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QUARK-ANTIQUARK STUDY WITH INVERSELY QUADRATIC YUKAWA POTENTIAL USING THE NIKIFOROV-UVAROV-FUNCTIONAL-ANALYSIS METHOD[†]

Etido P. Inyang[a,b]*, Prince C. Iwuji[b], Joseph E. Ntibi[b], E. Omugbe[c], Efiong A. Ibanga[a], Eddy S. William[b]

[a]*Department of Physics, National Open University of Nigeria, Jabi-Abuja, Nigeria*
[b]*Theoretical Physics Group, Department of Physics, University of Calabar, P.M.B 1115 Calabar, Nigeria*
[c]*Department of Physics, Federal University of Petroleum Resources, Effurun, Delta State, Nigeria*
**Corresponding Author: etidophysics@gmail.com, einyang@noun.edu.ng*
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The solutions of the Schrödinger equation are obtained with an inversely quadratic Yukawa potential using the Nikiforov-Uvarov-Functional-analysis method. The energy spectrum and wave function were obtained in closed form. The energy equation was used to predict the masses of the heavy mesons such as charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$) for different quantum numbers. The results obtained agreed with other theoretical predictions and experimental data with a percentage error of 1.68 % and 0.50 % for charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$) respectively.
Keywords: Schrödinger equation; Nikiforov-Uvarov-Functional-analysis method; inversely quadratic Yukawa potential; heavy mesons
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The solutions of the Schrodinger equation (SE) with spherically symmetric potentials are of foremost concern in describing the mass spectra (MS) of the heavy mesons (HMs) such as bottomonium ($b\bar{b}$), and charmonium ($c\bar{c}$) [1-3]. In describing this system an interaction potential is used to simulate the system. The Cornell potential (CP) which includes a short-range Coulomb term and a linear confinement term is generally used [4]. More so, in solving this SE with any chosen potential, an analytical method is employed, such as, the Nikiforov-Uvarov (NU) method [5-8], the NU Functional Analysis (NUFA) method [9,10], the series expansion method (SEM) [11], Laplace transformation method (LTM) [12], WKB approximation method [13], exact quantization rule [14,15], proper quantization rule [16,17], group theory approach [18] and so on [19]. The study of MS with CP has gained remarkable attention and has attracted the attention of many scholars [20-23]. For instance, Kumar et al. [24] used the NUFA method to solve the SE with generalized Cornell potential. The result was used to determine the MS of the HMs. Using the vibrational method (VM) and supersymmetric quantum mechanics, Vega and Flores, [25] obtained the analytical solutions of the SE with CP. The eigenvalues were used to calculate the MS of the HMs. Also, Mutuk [26] solved the SE with CP using a neural network approach. The bottomonium, charmonium, and bottom-charmed spin-averaged spectra were calculated. Furthermore, Hassanabadi et al. [27], used the VM to solve the SE with CP. The eigenvalues were used to calculate the mesonic wave function. In recent times, the study of MS of the HMs with exponential-type potentials has aroused the interest of scholars [28,29]. Potential models such as Yukawa potential [30], Varshni [31], screened Kratzer potential [32], Hulthen plus Hellmann potential [33], and so on [34] have been used in the prediction of the MS of the HMs. For instance, Purohit et al [35] combined linear plus modified Yukawa potential to obtain the masses of the HMs. Also, Al-Jamel [36], studied the MS with Coulomb plus inverse quadratic term. It was found that the MS exhibit a saturation effect. Furthermore, in 2019 Al-Jamel, solved the SE with a combination of a cotangent with a square co-secant function using the NU method. The energy equation was used to predict the MS of heavy quarkonia [37].

The SE for most of the potentials with spin addition cannot be solved analytically; hence, numerical solutions such as Runge-Kutte approximation [38], Numerov matrix method [39], Fourier grid Hamiltonian method [40], and so on [41] are employed. Also, adding spin enables one to determine other properties of the mesons like decay properties and root mean square radii. However, we have considered our mesons as spinless particles for easiness [1, 25, 42-44]. The potential of interest is the inversely quadratic Yukawa potential (IQYP) suggested by Hamzavi et al [45]. The IQYP is a short-range potential, Onate [46] studied the Klein Gordon equation and SE with the IQYP. Since then, researchers have carried out many studies on relativistic and non-relativistic regimes with this potential [47-49]. The potential model is of the form [45]:

$$V(r) = -\frac{pe^{-2\theta r}}{r^2}, \tag{1}$$

where p is potential strength and θ is the screening parameter.

