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USE OF NONLINEAR OPERATORS FOR SOLVING GEOMETRIC OPTICS PROBLEMS†

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The aim of this work is to develop and apply a mathematical apparatus based on nonlinear operators for solving problems of geometric optics, namely the construction of images of objects in systems of thin lenses. The problem of constructing the image of a point in a thin lens was considered, on the basis of which the concept of the lensing operator was defined. The mathematical properties of the operator were investigated. The model problem of constructing an image in thin lenses folded together was investigated, on the basis of which it became possible to establish a physical interpretation of the previously determined properties. The problem of a system of lenses located at a distance was also considered, which resulted in the introduction of the concept of shift operator. The properties of the shift operator were studied, which together with the properties of the lens operator made it possible to determine the rules for using the created operators for solving the problems. In addition to solving the model problems, the following problems were considered: the speed of the moving point image, the magnification factor and the construction of the curve image. As an example, images of a segment and an arc of the circle were constructed. The segment was transformed into the segment, and the arc of the circle into the arc of the curve of the second order. The presented mathematical apparatus is very convenient for implementation in computer programs, as well as for the study of images of different curves.
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Currently, there are many different mathematical methods used to simulate the propagation of light. One of the classical methods is matrix optics [1]. Among the new methods of image modeling in lens systems are the use of ray tracing methods [2] and complex vectors [3]. Operators are also used in optics. For example, in geometric optics in the general theory of relativity bilocal operators are used [4]. The object of this work is to develop a new mathematical apparatus based on nonlinear operators, which allow to build easily images of objects in thin lens systems. In the course of the work, the following tasks were set: determination of the necessary operators and their mathematical properties; solving model problems to improve the theory; solving problems of curves images construction, etc. The idea of this work was to represent a point in the form of a radius vector and to define a reflection, i.e. an operator that transforms the radius vector of an object into the radius vector of the image. As an example, an image of the segment and an arc of the circle were constructed with a graphic demonstration of these images. The presented mathematical apparatus requires an additional study of possible areas of application, what creates some opportunities for other scientists in this area.

RESEARCH METHODS AND PROCEDURE

Lensing operator

Definition of the lensing operator. Let's consider the point image, constructed using a condenser lens. We place the optical center of the thin condenser lens, whose focal length is equal to F, in the center of the coordinate system, the OX axis of which is directed along the main optical axis of the lens, and the OY axis is directed along the plane of the lens (see Fig.1). From now on we will work only in the orthonormal basis.

The position of point A will be described by the radius vector:

$$\overrightarrow{r_A}(x_A, y_A). \tag{1}$$

The lens forms an image of this point, which will be described by the radius vector:

$$\overrightarrow{r_{A'}}(x_{A'}, y_{A'}). \tag{2}$$

Since the lens converts vector $\overrightarrow{r_A}(x_A, y_A)$ into vector $\overrightarrow{r_{A'}}(x_{A'}, y_{A'})$ $\overrightarrow{r_{A'}}(x_{A'}, y_{A'})$, there is an operator, which hereinafter will be called as lensing operator $\hat{L}_+$:

$$\overrightarrow{r_{A'}} = \hat{L}_+\overrightarrow{r_A}. \tag{3}$$

Let's determine the expression for the lensing operator. From the formula of a thin lens we have [5]:

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$$\frac{1}{F} = \frac{1}{|x_A|} + \frac{1}{|x'_A|}. \tag{4}$$

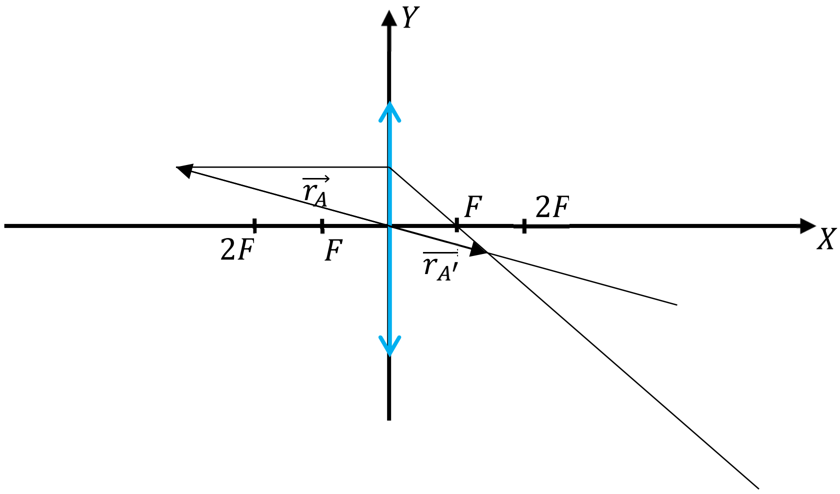

Figure 1. Image of a point in a thin lens.

Let's disclose the coordinate modules (see Fig.1):

$$\begin{cases} |x_A| = -x_A, \\ |y_A| = y_A, \\ |x'_A| = x'_A, \\ |y'_A| = -y'_A. \end{cases} \tag{5}$$

From expressions (4) and (5) we have the expression for the coordinate x'_A:

$$x'_A = x_A \frac{F}{x_A + F}. \tag{6}$$

From the similarity of triangles, we find the expression for the module of coordinate y'_A:

$$\frac{|y'_A|}{|y_A|} = \frac{|x'_A|}{|x_A|}. \tag{7}$$

From expressions (6) and 7 we get:

$$y'_A = y_A \frac{F}{x_A + F}. \tag{8}$$

Thus, the coordinates of the vector $\overrightarrow{r_{A'}}(x_{A'}, y_{A'})$ through the coordinates of the vector $\overrightarrow{r_A}(x_A, y_A)$ are equal to:

$$\begin{cases} x'_A = x_A \dfrac{F}{x_A + F}, \\ y'_A = y_A \dfrac{F}{x_A + F}. \end{cases} \tag{9}$$

We can record the effect of the lensing operator as follows:

$$\hat{L}_+ \vec{r} = \frac{F}{(\vec{r}, \overrightarrow{e_x}) + F} \vec{r}. \tag{10}$$

where $(\vec{r}, \overrightarrow{e_x}) = x$, $\overrightarrow{e_x}$ – orth in the direction of the OX axis.
Now, we consider an image of the point constructed using a scattering lens.

