\alpha)}+1\right) + 2MKr + 2M\left(E_{nl}-d\right)r^2 - 2Mar^3 - 2Mbr^4\right]\psi_{nl}(r) = 0 , \tag{34}$$

that we put in a simple form by introducing the following constants:

$$\begin{aligned} &\xi_0 = l_{(\alpha)}\left(l_{(\alpha)}+1\right) + 2Mg, \quad \xi_1 = 2MK, \quad \xi_2 = 2M\left(E_{nl}-d\right) \\ &\xi_3 = 2Ma, \quad \xi_4 = 2Mb \end{aligned} . \tag{35}$$

The functions from the Nikiforov-Uvarov method are:

$$\bar{\tau}(r) = 0, \tag{36}$$

$$\sigma(r) = r^2, \tag{37}$$

$$\bar{\sigma}(r) = -\xi_0 + \xi_1 r + \xi_2 r^2 - \xi_3 r^3 - \xi_4 r^4. \tag{38}$$

Using equation Eq. (10), $\pi(r)$ is found as:

$$\pi(r) = \frac{1}{2} \pm \sqrt{\xi_0 + \frac{1}{4} - \xi_1 r - \xi_2 r^2 + \xi_3 r^3 + \xi_4 r^4 + rG(r)}\,. \tag{39}$$

Assuming that the term under the square root of the above expression becomes quadratic,

$$\pi(r) = \frac{1}{2} \pm \left(\alpha_0 + \alpha_1 r + \alpha_2 r^2\right). \tag{40}$$

Let's take the function $G(r)$ in the form $G(r) = Ar + B$. By comparison between Eq. (39) and Eq. (40), we obtain the following four sets of solutions for the unknowns $\alpha_2,\ \alpha_1,\ \alpha_0$ $A$ and $B$ in terms of the problem parameters:

$$\mathbf{I}: \begin{cases} \alpha_2 = \sqrt{\xi_4} \\ \alpha_1 = \dfrac{\xi_3}{2\sqrt{\xi_4}} \\ \alpha_0 = \sqrt{\xi_0 + \dfrac{1}{4}} \\ A = \xi_2 + \dfrac{[\xi_3]^2}{4\sqrt{\xi_4}} + 2\sqrt{\xi_4}\sqrt{\xi_0 + \dfrac{1}{4}} \\ B = \xi_1 + \dfrac{\xi_3}{\sqrt{\xi_4}}\sqrt{\xi_0 + \dfrac{1}{4}} \end{cases} \qquad \mathbf{II}: \begin{cases} \alpha_2 = -\sqrt{\xi_4} \\ \alpha_1 = -\dfrac{\xi_3}{2\sqrt{\xi_4}} \\ \alpha_0 = \sqrt{\xi_0 + \dfrac{1}{4}} \\ A = \xi_2 + \dfrac{[\xi_3]^2}{4\sqrt{\xi_4}} - 2\sqrt{\xi_4}\sqrt{\xi_0 + \dfrac{1}{4}} \\ B = \xi_1 - \dfrac{\xi_3}{\sqrt{\xi_4}}\sqrt{\xi_0 + \dfrac{1}{4}} \end{cases}$$

$$\mathbf{III}: \begin{cases} \alpha_2 = \sqrt{\xi_4} \\ \alpha_1 = \dfrac{\xi_3}{2\sqrt{\xi_4}} \\ \alpha_0 = -\sqrt{\xi_0 + \dfrac{1}{4}} \\ A = \xi_2 + \dfrac{[\xi_3]^2}{4\sqrt{\xi_4}} - 2\sqrt{\xi_4}\sqrt{\xi_0 + \dfrac{1}{4}} \\ B = \xi_1 - \dfrac{\xi_3}{\sqrt{\xi_4}}\sqrt{\xi_0 + \dfrac{1}{4}} \end{cases} \qquad \mathbf{IV}: \begin{cases} \alpha_2 = -\sqrt{\xi_4} \\ \alpha_1 = -\dfrac{\xi_3}{2\sqrt{\xi_4}} \\ \alpha_0 = -\sqrt{\xi_0 + \dfrac{1}{4}} \\ A = \xi_2 + \dfrac{[\xi_3]^2}{4\sqrt{\xi_4}} + 2\sqrt{\xi_4}\sqrt{\xi_0 + \dfrac{1}{4}} \\ B = \xi_1 + \dfrac{\xi_3}{\sqrt{\xi_4}}\sqrt{\xi_0 + \dfrac{1}{4}} \end{cases}. \tag{41}$$

Considering Eq. (4), we have two solutions for $\phi(r)$ which are given by:

$$\phi_\pm(r) = K_\pm r^{\frac{1}{2}\mp\alpha_0} \exp\left(\pm\frac{1}{2}\left(\alpha_2 r^2 + 2\alpha_1 r\right)\right). \tag{42}$$