To the best of our knowledge, no report has been made in the literature using this potential to predict the MS of the HMs, hence this study.

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This study aims to solve the SE with IQYP, using a newly proposed method called the NUFA method. The obtained energy equation will be used to predict the MS of the HMs. This paper is organized in the following order. In section 2, the solutions of the SE with the IQYP are presented. In section 3, the results and discussion are presented. Finally, the conclusion is presented in section 4.

THE SOLUTIONS OF THE SE WITH IQYP

The SE reads [50]

$$\frac{d^2\psi(r)}{dr^2}+\left[\frac{2\mu}{\hbar^2}\left(E_{nl}-V(r)\right)-\frac{l(l+1)}{r^2}\right]\psi(r)=0\,, \tag{2}$$

where $\psi(r)$ is the eigenfunctions, E_{nl} is the energy eigenvalues, μ is the reduced mass of the system, $\hbar$ is the reduced Planck's constant and r is the inter-nuclear separation.

The exponential term in Eq. (1) is subjected to series expansion, to model the potential to interact in the quark-antiquark system, this yields:

$$V(r)=\frac{A}{r^2}-\frac{B}{r}+Cr-D\,. \tag{3}$$

where

$$\left.\begin{aligned}&A=-p,\ \mathrm{B}=2\theta p\\&C=-1.33p\theta^3,\ D=2p\theta^2\end{aligned}\right\} \tag{4}$$

Plugging Eq. (3) into Eq. (2) gives

$$\frac{d^2\psi(r)}{dr^2}+\left[\frac{2\mu}{\hbar^2}\left(E_{nl}-\frac{A}{r^2}+\frac{B}{r}-Cr+D\right)-\frac{l(l+1)}{r^2}\right]\psi(r)=0\,. \tag{5}$$

To solve Eq. (5), we introduced the Greene-Aldrich approximation to deal with the inverse square term, valid for small screening parameter values [51-54].

$$\frac{1}{r^2}\approx\frac{\theta^2}{\left(1-e^{-\theta r}\right)^2}. \tag{6}$$

Substituting Eqs. (6) into Eq.(5), gives Eq.(7) as

$$\frac{d^2\psi(r)}{dr^2}+\left[\frac{2\mu}{\hbar^2}\left(E_{nl}-\frac{A\theta^2}{\left(1-e^{-\theta r}\right)^2}+\frac{B\theta}{\left(1-e^{-\theta r}\right)}-C\frac{\left(1-e^{-\theta r}\right)}{\theta}-D\right)-\frac{l(l+1)\theta^2}{\left(1-e^{-\theta r}\right)^2}\right]\psi(r)=0\,. \tag{7}$$

Transmuting the coordinate of Eq. (7), we set, $y=e^{-\theta r}$ and Eq. (7) is rewritten as follows:

$$\frac{d^2\psi(y)}{dy^2}+\frac{1-y}{y(1-y)}\frac{d\psi(y)}{dy}+\frac{1}{y^2(1-y)^2}\left[-\varepsilon(1-y)^2-\chi_0+\chi_1(1-y)-\chi_2(1-y)^3-\gamma\right]\psi(y)=0, \tag{8}$$

where

$$\left.\begin{aligned}&-\varepsilon=\frac{2\mu E_{nl}}{\theta^2\hbar^2}+\frac{2\mu D}{\theta^2\hbar^2},\quad \chi_0=\frac{2\mu A}{\hbar^2},\\&\chi_1=\frac{2\mu B}{\theta\hbar^2},\quad \chi_2=\frac{2\mu C}{\theta^3\hbar^2},\ \gamma=l(l+1)\end{aligned}\right\}. \tag{9}$$

Expanding Eq. (8) and neglecting higher degree polynomials of y greater than two [55], we have

$$\frac{d^2\psi(y)}{dy^2}+\frac{1-y}{y(1-y)}\frac{d\psi(y)}{dy}+\frac{1}{y^2(1-y)^2}\begin{bmatrix}-(\varepsilon+3\chi_2)y^2\\+(2\varepsilon-\chi_1+3\chi_2)y-(\varepsilon+\chi_0-\chi_1+\chi_2+\gamma)\end{bmatrix}\psi(y)=0, \tag{10}$$