Let's place the optical center of the thin scattering lens, whose focal length is F, in the center of the coordinate system, the OX axis of which is directed along the main optical axis of the lens, and OY axis, which is perpendicular to it the main optical axis, lies in the lens plane (see Fig.2).

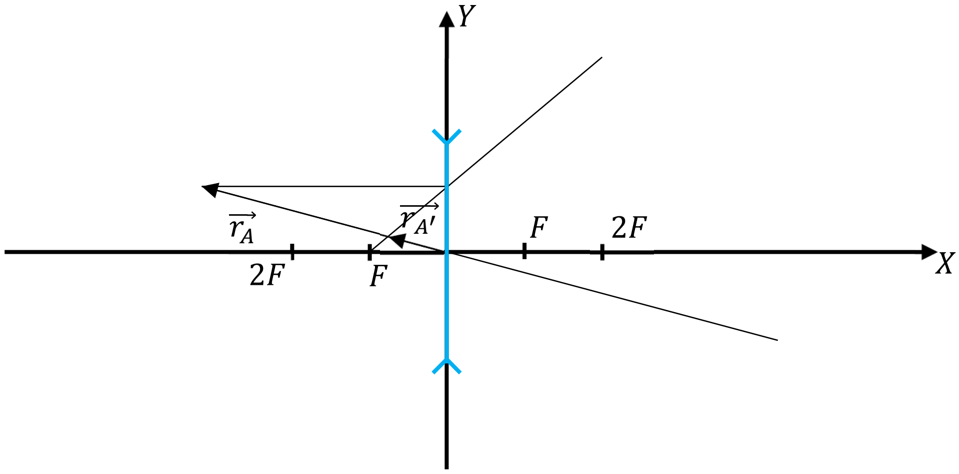

Figure 2. Image of the point in a thin scattering lens.

The position of point A will be described by the radius vector:

$$\overrightarrow{r_A}(x_A, y_A). \tag{11}$$

The lens forms an image of this point, which will be described by the radius vector:

$$\overrightarrow{r_{A'}}(x_{A'}, y_{A'}). \tag{12}$$

Since the lens converts vector $\overrightarrow{r_A}(x_A, y_A)$ into vector $\overrightarrow{r_{A'}}(x_{A'}, y_{A'})$, there is an operator, which hereinafter will be called as lensing operator $\hat{L}_-$:

$$\overrightarrow{r_{A'}} = \hat{L}_- \overrightarrow{r_A}. \tag{13}$$

Let's determine the expression for the lensing operator $\hat{L}_-$. From the formula of the thin lens:

$$-\frac{1}{F} = \frac{1}{|x_A|} - \frac{1}{|x_A'|}. \tag{14}$$

Let's disclose the coordinate modules (see Fig.2):

$$\begin{cases} |x_A| = -x_A, \\ |y_A| = y_A, \\ |x_A'| = -x_A', \\ |y_A'| = y_A'. \end{cases} \tag{15}$$

From expressions (14) and (15) the coordinate x_A' is:

$$x_A' = -x_A \frac{F}{x_A - F}. \tag{16}$$

Similar to the previous paragraph, we have an expression for y_A' coordinate:

$$y_A' = -y_A \frac{F}{x_A - F}. \tag{17}$$

Thus, we have the expressions of the coordinates of vector $\overrightarrow{r_{A'}}(x_{A'}, y_{A'})$ through the coordinates of vector $\overrightarrow{r_A}(x_A, y_A)$:

$$\begin{cases} x_A' = -x_A \dfrac{F}{x_A - F}, \\ y_A' = -y_A \dfrac{F}{x_A - F}. \end{cases} \tag{18}$$

We can record the action of the lensing operator as follows:

$$\hat{L}_{-}\vec{r} = \frac{-F}{(\vec{r},\overrightarrow{e_x}) - F}\vec{r}. \tag{19}$$

If assuming that the focal length has a sign, subject to the type of lens, one can introduce a generalized lensing operator:

$$\hat{L}\vec{r} = \frac{F}{(\vec{r},\overrightarrow{e_x}) + F}\vec{r}. \tag{20}$$

In this case, if $F > 0$, then operator $\hat{L}$ becomes operator $\hat{L}_{+}$, and if $F < 0$, then operator $\hat{L}$ becomes operator $\hat{L}_{-}$.

Properties of the lensing operator.

