UDC 539.12

POLYADIC SYSTEMS, REPRESENTATIONS AND QUANTUM GROUPS

S.A. Duplij*
Center for Mathematics, Science and Education Rutgers University,
118 Frelinghuysen Rd., Piscataway, NJ 08854-8019
E-mail: duplij@math.rutgers.edu, http://homepages.spa.umn.edu/˜duplij
Received 20 May 2012

A review of polyadic systems and their representations is given. The classification of general polyadic systems is done. The multiplace generalization of homomorphisms, preserving associativity, is presented. The multiplace representations and multiactions are defined, concrete examples of matrix representations for some ternary groups are given. The ternary algebras and Hopf algebras are defined, their properties are studied. At the end some ternary generalizations of quantum groups and the Yang-Baxter equation are presented.
KEY WORDS: n-ary group, Post theorem, commutativity, homomorphism, group action, Yang-Baxter equation

ПОЛИАДИЧЕСКИЕ СИСТЕМЫ, ПРЕДСТАВЛЕНИЯ И КВАНТОВЫЕ ГРУППЫ

С.А. Дуплий
Центр математики, науки и образования, университет Ратгерса, Пискатавэй, 08854-8019, США

Приведен обзор полиадических систем и их представлений, дана классификация общих полиадических систем. Построены многоместные обобщения гомоморфизмов, сохраняющие ассоциативность. Определены мультидействия и мультиместные представления. Приведены конкретные примеры матричных представлений для некоторых тернарных групп. Определены тернарные алгебры и Хопф алгебры, изучены их свойства. В заключение, предствлены некоторые тернарные обобщения квантовых групп и уравнения Янга-Бакстера.
КЛЮЧЕВЫЕ СЛОВА: n-арная группа, теорема Поста, коммутативность, гомоморфизм, групповое действие, уравнение Янга-Бакстера

ПОЛIАДИЧНI СИСТЕМИ, ПРЕДСТАВЛЕННЯ I КВАНТОВI ГРУПИ

С.А. Дуплiй
Центр математики, науки та освiти, унiверситет Ратгерсу, Пiскатавей, 08854-8019, США

Зроблено огляд полiадичних систем та їх представлень, дана класифiкацiя загальних полiадичних систем. Побудованi багатомiснi узагальнення гомоморфiзмив, що зберiгають асоцiативнiсть. Визначенi мультидiї i мультимiснi представлення. Наведенi конкретнi приклади матричних представлень для деяких тернарних груп. Визначенi тернарна алгебра i алгебри Хопфа, вивченi їх властивостi. На закiнчення, предствленi деякi тернарнi узагальнення квантових груп та рiвняння Янга-Бакстера.
КЛЮЧОВI СЛОВА: n-арна група, теорема Поста, комутативнiсть, гомоморфiзм, групова дiя, рiвняння Янга-Бакстера

One of the most promising steps in generalizing physical theories is consideration of higher arity algebras [1], in other words ternary and n-ary algebras, in which the binary composition law is substituted by ternary or n-ary one [2].

Firstly ternary algebraic operations (with the arity $n = 3$) were introduced already in the XIX-th century by A. Cayley in 1845 and later by J. J. Silvester in 1883. The notion of an n-ary group was introduced in 1928 by [3] (inspired by E. Nöther) which is a natural generalization of the notion of a group. Even before in 1924, a particular case, that is, ternary group of idempotents, was used in [4] to study infinite abelian groups. The important Post's coset theorem explained the connection between n-ary groups and their covering binary groups [5]. The next step in study of n-ary groups was the Gluskin-Hosszú theorem [6, 7]. Another definition of n-ary group can be given as a universal algebra with additional laws [8] or identities containing special elements [9].

The representation theory of (binary) groups [10, 11] plays an important role in their physical applications [12]. It is initially based on a matrix realization of group elements and abstract group action as a usual matrix multiplication [13, 14]. The cubic and n-ary generalizations of matrices and determinants were made is [15, 16], and their physical application appeared in [17, 18]. In general, particular questions of n-ary group representations were considered in and matrix representations were derived by the author [19], and some general theorems

*On leave of absence from V.N. Karazin Kharkov National University, Svoboda Sq. 4, Kharkov 61022, Ukraine.

connecting representations of binary and n-ary groups were presented in [20]. Here to generalize the above constructions of n-ary group representations to more complicated and nontrivial cases.