The wave function will be physically acceptable if $\alpha_1 < 0,\ \alpha_2 < 0$ and $\alpha_0 > 0$ and then the set **II** of parameters is the suitable one in the determination of the eigenvalue and the eigenfunction of the problem.
As the functions $h(r)$ and $h_n(r)$ are equal, we use Eq. (5) and Eq. (9) to obtain the energy equation as

$$Ar + B \pm \left(2\alpha_2 r + \alpha_1\right) = C_n - n\left[\pm\left(2\alpha_2 r + \alpha_1\right)\right], \tag{43}$$

where $C_n$ is an integration constant which couples with the parameters of potential. Eq. (43) leads us to have two distinct choices $++$ and $--$, which are given by **I** and **II** as follows

$$\mathbf{I}:\begin{cases}2\alpha_2+A=-2n\alpha_2\\ B+\alpha_1=C_n-n\alpha_1\end{cases}\quad \mathbf{II}:\begin{cases}-2\alpha_2+A=2n\alpha_2\\ B-\alpha_1=C_n+n\alpha_1\end{cases}. \tag{44}$$

The choice of $--$ and set **II** results on the relation $A+2(n+1)\alpha_2=0$ from which we can write:

$$\frac{4\xi_4\xi_2+[\xi_3]^2-4\xi_4\sqrt{\xi_4}\sqrt{\xi_0+1}}{4\xi_4}=2(n+1)\sqrt{\xi_4}\,. \tag{45}$$

Using Eq. (35) and Eq. (45), we obtain the expression below:

$$E_{nlm}=d-\frac{a^2}{4b}+2\sqrt{\frac{2b}{M}}\left(2(n+1)+2\sqrt{2Mg+\left[l-\left(1-\frac{1}{\alpha}\right)m\right]\left[l-\left(1-\frac{1}{\alpha}\right)m+1\right]+\frac{1}{4}}\right). \tag{46}$$

We note that the presence of the cosmic string parameter $\alpha$ breaks the degeneracy in all the states with $l\neq 0$. When $\alpha\to 1$ and $d\to 0$, we recover the results obtained in [26, 33]. Moreover, for $\alpha\to 1$, $d\to 0$, $a=0$ and $g=0$ we obtain the energy spectrum of a spherical quantum harmonic oscillator [26].

For the wave function, it is easy to obtain $\phi(r)$ from Eq. (4) as

$$\phi(r)=r^{\left(\alpha_0+\frac{1}{2}\right)}e^{\left(\alpha_1 r+\frac{1}{2}\alpha_2 r^2\right)}, \tag{47}$$

and $\rho(r)$ from Eq. (7) as

$$\rho(r)=r^{2\alpha_0}e^{2\left(\alpha_1 r+\frac{1}{2}\alpha_2 r^2\right)}. \tag{48}$$

Then we use Eq. (6) to obtain the function $Y_n(r)$ as

$$Y_n(r)=B_n r^{-2\alpha_0}e^{-2\left(\alpha_1 r+\frac{1}{2}\alpha_2 r^2\right)}\frac{d^n}{dr^n}\left[r^{n+2\alpha_0}e^{2\left(\alpha_1 r+\frac{1}{2}\alpha_2 r^2\right)}\right]. \tag{49}$$

From Eq. (2) the radial eigenfunctions are then

$$\psi_{nl}(r)=N_{nl}r^{-\alpha_0+\frac{1}{2}}e^{-\left(\alpha_1 r+\frac{1}{2}\alpha_2 r^2\right)}\frac{d^n}{dr^n}\left[r^{n+2\alpha_0}e^{2\left(\alpha_1 r+\frac{1}{2}\alpha_2 r^2\right)}\right], \tag{50}$$

where $N_{nl}$ is the normalization factor, and $\alpha_0,\alpha_1,\alpha_2$ are given from set **II**

## 4. THERMODYNAMIC PROPERTIES OF THE $q\bar{q}$ SYSTEM

To consider the thermodynamic properties of heavy quarkonia within cosmic-string framework, the starting point is the partition function [34, 35]. From a statistical mechanical point of view, the partition function can be constructed as follow:

$$\begin{aligned}Z(\beta)&=\sum_{n=0}^{\infty}e^{-\beta E_n}\\ &=e^{-\beta\left[-\frac{\alpha^2}{4b}+4\sqrt{\frac{2b}{\mu}}\left(1+\sqrt{2\mu g+\left[l-\left(1-\frac{1}{\alpha}\right)m\right]\left[l-\left(1-\frac{1}{\alpha}\right)m+1\right]+\frac{1}{4}}\right)\right]}\sum_{n=0}^{\infty}e^{-4\beta\sqrt{\frac{2b}{\mu}}n}\\ &=\frac{1}{2}e^{-\beta\left(2\sqrt{\frac{2b}{\mu}}+\left[-\frac{a^2}{4b}+4\sqrt{\frac{2b}{\mu}}\left(1+\sqrt{2\mu g+\left[l-\left(1-\frac{1}{\alpha}\right)m\right]\left[l-\left(1-\frac{1}{\alpha}\right)m+1\right]+\frac{1}{4}}\right)\right]\right)}\operatorname{csch}\left(2\beta\sqrt{\frac{2b}{\mu}}\right)\end{aligned}, \tag{51}$$

here $\beta=k_B T$, where $k_B$ is the Boltzmann constant and $T$ is the absolute temperature of the system. Once the partition function is obtained, we can have the Helmholtz free energy as:

$$F=-\frac{1}{\beta}\ln Z(\beta)$$
$$=\left[-\frac{a^2}{4b}+4\sqrt{\frac{2b}{\mu}}\left(1+\sqrt{2\mu g+\left[l-\left(1-\frac{1}{\alpha}\right)m\right]\left[l-\left(1-\frac{1}{\alpha}\right)m+1\right]+\frac{1}{4}}\right)\right]+\frac{1}{\beta}\ln\left(1-e^{-4\beta\sqrt{\frac{2b}{\mu}}}\right)$$
$$=\left[-\frac{a^2}{4b}+4\sqrt{\frac{2b}{\mu}}\left(1+\sqrt{2\mu g+\left[l-\left(1-\frac{1}{\alpha}\right)m\right]\left[l-\left(1-\frac{1}{\alpha}\right)m+1\right]+\frac{1}{4}}\right)\right]-2\sqrt{\frac{2b}{\mu}}$$
$$+\frac{1}{\beta}\ln\left(2\sinh\left(2\beta\sqrt{\frac{2b}{\mu}}\right)\right) \quad . \quad (52)$$