Linking Eq. (10) and Eq. (A2), gives the following:

$$\beta_1=\beta_2=\beta_3=1,\xi_1=\varepsilon+3\chi_2,\xi_2=2\varepsilon-\chi_1+3\chi_2,\xi_3=\varepsilon+\chi_0-\chi_1+\chi_2+\gamma\}. \tag{11}$$

Inserting Eq. (11) into Eqs. (A8) and (A9), gives

$$\lambda=\frac{1}{2}\sqrt{4(\varepsilon+\chi_0-\chi_1+\chi_2+\gamma)}, \tag{12}$$

and

$$v=\frac{1+\sqrt{1+4(\chi_0+\gamma)}}{2}. \tag{13}$$

Substituting Eqs. (4), (9), (12), and (13) in Eq. (A11), we obtain the energy spectrum of the IQYP as

$$E_{nl}=-2p\theta^2-\frac{4\mu p^2\theta^2/2\hbar^2}{\left[n+\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{2\mu p}{\hbar^2}-\frac{2.66\mu p\theta^3}{\hbar^2}}\right]^2}. \tag{14}$$

Inserting Eq. (11) into Eq. (A12) the corresponding un-normalized wave function is given as

$$\psi(y)=N_y\frac{\sqrt{4(\varepsilon+\chi_0-\chi_1+\chi_2+\gamma)}}{2}(1-y)\frac{1+\sqrt{1+4(\chi_0+\gamma)}}{2}\,{}_2F_1(a,b,c;y), \tag{15}$$

where,

$$a=\frac{1}{2}\sqrt{4(\varepsilon+\chi_0-\chi_1+\chi_2+\gamma)}+\frac{1+\sqrt{1+4(\chi_0+\gamma)}}{2}+\sqrt{\varepsilon+3\chi_2}, \tag{16}$$

$$b=\frac{1}{2}\sqrt{4(\varepsilon+\chi_0-\chi_1+\chi_2+\gamma)}+\frac{1+\sqrt{1+4(\chi_0+\gamma)}}{2}-\sqrt{\varepsilon+3\chi_2}, \tag{17}$$

$$c=1+\sqrt{4(\varepsilon+\chi_0-\chi_1+\chi_2+\gamma)}. \tag{18}$$

RESULTS AND DISCUSSION

The prediction of the MS of bottomonium and charmonium is carried out using the relation [56,57]

$$\mathrm{M}=2m+E_{nl} \tag{19}$$

where m is quark-antiquark mass and E_{nl} is eigenvalues.
Putting Eq. (14) into Eq. (19) gives

$$\mathrm{M}=2m-2p\theta^2-\frac{4\mu p^2\theta^2/2\hbar^2}{\left[n+\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{2\mu p}{\hbar^2}-\frac{2.66\mu p\theta^3}{\hbar^2}}\right]^2}. \tag{20}$$

The reduced mass (RM) is defined as $\mu=\frac{m}{2}$. For bottomonium and charmonium, the numerical values are $m_b=4.823\,GeV$ and $m_c=1.209\,GeV$, and the corresponding RM is $\mu_b=$ $2.4115\,GeV$ and $\mu_c=0.6045\,GeV$

correspondingly [58]. In other to obtain the potential parameters. Equation 20 is fitted with experimental data (ED). The ED is taken from Ref. [59].

We observed that the results obtained from the prediction of MS of charmonium and bottomonium for different quantum states are in agreement with ED and other theoretical predictions with different analytical methods as shown in Tables 1 and 2.

Table 1. Mass spectra of charmonium in (GeV) (m_c =1.209 GeV, μ = 0.6045 GeV, $\theta = 0.05$, p = - 234.2385 GeV, $\hbar$ = 1)

State	Present work [NUFA]	AIM [1]	NU [57]	SEM [32]	Experiment [59]	Absolute Relative deviation (ARD)
1S	3.096	3.096	3.096	3.095922	3.096	0.000
2S	3.686	3.686	3.686	3.685893	3.686	0.000
1P	3.327	3.214	3.433	-	3.525	0.198
2P	3.774	3.773	3.910	3.756506	3.773	0.001
3S	4.140	4.275	3.984	4.322881	4.040	0.100
4S	4.364	4.865	4.150	4.989406	4.263	0.101
1D	3.761	3.412	3.767	-	3.770	0.009
2D	4.058	-	-	-	4.159	0.101