1. The lensing operators are nonlinear operators.

$$\hat{L}(\alpha\vec{x} + \beta\vec{y}) \neq \alpha\hat{L}\vec{x} + \beta\hat{L}\vec{y}. \tag{21}$$

2. For different values of focal length for two operators, the equality holds:

$$[\hat{L}_1, \hat{L}_2] = 0. \tag{22}$$

Let's prove the commutativity of the lensing operators corresponding to different lenses.

$$\begin{aligned}\hat{L}_1\hat{L}_2\vec{r} = \hat{L}_1\left(\frac{F_2}{(\vec{r},\overrightarrow{e_x}) + F_2}\vec{r}\right) &= \frac{F_1}{\frac{F_2}{(\vec{r},\overrightarrow{e_x}) + F_2}(\vec{r},\overrightarrow{e_x}) + F_1}\left(\frac{F_2}{(\vec{r},\overrightarrow{e_x}) + F_2}\vec{r}\right)\\ &= \frac{F_1F_2}{F_2(\vec{r},\overrightarrow{e_x}) + F_1(\vec{r},\overrightarrow{e_x}) + F_1F_2}\vec{r}.\end{aligned} \tag{23}$$

$$\begin{aligned}\hat{L}_2\hat{L}_1\vec{r} = \hat{L}_2\left(\frac{F_1}{(\vec{r},\overrightarrow{e_x}) + F_1}\vec{r}\right) &= \frac{F_2}{\frac{F_1}{(\vec{r},\overrightarrow{e_x}) + F_1}(\vec{r},\overrightarrow{e_x}) + F_2}\left(\frac{F_1}{(\vec{r},\overrightarrow{e_x}) + F_1}\vec{r}\right)\\ &= \frac{F_1F_2}{F_2(\vec{r},\overrightarrow{e_x}) + F_1(\vec{r},\overrightarrow{e_x}) + F_1F_2}\vec{r}.\end{aligned} \tag{24}$$

The property of the scalar product homogeneity was used here, namely:

$$\left(\frac{F}{(\vec{r},\overrightarrow{e_x}) + F}\vec{r},\overrightarrow{e_x}\right) = \frac{F}{(\vec{r},\overrightarrow{e_x}) + F}(\vec{r},\overrightarrow{e_x}). \tag{25}$$

because the expression

$$\frac{F}{(\vec{r},\overrightarrow{e_x}) + F}. \tag{26}$$

is a scalar that we can derive from the scalar product.

As one can see, expressions (23) and (24) are equal, i.e., there is such an equality here:

$$\hat{L}_1\hat{L}_2\vec{r} = \hat{L}_2\hat{L}_1\vec{r}. \tag{27}$$

And, therefore, the operator junction is equal to 0.

3. The lensing operator is nondegenerate.

Let's consider the generalized lensing operator. By definition, a nondegenerate operator is an operator, whose kernel dimension is 0. We investigate the kernel of the generalized lensing operator.

Suppose, there exists such a nonzero vector $\vec{r}$, where:

$$\hat{L}\vec{r} = 0. \tag{28}$$

We describe expression (28) more specifically:

$$\hat{L}\vec{r} = \frac{F}{(\vec{r},\overrightarrow{e_x}) + F}\vec{r} = 0. \tag{29}$$

From the fact that vector $\vec{r}$ is nonzero it follows:

$$\frac{F}{(\vec{r},\overrightarrow{e_x})+F}=0. \tag{30}$$

where the focal length is a nonzero value and $(\vec{r},\overrightarrow{e_x})$ is a finite value, so we have:

$$\forall\vec{r}\neq 0,\frac{F}{(\vec{r},\overrightarrow{e_x})+F}\neq 0. \tag{31}$$

That is

$$\forall\vec{r}\neq 0,\hat{L}\vec{r}\neq 0\rightarrow\dim\left(Ker(\hat{L})\right)=0. \tag{32}$$

Thus, the generalized operator is nondegenerate.

4. The lensing operator is an injection.

We prove the injectivity of the mapping. Suppose, there are two different vectors $\overrightarrow{r_1}$ and $\overrightarrow{r_2}$ such that $\hat{L}\overrightarrow{r_1}=\hat{L}\vec{r}_2$, then we have:

$$\hat{L}\overrightarrow{r_1}-\hat{L}\vec{r}_2=0. \tag{33}$$

$$\frac{F}{(\overrightarrow{r_1},\overrightarrow{e_x})+F}\vec{r}_1-\frac{F}{(\overrightarrow{r_2},\overrightarrow{e_x})+F}\vec{r}_2=\frac{F\vec{r}_1(\overrightarrow{r_2},\overrightarrow{e_x})+F^2\vec{r}_1-F\vec{r}_2(\overrightarrow{r_1},\overrightarrow{e_x})-F^2\vec{r}_2}{\left((\overrightarrow{r_1},\overrightarrow{e_x})+F\right)\left((\overrightarrow{r_2},\overrightarrow{e_x})+F\right)}=0. \tag{34}$$

$$F\vec{r}_1(\overrightarrow{r_2},\overrightarrow{e_x})+F^2\vec{r}_1-F\vec{r}_2(\overrightarrow{r_1},\overrightarrow{e_x})-F^2\vec{r}_2=0. \tag{35}$$

$$(x_1x_2-x_2x_1,y_1x_2-y_2x_1)+F(x_1-x_2,y_1-y_2)=0. \tag{36}$$

$$(F(x_1-x_2),y_1x_2-y_2x_1+Fy_1-Fy_2)=0. \tag{37}$$

$$\begin{cases}x_1-x_2=0;\\ y_1x_2-y_2x_1+Fy_1-Fy_2=0.\end{cases} \tag{38}$$

$$y_1x_1-y_2x_1+Fy_1-Fy_2=0. \tag{39}$$

$$x_1(y_1-y_2)+F(y_1-y_2)=0. \tag{40}$$

$$(x_1+F)(y_1-y_2)=0. \tag{41}$$

We have two cases. If $y_1-y_2=0$, then our assumption is violated. There remains the case $x_1+F=0$, which does not make sense, because in this case the expression of the operator is impossible (division by 0).

