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NON-RELATIVISTIC STUDY OF MASS SPECTRA AND THERMAL PROPERTIES OF A QUARKONIUM SYSTEM WITH ECKART-HELLMANN POTENTIAL[†]

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In this present study, we model Eckart-Hellmann Potential (EHP) to interact in a quark-antiquark system. The solutions of the Schrödinger equation are obtained with EHP using the Nikiforov-Uvarov method. The energy equation and normalized wave function were obtained. The masses of the heavy mesons such as charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$) for different quantum numbers were predicted using the energy equation. Also, the partition function was calculated from the energy equation, thereafter other thermal properties such as mean energy, free energy, entropy, and specific heat capacity were obtained. The results obtained showed an improvement when compared with the work of other researchers and excellently agreed with experimental data.
Keywords: Schrödinger equation; Nikiforov-Uvarov method; Eckart-Hellmann Potential; heavy mesons; Thermal properties
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The mass spectra (MS) of the heavy mesons (HMs) interactions can be well studied by the Schrodinger equation (SE) [1--3]. In describing the interaction of the HMs, confining-type potentials are generally used, which is the Cornell potential (CP) with two terms of Coulomb interaction and a confining term [4]. More so, in solving this equation with any chosen potential, an analytical method is employed. Most of the analytical methods used are as follows, the Nikiforov-Uvarov (NU) method [5-8], the Nikiforov-Uvarov Functional Analysis (NUFA) method [9,10], series expansion method (SEM) [11,12], Laplace transformation method (LTM)[13],WKB approximation method [14,15] and so on[16]. The study of MS with CP has gained remarkable interest and has attracted the attention of many scholars [17-20]. For instance, Kumar et al.,[21] used the NUFA method to solve the SE with generalized Cornell potential. The result was used to predict the MS of the HMs. Using, the vibrational method and supersymmetric quantum mechanics Vega and Flores, [22] obtained the analytical solutions of the SE with CP. The eigenvalues were used to calculate the MS of the HMs. Also, Mutuk [23] solved the SE with CP using a neural network approach. The bottomonium, charmonium, and bottom-charmed spin-averaged spectra were calculated. Furthermore, Hassanabadi et al. [24], used the variational method to solve the SE with CP. The eigenvalues were used to calculate the mesonic wave function.

In recent times, the study of MS of the HMs with exponential-type potentials has aroused the interest of scholars [25, 26]. Potential models such as Yukawa potential [27], Varshni [28], screened Kratzer potential [29], Hulthen plus Hellmann potential [30], and so on have been used in the prediction of the MS of the HMs. For instance, Purohit et al [31] combined linear plus modified Yukawa potential to obtain the masses of the HMs through the solutions of the Klein-Gordon equation. The SE for most of the potentials with spin addition cannot be solved analytically; hence, numerical solutions such as Runge-Kutte approximation [32], Numerov matrix method [33], Fourier grid Hamiltonian method [34], and so on [35] are employed. Also, adding spin enables one to determine other properties of the mesons like decay properties and root mean square radii. However, we have considered our mesons as spinless particles for easiness [1, 25, 36-38]. Furthermore, the thermal properties (TPs) of the HMs have been calculated recently [39-41].

The Eckart potential [42], is a potential model that has great significance in physics. Also, Hellmann potential [43], has been widely utilized in physics [44]. Hence, Inyang et al [45], proposed the Eckart-Hellmann potential (EHP) model through their combination to study selected diatomic molecules.

The combination of at least two potential models has a propensity to fit experimental data more than a single potential [40], hence this study. This study aims to model EHP to fit in the Cornell potential, and to predict the mass spectra of the heavy mesons through the solutions of the SE using the NU method.

The EHP takes the form [45],

$$V(q) = -\frac{R_0 e^{-\sigma q}}{1-e^{-\sigma q}} + \frac{R_1 e^{-\sigma q}}{\left(1-e^{-\sigma q}\right)^2} - \frac{R_2}{q} + \frac{R_3 e^{-\sigma q}}{q}, \tag{1}$$

where $R_0, R_1, R_2,$ and R_3 are the strength of the potential, σ is the screening parameter to be determined later and q is inter-nuclear distance.

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The exponential terms in Eq. (1) are expanded with the power series up to order three, so the potential can be used to study quarkonia system. Equations (2), (3), and (4) are obtained.

$$\frac{e^{-\sigma q}}{q}=\frac{1}{q}-\sigma+\frac{\sigma^2 q}{2}-\frac{\sigma^3 q^2}{6}+\dots, \tag{2}$$

$$\frac{e^{-\sigma q}}{1-e^{-\sigma q}}=\frac{1}{\sigma q}-\frac{1}{2}+\frac{\sigma q}{12}+\dots, \tag{3}$$

$$\frac{e^{-\sigma q}}{\left(1-e^{-\sigma q}\right)^2}=\frac{1}{\sigma^2 q^2}-\frac{1}{12}+\frac{\sigma^2 q^2}{240}+\dots \tag{4}$$

Putting Eqs. (2), (3)and (4) into Eq. (1) we have

$$V(q)=-\frac{G_0}{q}+G_1 q+G_2 q^2+\frac{G_3}{q^2}+G_4, \tag{5}$$

where

$$\left.\begin{aligned} G_0&=\frac{R_0}{\sigma}+R_2-R_3,\ \mathrm{G}_1=-\frac{\sigma R_0}{12}+\frac{\sigma^2 R_3}{2}\\ G_2&=\frac{R_1\sigma^2}{240}-\frac{R_3\sigma^3}{6},\ \mathrm{G}_3=\frac{R_1}{\sigma^2},\ \mathrm{G}_4=\frac{R_0}{2}-\frac{R_1}{12}-\sigma R_3 \end{aligned}\right\} \tag{6}$$

2. The solutions of the SE with EHP

The NU method is adopted with details found in Ref. [46]. The SE of the form is used [47]

$$\frac{d^2\,\mathrm{U}(q)}{dq^2}+\left[\frac{2\mu}{\hbar^2}\left(E_{nl}-V(q)\right)-\frac{l\left(l+1\right)}{q^2}\right]\mathrm{U}(q)=0 \tag{7}$$

where l, is the angular momentum quantum number, μ, is the reduced mass for the quark-antiquark particle, q is the inter-particle distance, and $\hbar$ is reduced plank constant.