From the Helmholtz free energy, we can obtain the other statistical quantities in a straightforward manner. The entropy of the system can be obtained as

$$S=-k_B\beta^2\frac{\partial F}{\partial\beta}$$
$$=-\ln\left[1-e^{-4\beta\sqrt{\frac{2b}{\mu}}}\right]+4\beta\sqrt{\frac{2b}{\mu}}\frac{e^{-4\beta\sqrt{\frac{2b}{\mu}}}}{\left(1-e^{-4\beta\sqrt{\frac{2b}{\mu}}}\right)} \quad . \quad (53)$$
$$=-\ln\left[2e^{-2\beta\sqrt{\frac{2b}{\mu}}}\sinh\left(2\beta\sqrt{\frac{2b}{\mu}}\right)\right]+2\beta\sqrt{\frac{2b}{\mu}}e^{-2\beta\sqrt{\frac{2b}{\mu}}}\operatorname{csch}\left(2\beta\sqrt{\frac{2b}{\mu}}\right)$$

The internal energy U is defined as:

$$U=-\frac{\partial}{\partial\beta}(\beta F)=-\frac{\partial\ln Z(\beta)}{\partial\beta}$$
$$=\left[-\frac{a^2}{4b}+\sqrt{\frac{2b}{\mu}}\left(1+\sqrt{2\mu g+\left[l-\left(1-\frac{1}{\alpha}\right)m\right]\left[l-\left(1-\frac{1}{\alpha}\right)m+1\right]+\frac{1}{4}}\right)\right]+\frac{4\sqrt{\frac{2b}{\mu}}e^{-4\beta\sqrt{\frac{2b}{\mu}}}}{\left(1-e^{-4\beta\sqrt{\frac{2b}{\mu}}}\right)} \quad . \quad (54)$$
$$=\left[-\frac{a^2}{4b}+4\sqrt{\frac{2b}{\mu}}\left(1+\sqrt{2\mu g+\left[l-\left(1-\frac{1}{\alpha}\right)m\right]\left[l-\left(1-\frac{1}{\alpha}\right)m+1\right]+\frac{1}{4}}\right)\right]+\sqrt{\frac{2b}{\mu}}e^{-2\beta\sqrt{\frac{2b}{\mu}}}\operatorname{csch}\left(2\beta\sqrt{\frac{2b}{\mu}}\right)$$

Then from the internal energy we can compute the specific heat capacity at constant volume $C_v$ as

$$C_v=-\beta^2\frac{\partial U}{\partial\beta}$$
$$=16k_B\beta^2\sqrt{\frac{2b}{\mu}}\frac{e^{-4\beta\sqrt{\frac{2b}{\mu}}}\left[1+\left(\sqrt{\frac{2b}{\mu}}-1\right)e^{-4\beta\sqrt{\frac{2b}{\mu}}}\right]}{\left(1-e^{-4\beta\sqrt{\frac{2b}{\mu}}}\right)^2} \quad . \quad (55)$$
$$=4k_B\beta^2\sqrt{\frac{2b}{\mu}}\left[1+\left(\sqrt{\frac{2b}{\mu}}-1\right)e^{-4\beta\sqrt{\frac{2b}{\mu}}}\right]\operatorname{csch}^2\left(2\beta\sqrt{\frac{2b}{\mu}}\right)$$

## 5. RESULTS AND DISCUSSION

In this section, we discuss the effect of cosmic-string geometry on thermodynamic properties and spectrum of heavy quarkonia. Fig. 1,2 and 3 show the plots of the radial wave functions of $c\bar{c}$ mesons for the $1P$, $2P$ and $3P$ states respectively, at different values of the parameter of the topology. Whereas, in Fig. 4,5 and 6, variations of the radial wave functions of $b\bar{b}$ mesons are plotted for the $1P$, $2P$ and $3P$ states respectively, at different values of the parameter $\alpha$.