This is how we proved the injectivity of the mapping.

Note:

In the course of the analysis of the effect of lensing operators, there is a problem with the fact that this type of operator gives correct results, only if the point is located to the left of the lens. This problem can be solved in two ways:

1. We consider only the images of the points located on the left for the right orientation of the basis and on the right for the left orientation of the basis [1].

2. We modify the expression of the lensing operator by introducing the concept of the focal vector that will meet the following conditions:

a) For the condenser lens the focus vector is oriented in the direction from the point, whose image we construct, and for the scattering lens – in the direction of the point.

b)

$$\begin{cases}|x|=F,\\ y=0.\end{cases} \tag{42}$$

Then, the expression for the lensing operator will have the following form:

$$\hat{L}\vec{r}=\frac{(\vec{F},\overrightarrow{e_x})}{(\vec{r},\overrightarrow{e_x})+(\vec{F},\overrightarrow{e_x})}\vec{r}. \tag{43}$$

And in this case, it is easy to make sure that all the properties of this operator coincide with the previously introduced lensing operator.

However, for the convenience of calculations we use the first method.

5. Inverted operators to lensing operators.

[1] The following wording is stricter: for any basis orientation the following condition must be met: $(\vec{r},\overrightarrow{e_x})<0$

From the injectivity of the generalized lensing operator follows the injectivity of the operators $\hat{L}_+$ and $\hat{L}_-$. The fact of the lens operators' injectivity indicates the existence of inverse lens operators [6]. Let's consider $\hat{L}_+$ and $\hat{L}_-$. We return to expression (23) and make the following substitutions:

$$\begin{cases} F_1 = F; \\ F_2 = -F. \end{cases} \tag{44}$$

We get such an expression:

$$\hat{L}_+\hat{L}_-\vec{r} = \hat{L}_+\left(\frac{-F}{(\vec{r},\overrightarrow{e_x}) - F}\vec{r}\right) = \frac{F}{\frac{-F}{(\vec{r},\overrightarrow{e_x}) - F}(\vec{r},\overrightarrow{e_x}) + F}\left(\frac{-F}{(\vec{r},\overrightarrow{e_x}) - F}\vec{r}\right) = \frac{-F^2}{-F(\vec{r},\overrightarrow{e_x}) + F(\vec{r},\overrightarrow{e_x}) - F^2}\vec{r} = \vec{r}. \tag{45}$$

From expression (45) it follows that

$$\hat{L}_+\hat{L}_-\vec{r} = \vec{r}. \tag{46}$$

That is

$$\hat{L}_+\hat{L}_- = \hat{E}. \tag{47}$$

From the above it follows that

$$\hat{L}^{-1}{}_+ = \hat{L}_-. \tag{48}$$

Now we enter the indication:

$$\begin{cases} \hat{L}[F] = \hat{L}_+; \\ \hat{L}[-F] = \hat{L}_-. \end{cases} \tag{49}$$

From expressions (48) and (49) it follows:

$$\hat{L}[F]^{-1} = \hat{L}[-F]. \tag{50}$$

Systems of thin centered lenses, whose optical centers coincide. Let's consider the construction of images using the lens operators in the system of two thin lenses folded together, so that their optical centers coincide. Assume, that the construction of images in such a complex system of lenses corresponds to the consistent use of lens operators, what is described in (23).

$$\hat{L}_1\hat{L}_2\vec{r} = \frac{F_1F_2}{F_2(\vec{r},\overrightarrow{e_x}) + F_1(\vec{r},\overrightarrow{e_x}) + F_1F_2}\vec{r}. \tag{51}$$

Let's make some transformations:

$$\hat{L}_1\hat{L}_2\vec{r} = \frac{F_1F_2}{F_2(\vec{r},\overrightarrow{e_x}) + F_1(\vec{r},\overrightarrow{e_x}) + F_1F_2}\vec{r} = \frac{\frac{F_1F_2}{F_1+F_2}}{(\vec{r},\overrightarrow{e_x}) + \frac{F_1F_2}{F_1+F_2}}\vec{r}. \tag{52}$$

We make such a replacement:

$$\frac{F_1F_2}{F_1+F_2} = F_3. \tag{53}$$

Or, introducing the concept of optical power of the lens $D_i = \frac{1}{F_i}$ from (53) we obtain:

$$D_1 + D_2 = D_3. \tag{54}$$

Thus, according to (52) - (54), the sequential use of two lens operators $\hat{L}_1$ and $\hat{L}_2$ is equivalent to the action of one lens operator $\hat{L}_3 = \hat{L}_1\hat{L}_2$. From the physical point of view, this corresponds to the fact that the image of a point in two consecutive thin lenses with optical powers D_1 та D_2 coincides with its image in a thin lens with optical power $D_3 = D_1 + D_2$.

Physical interpretation of lensing operator properties. In the previous section, the concept of the lensing operator – a nonlinear operator, whose action on the radius-vector of a point gives the image of this point in a thin lens with the given optical power, was determined. Basing on of the expression for this operator, some peculiar mathematical properties were identified, which require the analysis from the physical point of view, because the mathematical apparatus, created in this work, is a method of describing physical phenomena. Therefore, this section will be devoted to the interpretation of the lens operator mathematical properties from the point of view of physics.

First of all, a very interesting feature is the commutativity of lens operators corresponding to different lenses. In the previous section, it was not defined, what the consistent action of several lens operators corresponds to. As it was shown earlier, the sequential use of two (or more) lens operators corresponds to the combination of two (or more) thin lenses. From this it follows that the commutativity of the operators means that it does not matter in what order the lenses will be used.

