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FAMILY OF THE ATOMIC RADIAL BASIS FUNCTIONS OF THREE INDEPENDENT VARIABLES GENERATED BY HELMHOLTZ-TYPE OPERATOR†

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The paper presents an algorithm for constructing the family of the atomic radial basis functions of three independent variables $AHorp_k(x_1,x_2,x_3)$ generated by Helmholtz-type operator, which may be used as basis functions for the implementation of meshless methods for solving boundary-value problems in anisotropic solids. Helmholtz-type equations play a significant role in mathematical physics because of the applications in which they arise. In particular, the heat equation in anisotropic solids in the process of numerical solution is reduced to the equation that contains the differential operator of the special form (Helmholtz-type operator), which includes components of the tensor of the second rank, which determines the anisotropy of the material. The family of functions $AHorp_k(x_1,x_2,x_3)$ is infinitely differentiable and finite (compactly supported) solutions of the functional-differential equation of the special form. The choice of compactly supported functions as basis functions makes it possible to consider boundary-value problems on domains with complex geometric shapes. Functions $AHorp_k(x_1,x_2,x_3)$ include the shape parameter k, which allows varying the size of the support and may be adjusted in the process of solving the boundary-value problem. Explicit formulas for calculating the considered functions and their Fourier transform are obtained. Visualizations of the atomic functions $AHorp_k(x_1,x_2,x_3)$ and their first derivatives with respect to the variables x_1 and x_2 at the fixed value of the variable $x_3=0$ for isotropic and anisotropic cases are presented. The efficiency of using atomic functions $AHorp_k(x_1,x_2,x_3)$ as basis functions is demonstrated by the solution of the non-stationary heat conduction problem with the moving heat source. This work contains the results of the numerical solution of the considered boundary-value problem, as well as average relative error, average absolute error and maximum error are calculated using atomic radial basis functions $AHorp_k(x_1,x_2,x_3)$ and multiquadric radial basis functions.
Keywords: atomic radial basis function, Helmholtz-type operator, meshless methods, boundary-value problems, anisotropic thermal conductivity.
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Currently, meshless methods for the numerical solution of boundary-value problems are being actively developed [1-5]. In particular, methods that implement the approximation of a differential equation in the strong form (collocation methods) using compactly supported radial functions as basis [6-10]. The use of compactly supported radial basis functions leads to a sparse interpolation matrix and allows effectively avoiding ill-conditioning, and therefore, reduces computational costs. However, the lower order of accuracy of compactly supported radial basis functions compared to global supported functions is a serious obstacle to their practical use.

New opportunities for the practical implementation of meshless schemes appear with the use of atomic radial basis functions. The discovery of classes of atomic functions is due to Rvachev V. L. and Rvachev V. A. [11], who constructed the simplest one-dimensional atomic function $up(x)$ in 1971. The special properties of the function $up(x)$ (infinite differentiability and compact support) made it possible to construct algorithmically simple computational schemes for solving problems of interpolation and approximation of functions [12]. These functions were used to solve boundary-value problems through the application of variational methods. The expansion of the concept of atomic function in case of many independent variables was presented in the works of Kolodyazhny V. M., Rvachev V. A. and Lisina O. Yu. [13-17]. Atomic functions generated by various differential operators such as the Laplace, Helmholtz, Klein-Gordon, biharmonic operator, etc. have been constructed. The obtained atomic radial basis functions have demonstrated their efficiency in the numerical solution of unsteady heat conduction problems in isotropic solids using meshless schemes [18,19].

Currently, there are many natural and synthetic materials, whose thermophysical properties depend on the direction; they are called anisotropic materials. Common examples of anisotropic materials are crystals and single crystals, steel and alloy billets (rolling, stamping), fibrous materials and thin films, fiber reinforced plastics, quartz, graphite, etc. In this case the heat equation in anisotropic solids in the process of numerical solution is reduced to the equation that contains the differential operator of the special form (Helmholtz-type operator), which includes components of the tensor of the second rank, which determines the anisotropy of the material.

The study of heat conduction processes in anisotropic solids is a major focus of modern engineering research in the energy, machine-building, nuclear and other industries.

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Goals of article is constructing family of the atomic radial basis functions generated by Helmholtz-type operator, which extends the subclass of functions used as basis functions in the implementation of meshless methods for solving boundary-value problems in anisotropic solids.

THE CONSTRUCTION ALGORITHM

Consider algorithm for constructing family of the atomic radial basis functions of three independent variables, which are the solution of the functional-differential equation of the following form:

$$L(K)u(x_1,x_2,x_3)-\delta^2 u(x_1,x_2,x_3)=\lambda\iint_{\partial\Omega}u(k(x_1-\xi_1),k(x_2-\xi_2),k(x_3-\xi_3))d\omega+\mu u(kx_1,kx_2,kx_3), \quad (1)$$

where $L(K)-\delta^2=\sum_{i,j=1}^{3}K_{ij}\frac{\partial^2}{\partial x_i\partial x_j}-\delta^2$ – Helmholtz-type operator; $K=\left[K_{ij}\right]_{1\le i,j\le 3}$ – symmetric positive definite tensor of the second rank, which determines the anisotropy of the material; $\partial\Omega$ – boundary of the sphere of radius $r_k:\sum_{i=1}^{3}\xi_i^2=r_k^2$, $r_k=r_k(k)=\frac{k+1}{2k}$; k – shape parameter; λ , μ – parameters whose values are determined from the condition guaranteeing the existence of the compactly supported solution of equation (1); δ^2 – parameter of the Helmholtz-type operator.