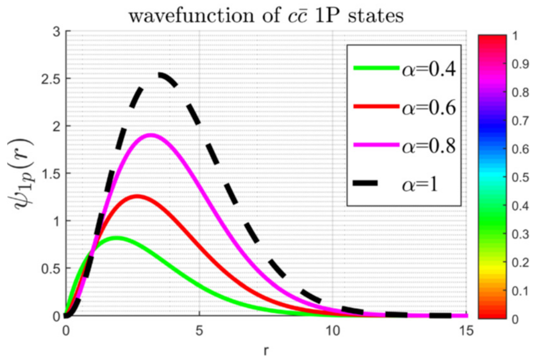


**Figure 1.** Ground states wave functions of $c\bar{c}$ meson plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

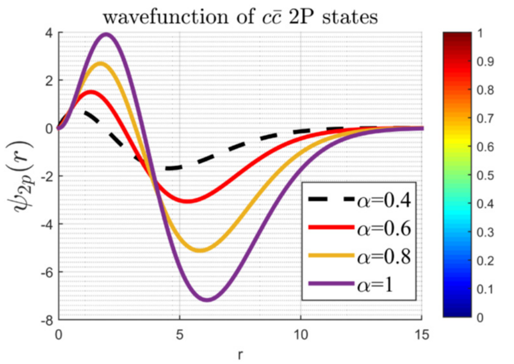


**Figure 2.** Radial wave functions of first excited states of $c\bar{c}$ meson plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

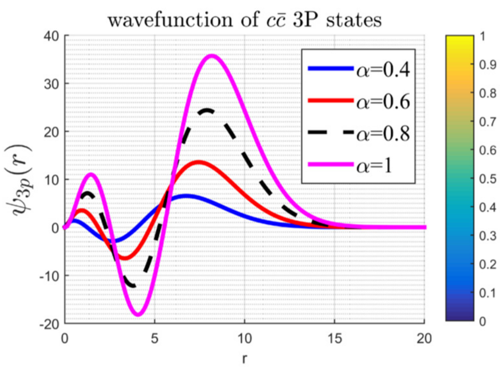


**Figure 3**. Radial wave functions of second excited states of $c\bar{c}$ meson plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

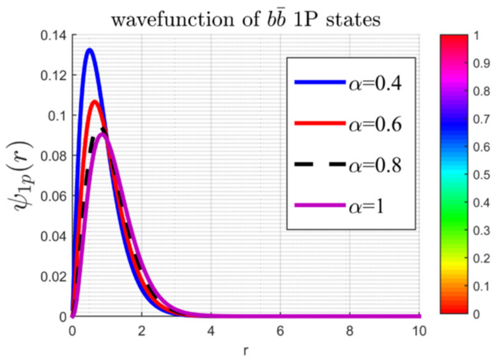


**Figure 4**. Ground states wave functions of $b\bar{b}$ meson plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

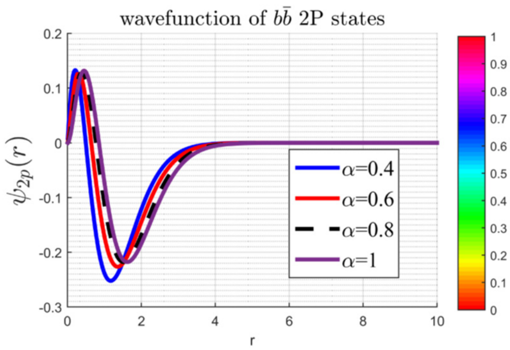


**Figure 5**. Radial wave functions of first excited states of $b\bar{b}$ mesons plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

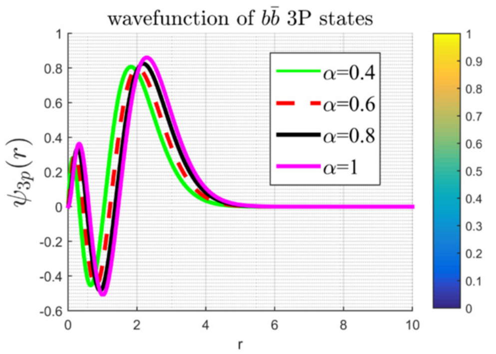


**Figure 6**. Radial wave functions of second excited states of $b\bar{b}$ mesons plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

It appears that the peaks of the wave functions are shifted backward for small values of $\alpha$, and its values shift to higher values by increasing $\alpha$ for $c\bar{c}$, and by decreasing $\alpha$ for $b\bar{b}$. In Fig. 7,8 and 9 are plotted the radial functions of probability densities of $c\bar{c}$ mesons for the $1P$, $2P$ and $3P$ states respectively, considering different values of the topological parameter $\alpha$. Whereas, in Fig. 10,11 and 12 are plotted the radial functions of probability densities of $b\bar{b}$ mesons for the $1P$, $2P$ and $3P$ states respectively It is observed that the behavior is the same for any $P-$state, but with some shift in the peaks toward the origin. Considering $c\bar{c}$ and $b\bar{b}$ mesons, the peaks are shifted toward the origin as $\alpha$ decrease.

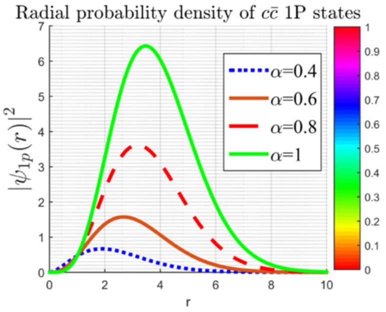


**Figure 7**. Density functions of radial probability for ground states of $c\bar{c}$ meson plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

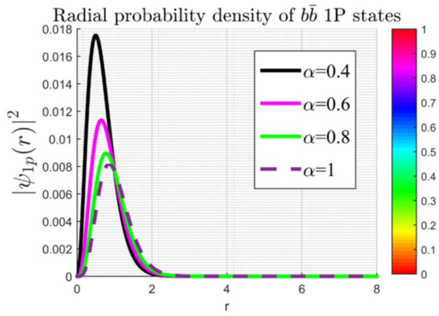


**Figure 8**. Density functions of radial probability for ground states of $b\bar{b}$ mesons plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