As well, from the fact that the sequential use of operators corresponds to several thin lens folded together, it follows a very simple explanation to the fact, that the inverse lensing operator to the given one is an operator with the opposite value of the optical power. As long as the total optical power of such a system will be zero, no lensing will occur.

The injectivity of the mapping, in its turn, corresponds to the fact that two different points cannot give the same point as an image.

It is important to check the following property: if point B is an image of point A, then point A is an image of point B [5]. Suppose, we have a lens that is specified by operator $\hat{L}$, then, taking into account the properties of the operator, it is necessary to check the following equality:

$$\hat{L}[-F]\overrightarrow{r'} = \vec{r}. \tag{55}$$

The sign minus before focus is due to the fact that the object, whose image we are constructing, is located to the right of the lens:

$$\hat{L}[-F]\hat{L}\vec{r} = \hat{L}^{-1}\hat{L}\vec{r} = \vec{r}. \tag{56}$$

From this, it follows that the above-mentioned property is fulfilled.

Lensing operators' group.

1. Associativity.

In terms of optical power, the lens operator can be rewritten:

$$\hat{L}\vec{r} = \frac{1}{1 + D(\vec{r}, \overrightarrow{e_x})}\vec{r}. \tag{57}$$

Let's write the operator as follows:

$$\hat{L}[D]\vec{r} \equiv \frac{1}{1 + D(\vec{r}, \overrightarrow{e_x})}\vec{r}. \tag{58}$$

It follows from expressions (51) and (54) that a certain operator $\hat{L}(D_1)\hat{L}(D_2)\hat{L}(D_3)$ can be written as one operator $\hat{L}(D_1 + D_2 + D_3)$. Thus, from the associativity of the operation of adding real numbers (which here are optical forces) follows the associativity of lensing operators.

2. Lensing operators' group.

From the existence of the internal binary operation, its associativity, commutativity, the existence of a neutral and inverse element, it follows that the lensing operators form an infinite Abelian group [7]. Let's call the group of lensing operators G_L. Thus, the properties of the lensing operator can be written as follows:

1) $\forall\ \hat{L}[D_1], \hat{L}[D_2] \in G_L\ \exists\ \hat{L}[D_1 + D_2] \in G_L: \hat{L}[D_1]\hat{L}[D_2] = \hat{L}[D_1 + D_2]$,

2) $\forall\ \hat{L}[D_1], \hat{L}[D_2] \in G_L\ \hat{L}[D_1]\hat{L}[D_2] = \hat{L}[D_2]\hat{L}[D_1]$,

3) $\forall\ \hat{L}[D_1], \hat{L}[D_2], \hat{L}[D_3] \in G_L\ \hat{L}[D_1]\big(\hat{L}[D_2]\hat{L}[D_3]\big) = \big(\hat{L}[D_1]\hat{L}[D_2]\big)\hat{L}[D_3]$,

4) $\exists\ \hat{E} = \hat{L}[0] \in G_L: \forall\ \hat{L}[D] \in G_L\ \hat{E}\hat{L}[D] = \hat{L}[D]\hat{E} = \hat{L}[D]$,

5) $\exists\ \hat{L}^{-1} = \hat{L}[-D] \in G_L: \forall\ \hat{L}[D] \in G_L\ \hat{L}[D]\hat{L}[-D] = \hat{E}$.

3. Generator of G_L group.

As mentioned in the previous paragraph, lensing operators form an Abelian infinite group. Let's define its generator. By definition, the following operator is called as the generator (infinitesimal operator) of the s-parametric group [7]:

$$\hat{x}_i \equiv \frac{\partial g(a_1, a_2, \dots, a_s)}{\partial a_i}|_{a_i=0}\,. \tag{59}$$

In our case, the group is one-parametric, and the parameter itself is the optical power. Let's define the generator of G_L group:

$$\hat{x}\vec{r} = \frac{\partial}{\partial D}\left(\frac{1}{1 + D(\vec{r}, \overrightarrow{e_x})}\right)\vec{r}|_{D=0} = -(\vec{r}, \overrightarrow{e_x})\vec{r}\,. \tag{60}$$

Thus, any element of the G_L group in the vicinity of zero can be decomposed as follows:

$$\hat{L}[D]\vec{r} = \left(\hat{E} + D\hat{x}\right)\vec{r} = \hat{E}\vec{r} - D(\vec{r}, \overrightarrow{e_x})\vec{r}. \tag{61}$$

Shift operator in centered thin lens systems, whose optical centers do not coinside

Definition of the shift operator. Let's consider a centered system of two lenses located at distance d[2]. It is necessary to identify the lensing operator of this system from the lensing operators of the specified lenses. That is, we are looking for a lens, whose optical center is located in the coordinate center, which will replace the above-mentioned lens system.

In the course of the search for a solution to this problem, it was determined that a shift operator $\hat{S}$ must be introduced.