In physics the most applicable structures are nonassociative Grassmann, Clifford and Lie algebras [21–23], and so their higher arity generalizations play the key role in further applications. Indeed, the ternary analog of Clifford algebra was considered in [24], the ternary analog of Grassmann algebra [25] was exploited to construct various ternary extensions of supersymmetry [26].

The construction of realistic physical models is based on the Lie algebras, such that the fields take their values in the concrete binary Lie algebra [23]. In the higher arity considerations the standard Lie bracket is replaced by a linear n-ary bracket, and the algebraic structure of the corresponding model is defined by the additional characteristic identity for this generalized bracket, the Jacobi identity [2]. There are two possibilities to construct the generalized Jacobi identity: 1) The Lie bracket is a derivation by itself; 2) A double Lie bracket vanishes, when antisymmetrized with respect of its entries. The first case leads to so called Filippov algebras [27] (or n-Lie algebra) and second case corresponds to generalized Lie algebras [28] (or higher order Lie algebras).

The infinite-dimensional version of n-Lie algebras are the Nambu algebras [29, 30], and their n-bracket is given by the Jacobian determinant of n functions, the Nambu bracket, which in fact satisfies the Filippov identity [27]. Recently, the ternary Filippov algebras were successfully applied to a three-dimensional superconformal gauge theory describing effective worldvolume theory of coincident $M2$-branes of M-theory [31–33]. The infinite-dimensional Nambu bracket realization [34] gave possibility to describe a condensate of nearly coincident $M2$-branes [35].

From another side, Hopf algebras [36–38] play a fundamental role in the quantum group theory [39, 40]. Previously, it was introduced their Von Neumann generalization in [41–43], also their actions on quantum plane were classified in [44], and the ternary Hopf algebras were defined and studied in [45, 46].

The goal of this paper is to give a comprehensive review of polyadic systems and their representations. First, we classify general polyadic systems and introduce n-ary semigroups and groups. Then we consider their homomorphisms and the multiplace generalizations, paying attention on their associativity. We define multiplace representations and multiactions, give examples of matrix representations for some ternary groups. We define and investigate ternary algebras and Hopf algebras, study their properties and give some examples. At the end we consider some ternary generalizations of quantum groups and the Yang-Baxter equation.

PRELIMINARIES

Let G be a non-empty set (underlying set, universe, carrier), its elements we denote by lower-case Latin letters $g_i \in G$. The n-`tuple` (or `polyad`) $g_1, \ldots, g_n$ of elements from G is denoted by $(g_1, \ldots, g_n)$. The Cartesian product[1] $\overbrace{G \times \ldots \times G}^{n} = G^{\times n}$ consists of all n-tuples $(g_1, \ldots, g_n)$, such that $g_i \in G$, $i = 1, \ldots, n$. For all equal elements $g \in G$, we denote n-tuple (polyad) by power (g^n). If the number of elements in the n-tuple is clear from the context or is not important, we denote it in one bold letter $(\boldsymbol{g})$, in other case we use power in brackets $\left(\boldsymbol{g}^{(n)}\right)$. Introduce two important constructions on sets.

The i-`projection` of the Cartesian product $G^{\times n}$ on its i-th "axis" is the map $\mathsf{Pr}_i^{(n)} : G^{\times n} \to G$ such that $(g_1, \ldots g_i, \ldots, g_n) \longmapsto g_i$.

The i-`diagonal` $\mathsf{Diag}_n : G \to G^{\times n}$ sends one element to the equal element n-tuple $g \longmapsto (g^n)$.

The one-point set $\{\bullet\}$ can be treated as a unit for the Cartesian product, since there are bijections between G and with $G \times \{\bullet\}^{\times n}$, where G can be on any place. On the Cartesian product $G^{\times n}$ one can define a polyadic (n-ary, n-adic, if it is necessary to specify n, its arity or rank) operation $\mu_n : G^{\times n} \to G$. For operations we use small Greek letters and place arguments in square brackets $\mu_n [\boldsymbol{g}]$. The operations with $n = 1, 2, 3$ are called `unary, binary and ternary`. The case $n = 0$ is special and corresponds to fixing a distinguished element of G, a "constant" $c \in G$, and it is called a `0-ary operation` $\mu_0^{(c)}$, which maps the one-point set $\{\bullet\}$ to G, such that $\mu_0^{(c)} : \{\bullet\} \to G$, and formally has the value $\mu_0^{(c)} [\{\bullet\}] = c \in G$. The 0-ary operation "kills" arity, which can be seen from the following [47]: the composition of n-ary and m-ary operations $\mu_n \circ \mu_m$ gives $(n + m - 1)$-ary operation by

$$\mu_{n+m-1} [\boldsymbol{g}, \boldsymbol{h}] = \mu_n [\boldsymbol{g}, \mu_m [\boldsymbol{h}]] . \tag{1}$$