Then, we substitute Eq. (5) into Eq. (7), the radial wave equation is obtained as

$$\frac{d^2\,\mathrm{U}(q)}{dq^2}+\left[\frac{2\mu E}{\hbar^2}+\frac{2\mu G_0}{\hbar^2 q}-\frac{2\mu G_1 q}{\hbar^2}-\frac{2\mu G_2 q^2}{\hbar^2}-\frac{2\mu G_3}{\hbar^2 q^2}-\frac{2\mu G_4}{\hbar^2}-\frac{l(l+1)}{q^2}\right]\mathrm{U}(q)=0\,. \tag{8}$$

Transformation of q (in Eq. (8)) to w coordinates yields Eq.(9),

$$w=\frac{1}{q},q\succ 0\,. \tag{9}$$

The second derivatives of Eq. (9) is given as,

$$\frac{d^2U(q)}{dq^2}=2w^3\frac{dU(w)}{dw}+w^4\frac{d^2U(w)}{dw^2}\,. \tag{10}$$

Substituting Eqs. (9) and (10) in Eq. (8) gives;

$$\frac{d^2\,\mathrm{U}(w)}{dw^2}+\frac{2w}{w^2}\frac{dU}{dw}+\frac{1}{w^4}\left[\begin{aligned}&\frac{2\mu E}{\hbar^2}+\frac{2\mu G_0 w}{\hbar^2}-\frac{2\mu G_1}{\hbar^2 w}\\ &-\frac{2\mu G_2}{\hbar^2 w^2}-\frac{2\mu G_3 w^2}{\hbar^2}-\frac{2\mu G_4}{\hbar^2}-l\left(l+1\right)w^2\end{aligned}\right]\mathrm{U}(w)=0\,. \tag{11}$$

The approximation scheme (AS) on the terms is introduced by assuming that there is a characteristic radius r_0 of the meson. The AS is achieved by the expansion of $\frac{G_1}{w}$ and $\frac{G_2}{w^2}$ in a power series around r_0; i.e. around $\delta \equiv \frac{1}{r_0}$, up to the second-order [48].

By setting $y = w - \delta$ and around $y = 0$ we expand it in powers of series as;

$$\frac{G_1}{w} = \frac{G_1}{y+\delta} = \frac{G_1}{\delta\left(1+\frac{y}{\delta}\right)} = \frac{G_1}{\delta}\left(1+\frac{y}{\delta}\right)^{-1}. \tag{12}$$

Equation (12) yields

$$\frac{G_1}{w} = G_1\left(\frac{3}{\delta} - \frac{3z}{\delta^2} + \frac{z^2}{\delta^3}\right). \tag{13}$$

Similarly,

$$\frac{G_2}{w^2} = G_2\left(\frac{6}{\delta^2} - \frac{8w}{\delta^3} + \frac{3w^2}{\delta^4}\right). \tag{14}$$

Also, stroking Eqs. (13) and (14) into Eq. (11) gives:

$$\frac{d^2\,\mathrm{U}(w)}{dw^2} + \frac{2w}{w^2}\frac{dU(w)}{dw} + \frac{1}{w^4}\left[-\varepsilon + X_0 w - X_1 w^2\right]\mathrm{U}(w) = 0, \tag{15}$$

where

$$\left.\begin{aligned} -\varepsilon &= \left(\frac{2\mu E}{\hbar^2} - \frac{6\mu G_1}{\hbar^2\delta} - \frac{12\mu G_2}{\hbar^2\delta^2} - \frac{2\mu G_4}{\hbar^2}\right),\ X_0 = \left(\frac{2\mu G_0}{\hbar^2} + \frac{6\mu G_1}{\hbar^2\delta^2} + \frac{16\mu G_2}{\hbar^2\delta^3}\right) \\ X_1 &= \left(\frac{2\mu G_1}{\hbar^2\delta^3} + \frac{6\mu G_2}{\hbar^2\delta^4} + \frac{2\mu G_3}{\hbar^2} + \gamma\right),\ \gamma = l(l+1) \end{aligned}\right\}. \tag{16}$$

Linking Eq. (15) and Eq. (1) of Ref. [46], gives

$$\left.\begin{aligned} &\tilde{\tau}(w) = 2w,\ \sigma(w) = z^2 \\ &\tilde{\sigma}(w) = -\varepsilon + \alpha w - \beta w^2 \\ &\sigma'(w) = 2w,\ \sigma''(w) = 2 \end{aligned}\right\}. \tag{17}$$

Plugging Eq. (17) into Eq. (11) of Ref. [46],

$$\pi(w) = \pm\sqrt{\varepsilon - X_0 w + \left(X_1 + k\right)w^2}. \tag{18}$$

To determine k, in Eq. (18), the discriminant of the function (Eq. (19)) and Eq. (20) were obtained,

$$k = \frac{X_0^{\ 2} - 4X_1\varepsilon}{4\varepsilon}, \tag{19}$$

$$\pi(w) = \pm\left(\frac{X_0 w}{2\sqrt{\varepsilon}} - \frac{\varepsilon}{\sqrt{\varepsilon}}\right). \tag{20}$$

For acceptable solution, the negative part of Eq. (20) is essential for bound state problems, upon differentiating we get.

$$\pi_-'(w) = -\frac{X_0}{2\sqrt{\varepsilon}}. \tag{21}$$

By placing Eqs. (17) and (20) into Eq. (6) of Ref. [46]

$$\tau(w) = 2w - \frac{X_0 w}{\sqrt{\varepsilon}} + \frac{2\varepsilon}{\sqrt{\varepsilon}} \tag{22}$$

Differentiating Eq. (22) gives,

$$\tau'(w) = 2 - \frac{X_0}{\sqrt{\varepsilon}} \tag{23}$$

Usingto Eq. (19) and Eq. (21) of Ref. [46], gives,

$$\lambda = \frac{X_0^{\ 2} - 4X_1\varepsilon}{4\varepsilon} - \frac{X_0}{2\sqrt{\varepsilon}} \tag{24}$$

$$\lambda_n = \frac{nX_0}{\sqrt{\varepsilon}} - n^2 - n \tag{25}$$

Equating Eqs. (24) and (25), followed by the substitution of Eqs. (6) and (16) yielded the energy eigenvalue equation of the EHP

$$\begin{aligned} E_{nl} &= \frac{R_0}{2} - \frac{R_1}{12} - R_0R_3 + \frac{3}{\delta}\left(-\frac{\sigma R_0}{12} + \frac{R_3\sigma^2}{2}\right) + \frac{6}{\delta^2}\left(\frac{R_1\sigma^2}{240} - \frac{R_3\sigma^3}{6}\right) \\ &- \frac{\hbar^2}{8\mu}\left[\frac{\frac{2\mu}{\hbar^2}\left(\frac{R_0}{\sigma} + R_2 - R_3\right) + \frac{6\mu}{\hbar^2\delta^2}\left(-\frac{\sigma R_0}{12} + \frac{R_3\sigma^2}{2}\right) + \frac{16\mu}{\hbar^2\delta^3}\left(\frac{R_1\sigma^2}{240} - \frac{R_3\sigma^3}{6}\right)}{n + \frac{1}{2} + \sqrt{\left(\frac{1}{2} + l\right)^2 + \frac{2\mu}{\hbar^2\delta^3}\left(-\frac{\sigma R_0}{12} + \frac{R_3\sigma^2}{2}\right) + \frac{6\mu}{\hbar^2\delta^4}\left(\frac{R_1\sigma^2}{240} - \frac{R_3\sigma^3}{6}\right) + \frac{2\mu R_1}{\hbar^2\sigma^2}}}\right]^2 \end{aligned} \tag{26}$$