We determine the shift operator $\hat{S}$ using the ratio

$$\hat{L}_2\hat{S}\hat{L}_1\vec{r} = \hat{L}\vec{r}. \tag{62}$$

The optical power of the system of two thin lenses under research is determined as follows [5]:

$$D = D_1 + D_2 - dD_1D_2. \tag{63}$$

From expression (63) it follows that the lens operator of this system will have the following form:

$$\hat{L}\vec{r} = \frac{F_1F_2}{F_1F_2 + (\vec{r}, \overrightarrow{e_x})(F_1 + F_2 - d)}. \tag{64}$$

From expression (62) follows the expression for the shift operator:

$$\hat{S}\vec{r} = \hat{L}_2^{-1}\hat{L}\hat{L}_1^{-1}\vec{r}. \tag{65}$$

Let's find the analytical expression for the shift operator:

$$\hat{L}_1^{-1}\vec{r} = \frac{-F_1}{-F_1 + (\vec{r}, \overrightarrow{e_x})}\vec{r}. \tag{66}$$

$$\hat{L}\hat{L}_1^{-1}\vec{r} = \frac{F_1F_2\left(\frac{-F_1}{-F_1 + (\vec{r}, \overrightarrow{e_x})}\right)}{F_1F_2 + (F_1 + F_2 - d)\frac{-F_1}{-F_1 + (\vec{r}, \overrightarrow{e_x})}(\vec{r}, \overrightarrow{e_x})}\vec{r} = \frac{-F_1F_2}{-F_1F_2 + (d - F_1)(\vec{r}, \overrightarrow{e_x})}\vec{r}. \tag{67}$$

$$\hat{L}_2^{-1}\hat{L}\hat{L}_1^{-1}\vec{r} = \frac{-F_2\frac{-F_1F_2}{-F_1F_2 + (d - F_1)(\vec{r}, \overrightarrow{e_x})}}{-F_2 + \frac{-F_1F_2(\vec{r}, \overrightarrow{e_x})}{-F_1F_2 + (d - F_1)(\vec{r}, \overrightarrow{e_x})}}\vec{r} = \frac{F_1F_2}{F_1F_2 - d(\vec{r}, \overrightarrow{e_x})}\vec{r}. \tag{68}$$

So, the expression for the shift operator can be written as follows:

$$\hat{S}\vec{r} = \frac{1}{1 - dD_1D_2(\vec{r}, \overrightarrow{e_x})}\vec{r}. \tag{69}$$

For certainty, we will denote the shift operator as follows:

[2] The optical center of one of the lenses is located at the origin.

$$\hat{S}[D_1, D_2]\vec{r} \equiv \frac{1}{1 - dD_1D_2(\vec{r}, \overrightarrow{e_x})}\vec{r}. \tag{70}$$

Then, the solution of the problem, which we formulated at the beginning of the section, will look like this:

$$\overrightarrow{r'} = \hat{L}_2\hat{S}[D_1, D_2]\hat{L}_1\vec{r}. \tag{71}$$

In this way we obtained the coordinates of the point image in the system of centered thin lenses with optical power D_1 and D_2, the distance between the optical centers of which is equal to d.

The shift operator $\hat{S}[D_1, D_2]$ together with the lens operators $\hat{L}_1$ and $\hat{L}_2$ of two thin lenses allows obtaining a lensing operator $\hat{L}_2\hat{S}[D_1, D_2]\hat{L}_1$ for the system of centered lenses, which are located at a finite distance from each other.

Shift operator properties.

1. The shift operator is a nonlinear operator.

$$\hat{S}(\alpha\vec{x} + \beta\vec{y}) \neq \alpha\hat{S}\vec{x} + \beta\hat{S}\vec{y}. \tag{72}$$

2. The switch of the shift operator and the lensing operator is equal to 0.

Let's prove this property:

$$\hat{L}\hat{S}\vec{r} = \hat{L}\frac{1}{1 - dD_1D_2(\vec{r}, \overrightarrow{e_x})}\vec{r} = \frac{\frac{1}{1 - dD_1D_2(\vec{r}, \overrightarrow{e_x})}\vec{r}}{1 + D\frac{1}{1 - dD_1D_2(\vec{r}, \overrightarrow{e_x})}(\vec{r}, \overrightarrow{e_x})} = \frac{1}{1 - dD_1D_2(\vec{r}, \overrightarrow{e_x}) + D(\vec{r}, \overrightarrow{e_x})}\vec{r}. \tag{73}$$

$$\hat{S}\hat{L}\vec{r} = \hat{S}\frac{1}{1 + D(\vec{r}, \overrightarrow{e_x})}\vec{r} = \frac{\frac{1}{1 + D(\vec{r}, \overrightarrow{e_x})}\vec{r}}{1 - dD_1D_2\frac{1}{1 + D(\vec{r}, \overrightarrow{e_x})}(\vec{r}, \overrightarrow{e_x})} = \frac{1}{1 + D(\vec{r}, \overrightarrow{e_x}) - dD_1D_2(\vec{r}, \overrightarrow{e_x})}\vec{r}. \tag{74}$$

Expressions (74) and (75) are the same, so we have:

$$[\hat{L}, \hat{S}] = 0. \tag{75}$$

3. Shift operators for different pairs of lenses are commutative.

Prove this property:

$$\begin{aligned}\hat{S}[D_1, D_2]\hat{S}[D_3, D_4]\vec{r} &= \hat{S}[D_1, D_2]\frac{1}{1 - d_2D_3D_4(\vec{r}, \overrightarrow{e_x})}\vec{r} = \frac{\frac{1}{1 - d_2D_3D_4(\vec{r}, \overrightarrow{e_x})}\vec{r}}{1 - d_1D_1D_2\frac{1}{1 - d_2D_3D_4(\vec{r}, \overrightarrow{e_x})}(\vec{r}, \overrightarrow{e_x})} \\ &= \frac{1}{1 - d_2D_3D_4(\vec{r}, \overrightarrow{e_x}) - d_1D_1D_2(\vec{r}, \overrightarrow{e_x})}\vec{r}.\end{aligned} \tag{76}$$

$$\begin{aligned}\hat{S}[D_3, D_4]\hat{S}[D_1, D_2]\vec{r} &= \hat{S}[D_3, D_4]\frac{1}{1 - d_1D_1D_2(\vec{r}, \overrightarrow{e_x})}\vec{r} = \frac{\frac{1}{1 - d_1D_1D_2(\vec{r}, \overrightarrow{e_x})}\vec{r}}{1 - d_2D_3D_4\frac{1}{1 - d_2D_3D_4(\vec{r}, \overrightarrow{e_x})}(\vec{r}, \overrightarrow{e_x})} \\ &= \frac{1}{1 - d_1D_1D_2(\vec{r}, \overrightarrow{e_x}) - d_2D_3D_4(\vec{r}, \overrightarrow{e_x})}\vec{r}.\end{aligned} \tag{77}$$