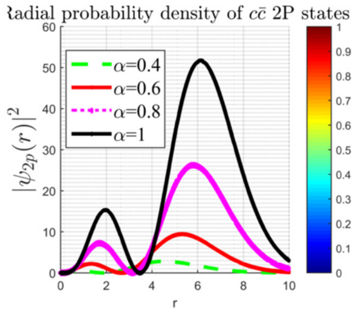


**Figure 9**. Density functions of radial probability for first excited states of $c\bar{c}$ meson plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

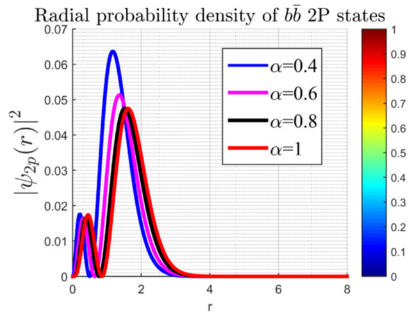


**Figure 10**. Density functions of radial probability for first excited states of $b\bar{b}$ meson plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

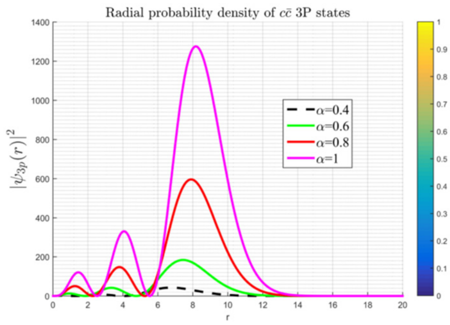


**Figure 11**. Density functions of radial probability for second excited states of $c\bar{c}$ meson plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

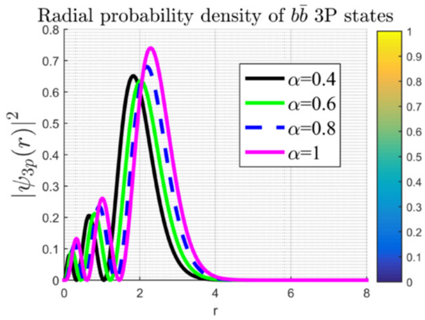


**Figure 12**. Density functions of radial probability for second excited states of $b\bar{b}$ meson plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

To calculate the thermodynamic properties of quark-antiquark systems, we have employed tools from statistical mechanics, taking as a starting point the canonical partition function of the system, from which the other statistical quantities were calculated in a straightforward manner. Next, we depict the statistical properties of $c\bar{c}$ and $b\bar{b}$ mesons in Figs. 13-22. In our model, the effect of the cosmic-string appears only when $l \neq 0$. Indeed, from Eq. (51) we clearly see that $S-$states are not influenced by the cosmic-string geometry but by the usual Minkowski geometry because when $l = 0$ we equally have $l_{(\alpha)} = 0$, and all the quantities become $\alpha-$ independent. In fact, setting the cosmic-string parameter to 1 automatically eliminate the effect of topological defect. For this reason, much emphasis were placed on the study of states with $l \neq 0$, namely the $P-$states.

In Fig. 13,14, we have plotted the canonical partition function respectively for $c\bar{c}$ and $b\bar{b}$ quarkonia in terms of $\beta$ for different values of the cosmic-string parameter $\alpha$. It is clearly seen that by decreasing the value of $\alpha$, all the curves are separated from the classical Minkowski curve ($\alpha = 1$). For all the values of the topological parameter the behavior is the same but with different magnitudes. Moreover, the splitting increases as $\alpha$ decreases and then the partition function decreases with increasing $\beta$ which is in agreement with [36, 37]. In [36], the author applied the deformed five parameter exponential potential and observed that with increasing $\beta$, the canonical partition function for the Minkowskian case decreases monotonically.

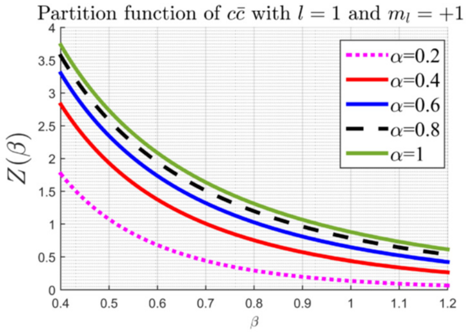


**Figure 13**. Density functions of radial probability for second excited states of $c\bar{c}$ meson plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

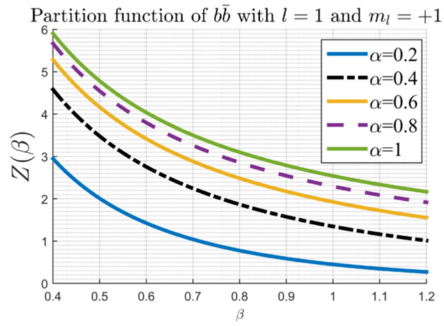


**Figure 14**. Density functions of radial probability for second excited states of $b\bar{b}$ meson plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

In Fig. 15,16 The Helmholtz free energy is plotted as a function of $\beta$ respectively for $c\bar{c}$ and $b\bar{b}$ mesons.