The wave function, is obtained by putting Eqs. (17) and (20) into Eq. (4) of Ref. [46]

$$\frac{d\phi}{\phi} = \left(\frac{\varepsilon}{w^2\sqrt{\varepsilon}} - \frac{X_0}{2w\sqrt{\varepsilon}}\right)dw \tag{27}$$

Integration of Eq. (27) gives

$$\phi(w) = w^{-\frac{X_0}{2\sqrt{\varepsilon}}} e^{-\frac{\varepsilon}{w\sqrt{\varepsilon}}} \tag{28}$$

2.1 Determination of the weight function

Upon differentiating the left-hand of Eq. (6) of Ref. [46] we have,

$$\frac{\rho'(w)}{\rho(w)} = \frac{\tau(w) - \sigma'(w)}{\sigma(w)} \tag{29}$$

The substitution of Eqs. (17) and (22) into Eq. (29) and thereafter integrate, gave

$$\rho(w) = w^{-\frac{X_0}{\sqrt{\varepsilon}}} e^{-\frac{2\varepsilon}{w\sqrt{\varepsilon}}} \tag{30}$$

The substitution of Eqs. (17) and(30) into Eq. (5) of Ref. [46] gave

$$y_n(w) = B_n e^{\frac{2\varepsilon}{w\sqrt{\varepsilon}}} z^{\frac{X_0}{\sqrt{\varepsilon}}} \frac{d^n}{dw^n}\left[e^{-\frac{2\varepsilon}{w\sqrt{\varepsilon}}} w^{n-\frac{X_0}{\sqrt{\varepsilon}}}\right] \tag{31}$$

The Rodrigues' formula of the associated Laguerre polynomials is

$$L_n^{\frac{X_0}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{w\sqrt{\varepsilon}}\right) = \frac{1}{n!} e^{\frac{2\varepsilon}{w\sqrt{\varepsilon}}} w^{\frac{X_0}{\sqrt{\varepsilon}}} \frac{d^n}{dw^n}\left(e^{-\frac{2\varepsilon}{w\sqrt{z}}} w^{n-\frac{X_0}{\sqrt{\varepsilon}}}\right) \tag{32}$$

where $B_n = \frac{1}{n!}$.

Hence,

$$y_n(w) \equiv L_n^{\frac{X_0}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{w\sqrt{\varepsilon}}\right) \tag{33}$$

The substitution of Eqs. (28) and (33) into Eq. (2) of Ref. [46], gives the wave function in terms of associated Laguerre polynomials as

$$\psi(w) = N_{nl} w^{-\frac{X_0}{2\sqrt{\varepsilon}}} e^{-\frac{\varepsilon}{w\sqrt{\varepsilon}}} L_n^{\frac{X_0}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{w\sqrt{\varepsilon}}\right) \tag{34}$$

where N_{nl} is normalization constant, which can be obtained from

$$\int_0^\infty |\psi_{nl}(r)|^2 dr = 1 \tag{35}$$

Inserting (34) into (35) with $w = 1/r$ gives

$$N_{nl}^2 \int_0^\infty r^{X_0/\sqrt{\varepsilon}} e^{-2\sqrt{\varepsilon} r}\left[L_n^{X_0/\sqrt{\varepsilon}}\left(2\sqrt{\varepsilon} r\right)\right]^2 dr = 1 \tag{36}$$

By using the transformation $x = 2\sqrt{\varepsilon} r$ we obtained the well-known standard integral of the Laguerre polynomials

$$\frac{N_{nl}^2}{\left(2\sqrt{\varepsilon}\right)^{X_0/\sqrt{\varepsilon}+1}} \int_0^\infty x^{X_0/\sqrt{\varepsilon}} e^{-x}\left[L_n^{X_0/\sqrt{\varepsilon}}(x)\right]^2 dx = 1 \tag{37}$$

The solution of the standard integral [49] is given as

$$\int_0^\infty x^{X_0/\sqrt{\varepsilon}} e^{-x}\left[L_n^{X_0/\sqrt{\varepsilon}}(x)\right]^2 dx = \frac{\Gamma\left(n + X_0/\sqrt{\varepsilon} + 1\right)}{\Gamma(n+1)} \tag{38}$$

Comparing Eqs. (38) and (37) we obtained the normalization factor such that the total wave function of the mesons can be written in closed form as

$$\psi_{nl}(r) = \sqrt{\frac{\left(2\sqrt{\varepsilon}\right)^{X_0/\sqrt{\varepsilon}+1}\Gamma(n+1)}{\Gamma\left(n + X_0/\sqrt{\varepsilon} + 1\right)}}\, r^{X_0/2\sqrt{\varepsilon}} e^{-\sqrt{\varepsilon} r} L_n^{X_0/\sqrt{\varepsilon}}\left(2\sqrt{\varepsilon} r\right) \tag{39}$$

Special cases

1. When we set $R_0 = R_1 = 0$ Eq. (26) reduces to HP energy

$$E_{nl} = \frac{3R_3\sigma^2}{2\delta} - \frac{R_3\sigma^3}{\delta^2} - \frac{\hbar^2}{8\mu}\left[\frac{\frac{2\mu}{\hbar^2}\left(R_2 - R_3\right) + \frac{3\mu R_3\sigma^2}{\hbar^2\delta^2} - \frac{16\mu R_3\sigma^3}{6\hbar^2\delta^3}}{n + \frac{1}{2} + \sqrt{\left(\frac{1}{2}+l\right)^2 + \frac{\mu R_3\sigma^2}{\hbar^2\delta^3} - \frac{\mu R_3\sigma^3}{\hbar^2\delta^4}}}\right]^2 \tag{40}$$