So, we have:

$$[\hat{S}(D_1, D_2), \hat{S}(D_3, D_4)] = 0. \tag{78}$$

Rules for the use of lensing and shift operators. In the previous paragraphs, the definition of lensing and shift operators was formulated. Now let's outline the rules for using these operators to describe the lens system.

Suppose there is a given system of lenses with optical forces $\{D_i\}_{i=1}^N$, which are located at distances $\{d_i\}_{i=1}^N$ from the origin, and $d_1 = 0$. Then, in order to describe this system, one must use the following rules:

1. The operators must be recorded starting with the lensing operator of the first lens.
2. If there are more than two lenses, the shift operators for the following lenses must indicate the optical power of the system of lenses, which are located before the lens under consideration, but not the optical power of the previous lens.
3. Once all the operators have been identified, they can be rearranged in any order, because all of them are commutative.

RESULTS

The lens systems, to which operators can be applied, have previously been considered as basic problems for constructing a theory. Let's consider some other possibilities of using nonlinear operators in geometric optics.

Speed of the moving point image

Let the point, whose image we are building, moves. Then its image will also move. We find the speed of the image movement.

By definition:

$$\vec{v} = \frac{d\vec{r}}{dt}. \tag{79}$$

$$\overrightarrow{v'} = \frac{d}{dt}\overrightarrow{r'} = \frac{d}{dt}\hat{L}\vec{r}. \tag{80}$$

$$\overrightarrow{v'} = \frac{d}{dt}\left(\frac{F}{(\vec{r},\overrightarrow{e_x}) + F}\vec{r}\right). \tag{81}$$

$$\overrightarrow{v'} = \frac{F}{(\vec{r},\overrightarrow{e_x}) + F}\vec{v} + \frac{F(\vec{v},\overrightarrow{e_x})}{\left((\vec{r},\overrightarrow{e_x}) + F\right)^2}\vec{r}. \tag{82}$$

Construction of the curves image

Suppose, there is a curve given in a parametric form:

$$\gamma: \vec{r} = \vec{r}\big(x(t), y(t)\big), t \in [t_1, t_2]. \tag{83}$$

Consider its image in a thin lens. To build the image, we'll use the lensing operator:

$$\gamma': \overrightarrow{r'} = \hat{L}\vec{r}\big(x(t), y(t)\big), t \in [t_1, t_2]. \tag{84}$$

$$\gamma': \overrightarrow{r'} = \left(\frac{Fx(t)}{F + x(t)}, \frac{Fy(t)}{F + x(t)}\right), t \in [t_1, t_2]. \tag{85}$$

Let's consider several examples of building the curves image in a thin lens.

1. Segment

Let the segment be a part of the line given by the equation:

$$y = kx + b. \tag{86}$$

We write the equation of the line in a parametric form:

$$\begin{cases} x = t, \\ y = kt + b. \end{cases} \tag{87}$$

Then, its image will have the form:

$$\gamma': \overrightarrow{r'} = \left(\frac{Ft}{F + t}, \frac{F(kt + b)}{F + t}\right). \tag{88}$$

Let's denote the parameter t through x:

$$t = \frac{Fx}{F - x}. \tag{89}$$

Substitute (87) into the expression for the image coordinate:

$$y = \frac{Fk\frac{Fx}{F - x} + Fb}{F + \frac{Fx}{F - x}}. \tag{90}$$

After simplification, we obtain that the image of the segment is a segment:

$$y = \left(k - \frac{b}{F}\right)x + b. \tag{91}$$

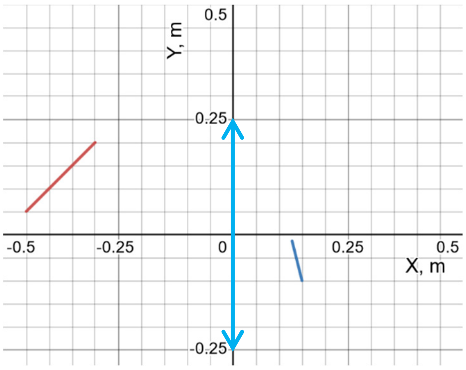

Fig.3. Image of a straight-line segment $y = x + 0.5\ m$ when $x \in [-0.45, -0.3]$ in a thin condenser lens with focal length $F = 0.1\ m$.