Apply the three-dimensional Fourier transform to equation (1):

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[L(K)u(x_1,x_2,x_3)-\delta^2u(x_1,x_2,x_3)\right]e^{-i(t_1x_1+t_2x_2+t_3x_3)}dx_1dx_2dx_3=$$
$$=\lambda\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{\iint_{\partial\Omega}u(k(x_1-\xi_1),k(x_2-\xi_2),k(x_3-\xi_3))d\omega+\mu u(kx_1,kx_2,kx_3)\right\}e^{-i(t_1x_1+t_2x_2+t_3x_3)}dx_1dx_2dx_3. \quad (2)$$

Denote by $U(t_1,t_2,t_3)$ the result of applying the three-dimensional Fourier transform to the function $u(x_1,x_2,x_3)$:

$$U(t_1,t_2,t_3)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}u(x_1,x_2,x_3)e^{-i(t_1x_1+t_2x_2+t_3x_3)}dx_1dx_2dx_3.$$

Let $k(x_i-\xi_i)=\eta_i$, $i=1,2,3,$ in this case $x_i=\frac{\eta_i}{k}+\xi_i$. On the right-hand side of equation (2), we change the order of applying the operation of integration over the surface of the sphere and the operation of the three-dimensional Fourier transform. As a result, equation (2) can be rewritten as

$$-\left(K_{11}t_1^2+K_{22}t_2^2+K_{33}t_3^2+2K_{12}t_1t_2+2K_{13}t_1t_3+2K_{23}t_2t_3\right)U(t_1,t_2,t_3)-\delta^2U(t_1,t_2,t_3)=$$
$$=\lambda\iint_{\partial\Omega}\left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}u\left(k\frac{\eta_1}{k},k\frac{\eta_2}{k},k\frac{\eta_3}{k}\right)e^{-i\left[t_1\left(\frac{\eta_1}{k}+\xi_1\right)+t_2\left(\frac{\eta_2}{k}+\xi_2\right)+t_3\left(\frac{\eta_3}{k}+\xi_3\right)\right]}d\frac{\eta_1}{k}d\frac{\eta_2}{k}d\frac{\eta_3}{k}\right\}d\omega+\frac{\mu}{k^3}U\left(\frac{t_1}{k},\frac{t_2}{k},\frac{t_3}{k}\right). \quad (3)$$

After applying the three-dimensional Fourier transform on the right-hand side of equation (3), we obtain

$$-\left(K_{11}t_1^2+K_{22}t_2^2+K_{33}t_3^2+2K_{12}t_1t_2+2K_{13}t_1t_3+2K_{23}t_2t_3+\delta^2\right)U(t_1,t_2,t_3)=$$
$$=\frac{1}{k^3}U\left(\frac{t_1}{k},\frac{t_2}{k},\frac{t_3}{k}\right)\left[\lambda\iint_{\partial\Omega}e^{-i\left(t_1\xi_1+t_2\xi_2+t_3\xi_3\right)}d\omega+\mu\right]. \quad (4)$$

For further solution, it is necessary to consider the integral over the surface of the sphere $\partial\Omega:\xi_1^2+\xi_2^2+\xi_3^2=r_k^2$ on the right-hand side of equation (4). It should be noted that the exponent of the integrand represents the dot product of two vectors $\vec{T}=(t_1,t_2,t_3)$, $\vec{\Xi}=(\xi_1,\xi_2,\xi_3)$.

We will assume that the vector $\vec{T}$ is directed along the z-axis of the Cartesian coordinate system in which the sphere $\partial\Omega$ is defined, and the vector $\vec{\Xi}$ is directed along the radius vector that describes this sphere. To simplify the integration procedure, we introduce spherical coordinates as $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\varphi$. This representation allows the surface integral to be rewritten in the form

$$\iint_{\partial\Omega} e^{-i(t_1\xi_1+t_2\xi_2+t_3\xi_3)}d\omega = r_k^2\int_0^{2\pi}\int_0^{\pi} e^{-i\sqrt{t_1^2+t_2^2+t_3^2}\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}\cos\theta}\sin\theta d\theta d\varphi = r_k^2\int_0^{2\pi}\int_0^{\pi} e^{-ir_k\sqrt{t_1^2+t_2^2+t_3^2}\cos\theta}\sin\theta d\theta d\varphi. \quad (5)$$

The implementation of the integration procedure in (5) leads to the representation of the integral as the elementary function

$$\iint_{\partial\Omega} e^{-i(t_1\xi_1+t_2\xi_2+t_3\xi_3)}d\omega = 4\pi r_k^2\frac{\sin r_k\sqrt{t_1^2+t_2^2+t_3^2}}{r_k\sqrt{t_1^2+t_2^2+t_3^2}}.$$

Based on the above, equation (4) can be rewritten as follows

$$U(t_1,t_2,t_3) = -\frac{U\left(\frac{t_1}{k},\frac{t_2}{k},\frac{t_3}{k}\right)\left[\lambda 4\pi r_k^2\frac{\sin r_k\sqrt{t_1^2+t_2^2+t_3^2}}{r_k\sqrt{t_1^2+t_2^2+t_3^2}}+\mu\right]}{k^3\left(K_{11}t_1^2+K_{22}t_2^2+K_{33}t_3^2+2K_{12}t_1t_2+2K_{13}t_1t_3+2K_{23}t_2t_3+\delta^2\right)}. \quad (6)$$

In order for the expression in braces to be an entire function, we will use the possibility of choosing the parameter μ, considering that $t_1^2+t_2^2+t_3^2 \to 0$, $K_{11}t_1^2+K_{22}t_2^2+K_{33}t_3^2+2K_{12}t_1t_2+2K_{13}t_1t_3+2K_{23}t_2t_3 \to 0$. In this case $\mu = -\frac{4\pi}{i\delta}\lambda r_k\sin(r_k i\delta)$.