Table 2. Mass spectra of bottomonium in (GeV) (m_b = 4.823 GeV, μ = 2.4115 GeV, p = -333.87617, GeV, $\theta = 0.05$, $\hbar$ = 1)

State	Present work[NUFA]	AIM [1]	NU[57]	SEM[32]	Experiment [59]	Absolute Relative deviation (ARD)
1S	9.460	9.460	9.460	9.515194	9.460	0.000
2S	10.023	10.023	10.023	10.01801	10.023	0.000
1P	9.841	9.492	9.840	-	9.899	0.058
2P	10.160	10.038	10.160	10.09446	10.260	0.100
3S	10.341	10.585	10.280	10.44142	10.355	0.014
4S	10.422	11.148	10.420	10.85777	10.580	0.158
1D	10.142	9.551	10.140	-	10.164	0.022

The absolute percentage deviation in the predicted results and ED is calculated using the following relation

$$\sigma = \frac{100}{N}\sum\left|\frac{T_P - T_{EXP.}}{T_{EXP.}}\right|, \tag{21}$$

where $T_{EXP.}$ is the ED, T_P is the predicted results and N is the number of ED and also the percentage error is computed with the given relation [60]

$$error = \frac{\sum ARD}{\sum T_{EXP.}} \times 100\% . \tag{22}$$

The absolute percentage deviation was calculated using Eq. (21) and the values for charmonium and bottomonium are 1.644 GeV and 0.487 GeV respectively. Also, the percentage error (PE) of the predicted results to the experimental data, was calculated using Eq. (22) and the results show that for charmonium we have 1.68 % while that of the bottomonium is 0.50 %.

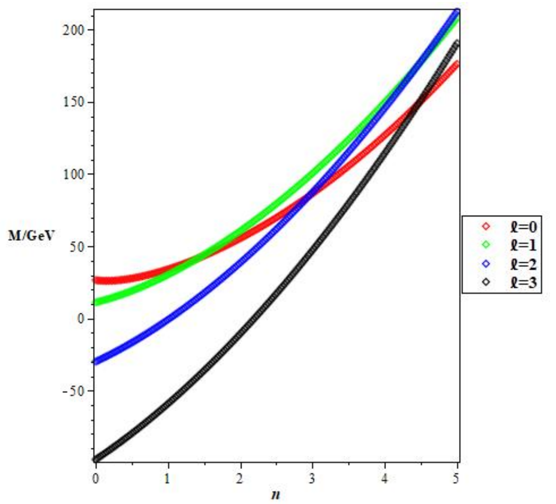

Figure 1. Mass spectra of bottomonium with a principal quantum number for a different angular quantum number

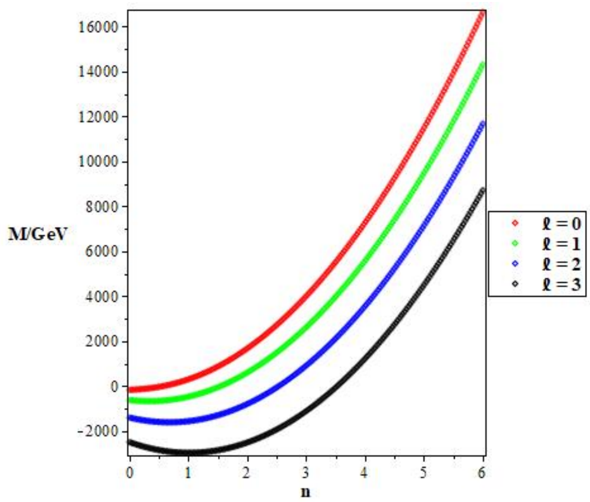

Figure 2. Mass spectra of charmonium with a principal quantum number for different angular quantum number

Figure 1 shows the variation of MS of bottomonium with a principal quantum number (PQN) for different angular quantum numbers (AQNs). It was observed that the MS increases with an increase in PQN for different AQNs which is in agreement with the ED. In Fig. 2, we plotted the MS of charmonium with PQN for different AQN. It was noticed that the MS increases as the PQN increase. This indicates that the binding energy increases with an increase in PQN.