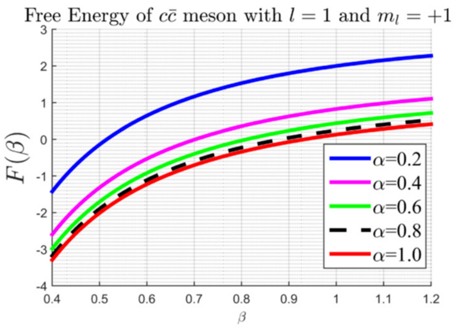


**Figure 15**. Density functions of radial probability for second excited states of $c\bar{c}$ meson plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

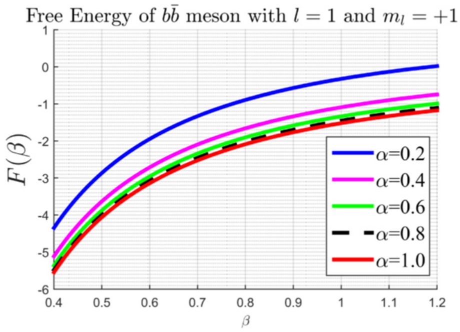


**Figure 16**. Density functions of radial probability for second excited states of $b\bar{b}$ meson plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

The effect of the cosmic-string geometry clearly appears when $\alpha$ takes values different from $\alpha = 1$ (classical Minkowski space-time). It is found that the Helmholtz function increases monotonically with increasing $\beta$, and the curves with $\alpha \neq 1$ are separated from the classical Minkowski curve. Moreover, the splitting is important for small values of $\alpha$. In [34], the author treated quark-gluon plasma as composed of light quarks only, which interact weakly. They observed that the Helmholtz free energy decreases as the temperature increases. In [35], Modarres and Gholizade calculated the Helmholtz function of a neutral particle and observed that the free energy of the system decreases as the temperature increases. In the present model, the behavior of charm and bottom quark matter is in agreement with [34, 35, 38].

In Fig. 17,18 the internal energy is plotted against $\beta$ for different values of $\alpha$, respectively for $c\bar{c}$ and $b\bar{b}$ mesons. It reveals that, $U(\beta)$ decreases with increasing $\beta$, and its values shift to lower values by increasing the parameter of the topology $\alpha$.

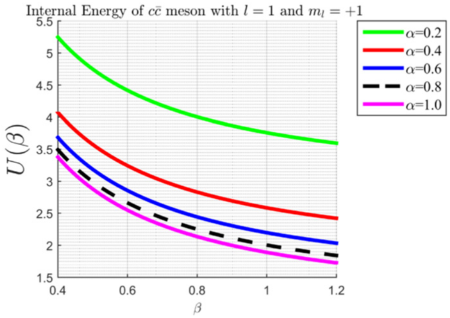


**Figure 17**. Density functions of radial probability for second excited states of $c\bar{c}$ meson plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

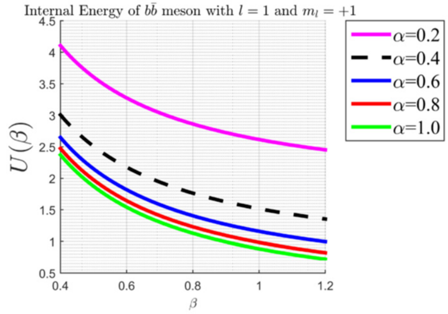


**Figure 18**. Density functions of radial probability for second excited states of $b\bar{b}$ meson plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

In [38], the authors studied thermodynamic properties of neutral particles in a cosmic-string background from non-relativistic Schrödinger-Pauli equation. They observed that the internal energy of the system increases as the temperature increases. Thus, the conclusion of the present work for internal energy is the same with recent works [36, 37, 38].

The entropy $S(\beta)$ and the specific heat capacity $C_v(\beta)$ are plotted against $\beta$ respectively in Fig. 19 and Fig. 21 $c\bar{c}$ meson and in Fig. 20 and Fig. 22 $b\bar{b}$ meson. These figures show that the entropy and the specific heat capacity of $c\bar{c}$ and $b\bar{b}$ mesons are not influenced by the parameters of topology, and are similar to the solution for a at Minkowski space-time. In [38] a similar result was obtained for neutral particles, showing the non-dependence on topological defect of heat capacity and entropy, which is in total agreement with the present work.

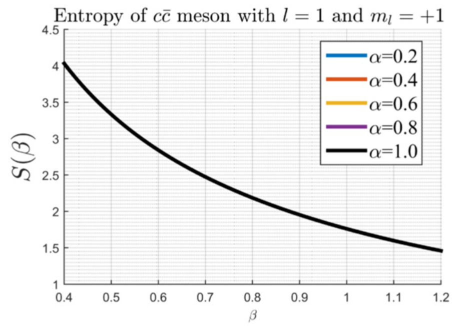


**Figure 19**. Density functions of radial probability for second excited states of $c\bar{c}$ meson plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

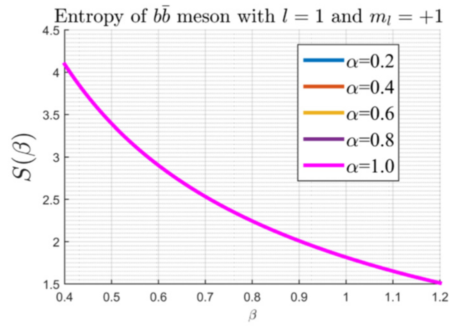


**Figure 20**. Density functions of radial probability for second excited states of $b\bar{b}$ meson plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