The structure of equation (6) makes it possible to represent the ratio

$$f(x) = C(x)f\left(\frac{x}{a}\right),$$

where $f\left(\frac{x}{a}\right)$, $C(x)$ – functions which are analytic everywhere on the numerical axis, $a>0$, $a = const$, $C(0)=1$, $f(0)=1$, in the form of the infinite product [16]: $f(x)=\prod_{h=0}^{+\infty}C\left(\frac{x}{a^h}\right)$. Thus, equation (6) can be written in the following form:

$$U(t_1,t_2,t_3) = \prod_{h=0}^{\infty}\frac{\mu - 4\pi r_k^2\lambda\frac{\sin\frac{r_k}{k^h}\sqrt{t_1^2+t_2^2+t_3^2}}{\frac{r_k}{k^h}\sqrt{t_1^2+t_2^2+t_3^2}}}{k^3\left(\frac{K_{11}t_1^2+K_{22}t_2^2+K_{33}t_3^2+2K_{12}t_1t_2+2K_{13}t_1t_3+2K_{23}t_2t_3}{k^{2h}}+\delta^2\right)}. \quad (7)$$

To ensure the convergence of the infinite product (7), we choose the parameter λ from the conditions: $h=0$, $t_1^2+t_2^2+t_3^2 \to 0$, $K_{11}t_1^2+K_{22}t_2^2+K_{33}t_3^2+2K_{12}t_1t_2+2K_{13}t_1t_3+2K_{23}t_2t_3 \to 0$, in this case $\lambda = -\frac{(k\delta)^3 i}{4\pi r_k\left(\sin(r_k i\delta)+r_k i\delta\right)}$.

Based on the generalization of the Paley-Wiener theorem [20] for the multidimensional case and the Polya-Plancherel theorem [21], we establish that the function $u(x_1,x_2,x_3)$ is an infinitely differentiable compactly supported function, for which the Fourier transform $U(t_1,t_2,t_3)$ is represented by the rapidly decreasing entire function of exponential type at $t_1^2+t_2^2+t_3^2\to\infty$, $K_{11}t_1^2+K_{22}t_2^2+K_{33}t_3^2+2K_{12}t_1t_2+2K_{13}t_1t_3+2K_{23}t_2t_3\to\infty$. Thus, as a result of applying the inverse Fourier transform to expression (7), we obtain the required finite function (the support of this function will be the sphere of unit radius). This function will be denoted by $AHorp_k(x_1,x_2,x_3)$, and will be called the atomic function.

From the above it is clear that the Fourier transform of the function $AHorp_k(x_1,x_2,x_3)$ is

$$AH\tilde{o}\tilde{r}p_k(t_1,t_2,t_3)=\prod_{h=0}^{\infty}\frac{\mu-4\pi r_k^2\lambda\dfrac{\sin\frac{r_k}{k^h}\sqrt{t_1^2+t_2^2+t_3^2}}{\frac{r_k}{k^h}\sqrt{t_1^2+t_2^2+t_3^2}}}{k^3\left(\dfrac{K_{11}t_1^2+K_{22}t_2^2+K_{33}t_3^2+2K_{12}t_1t_2+2K_{13}t_1t_3+2K_{23}t_2t_3}{k^{2h}}+\delta^2\right)}. \tag{8}$$

Function $AHorp_k(x_1,x_2,x_3)$ is even with respect to its variables and can be expanded in the triple Fourier series

$$AHorp_k(x_1,x_2,x_3)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}a_{pqr}\cos(p\pi x_1)\cos(q\pi x_2)\cos(r\pi x_3), \tag{9}$$

in which the Fourier coefficients are calculated by the following formulas:

$$\begin{aligned}
a_{000}&=\frac{1}{8};\\
a_{p00}&=\frac{1}{4}\int_{-\infty}^{+\infty}AHorp_k(\xi_1,0,0)\cos(p\pi\xi_1)\,d\xi_1;\\
a_{pq0}&=\frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}AHorp_k(\xi_1,\xi_2,0)\cos(p\pi\xi_1)\cos(q\pi\xi_2)\,d\xi_1d\xi_2;\\
a_{0q0}&=\frac{1}{4}\int_{-\infty}^{+\infty}AHorp_k(0,\xi_2,0)\cos(q\pi\xi_2)\,d\xi_2;\\
a_{0qr}&=\frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}AHorp_k(0,\xi_2,\xi_3)\cos(q\pi\xi_2)\cos(r\pi\xi_3)\,d\xi_2d\xi_3;\\
a_{p0r}&=\frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}AHorp_k(\xi_1,0,\xi_3)\cos(p\pi\xi_1)\cos(r\pi\xi_3)\,d\xi_1d\xi_3;\\
a_{00r}&=\frac{1}{4}\int_{-\infty}^{+\infty}AHorp_k(0,0,\xi_3)\cos(r\pi\xi_3)\,d\xi_3;\\
a_{pqr}&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}AHorp_k(\xi_1,\xi_2,\xi_3)\cos(p\pi\xi_1)\cos(q\pi\xi_2)\cos(r\pi\xi_3)\,d\xi_1d\xi_2d\xi_3,
\end{aligned} \tag{10}$$