2. When we set $R_2 = R_3 = 0$ Eq. (26) Echart potential energy

$$E_{nl} = E_{nl} = \frac{R_0}{2} - \frac{R_1}{12} - \frac{\sigma R_0}{4\delta} + \frac{R_1\sigma^2}{40\delta^2} - \frac{\hbar^2}{8\mu}\left[\frac{\frac{2\mu R_0}{\sigma\hbar^2} - \frac{\mu\sigma R_0}{2\hbar^2\delta^2} + \frac{\mu R_1\sigma^2}{40\hbar^2\delta^3}}{n + \frac{1}{2} + \sqrt{\left(\frac{1}{2}+l\right)^2 - \frac{\mu\sigma R_0}{6\hbar^2\delta^3} + \frac{6\mu R_1\sigma^2}{40\hbar^2\delta^4} + \frac{2\mu R_1}{\hbar^2\sigma^2}}}\right]^2 \tag{41}$$

3. When we set $R_0 = R_1 = R_3 = \sigma = 0$, Eq. (26) reduces to Coulomb potential energy

$$E_{nl} = -\frac{\mu R_2^{\,2}}{2\hbar^2 \left(n+l+1\right)^2} \tag{42}$$

The result of Eq. (42) is the same as reported by Ref. [31] in Eq. (36).

3. Thermal Properties of the SE with EHP

To obtain the TPs of the heavy mesons, we first calculate the partition function.
Equation 26 can be written in the form

$$E_{nl} = P_1 - \frac{\hbar^2}{8\mu}\left[\frac{P_2}{\left(n+\theta\right)}\right]^2 \tag{43}$$

where,

$$\theta = \frac{1}{2} + \sqrt{\left(\frac{1}{2}+l\right)^2 + \frac{2\mu}{\hbar^2\delta^3}\left(-\frac{\sigma R_0}{12}+\frac{R_3\sigma^2}{2}\right) + \frac{6\mu}{\hbar^2\delta^4}\left(\frac{R_1\sigma^2}{240}-\frac{R_3\sigma^3}{6}\right) + \frac{2\mu R_1}{\hbar^2\sigma^2}} \tag{44}$$

$$P_1 = \frac{R_0}{2} - \frac{R_1}{12} - R_0R_3 + \frac{3}{\delta}\left(-\frac{\sigma R_0}{12}+\frac{R_3\sigma^2}{2}\right) + \frac{6}{\delta^2}\left(\frac{R_1\sigma^2}{240}-\frac{R_3\sigma^3}{6}\right) \tag{45}$$

$$P_2 = \frac{2\mu}{\hbar^2}\left(\frac{R_0}{\sigma}+R_2-R_3\right) + \frac{6\mu}{\hbar^2\delta^2}\left(-\frac{\sigma R_0}{12}+\frac{R_3\sigma^2}{2}\right) + \frac{16\mu}{\hbar^2\delta^3}\left(\frac{R_1\sigma^2}{240}-\frac{R_3\sigma^3}{6}\right) \tag{46}$$

3.1 Partition function $Z(\beta)$

The partition function (PF) takes the form [39],

$$Z(\beta) = \sum_{n=0}^{\lambda} e^{-\beta E_{nl}} \tag{47}$$

where, $\beta = \frac{1}{KT}$, K is the Boltzmann constant, T is the absolute temperature, n is the principal quantum number, and λ is the maximum quantum number.
Replacing Eq. (43) into Eq. (47) gives

$$Z(\beta) = \sum_{n=0}^{\lambda} e^{-\beta\left(P_1 - \frac{\hbar^2}{8\mu}\left[\frac{P_2}{(n+\theta)}\right]^2\right)} \tag{48}$$

In the classical limit, at high temperature T, the summation is replaced by an integral,

$$Z(\beta) = \int_{\theta}^{\lambda+\theta} e^{M_1\beta + \frac{N_1\beta}{\rho^2}}\, d\rho \tag{49}$$

where,

$$\left.\begin{aligned} n+\theta &= \rho \\ M_1 &= -P_1 \\ N_1 &= \frac{\hbar^2 P_2^2}{8\mu} \end{aligned}\right\} \tag{50}$$

Integrating Eq. (49) gives the PF as,

$$Z(\beta) = e^{M_1\beta}\left(\frac{\rho e^{\frac{N_1\beta}{\rho^2}} - N_1\beta\sqrt{\pi}\, erfi\left(\frac{\sqrt{N_1\beta}}{\rho}\right)}{\sqrt{N_1\beta}}\right), \qquad \theta \le \rho \le \lambda+\theta \tag{51}$$

The imaginary error function $erfi(y)$ is given as [40],

$$erfi(\mathrm{y})=\frac{erf\,(iy)}{i}=\frac{2}{\sqrt{\pi}}\int_0^y e^{t^2}dt. \tag{52}$$

Other TPs can be obtained as follows:

3.2 Mean energy $U(\beta)$

$$U(\beta)=-\frac{\partial}{\partial\beta}InZ(\beta), \tag{53}$$

3.3 Free energy $\mathrm{F}(\beta)$

$$F(\beta)=-KT\ln Z(\beta) \tag{54}$$

3.4 Entropy $\mathrm{S}(\beta)$

$$S(\beta)=K\ln Z(\beta)-K\beta\frac{\partial}{\partial\beta}\ln Z(\beta) \tag{55}$$

3.5 Specific heat capacity $\mathrm{C}(\beta)$

$$C(\beta)=\frac{\partial U}{\partial T}=-K\beta^2\frac{\partial U}{\partial\beta} \tag{56}$$

4. Results and discussion

The prediction of the MS of the HMs is carried out using the relation [50,51]

$$M=2m+E_{nl} \tag{57}$$

where m is quarkonium mass and E_{nl} is energy eigenvalues.

Plugging Eq. (26) into Eq. (57) gives,

$$\begin{gathered}M=2m+\frac{R_0}{2}-\frac{R_1}{12}-R_0R_3+\frac{3}{\delta}\left(-\frac{\sigma R_0}{12}+\frac{R_3\sigma^2}{2}\right)+\frac{6}{\delta^2}\left(\frac{R_1\sigma^2}{240}-\frac{R_3\sigma^3}{6}\right)\\-\frac{\hbar^2}{8\mu}\left[\frac{\frac{2\mu}{\hbar^2}\left(\frac{R_0}{\sigma}+R_2-R_3\right)+\frac{6\mu}{\hbar^2\delta^2}\left(-\frac{\sigma R_0}{12}+\frac{R_3\sigma^2}{2}\right)+\frac{16\mu}{\hbar^2\delta^3}\left(\frac{R_1\sigma^2}{240}-\frac{R_3\sigma^3}{6}\right)}{n+\frac{1}{2}+\sqrt{\left(\frac{1}{2}+l\right)^2+\frac{2\mu}{\hbar^2\delta^3}\left(-\frac{\sigma R_0}{12}+\frac{R_3\sigma^2}{2}\right)+\frac{6\mu}{\hbar^2\delta^4}\left(\frac{R_1\sigma^2}{240}-\frac{R_3\sigma^3}{6}\right)+\frac{2\mu R_1}{\hbar^2\sigma^2}}}\right]^2\end{gathered} \tag{58}$$

The reduced mass is defined as $\mu=\frac{m}{2}$. For bottomonium and charmonium, the numerical values of these masses are $m_b=4.823\,GeV$ and $m_c=1.209\,GeV$, and the corresponding reduced mass is $\mu_b=2.4115\,GeV$ and $\mu=0.6045\,GeV$ correspondingly [52]. The potential parameters were also calculated by fitting with experimental data. Experimental data are taken from [53].