CONCLUSION

In the present study, the solutions of the SE with IQYP using the NUFA method were solved. The energy spectrum and un-normalized wave function were obtained. The obtained energy equation was used to predict the MS of the HMs for different quantum numbers. The results obtained agreed with other predictions and with ED. The PE shows that the IQYP is fitted in the prediction of the MS of the HMs since the PE is less.

ORCID IDs
Etido P. Inyang, https://orcid.org/0000-0002-5031-3297; **Prince C. Iwuji,** https://orcid.org/0000-0001-5715-9336
Joseph E. Ntibi, https://orcid.org/0000-0002-7908-2840

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APPENDIX A: NUFA METHOD

The NUFA method proposed by Ikot et al. [10] is a simple and elegant method for solving a second-order differential equation. The method is a combination of the NU [61], parametric NU [62], and Factorization [63] methods. The NU is well-known for solving a second-order differential equation of the form:

$$\psi''(y)+\frac{\tilde{\tau}(y)}{\sigma(y)}\psi'(y)+\frac{\tilde{\sigma}(y)}{\sigma^2(y)}\psi(y)=0, \tag{A1}$$

where $\tilde{\sigma}(y)$ and $\sigma(y)$ are polynomials of the maximum second degree and $\tilde{\tau}(y)$ is a polynomial of the maximum first degree? Tezcan and Sever [70] later introduced the parametric form of the NU method in the form

$$\psi''+\frac{\beta_1-\beta_2 y}{y(1-\beta_3 y)}\psi'+\frac{1}{y^2(1-\beta_3 y)^2}\left[-\xi_1 y^2+\xi_2 y-\xi_3\right]\psi(y)=0, \tag{A2}$$

where β_i and $\xi_i\,(i=1,2,3)$ are all parameters. It can be observed in Eq. (A2) that the differential equation has two singularities at $y\to 0$ and $y\to 1$, thus we take the wave function in the form,

$$\psi(y)=y^{\lambda}(1-\beta_3 y)^{v}f(y). \tag{A3}$$

Replacing Eq. (4) with Eq. (3) leads to the Eq. (A4),

$$\begin{aligned}&y(1-\beta_3 y)f''(y)+\left[\beta_1+2\lambda-(2\lambda\beta_3+2v\beta_3+\beta_2)y\right]f'(y)\\&-\beta_3\left(\lambda+v+\frac{\beta_2}{\beta_3}-1+\sqrt{\left(\frac{\beta_2}{\beta_3}-1\right)^2+\frac{\xi_1}{\beta_3}}\right)\\&\left(\lambda+v+\frac{\beta_2}{\beta_3^2}-1+\sqrt{\left(\frac{\beta_2}{\beta_3}-1\right)^2+\frac{\xi_1}{\beta_3^2}}\right)f(y)\\&+\left[\frac{\lambda(\lambda-1)+\beta_1\lambda-\xi_3}{y}+\frac{\beta_2 v-\beta_1\beta_3 v+v(v-1)\beta_3-\frac{\xi_1}{\beta_3}+\xi_2-\xi_3\beta_3}{(1-\beta_3 y)}\right]f(y)=0.\end{aligned} \tag{A4}$$

Equation (A4) can be reduced to a Gauss hypergeometric equation if and only if the following functions are gone,

$$\lambda(\lambda-1)+\beta_1\lambda-\xi_3=0, \tag{A5}$$

and

$$\beta_2 v - \beta_1\beta_3 v + v(v-1)\beta_3 - \frac{\xi_1}{\beta_3} + \xi_2 - \xi_3\beta_3 = 0. \quad \text{(A6)}$$

Thus Eq. (A4) becomes

$$y(1-\beta_1 y) f''(y) + \left[\beta_1 + 2\lambda - (2\lambda\beta_3 + 2v\beta_3 + \beta_2)y\right] f'(y)$$
$$-\beta_3\left(\lambda + v + \frac{\beta_2}{\beta_3} - 1 + \sqrt{\left(\frac{\beta_2}{\beta_3}-1\right)^2 + \frac{\xi_1}{\beta_3}}\right)\left(\lambda + v + \frac{\beta_2}{\beta_3^2} - 1 + \sqrt{\left(\frac{\beta_2}{\beta_3}-1\right)^2 + \frac{\xi_1}{\beta_3^2}}\right) f(y) = 0. \quad \text{(A7)}$$