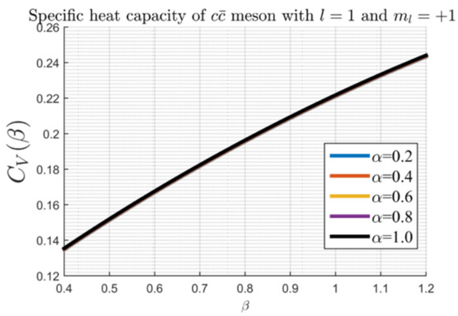


**Figure 21**. Density functions of radial probability for second excited states of $c\bar{c}$ meson plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

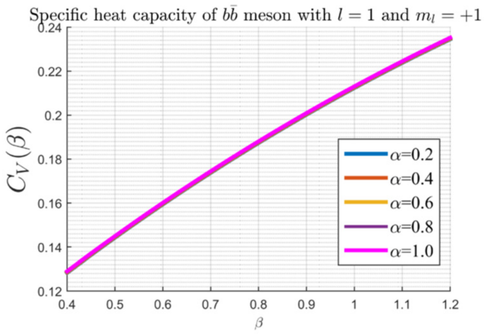


**Figure 22**. Density functions of radial probability for second excited states of $b\bar{b}$ meson plotted against of the radius $r$ at different values of $\alpha$ with $l$=1, $m_l = +1$

## 6. CONCLUSION

In this work, we have investigated the effects of the gravitational field of a topological defect on thermodynamical properties of heavy quarkonia. We have used the extended Nikiforov-Uvarov method to solve the radial Schrödinger wave equation in the space-time of a cosmic string, for the potential $V(r) = -\frac{K}{r} + \alpha r + \frac{g}{r^2} + br^2$. We obtain the radial wave functions of $c\bar{c}$ and $b\bar{b}$ mesons as well as the energy eigenvalues, which are shifted from the usual Minkowski energies. From this, we have considered all the statistical quantities using the canonical partition function of the system. The wave functions and the thermodynamic properties were analyzed graphically. It was observed that the peaks of radial functions of probability densities are shifted toward the origin as $\alpha$ decrease. The thermodynamic quantities present a shift compared to the classic limit; this difference becomes more important for small values of the cosmic-string parameter. The partition function shifts to lower values as $\alpha$ decrease, the Helmholtz function increases monotonically with increasing $\beta$, and the curves with $\alpha \neq 1$ are separated from the Minkowski curve, then the values of the internal energy are shifted to lower values by increasing the parameter of the topology. Meanwhile the specific heat capacity and the entropy are not influenced by the topological defect.

**Declaration of competing interest**. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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**REFERENCES**