We observed that the results obtained from the prediction of mass spectra of charmonium and bottomonium for different quantum states are in agreement with experimental data and are seen to be improved when compared with other theoretical predictions with different analytical methods from literature as shown in Tables 1 and 2.

Table 1. Mass spectra of charmonium in (GeV)
$m_c=1.209\ GeV, \mu=0.6045\ GeV$, $\mathrm{R}_0=89960.89\ GeV$, $\mathrm{R}_1=0.230\ GeV, R_2=-8.995999582\times10^6\mathrm{GeV}$,
$\mathrm{R}_3=0.5014478276$ GeV, σ=0.01, $\delta=1.7\ GeV$, $\hbar=1$

State	Present work	AIM [21]	LTM [17]	SEM [15]	Experiment [53]
1S	3.096	3.096	3.0963	3.095922	3.096
2S	3.686	3.686	3.5681	3.685893	3.686
1P	3.255	3.214	3.5687	-	3.525
2P	3.779	3.773	3.5687	3.756506	3.773
3S	4.040	4.275	4.0400	4.322881	4.040
4S	4.269	4.865	4.5119	4.989406	4.263
1D	3.504	3.412	4.0407	-	3.770
2D	4.146	-	-	-	4.159

Table 2: Mass spectra of bottomonium in (GeV)
$\left(m_b = 4.823\ GeV, \mu = 2.4115\ GeV,\ R_0 = 1.805186081 \times 10^6\ GeV,\ R_1 = 3.084\ GeV, \atop R_2 = -1.805170402 \times 10^8 GeV, R_3 = 0.5014694079\ GeV,\ \sigma = 0.01, \delta = 1.70\ GeV, \hbar = 1 \right)$

State	Present work	AIM [21]	LTM [17]	SEM [15]	Experiment [53]
1S	9.460	9.460	9.745	9.515194	9.460
2S	10.023	10.023	10.023	10.01801	10.023
1P	9.619	9.492	10.025	-	9.899
2P	10.114	10.038	10.303	10.09446	10.260
3S	10.355	10.585	10.302	10.44142	10.355
4S	10.567	11.148	10.580	10.85777	10.580
1D	9.864	9.551	10.303	-	10.164

In Fig. 1, we plotted the MS against the principal quantum number (PQN) for different values of angular quantum number. It was noticed that the MS first increases as the PQN increases and the latter tends to converge towards a point. The plots of the TPs are shown in Figs. (2-6). The partition function (PF) is plotted against temperature (β) at various values of maximum quantum number (λ) of 10 and 20. It was observed that the PF increases linearly as the β is increased. Figure 3 depict the variation of free energy (FE) with temperature at different values of λ. The FE increase at the beginning at the same rate as the temperature increases and then decreases and converge at a point when the FE is equal to 1. The plot of internal energy (IE) with temperature is shown in Fig 4. The IE is seen to increase exponentially at $\lambda = 10$ and when $\lambda = 20$ no increment was noticed. In Fig 5, the entropy is plotted against temperature. It was observed that the entropy increases with an increase in temperature for both values of λ. In Fig. 6, the plot of specific heat capacity with temperature is shown. A decrease is noticed when the temperature increases for different values of λ.

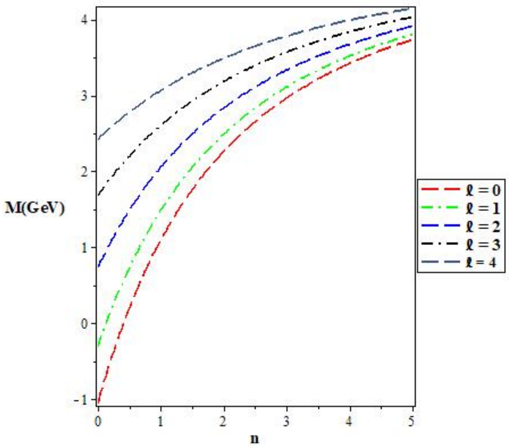

Figure 1. Variation of mass spectra with a principal quantum number for different angular quantum number

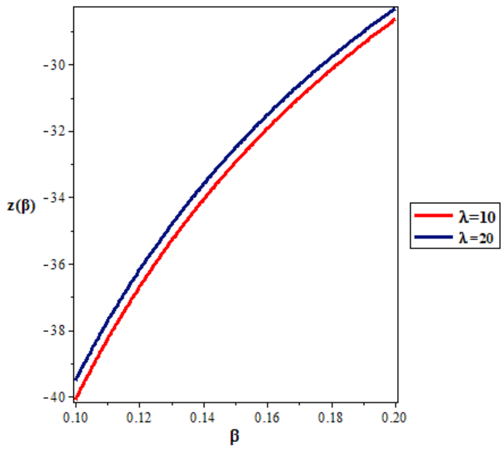

Figure 2. Variation of partition function with temperature for different values of λ