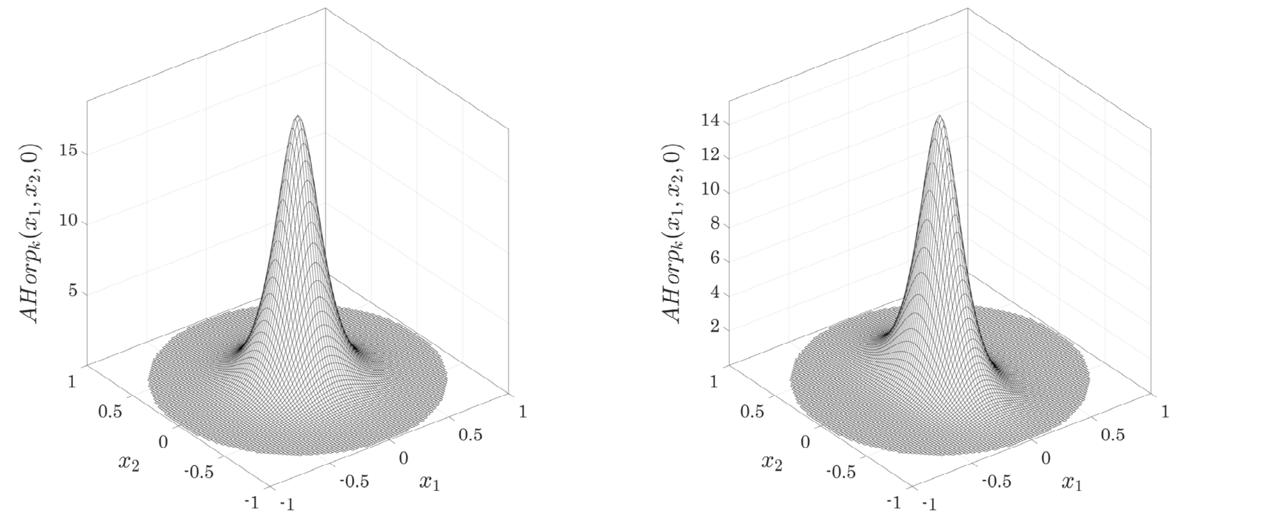
where $p,q,r=1,2,...$

It is clear that, since the function $AHorp_k(x_1,x_2,x_3)$ is finite, $\operatorname{supp}AHorp_k\in[-1,1]\times[-1,1]\times[-1,1]$ and even with respect to variables x_1,x_2,x_3, in the expressions for the Fourier coefficients (10), improper integrals can be replaced by definite integrals, and integrands can be replaced by exponential functions. These transformations make it possible to rewrite the Fourier coefficients (10) of series (9) in the following form:

$$
\begin{aligned}
&a_{000} = \frac{1}{8};\\
&a_{p00} = \frac{1}{4}\int_{-1}^{1} AHorp_k\left(\xi_1,0,0\right)e^{-ip\pi\xi_1}d\xi_1 = \frac{1}{4}AH\tilde{o}\tilde{r}p_k\left(p\pi,0,0\right);\\
&a_{pq0} = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} AHorp_k\left(\xi_1,\xi_2,0\right)e^{-ip\pi\xi_1}e^{-iq\pi\xi_2}d\xi_1 d\xi_2 = \frac{1}{2}AH\tilde{o}\tilde{r}p_k\left(p\pi,q\pi,0\right);\\
&a_{0q0} = \frac{1}{4}\int_{-1}^{1} AHorp_k\left(0,\xi_2,0\right)e^{-iq\pi\xi_2}d\xi_2 = \frac{1}{4}AH\tilde{o}\tilde{r}p_k\left(0,q\pi,0\right);\\
&a_{0qr} = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} AHorp_k\left(0,\xi_2,\xi_3\right)e^{-iq\pi\xi_2}e^{-ir\pi\xi_3}d\xi_2 d\xi_3 = \frac{1}{2}AH\tilde{o}\tilde{r}p_k\left(0,q\pi,r\pi\right);\\
&a_{p0r} = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} AHorp_k\left(\xi_1,0,\xi_3\right)e^{-ip\pi\xi_1}e^{-ir\pi\xi_3}d\xi_1 d\xi_3 = \frac{1}{2}AH\tilde{o}\tilde{r}p_k\left(p\pi,0,r\pi\right);\\
&a_{00r} = \frac{1}{4}\int_{-1}^{1} AHorp_k\left(0,0,\xi_3\right)e^{-ir\pi\xi_3}d\xi_3 = \frac{1}{4}AH\tilde{o}\tilde{r}p_k\left(0,0,r\pi\right);\\
&a_{pqr} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} AHorp_k\left(\xi_1,\xi_2,\xi_3\right)e^{-ip\pi\xi_1}e^{-iq\pi\xi_2}e^{-ir\pi\xi_3}d\xi_1 d\xi_2 d\xi_3 = AH\tilde{o}\tilde{r}p_k\left(p\pi,q\pi,r\pi\right),
\end{aligned} \tag{11}
$$

where $p,q,r = 1,2,...$

Functions $AHorp_k(x_1,x_2,x_3)$ form the family of atomic functions that are generated by the differential operator $L(K)-\delta^2$. Fig. 1 shows the visualization of the function $AHorp_k(x_1,x_2,x_3)$ at the fixed value of the variable $x_3 = 0$ for isotropic (*a*) and anisotropic $\left(K_{11} = 0.5, K_{22} = 1.5, K_{33} = 2.0, K_{12} = K_{13} = K_{23} = 0\right)$ (*b*) cases.