2. Arc of the circle

Consider an arc of the circle given by the equation:

$$x^2 + y^2 = R^2. \tag{92}$$

Write the equation of the circle in a parametric form:

$$\begin{cases} x = R\cos t\,, \\ y = R\sin t\,. \end{cases} \tag{93}$$

Then, the image takes the following form:

$$\gamma' : \overrightarrow{r'} = \left(\frac{FR\cos t}{F + R\cos t}, \frac{FR\sin t}{F + R\cos t}\right). \tag{94}$$

Denote the trigonometric functions of the parameter t through x:

$$\cos t = \frac{Fx}{(F - x)R}, \sin t = \sqrt{1 - \frac{x^2F^2}{(F - x)^2R^2}}. \tag{95}$$

and substitute them into the expression for the coordinate y from (90):

$$y = \frac{FR\sqrt{1 - \frac{x^2F^2}{(F - x)^2R^2}}}{F + R\frac{Fx}{(F - x)R}}. \tag{96}$$

After simplification we have:

$$x^2\left(1 - \frac{R^2}{F^2}\right) + \frac{2R^2}{F}x + y^2 - R^2 = 0. \tag{97}$$

We obtained a second-order curve. Define its type [8]. The matrix of coefficients looks like this:

$$\begin{pmatrix} 1 - \frac{R^2}{F^2} & 0 & \frac{R^2}{F} \\ 0 & 1 & 0 \\ \frac{R^2}{F} & 0 & -R^2 \end{pmatrix}. \tag{98}$$

Let us determine the invariants of this curve. Invariants of the second order curve of the form $a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0$ are defined as follows [8]:

$$\begin{cases} I_1 = sp\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}, \\ I_2 = det\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}, \\ I_3 = det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}. \end{cases} \tag{99}$$

For the curve under study, the invariants have the following values:

$$\begin{cases} I_1 = 2 - \frac{R^2}{F^2}, \\ I_2 = 1 - \frac{R^2}{F^2}, \\ I_3 = -R^2. \end{cases} \tag{100}$$

The type of the second-order curve depends on the value of the circle radius and the focal length of the lens.

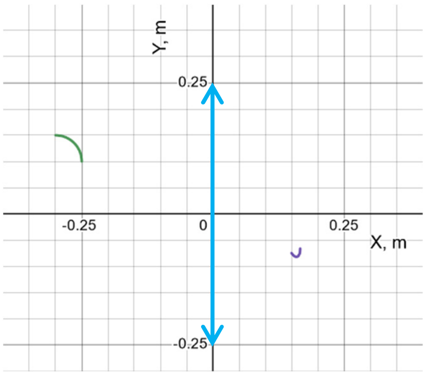

Figure 4. Image of the arc of the circus $(x + 0.3\ m)^2 + (y - 0.1\ m)^2 = (0.05\ m)^2$ with radius $R = 0.05\ m$ for angles from 0 to $\frac{\pi}{2}$ in a thin condenser lens with the focal length $F = 0.1\ m$. According to system (101) the invariants are equal to: $I_1 = 1.75,\ I_2 = 0.75, I_3 = -0.0025\ m^2$. From the invariants values it follows that the image is an ellipse arc [8].

Magnification factor

Let’s consider how to find an expression for the coefficient of linear magnification Γ from the lensing operator. By definition, the coefficient of linear magnification is the ratio of the height of the image to the height of the object [5]:

$$\Gamma = \frac{|(\overrightarrow{r'}, \overrightarrow{e_y})|}{|(\vec{r}, \overrightarrow{e_y})|}. \tag{101}$$

Let's describe it in more detail:

$$\Gamma = \frac{|(\hat{L}\vec{r}, \overrightarrow{e_y})|}{|(\vec{r}, \overrightarrow{e_y})|} = \left|\frac{F}{F + (\vec{r}, \overrightarrow{e_x})}\right| \frac{(\vec{r}, \overrightarrow{e_y})}{(\vec{r}, \overrightarrow{e_y})}. \tag{102}$$

Therefore, the magnification factor can be determined as follows:

$$\Gamma = \left|\frac{F}{F + (\vec{r}, \overrightarrow{e_x})}\right|. \tag{103}$$

CONCLUSION

The concept of lensing operator was introduced in the paper, its properties were investigated. A model problem of a centered system of folded thin lenses was solved, what helped to establish a physical interpretation of the properties of the lensing operator. The concept of shift operator was also introduced, what allowed considering the centered systems of thin lenses located at a certain distance, its properties were also investigated. Using the lens operator several problems were investigated, namely the speed of the moving point image, the coefficient of linear magnification of the lens and building the curves images. To solve the last problem, the general formula for the image of the curve, given in parametric form, was derived, and two examples were considered, i.e. the image of a segment and the arc of the circle in a thin lens.

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ВИКОРИСТАННЯ НЕЛІНІЙНИХ ОПЕРАТОРІВ ДЛЯ РОЗВ'ЯЗУВАННЯ ЗАДАЧ ГЕОМЕТРИЧНОЇ ОПТИКИ

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Метою цієї роботи є розробка та застосування математичного апарату, побудованого на нелінійних операторах, для розв'язування задач геометричної оптики, а саме побудови зображень предметів у системах тонких лінз. Було розглянуто задачу про побудову зображення точки в тонкій лінзі на основі чого було визначено поняття оператора лінзування. Були досліджені математичні властивості оператора. Було досліджено модельну задачу про побудову зображення у складених разом тонких лінзах, на основі чого стало можливо встановити фізичну інтерпретацію встановленим раніше властивостям. Також була розглянута задача про систему лінз, що розташовані на відстані, результатом чого було введення поняття оператора зсуву. Були досліджені властивості оператора зсуву, які разом із властивостями оператора лінзування дали змогу визначити правила застосування створених операторів для розв'язування задач. Окрім розв'язку модельних задач були розглянуті такі задачі: швидкість зображення рухомої точки, коефіцієнт збільшення та побудова зображення кривих. Як приклад були побудовані зображення відрізка та дуги кола. Відрізок перейшов у відрізок, а дуга кола в дугу кривої другого порядку. Представлений математичний апарат є дуже зручним для реалізації у вигляді комп'ютерних програм, а також для дослідження зображень різних кривих.
Ключові слова: геометрична оптика; тонка лінза; нелінійний оператор; системи лінз.