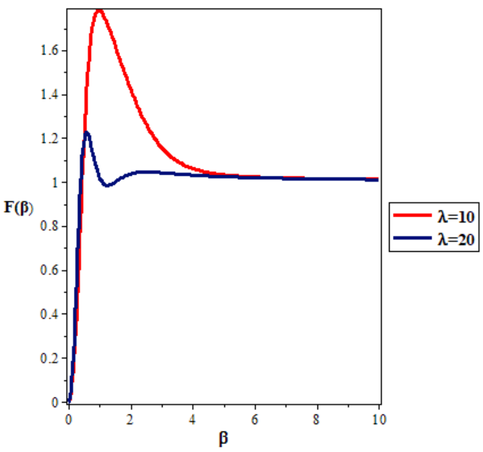

Figure 3. Variation of free energy with temperature for different values of λ

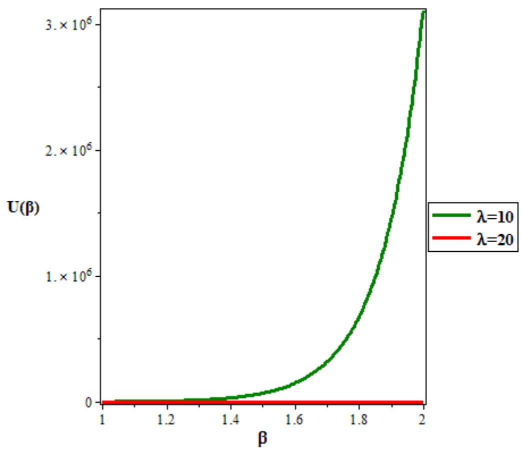

Figure 4. Variation of internal energy with temperature for different values of λ

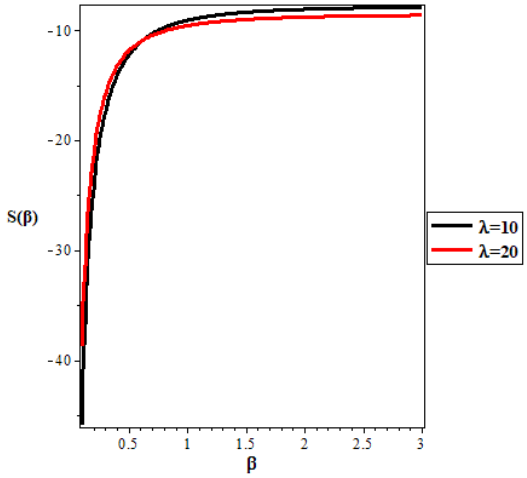

Figure 5. Variation of entropy with temperature for different values of λ

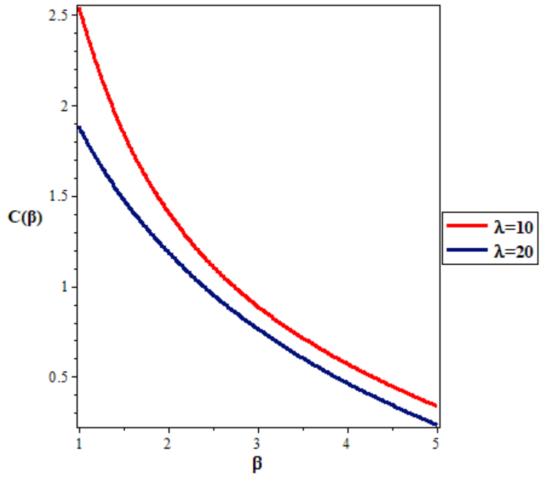

Figure 6. Variation of specific heart capacity with temperature for different values of λ

5. CONCLUSION

In this present study, the solutions of the SE were obtained with EHP using the NU method. The energy equation and normalized wave function were obtained. The energy spectrum was used to predict the MS of the HMs. Also, the PF was calculated from the energy equation, thereafter other TPs were obtained. The results obtained showed an improvement when compared with the work of other researchers and excellently agreed with experimental data.

Competing interests. The authors declare that they have no competing interests.

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REFERENCES

[1] H. Ciftci, and H.F. Kisoglu, "Nonrelativistic-Arbitrary l-states of quarkonium through Asymptotic Iteration method", Advances in High Energy Physics, 4549705 (2018). https://doi.org/10.1155/2018/4549705
[2] H. Mutuk, "Mass Spectra and Decay constants of Heavy-light Mesons: A case study of QCD sum Rules and Quark model", Advan. in High Energy Phys. 8095653 (2018). https://doi.org/10.1155/2018/8095653
[3] M. Allosh, Y. Mustafa, N.K. Ahmed, and A.S. Mustafa, "Ground and Excited state mass spectra and properties of heavy-light mesons", Few-Body Syst. **62**, 26 (2021). https://doi.org/10.1007/s00601-021-01608-1
[4] E. Omugbe, O.E. Osafile, I.B. Okon, E.P. Inyang, E.S. William, and A. Jahanshir, "Any L-state energy of the spinlessSalpeter equation under the Cornell potential by the WKB Approximation method: An Application to mass spectra of mesons", Few-Body Systems, **63**(1), 7 (2022). https://doi.org/10.1007/s00601-021-01705-1
[5] E.S. William, E.P. Inyang, and E.A. Thompson, "Arbitrary ℓ-solutions of the Schrödinger equation interacting with Hulthén-Hellmann potential model", Revista Mexicana Fisica, **66,** 730 (2020). https://doi.org/10.31349/RevMexFis.66.730
[6] E. Omugbe, "Non-relativistic eigensolutions of molecular and heavy quarkonia interacting potentials via the Nikiforov-Uvarov method", Canadian Journal of Physics, **98**, 1125 (2020). https://doi.org/10.1139/cjp-2020-0039
[7] E.S. William, E.P. Inyang, I.O. Akpan, J.A. Obu, A.N. Nwachukwu, and E.P. Inyang, "Ro-vibrational energies and expectation values of selected diatomic molecules via Varshni plus modified Kratzer potential model", Indian Journal of Physics, (2022). https://doi.org/10.1007/s12648-022-02308-0
[8] E.P. Inyang, and E.O. Obisung, "The study of electronic states of NI and ScI molecules with screened Kratzer Potential", East Eur. J. Phys. **3**, 32 (2022). https://doi.org/10.26565/2312-4334-2022-3-04
[9] E.P. Inyang, E.S. William, E. Omugbe, E.P. Inyang, E.A. Ibanga, F. Ayedun, I.O. Akpan, and J.E. Ntibi, "Application of Eckart-Hellmann potential to study selected diatomic molecules using Nikiforov-Uvarov-Functional analysis method", Revista Mexicana de Fisica, **68**, 14 (2022). https://doi.org/10.31349/RevMexFis.68.020401
[10] E.P. Inyang, E.S.William , J.E.Ntibi, J.A.Obu, P.C.Iwuji, and E.P.Inyang "Approximate solutions of the Schrodinger equation with Hulthen plus screened Kratzer potential using the Nikiforov-Uvarov-Functional analysis method: An Application to diatomic molecules"., Canadian Journal of Physics (2022). https://doi.org/10.1139/cjp-2022-0030
[11] E.P. Inyang, E.P. Inyang, I.O. Akpan, J.E. Ntibi, and E.S.William, "Analytical solutions of the Schrödinger equation with class of Yukawa potential for a quarkonium system via series expansion method", European Journal of Applied Physics **2**, 26 (2020). http://dx.doi.org/10.24018/ejphysics.2020.2.6.26
[12] E.P. Inyang, P.C. Iwuji, J.E. Ntibi, E.S. William, and E.A. Ibanga, "Solutions of the Schrodinger equation with Hulthen –screened Kratzer potential: Application to diatomic molecules", East. Eur. J. Phys. **1**, 11 (2022). https://doi.org/10.26565/2312-4334-2022-2-02
[13] M. Abu-Shady, T.A. Abdel-Karim, and E.M. Khokha, "Exact solution of the N-dimensional Radial Schrödinger Equation via Laplace Transformation method with the Generalized Cornell potential", J. Quantum Phys. **45**, 587 (2018). https://doi.org/10.48550/arXiv.1802.02092