(*a*) *Isotropic case* (*b*) *Anisotropic case*

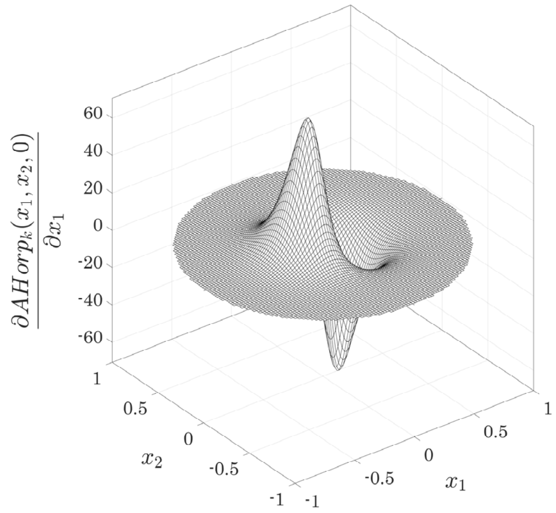
Figure 1. Visualization of the function $AHorp_k(x_1,x_2,x_3)$ at the fixed value of the variable $x_3 = 0$ for isotropic (*a*) and anisotropic (*b*) cases.

Theorem 1. *The family of atomic functions* $AHorp_k(x_1,x_2,x_3)$*, which are solutions of the functional-differential equation (1) with the values of the parameters*

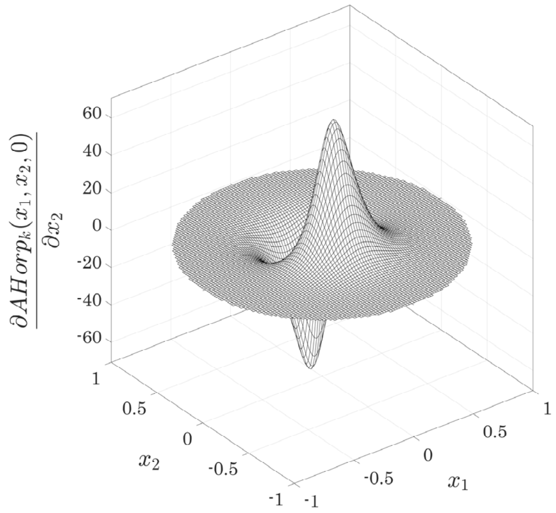
$$
\mu = -\frac{4\pi}{i\delta}\lambda r_k \sin\left(r_k i\delta\right); \quad \lambda = -\frac{\left(k\delta\right)^3 i}{4\pi r_k\left(\sin\left(r_k i\delta\right)+r_k i\delta\right)}
$$

are finite, infinitely differentiable functions with support in the form of the sphere of unit radius, normalized by the condition $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} AHorp_k(x_1,x_2,x_3)dx_1dx_2dx_3 = 1$, *which are represented in the cube:* $[-1,1]\times[-1,1]\times[-1,1]$ *by the Fourier series (9) with the coefficients (11). The Fourier transform of functions* $AHorp_k(x_1,x_2,x_3)$ *(8) is rapidly decreasing function of exponential type at* $t_1^2+t_2^2+t_3^2 \to \infty$, $K_{11}t_1^2+K_{22}t_2^2+K_{33}t_3^2+2K_{12}t_1t_2+2K_{13}t_1t_3+2K_{23}t_2t_3 \to \infty$.

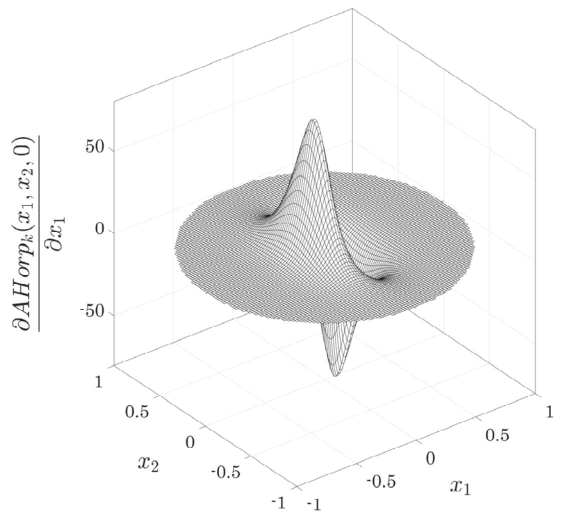
Fig. 2 shows the visualization of the first derivatives of the function $AHorp_k(x_1,x_2,x_3)$ with respect to the variables x_1 and x_2 at the fixed value of the variable $x_3 = 0$ for isotropic (*a*)-(*b*) and anisotropic $(K_{11}=0.5, K_{22}=1.5, K_{33}=2.0, K_{12}=K_{13}=K_{23}=0)$ (*c*)-(*d*) cases.



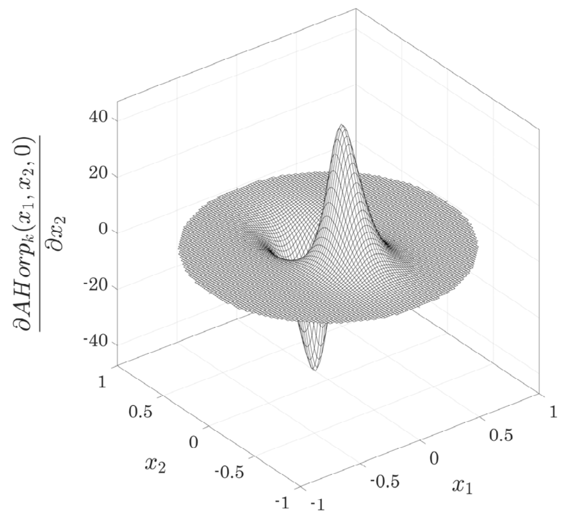
(*a*) *Isotropic case*



(*b*) *Isotropic case*



(*c*) *Anisotropic case*



(*d*) *Anisotropic case*

Figure 2. Visualization of the first derivatives of the function $AHorp_k(x_1,x_2,x_3)$ with respect to the variables x_1 and x_2 at the fixed value of the variable $x_3 = 0$ for isotropic (*a*)-(*b*) and anisotropic (*c*)-(*d*) cases.