[1] L.R. Ribeiro et al, Phys. Lett. A, **348**, 135 (2006). https://doi.org/10.1016/j.physleta.2005.08.046
[2] K. Bakke, Int. J. Mod. Phys. A, **26**, 4239 (2011). https://doi.org/10.1142/S0217751X11054437
[3] Y. Aharonov, and D. Bohm, Phys. Rev. **115**, 485 (1959). https://doi.org/10.1103/PhysRev.115.485
[4] K.W.B. Kibble, J. Phys. A: Math. Gen. **9**(8), 1387 (1976). https://doi.org/10.1088/0305-4470/9/8/029
[5] K.W.B. Kibble, Phys. Rev. D, **26**(2), 435 (1982). https://doi.org/10.1103/PhysRevD.26.435
[6] A. Vilenkin, and E.P.S. Shellard, *Cosmic Strings and Other Topological Defects*, (Cambridge University Press, Cambridge, U.K., 1994).
[7] J. Rocher, Thèse de Doctorat, Contraintes cosmologiques sur la physique de l'univers primordial, Universit de Paris XI-Orsay, 2005.
[8] L. Bergström, and A. Goobar, *Cosmology and Particle Astrophysics*, 2nd edition (Springer, 2006).
[9] A.-C. Davis, and T.W. Kibble, Contemp. Phys. **46**, 313 (2005). https://doi.org/10.1080/00107510500165204
[10] N.G. Marchuk, Nuovo. Cimento. B, **115**, 11 (2000). https://doi.org/10.48550/arXiv.math-ph/0010045
[11] C. Furtado, V.B. Bezerrae, and F. Moraes, Phys. Lett. A, **289**, 160 (2001). https://doi.org/10.1016/S0375-9601(01)00615-6
[12] C. Furtado, B.G.C. da Cunha, F. Moraes, E.R.B. Mello, and V.B. Bezzerra, Phys. Lett. A, **195**, 90 (1994), https://doi.org/10.1016/0375-9601(94)90432-4
[13] J. Audretsch, and A. Economou, Phys. Rev. D, **44**, 3774 (1991). https://doi.org/10.1103/PhysRevD.44.3774
[14] D.D. Harari, and V.D. Skarzhinsky, Phys. Lett. B, **240**, 322 (1990). https://doi.org/10.1016/0370-2693(90)91106-L
[15] J.R. Gott III, Astrophys. **288**, 422 (1985). https://adsabs.harvard.edu/pdf/1985ApJ...288..422G
[16] V.B. Bezerra, Phys. Rev. D, **35**, 2031 (1987). https://doi.org/10.1103/PhysRevD.35.2031
[17] W. Florkowski, *Phenomenology of Ultra-Relativistic Heavy-Ion Collisions*, (World Scientific Singapore, 2010), pp. 416.
[18] U. Kalade, and B.K. Patra, Phys. Rev. C, **92**, 024901 (2015). https://doi.org/10.1103/PhysRevC.92.024901
[19] T. Matsui, and H. Satz, Phys. Lett. B, **178**, 416 (1986). https://doi.org/10.1016/0370-2693(86)91404-8
[20] R.C. Hwa, and X.N. Wang, editors, *Quark-Gluon Plasma* 3, (World Scientific Publishing, 2004), pp. 788.
[21] V.S. Filinov, M. Bonitz, Y.B. Ivanov, M. Ilgenfritz, and V.E. Fortov, Contrib. Plasma Phys. **55**, 203 (2015). https://doi.org/10.1002/ctpp.201400056
[22] M. Schleif, and R. Wunsch, Eur. Phys. J. A, **1**, 171 (1998). https://doi.org/10.1007/s100500050046
[23] M. Abu-Shady, Inter. J. Theor. Phys. **52**, 1165 (2013). https://doi.org/10.1007/s10773-012-1432-z
[24] M. Abu-Shady, Inter. J. Theor. Phys. **54**, 1530 (2015). https://doi.org/10.1007/s10773-014-2352-x
[25] D. Nga Ongodo, J.M. Ema'a Ema'a, P. Ele Abiama, and G.H. Ben-Bolie, Int. J. Mod. Phys. E, **28**, 1950106 (2019). https://doi.org/10.1142/S0218301319501064
[26] Al-Jamel, Mod. Phys. Lett. A, **33**, 1850185 (2018). https://doi.org/10.1142/S0217732318501857
[27] H. Karayer, D. Demirhan, and F. Büyükilic, J. Math. Phys. **59**, 053501 (2008). https://doi.org/10.1063/1.5022008
[28] M.D. Katanaev, and I.V. Volovich, "Theory of defects in solids and three-dimensional gravity", Annals of Physics, **216**(1), 1 1992. https://doi.org/10.1016/0003-4916(52)90040-7
[29] C. Furtado, and F. Moraes, Phys. Lett. A, **188**(4-6), 394 (1994). https://doi.org/10.1016/0375-9601(94)90482-0
[30] C.R. Muniz, V.B. Bezerra and M.S. Cunha, Ann. Phys. **350**, 105 (2014). https://doi.org/10.1016/j.aop.2014.07.017
[31] V.F. Mukanov, H.A. Feldman, and R.H. Brandenberger, Physical Report **215**, 203 (1992). https://doi.org/10.1016/0370-1573(92)90044-Z
[32] J.L. Domenech-Garret, and M.A. Sanchis-Lozano, Physics Letters B, 669(1), 52 (2008). https://doi.org/10.1016/j.physletb.2008.09.021
[33] Y. Cançelik, and B. Gönül, Mod. Phys. Lett. A, **29**, 1450170 (2014). https://doi.org/10.1142/S0217732314501703
[34] M. Modarres, and A. Mohamadnejad, Phys. Part. Nucl. Lett. **10**, 99 (2013). https://doi.org/10.1134/S1547477113020106
[35] M. Modarres, and H. Gholizade, Int. J. Mod. Phys. E, **17**, 1335 (2008). https://doi.org/10.1142/S0218301308010453
[36] A.N. Ikot, B.C. Lutfuoglu, M.I. Ngweke, M.E. Udoh, S. Zare, and H. Hassanabadi, Eur. Phys. J. Plus, **131**, 419 (2016). https://doi.org/10.1140/epjp/i2016-16419-5
[37] W.A. Yahua, and K.J. Oyewumi, J. Asso. Arab. Univ. Bas. App. Scie, **21**, 53 (2016). https://doi.org/10.1016/j.jaubas.2015.04.001
[38] H. Hassanabadi, and M. Hosseinpoura, Eur. Phys. J. C, **76**, 553 (2016). https://doi.org/10.1140/epjc/s10052-016-4392-2

## ВПЛИВ ГРАВІТАЦІЙНОГО ПОЛЯ НА ТОПОЛОГІЧНИЙ ДЕФЕКТ НА СТАТИСТИЧНІ ВЛАСТИВОСТІ ВАЖКИХ КВАРК-АНТИКВАРКОВИХ СИСТЕМ

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У цій статті ми визначаємо власні енергії, власні функції та статистичні властивості нерелятивістського важкого кварконію, що взаємодіє з розширеним потенціалом Корнеля в просторі-часі, створеному космічною струною. Ми розширюємо потенціал Корнеля, додаючи обернений квадратний потенціал плюс квадратичний потенціал. Ми розрахували власні значення енергії та відповідні власні стани за допомогою розширеного методу Нікіфорова-Уварова (ENU). Потім на основі рівняння енергетичних спектрів обчислюються термодинамічні властивості, такі як статистична сума, ентропія, вільна енергія, середня енергія та питома теплоємність у просторі-часі космічної струни. На наступному кроці ми досліджуємо вплив параметра космічної струни на квантові стани важких кварконіїв та їх статистичні властивості.

**Ключові слова**: термодинамічні властивості, кварк, хвильове рівняння Шредінгера, топологічний дефект, простір-час, космічна струна, розширений потенціал Корнела, мезон, розширений метод Нікіфорова-Уварова.