[14] E. Omugbe, O.E. Osafile, I.B. Okon, E.S. Eyube, E.P. Inyang, U.S. Okorie, A. Jahanshir, and C.A. Onate, "Non-relativistic bound state solutions with α-deformed Kratzer-type potential using the super-symmetric WKB method: application to theoretic-information measures", European Physical Journal D, **76**, 72 (2022), https://doi.org/10.1140/epjd/s10053-022-00395-6
[15] E. Omugbe, O.E. Osafile, E.P. Inyang, and A. Jahanshir, "Bound state solutions of the hyper-radial Klein-Gordon equation under the Deng-Fan potential by WKB and SWKB methods", Physica Scripta, **96**(12), 125408 (2021). https://doi.org/10.1088/1402-4896/ac38d4
[16] C.O. Edet, S. Mahmoud, E.P. Inyang, N. Ali, S.A. Aljunid, R. Endut, A.N. Ikot, and M. Asjad, "Non-Relativistic Treatment of the 2D Electron System Interacting via Varshni-Shukla Potential Using the Asymptoptic Iteration Method", Mathematics, **10**, 2824 (2022). https://doi.org/10.3390/math10152824
[17] M. Abu-shady, C.O. Edet, and A.N. Ikot, "Non-relativistic Quark model under external magnetic and Aharanov-Bohm (AB) fields in the presence of Temperature-Dependent confined Cornell potential", Canadian Journal of Physics, (2021) https://doi.org/10.11139/cjp-2020-0101
[18] R. Kumar, and F. Chand, "Asymptotic study to the N-dimensional Radial Schrodinder Equation for the quark-antiquark system", Commun. In Theor. Phys .**59**, 467 (2013). https://doi.org/10.1088/0253-6102/59/5/02
[19] J.P. Prasanth, K. Sebastian, and V.M. Bannur, "Revisiting Cornell potential model of the Quark-Gluon plasma", Physica A, **558**,124921 (2020). https://doi.org/10.1016/j.physa.2020.124921
[20] H. S. Chung, J. Lee, and D. Kang, "Cornel potential parameters for S-wave heavy quarkonia", Journal of the Korean physical society, **52**, 1151 (2008). https://doi.org/10.3938/jkps.52.1151
[21] R. Kumar, R.M. Singh, S.B. Bhahardivaj, R. Rani, and F. Chand, "Analytical solutions to the Schrodinger equation for generalized Cornell potential and its application to diatomic molecules and heavy mesons", Mod. Phys. Lett. A. **37**, 2250010 (2022). https://doi.org/10.1142/S0217732322500109
[22] A. Vega, and J. Flores, "Heavy quarkonium properties from Cornell potential using variational method and supersymmetric quantum mechanics", Pramana – J. Phys. **87**, 73 (2016). https://doi.org/10.1007/s12043-016-1278-7
[23] H. Mutuk, "Cornell Potential: A Neural Network Approach", Advan. in High Energy Phys. **22,** 3105373 (2019). https://doi.org/10.1155/2019/3105373
[24] H. Hassanabadi, M. Ghafourian, and S. Rahmani, "Study of the Heavy-Light mesons properties via the Variational method for Cornell interaction", Few-Body Syst. **57**, 249 (2016). https://doi.org/10.1007/s00601-015-1040-6
[25] M. Abu-Shady, and S.Y. Ezz-Alarab, "Trigonometric Rosen–Morse Potential as a Quark–Antiquark Interaction Potential for Meson Properties in the Non-relativistic Quark Model Using EAIM". Few-Body Systems, **60**, 66 (2019). https://doi.org/10.1007/s00601-019-1531-y
[26] M. Abu-shady, H.M. Mansour, and A.I. Ahmador, "Dissociation of Quarkonium in Hot and Dense media in an Anisotropic plasma in the Non-relativistic Quark model", Advances in High Energy Physics, 4785615 (2019). https://doi.org/10.1155/2019/4785615
[27] E.P. Inyang, E.P. Inyang, J.E. Ntibi, E.E. Ibekwe, and E.S. William, **"**Approximate solutions of D-dimensional Klein-Gordon equation with Yukawa potential via Nikiforov-Uvarov method"**,** Indian Journal of Physics, **95**, 2733 (2021). https://doi.org/10.1007/s12648-020-01933-x
[28] E.P. Inyang, E.P. Inyang, E.S. William, and E.E. Ibekwe, "Study on the applicability of Varshni potential to predict the mass-spectra of the Quark-Antiquark systems in a non-relativistic framework", Jordan Journal of Physics, **14**, 337 (2021). https://journals.yu.edu.jo/jjp/Draft/Vol14No4/8-%20Study%20on%20the%20Applicability,%20Inyang%20et%20al.pdf
[29] E.E. Ibekwe, U.S. Okorie, J.B. Emah, E.P. Inyang, and S.A. Ekong, "Mass spectrum of heavy quarkonium for screened Kratzer potential (SKP) using series expansion method", Eur. Phys. J. Plus, **87**, 136 (2021). https://doi.org/10.1140/epjp/s13360-021-01090-y
[30] I.O. Akpan, E.P. Inyang, E.P. Inyang, and E.S. William, "Approximate solutions of the Schrödinger equation with Hulthen-Hellmann Potentials for a Quarkonium system", Revista Mexica De Fisica, **67**, 482 (2021). https://doi.org/10.31349/RevMexFis.67.482
[31] K.R. Purohit, P. Jakhad, and A.K. Rai, "Quarkonium spectroscopy of the linear plus modified Yukawa potential", Phys. Scripta, **97**, 044002 (2022). https://doi.org/10.1088/1402-4896/ac5bc2
[32] S. Patel, P.C. Vinodkumar, and S. Bhatnagar, "Decay rates of charmonia within a quark-antiquark confining potential", Chinese Physics C, **40**, 053102 (2016). https://doi.org/10.1088/1674-1137/40/5/053102
[33] V. Mateu, P.G. Ortega, D.R. Entem, and F. Fernadez, "Calibrating the nave Cornell model with NRQCD", The European Physical Journal C, **79**, 323 (2019). https://doi.org/10.1140/epjc/s10052-019-6808-2
[34] F. Brau, and C. Sernay, "The three-dimensional Fourier grid Hamiltonian method", Journal of computational physics, **139**, 136 (1998). https://doi.org/10.1006/jcph.1997.5866
[35] A. Bhaghyesh, "Charmonium properties using the Discrete variable representation (DVR) method", Advances in High Energy Physics, 9991152 (2021). https://doi.org/10.1155/2021/9991152
[36] S. Jacobs, M.G. Olsson, and C. Suchyta, "Comparing the Schrodinger and Spinless Salpeter equations for heavy-quark bound states", Physical Review D, **33**, 3338 (1986). https://doi.org/10.1103/physrevd.33.3338
[37] B. Grinstein, "A modern introduction to quarkonium theory", International Journal of Modern Physics, **15**, 461 (2000). https://doi.org/10.1142/S0217751X00000227
[38] W. Lucha, F. Schoberl, and D. Gromes, "Bound states of quarks", Physics Reports, **200**, 127 (1991). https://doi.org/10.1016/0370-1573(91)90001-3
[39] E. P. Inyang, A.N.Ikot, I.O.Akpan, J.E.Ntibi, E.Omugbe, and E. S. William, "Analytic study of thermal properties and masses of heavy mesons with quarkonium potential", Result in Physics. (2022) 105754. https://doi.org/10.1016/j.rinp.2022.105754
[40] E.P. Inyang, E.P. Inyang, I.O. Akpan, J.E. Ntibi, and E.S. William, "Masses and thermodynamic properties of a Quarkonium system", Canadian Journal Physics, **99**, 990 (2021). https://doi.org/10.1139/cjp-2020-0578
[41] E.P. Inyang, E.O. Obisung, P.C. Iwuji, J.E. Ntibi, J. Amajama, and E.S. William, "Masses and thermal properties of a charmonium and Bottomonium mesons", J. Nig. Soc. Phys. Sci. 4 (2022). https://doi.org/884.10.46481/jnsps.2022.884