Fig. 3 shows the visualization of the function $\left(L(K)-\delta^2\right)AHorp_k(x_1,x_2,x_3)$ at the fixed value of the variable $x_3 = 0$ for isotropic (*a*) and anisotropic $(K_{11}=0.5, K_{22}=1.5, K_{33}=2.0, K_{12}=K_{13}=K_{23}=0)$ (*b*) cases.

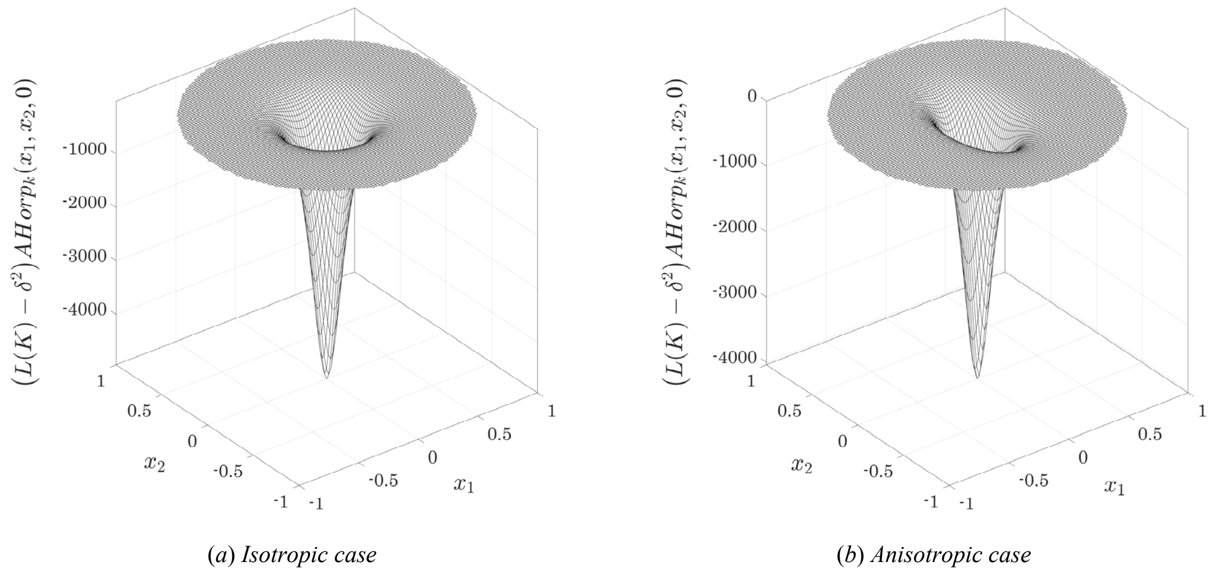


Figure 3. Visualization of the function $\left(L(K)-\delta^2\right)AHorp_k(x_1,x_2,x_3)$ at the fixed value of the variable $x_3=0$ for isotropic (*a*) and anisotropic (*b*) cases.

NUMERICAL RESULTS

We will illustrate the use of atomic functions $AHorp_k(x_1,x_2,x_3)$ as basis functions in the implementation of the meshless method for solving three-dimensional non-stationary heat conduction problems in materials with anisotropy, described in [1]. In this approach, the combination of the dual reciprocity method [22] using anisotropic radial basis function and the method of fundamental solutions [23] is used for solving boundary-value problem. The method of fundamental solutions is used for obtaining of homogenous part of the solution and the dual reciprocity method using anisotropic radial basis functions is used for obtaining of particular solution.

Problem statement

Consider the three-dimensional non-stationary heat conduction problem in the closed parallelepipedic domain $\Omega=\left[0,2\right]\times\left[0,2\right]\times\left[0,0.5\right]$ bounded by Γ. The unsteady heat equation in homogeneous anisotropic solids has the form:

$$\rho c_p\frac{\partial u}{\partial t}=K_{11}\frac{\partial^2 u}{\partial x^2}+K_{22}\frac{\partial^2 u}{\partial y^2}+K_{33}\frac{\partial^2 u}{\partial z^2}+2\left(K_{12}\frac{\partial^2 u}{\partial x\partial y}+K_{13}\frac{\partial^2 u}{\partial x\partial z}+K_{23}\frac{\partial^2 u}{\partial y\partial z}\right)+g$$

where $\rho=1$ – density, $c_p=1$ – specific heat at constant pressure, $u=u(x,y,z,t)$ – temperature, $g=g(x,y,z,t)$ – heat source, $t\in\left[0,2\right]$, $\Delta t=0.01$ – time step, $N=2646$ – the total number of interpolation nodes.