Then, if to compose μ_n with the 0-ary operation $\mu_0^{(c)}$, we obtain

$$\mu_{n-1}^{(c)} [\boldsymbol{g}] = \mu_n [\boldsymbol{g}, c] , \tag{2}$$

because $\boldsymbol{g}$ is a polyad of length $(n - 1)$. So, it is needed to make a clear distinction between the 0-ary operation $\mu_0^{(c)}$ and its value c in G, which will be seen and important below.

[1]We place the sign of the Cartesian product ($\times$) into the power, because the same abbreviation will be used below also for other types of products.

A `polyadic system` G is a set G which is closed under polyadic operations.

We will write G = ⟨set |operations ⟩ or G = ⟨set |operations |relations ⟩, where "relations" are some additional properties of operations (e.g., associativity conditions for semigroups or cancellation property). In such definition it is not needed to list the images of 0-ary operations (e.g. unit, zero in groups), as it is done in various other definitions. Here, we mostly consider concrete polyadic systems with one "chief" (fundamental) n-ary operation μ_n, which is called `polyadic multiplication` (or n-`ary multiplication`).

A n-`ary system` $\mathrm{G}_n = \langle G \mid \mu_n \rangle$ is a set G closed under one n-ary operation μ_n (without any other additional structure).

Note that a set with one closed binary operation without any other relations was called a groupoid by Hausmann and Ore [48] (see, also [49]). However, nowadays the term "groupoid" is widely used in the category theory and homotopy theory for a different construction with binary multiplication, the so-called Brandt groupoid [50] (see, also, [51]). Alternatively, and much later on, Bourbaki [52] introduced the term "magma" for binary systems. Then, the above terms were extended to the case of one fundamental n-ary operation as well. Nevertheless, we use some neutral notations "polyadic system" and "n-ary system" (when arity n is fixed/known/important), which adequately indicates all their main properties.

Let us consider the `changing arity problem`:

For a given n-ary system $\langle G \mid \mu_n \rangle$ to construct another polyadic system $\langle G \mid \mu'_{n'} \rangle$ over the same set G, which multiplication has different arity n'.

The formulas (1) and (2) give us the simplest examples, how to change arity of a polyadic system. In general, there are 3 ways:

1. *Iterating*. Using composition of the operation μ_n with itself, one can increase arity from n to n'_{iter} (as in (1)) without changing signature of the system. We denote the number of iterating multiplications by ℓ_μ, and use the bold Greek letters $\boldsymbol{\mu}_n^{\ell_\mu}$ to for the resulting composition of n-ary multiplications, such that
$$\mu'_{n'} = \boldsymbol{\mu}_n^{\ell_\mu} \stackrel{def}{=} \overbrace{\mu_n \circ \left(\mu_n \circ \ldots \left(\mu_n \times \mathrm{id}^{\times(n-1)}\right) \ldots \times \mathrm{id}^{\times(n-1)}\right)}^{\ell_\mu}, \tag{3}$$
where
$$n' = n_{iter} = \ell_\mu \left(n-1\right) + 1, \tag{4}$$
which gives the length of a polyad $(\boldsymbol{g})$ in the notation $\boldsymbol{\mu}_n^{\ell_\mu}[\boldsymbol{g}]$. Without assuming associativity there many variants of placing μ_n's among id's in r.h.s. of (3). The operation $\boldsymbol{\mu}_n^{\ell_\mu}$ is named a `long product` [3] or `derived` [53].

2. *Reducing*. Using n_c distinguished elements or constants (or n_c additional 0-ary operations $\mu_0^{(c_i)}$, $i = 1, \ldots n_c$), one can decrease arity from n to n'_{red} (as in (2)), such that[2]
$$\mu'_{n'} = \mu_{n'}^{(c_1 \ldots c_{n_c})} \stackrel{def}{=} \mu_n \circ \left(\overbrace{\mu_0^{(c_1)} \times \ldots \times \mu_0^{(c_{n_c})}}^{n_c} \times \mathrm{id}^{\times(n-n_c)} \right), \tag{5}$$
where
$$n' = n_{red} = n - n_c, \tag{6}$$
and the 0-ary operations $\mu_0^{(c_i)}$ can be on any places.