Solving Eqs. (A5) and (A6) completely give,

$$\lambda = \frac{(1-\beta_1)}{2} \pm \frac{1}{2}\sqrt{(1-\beta_1)^2 + 4\xi_3}\,, \quad \text{(A8)}$$

$$v = \frac{(\beta_3 + \beta_1\beta_3 - \beta_2) \pm \sqrt{(\beta_3 + \beta_1\beta_3 - \beta_2)^2 + 4\left(\frac{\xi_1}{\beta_3} + \beta_3\xi_3 - \xi_2\right)}}{2}. \quad \text{(A9)}$$

Equation (A7) is the hypergeometric equation type of the form,

$$y(1-y) f''(y) + \left[c - (a+b+1)y\right] f'(y) - abf(y) = 0. \quad \text{(A10)}$$

The energy equation and the related wave equation for the NUFA method are obtained using Eqs. (A3), (A7) and (A10), as follows:

$$\lambda^2 + 2\lambda\left(v + \frac{\beta_2}{\beta_3} - 1 + \frac{n}{\sqrt{\beta_3}}\right) + \left(v + \frac{\beta_2}{\beta_3} - 1 + \frac{n}{\sqrt{\beta_3}}\right)^2 - \left(\frac{\beta_2}{\beta_3} - 1\right)^2 - \frac{\xi_1}{\beta_3^2} = 0, \quad \text{(A11)}$$

$$\psi(y) = N y^{\frac{(1-\beta_1) + \sqrt{(1-\beta_1)^2 + 4\xi_3}}{2}} (1-\beta_3 y)^{\frac{(\beta_3 + \beta_1\beta_3 - \beta_2) + \sqrt{(\beta_3 + \beta_1\beta_3 - \beta_2)^2 + \left(\frac{\xi_1}{\beta_3^2} + \beta_3\xi_3 - \xi_2\right)}}{2}} {}_2F_1(a,b,c;y), \quad \text{(A12)}$$

where $a, b,$ and c are given as follows;

$$a = \sqrt{\beta_3}\left(\lambda + v + \frac{\beta_2}{\beta_3} - 1 + \sqrt{\left(\frac{\beta_2}{\beta_3}-1\right)^2 + \frac{\xi_1}{\beta_3}}\right), \quad \text{(A13)}$$

$$b = \sqrt{\beta_3}\left(\lambda + v + \frac{\beta_2}{\beta_3} - 1 - \sqrt{\left(\frac{\beta_2}{\beta_3}-1\right)^2 + \frac{\xi_1}{\beta_3}}\right), \quad \text{(A14)}$$

$$c = \beta_1 + 2\lambda. \quad \text{(A15)}$$

КВАРК-АНТИКВАРКОВЕ ДОСЛІДЖЕННЯ З ОБЕРНЕНО-КВАДРАТИЧНИМ ПОТЕНЦІАЛОМ ЮКАВИ З ВИКОРИСТАННЯМ МЕТОДУ ФУНКЦІОНАЛЬНОГО АНАЛІЗУ НІКИФОРОВА-УВАРОВА

Етідо П. Іньянг[a,b], Прінс К. Івуджі[b], Джозеф Е. Нтібі[b], Е. Омугбе[c], Ефіонг А. Ібанга[a], Едді С. Вільям[b]

[a]Фізичний факультет, Національний відкритий університет Нігерії, Джабі-Абуджа, Нігерія
[b]Група теоретичної фізики, факультет фізики, Університет Калабара, P.M.B 1115 Калабар, Нігерія
[c]Фізичний факультет, Федеральний університет нафтових ресурсів, Еффурун, штат Дельта, Нігерія

Методом функціонального аналізу Нікіфорова-Уварова отримано вирішення рівняння Шредінгера з обернено-квадратичним потенціалом Юкави. Енергетичний спектр і хвильова функція отримано в замкнутому вигляді. Рівняння енергії було використано для передбачення мас важких мезонів, таких як чармоній ($c\bar{c}$) і ботомоній ($b\bar{b}$) для різних квантових чисел. Отримані результати узгоджувалися з іншими теоретичними прогнозами та експериментальними даними з відсотковою похибкою 1,68 % та 0,50 % для чармонію ($c\bar{c}$) та ботомонію ($b\bar{b}$) відповідно.

Ключові слова: рівняння Шредінгера; метод функціонального аналізу Нікіфорова-Уварова; обернено квадратичний потенціал Юкави; важкі мезони