The initial condition is

$$u(x,y,z,0)=0,\quad (x,y,z)\in\Omega$$

The Dirichlet boundary conditions are

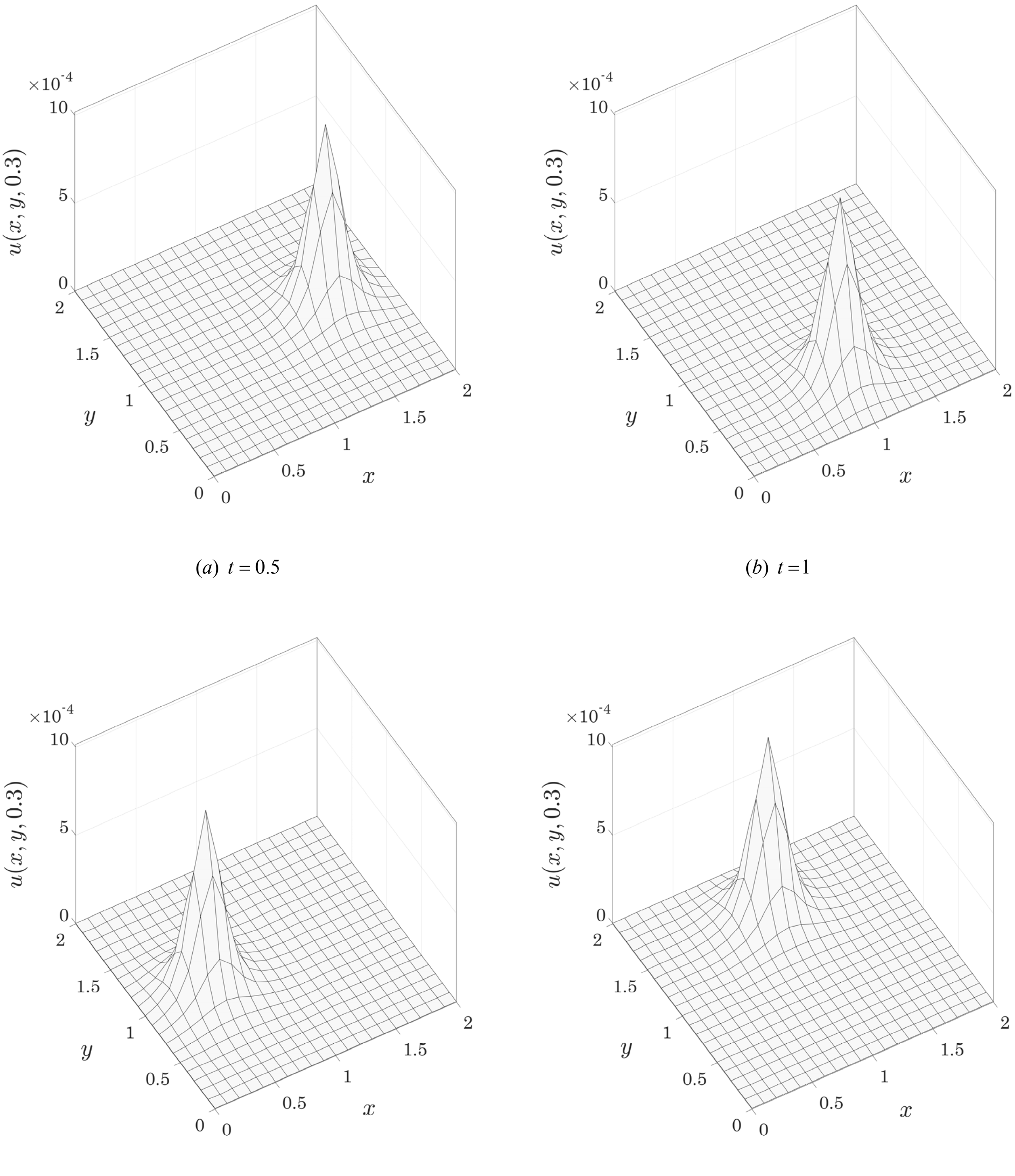
$$u(x,y,z,t)=0,\quad (x,y,z)\in\Gamma$$

Moving heat source is given by the equation:

$$g(x,y,z,t)=\exp\left(-80\left[\left(x-\frac{1}{2}\left(2+\sin(\pi t)\right)\right)^2+\left(y-\frac{1}{2}\left(2+\cos(\pi t)\right)\right)^2\right]\right),\quad (x,y,z)\in\Omega$$

The heat conduction tensor for this boundary-value problem has the form $K=\begin{bmatrix}5&0&0\\0&5&0\\0&0&10\end{bmatrix}$.

Fig. 4 shows the visualization of slices of the numerical solution by the plane $z = 0.3$ at different time moments.



(*a*) $t = 0.5$ (*b*) $t = 1$

(*c*) $t = 1.5$ (*d*) $t = 2$

Figure 4. Visualization of slices of the numerical solution by the plane $z = 0.3$ at different time moments.

The exact solution for this boundary-value problem is:

$$u(x,y,z,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} u_{m,n,k}(t)\sin\left(\frac{\pi m}{l_1}x\right)\sin\left(\frac{\pi n}{l_2}y\right)\sin\left(\frac{\pi k}{l_3}z\right),$$

where $u_{m,n,k}(t)=\int_0^t\left[\frac{8}{l_1 l_2 l_3 \rho c_p}\int_0^{l_1}\int_0^{l_2}\int_0^{l_3} g(\xi,\eta,\zeta,t)\sin\left(\frac{\pi m}{l_1}\xi\right)\sin\left(\frac{\pi n}{l_2}\eta\right)\sin\left(\frac{\pi k}{l_3}\zeta\right)d\xi\, d\eta\, d\zeta\right]\exp\left(-A_{m,n,k}\left(t-\tau\right)\right)d\tau$;

$A_{m,n,k}=a_{11}\left(\frac{\pi m}{l_1}\right)^2+a_{22}\left(\frac{\pi n}{l_2}\right)^2+a_{33}\left(\frac{\pi k}{l_3}\right)^2$; $a_{11}=\frac{K_{11}}{\rho c_p}$, $a_{22}=\frac{K_{22}}{\rho c_p}$, $a_{33}=\frac{K_{33}}{\rho c_p}$; $\{l_i\}_{i=1}^3$ – geometric dimensions of the parallelepipedic domain; $g(\xi,\eta,\zeta,t)$ – heat source.