3. *Mixing*. Changing (increasing or decreasing) arity by combining the iterating and reducing (maybe with additional operations of different arity). If we do not use additional operations the final arity can be presented in general form using (4) and (6). It will depend on the order of iterating and reducing, so we have two subcases:

   (a) *Iterating→Reducing*. We have
$$n' = n_{iter\to red} = \ell_\mu \left(n-1\right) - n_c + 1. \tag{7}$$
   The maximal number of constants (when $n'_{iter\to red} = 2$) is equal to
$$n_c^{\max} = \ell_\mu \left(n-1\right) - 1 \tag{8}$$
   and can be any by increasing the number of multiplications ℓ_μ.

[2]In [54] $\mu_n^{(c_1 \ldots c_{n_c})}$ is named a retract (which is already busy and widely used in the category theory for another construction).

(b) *Reducing→Iterating.* We obtain

$$n' = n_{red \to iter} = \ell_\mu (n - 1 - n_c) + 1. \tag{9}$$

Now the maximal number of constants is

$$n_c^{\max} = n - 2 \tag{10}$$

and is achieved, only when $\ell_\mu = 1$.

To give examples to the third (mixed) case we put $n = 4$, $\ell_\mu = 3$, $n_c = 2$ for both subcases of opposite ordering:

1. *Iterating→Reducing.* We can put

$$\mu_8^{(c_1,c_2)\prime} \left[\boldsymbol{g}^{(8)} \right] = \mu_4 \left[g_1, g_2, g_3, \mu_4 \left[g_4, g_5, g_6, \mu_4 \left[g_7, g_8, c_1, c_2 \right] \right] \right]. \tag{11}$$

2. *Reducing→Iterating.* We can have

$$\mu_4^{(c_1,c_2)\prime} \left[\boldsymbol{g}^{(4)} \right] = \mu_4 \left[g_1, c_1, c_2, \mu_4 \left[g_2, c_1, c_2, \mu_4 \left[g_3, c_1, c_2, g_4 \right] \right] \right]. \tag{12}$$

It is important to find conditions, when the iterating and reducing compensate each other, i.e. they do not change arity. Indeed, let the number of the iterating multiplications ℓ_μ is fixed, then we can find such number of reducing constants $n_c^{(0)}$, that the final arity will coincide with the initial arity n. The result will depend on the order of operations. There are two cases:

1. *Iterating→Reducing.* For the number of reducing constants $n_c^{(0)}$ we obtain from (4) and (6)

$$n_c^{(0)} = (n - 1)(\ell_\mu - 1), \tag{13}$$

such that there is no restriction on ℓ_μ.

2. *Reducing→Iterating.* For $n_c^{(0)}$ we get

$$n_c^{(0)} = \frac{(n - 1)(\ell_\mu - 1)}{\ell_\mu}, \tag{14}$$

and now $\ell_\mu \leq n - 1$. The requirement that $n_c^{(0)}$ should be integer gives two further possibilities

$$n_c^{(0)} = \begin{cases} \dfrac{n-1}{2}, & \ell_\mu = 2, \\ n - 2, & \ell_\mu = n - 1. \end{cases} \tag{15}$$

The above relations can be useful in the study of various n-ary multiplication structures and their presentation in special form needed in concrete problems.

SPECIAL ELEMENTS AND PROPERTIES OF POLYADIC SYSTEMS

Let us remind definitions of some standard algebraic systems and their special elements, which will be considered in this paper, using our notation.

A `zero` of a polyadic system is a distinguished element z (and the corresponding 0-ary operation $\mu_0^{(z)}$) such that for any $(n-1)$-tuple (polyad) $\boldsymbol{g} \in G^{\times(n-1)}$ we have

$$\mu_n [\boldsymbol{g}, z] = z, \tag{16}$$

where z can be on any place in the l.h.s. of (16).

There is only one zero (if not to fix its place) which can be possible in a polyadic system. As in the binary case, an analog of positive power of an element [5] should coincide with the number of multiplications ℓ_μ in the iterating (3).