[42] C. Eckart, "The Penetration of a potential Barrier by Electrons", Phys. Rev. **35**, 1303 (1930). https://doi.org/10.1103/PhysRev.35.1303
[43] H. Hellmann, "A New Approximation Method in the Problem of Many Electrons", J. Chem. Phys. **3**, 61 (1935). https://doi.org/10.1063/1.1749559
[44] C.A. Onate, J.O. Ojonubah, A. Adeoti, E.J. Eweh, and M. Ugboja, "Approximate Eigen Solutions of D.K.P. and Klein-Gordon Equations with Hellmann Potential", Afr. Rev. Phys. **9**, 497 (2014). http://eprints.lmu.edu.ng/1594/1/ONATE%2032.pdf
[45] E.P. Inyang, E.S. William, J.O. Obu, B.I. Ita, E.P. Inyang, and I.O. Akpan, "Energy spectra and expectation values of selected diatomic molecules through the solutions of Klein-Gordon equation with Eckart-Hellmann potential model", Molecular Physics, **119**(23), e1956615 (2021). https://doi.org/10.1080/00268976.2021.1956615
[46] S.K. Nikiforov, and V.B. Uvarov. *Special functions of Mathematical Physics*, (Birkhauser, Basel, 1988)
[47] E.P. Inyang, E.S. William, and J.A. Obu, "Eigensolutions of the N-dimensional Schrödinger equation` interacting with Varshni-Hulthen potential model", Rev. Mexi. Fisi. **67**, 193 (2021). https://doi.org/10.31349/RevMexFis.67.193
[48] M. Abu-Shady, "N-dimensional Schrödinger equation at finite temperature using the Nikiforov-Uvarov method", J. Egypt. Math. Soc. **23**, 4 (2016). https://doi.org/10.1016/j.joems.2016.06.006
[49] I.S. Gradshteyn, and I.M. Ryzhik, *Tables of integrals, series and product, 7th ed*. (Academic Press, Boston, 2007).
[50] E.P. Inyang, E.P. Inyang, J. Karniliyus, J.E. Ntibi, and E.S. William, "Diatomic molecules and mass spectrum of heavy quarkonium system with Kratzer-screened Coulomb potential (KSCP) through the solutions of the Schrödinger equation", European Journal of Applied Physics, **3**, 55 (2021). https://doi.org/10.24018/ejphysics.2021.3.2.61
[51] E.P. Inyang, P.C. Iwuji, J.E. Ntibi, E. Omugbe, E.A. Ibanga, and E.S. William "Quark-antiquark study with inversely quadratic Yukawa potential using Nikiforov-Uvarov-Functional analysis method", East Eur. J. Phys. **2**, 43 (2022), https://periodicals.karazin.ua/eejp/article/view/18535/16839
[52] R. Olive, D.E. Groom, and T.G. Trippe, "Particle Data Group", Chin. Phys. C, **38**, 60 (2014).
[53] M. Tanabashi, C.D. Carone, T.G. Trippe, and C.G. Wohl, Particle Data Group, Phys. Rev. D, **98**, 546 (2018).

НЕРЕЛЯТИВІСТСЬКЕ ДОСЛІДЖЕННЯ МАС-СПЕКТРІВ І ТЕПЛОВИХ ВЛАСТИВОСТЕЙ КВАРКОНІЄВОЇ СИСТЕМИ З ПОТЕНЦІАЛОМ ЕКАРТА-ГЕЛЬМАНА

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У цьому дослідженні ми моделюємо потенціал Екарта-Гельмана (EHP) для взаємодії в системі кварк-антикварк. Розв'язки рівняння Шредінгера отримані з ЕГП методом Нікіфорова-Уварова. Отримано рівняння енергії та нормовану хвильову функцію. Маси важких мезонів, таких як чармоній ($c\bar{c}$) і боттоній ($b\bar{b}$), для різних квантових чисел були передбачені за допомогою рівняння енергії. Крім того, розподільча функція була розрахована з рівняння енергії, після чого були отримані інші теплові властивості, такі як середня енергія, вільна енергія, ентропія та питома теплоємність. Отримані результати показали покращення порівняно з роботами інших дослідників і чудово узгоджувалися з експериментальними даними.

Ключові слова: рівняння Шредінгера; метод Нікіфорова-Уварова; потенціал Екарта-Гельмана; важкі мезони; теплові властивості