To estimate the accuracy of the approximation of the numerical solution, the average relative error $rerr(u)$, the average absolute error $aerr(u)$ and the maximum error $merr(u)$ are used, which are calculated by the formulas:

$$rerr(u)=\sqrt{\frac{1}{N}\frac{\sum_{j=1}^{N}(u_j-\tilde{u}_j)^2}{\sum_{j=1}^{N}u_j^2}}$$

$$aerr(u)=\sqrt{\frac{1}{N}\sum_{j=1}^{N}(u_j-\tilde{u}_j)^2}$$

$$merr(u)=\max_j\left|u_j-\tilde{u}_j\right|$$

where u_j and $\tilde{u}_j$ are exact and numerical solutions, respectively.

Table 1 shows the errors of the numerical solution of the boundary-value problem regarding to the exact solution obtained using the atomic radial basis function (ARBF) and multiquadric radial basis function (MQ) at different time moments.

Table 1. Errors of the numerical solution of the boundary-value problem.

Basis function	t	$rerr(u)$	$aerr(u)$	$merr(u)$
ARBF	0.5	$1.20529052 \times 10^{-3}$	$4.44930401 \times 10^{-6}$	$7.16979247 \times 10^{-5}$
	1	$1.16163349 \times 10^{-3}$	$4.28814625 \times 10^{-6}$	$6.90741195 \times 10^{-5}$
	1.5	$1.12912489 \times 10^{-3}$	$4.16814019 \times 10^{-6}$	$6.71658645 \times 10^{-5}$
	2	$1.16305499 \times 10^{-3}$	$4.29339239 \times 10^{-6}$	$6.93488164 \times 10^{-5}$
MQ	0.5	$3.41366269 \times 10^{-3}$	$1.26014623 \times 10^{-5}$	$1.05639294 \times 10^{-4}$
	1	$3.27575942 \times 10^{-3}$	$1.20923987 \times 10^{-5}$	$1.10745011 \times 10^{-4}$
	1.5	$2.98933244 \times 10^{-3}$	$1.10350563 \times 10^{-5}$	$1.04590253 \times 10^{-4}$
	2	$2.99707792 \times 10^{-3}$	$1.10636484 \times 10^{-5}$	$1.07216618 \times 10^{-4}$

CONCLUSIONS

This paper presents the algorithm for constructing family of the atomic radial basis functions of three independent variables generated by Helmholtz-type operator. The functions $AHorp_k(x_1,x_2,x_3)$ extend the subclass of functions used as basis in the implementation of meshless methods for solving boundary-value problems in anisotropic solids. The efficiency of using atomic functions as basis functions is demonstrated by the benchmark problem, for which the average relative, average absolute and maximum errors were calculated. It should be noted that the shape parameter k of the functions $AHorp_k(x_1,x_2,x_3)$ allows varying the size of the support and may be adjusted in the process of solving the boundary-value problem. Increase of the parameter k leads to decrease of the size of the support of the basis function and increase of the sparsity of the interpolation matrix. The choice of the optimal shape parameter k remains the subject for future research.

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СІМЕЙСТВО АТОМАРНИХ РАДІАЛЬНИХ БАЗИСНИХ ФУНКЦІЙ ТРЬОХ НЕЗАЛЕЖНИХ ЗМІННИХ, ЯКІ ПОРОДЖУЮТЬСЯ ОПЕРАТОРОМ ТИПУ ГЕЛЬМГОЛЬЦА

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У статті представлено алгоритм побудови сімейства атомарних радіальних базисних функцій трьох незалежних змінних $AHorp_k(x_1,x_2,x_3)$, що породжуються оператором типу Гельмгольца, які використовуються в якості базисних при реалізації безсіткових методів розв’язку крайових задач в анізотропних твердих тілах. Рівняння типу Гельмгольца відіграють значну роль в математичній фізиці завдяки додаткам, в яких вони виникають. Зокрема, рівняння теплопровідності для анізотропних твердих тіл в процесі чисельного розв’язку зводиться до рівняння, яке містить диференціальний оператор спеціального виду (оператор типу Гельмгольца), який включає в себе компоненти тензора другого рангу, що визначає анізотропію матеріалу. Сімейство атомарних радіальних базисних функцій $AHorp_k(x_1,x_2,x_3)$ є нескінченно диференційованими фінітними розв’язками функціонально-диференціального рівняння спеціального виду. Вибір фінітних функцій в якості базисних дає можливість розглядати крайові задачі на областях зі складною геометричною конфігурацією. Функції $AHorp_k(x_1,x_2,x_3)$ містять параметр форми k, який дозволяє варіювати розмір носія та може уточнюватися в процесі розв’язку крайової задачі. Отримано явні формули для обчислення функцій $AHorp_k(x_1,x_2,x_3)$ та їх перетворення Фур’є. В роботі представлені візуалізації атомарних функцій $AHorp_k(x_1,x_2,x_3)$ та їх перших похідних за змінними x_1 і x_2 при фіксованому значенні змінної $x_3 = 0$ для ізотропного та анізотропного випадків. Ефективність використання атомарних функцій $AHorp_k(x_1,x_2,x_3)$ в якості базисних демонструється на прикладі тривимірної нестаціонарної задачі теплопровідності з рухомим джерелом тепла. Наведено результати чисельного розв’язку тестової крайової задачі, а також обчислені середня відносна, середня абсолютна і максимальна похибки наближених розв’язків, які отримані за допомогою атомарних радіальних базисних функцій $AHorp_k(x_1,x_2,x_3)$ та мультиквадратичних радіальних базисних функцій.

Ключові слова: атомарна радіальна базисна функція, оператор типу Гельмгольца, безсіткові методи, крайові задачі, анізотропна теплопровідність.