A (positive) `polyadic power` of an element is

$$g^{\langle \ell_\mu \rangle} = \boldsymbol{\mu}_n^{\ell_\mu} \left[g^{\ell_\mu (n-1)+1} \right]. \tag{17}$$

An element of a polyadic system g is called ℓ_μ-`nilpotent` (or simply `nilpotent` for $\ell_\mu = 1$), if there exist such ℓ_μ that

$$g^{\langle \ell_\mu \rangle} = z. \tag{18}$$

A polyadic system with zero z is called ℓ_μ-`nilpotent`, if there exist such ℓ_μ that for any $(\ell_\mu (n-1)+1)$-tuple (polyad) $\boldsymbol{g}$ we have

$$\boldsymbol{\mu}_n^{\ell_\mu} [\boldsymbol{g}] = z. \tag{19}$$

Therefore, the `index of nilpotency` (number of elements whose product is zero) of ℓ_μ-nilpotent n-ary system is $(\ell_\mu (n-1)+1)$, while its polyadic power isℓ_μ .

A `polyadic (`n`-ary) identity` (or neutral element) of a polyadic system is a distinguished element e (and the corresponding 0-ary operation $\mu_0^{(e)}$) such that for any element $g \in G$ we have

$$\mu_n \left[g, e^{n-1}\right] = g, \tag{20}$$

where g can be on any place in the l.h.s. of (20).

In binary groups the identity is the only neutral element, while in polyadic systems, there exist `neutral polyads` $\boldsymbol{n}$ consisting of elements of G satisfying

$$\mu_n [g, \boldsymbol{n}] = g, \tag{21}$$

where g can be also on any place. The neutral polyads are determined not uniquely. It follows from (20) that the sequence of polyadic identities e^{n-1} is a neutral polyad.

An element of a polyadic system g is called ℓ_μ-`idempotent` (or simply `idempotent` for $\ell_\mu = 1$), if there exist such ℓ_μ that

$$g^{\langle \ell_\mu \rangle} = g. \tag{22}$$

Both zero and identity are ℓ_μ-idempotents with arbitrary ℓ_μ. We define the `(total) associativity` as invariance of the composition of two n-ary multiplications

$$\boldsymbol{\mu}_n^2 [\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{u}] = \mu_n [\boldsymbol{g}, \mu_n [\boldsymbol{h}], \boldsymbol{u}] = inv \tag{23}$$

under placement of the internal multiplication in r.h.s. with the fixed order of elements in the whole polyad of $(2n-1)$ elements $\boldsymbol{t}^{(2n-1)} = (\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{u})$. Informally, "internal brackets/multiplication can be moved on any place", which gives n relations

$$\mu_n \circ \left(\mu_n \times \mathrm{id}^{\times(n-1)}\right) = \ldots = \mu_n \circ \left(\mathrm{id}^{\times(n-1)} \times \mu_n\right). \tag{24}$$

There are many other particular kinds of associativity which were introduced in [55] and studied in [56,57]. Here we will confine ourselves the most general total associativity (23). In this case, the iterating does not depend of the placement of internal multiplications in the r.h.s of (3).

A `polyadic semigroup` (n`-ary semigroup`) is a n-ary system which operation is associative, or $\mathrm{G}_n^{semigrp} = \langle G \mid \mu_n \mid \text{associativity} \rangle$.

In a polyadic system with zero (16) one can have the `trivial associativity`, when all n terms is (23) are equal to zero, i.e.

$$\boldsymbol{\mu}_n^2 [\boldsymbol{g}] = z \tag{25}$$

for any $(2n-1)$-tuple $\boldsymbol{g}$. Therefore, we state that

Any 2-nilpotent n-ary system (having index of nilpotency $(2n-1)$) is a polyadic semigroup.

In the case of changing arity one should use in (25) not the changed final arity n', but the "real" arity which is n for the reducing case and $\ell_\mu (n-1)+1$ for all other cases. Let us give the examples.

In the mixed (interating-reducing) case with $n = 2$, $\ell_\mu = 3$, $n_c = 1$, we have a ternary system $\langle G \mid \mu_3 \rangle$ iterated from a binary system $\left\langle G \mid \mu_2, \mu_0^{(c)} \right\rangle$ with one distinguished element c (or an additional 0-ary operation)[3]

$$\mu_3^{(c)} [g, h, u] = (g \cdot (h \cdot (u \cdot c))), \tag{26}$$

where for binary multiplication we denote $g \cdot h = \mu_2 [g, h]$. Thus, if the ternary system $\left\langle G \mid \mu_3^{(c)} \right\rangle$ is nilpotent of index 7 (see 9), then it is a ternary semigroup (because $\mu_3^{(c)}$ is trivially associative) independently of associativity of μ_2 (see, e.g. [19]).

It is very important to find the `associativity preserving` conditions (constructions), when associative initial operation μ_n leads to associative final operation $\mu'_{n'}$ during the change of arity.

[3] This construction is named the b-derived groupoid in [54].

The associativity preserving reducing can be given by construction of the binary associative operation using $(n-2)$-tuple $\boldsymbol{c}$ consisting of $n_c = n-2$ different constants

$$\mu_2^{(\boldsymbol{c})}[g,h] = \mu_n[g,\boldsymbol{c},h]. \tag{27}$$

The associativity preserving mixing constructions with different arities and places were considered in [54, 57, 58].

An associative polyadic system with identity (20) is called a `polyadic monoid`.

The structure of any polyadic monoid is fixed [59]: it can be obtained by iterating of a binary operation [60] (for polyadic groups this was shown in [3]).

In polyadic systems, there are several analogs of binary commutativity. The most straightforward one comes from commutation of the multiplication with permutations.

A polyadic system is σ-`commutative`, if $\mu_n = \mu_n \circ \sigma$, or

$$\mu_n[\boldsymbol{g}] = \mu_n[\sigma \circ \boldsymbol{g}], \tag{28}$$

where $\sigma \circ \boldsymbol{g} = \left(g_{\sigma(1)}, \ldots, g_{\sigma(n)}\right)$ is a permutated polyad and σ is a fixed element of S_n, a permutation group of n elements. If (28) holds for all $\sigma \in S_n$, then a polyadic system is `commutative`.

A special type of the σ-commutativity

$$\mu_n[g,\boldsymbol{t},h] = \mu_n[h,\boldsymbol{t},g], \tag{29}$$

where $\boldsymbol{t}$ is any fixed $(n-2)$-polyad, is called `semicommutativity`. So for a n-ary semicommutative system we have

$$\mu_n\left[g,h^{n-1}\right] = \mu_n\left[h^{n-1},g\right]. \tag{30}$$

If a n-ary semigroup $\mathrm{G}^{semigrp}$ is iterated from a commutative binary semigroup with identity, then $\mathrm{G}^{semigrp}$ is semicommutative.

Let G be the set of natural numbers $\mathbb{N}$, and the 5-ary multiplication is defined by

$$\mu_5[\boldsymbol{g}] = g_1 - g_2 + g_3 - g_4 + g_5, \tag{31}$$

then $\mathrm{G}_5^{\mathbb{N}} = \langle \mathbb{N}, \mu_5 \rangle$ is a semicommutative 5-ary monoid having the identity $e_g = \mu_5\left[g^5\right] = g$ for each $g \in \mathbb{N}$. Therefore, $\mathrm{G}_5^{\mathbb{N}}$ is the idempotent monoid.

Another possibility is to generalize the binary `mediality` in semigroups

$$(g_{11} \cdot g_{12}) \cdot (g_{21} \cdot g_{22}) = (g_{11} \cdot g_{21}) \cdot (g_{12} \cdot g_{22}), \tag{32}$$

which, obviously, follows from the binary commutativity. But for n-ary systems they are different. It is seen that the mediality should contain $(n+1)$ multiplications, it is a relation between $n \times n$ elements, and therefore can be presented in a matrix from. The latter can be achieved by placing arguments of the external multiplication as a column.

A polyadic system is `medial` (or `entropic`), if [56, 61]

$$\mu_n\begin{bmatrix}\mu_n[g_{11},\ldots,g_{1n}]\\ \vdots \\ \mu_n[g_{n1},\ldots,g_{nn}]\end{bmatrix} = \mu_n\begin{bmatrix}\mu_n[g_{11},\ldots,g_{n1}]\\ \vdots \\ \mu_n[g_{1n},\ldots,g_{nn}]\end{bmatrix}. \tag{33}$$

For polyadic semigroups we use the notation (3) and can present the mediality as follows

$$\boldsymbol{\mu}_n^n[\boldsymbol{G}] = \boldsymbol{\mu}_n^n\left[\boldsymbol{G}^T\right], \tag{34}$$

where $\boldsymbol{G} = \|g_{ij}\|$ is the $n \times n$ matrix of elements and $\boldsymbol{G}^T$ is its transpose. The semicommutative polyadic semigroups are medial, as in the binary case, but, in general (except $n=3$) not vise versa [62]. A more general concept is σ-permutability [63], such that the mediality is its particular case with $\sigma = (1,n)$.

A polyadic system is `cancellative`, if

$$\mu_n[g,\boldsymbol{t}] = \mu_n[h,\boldsymbol{t}] \Longrightarrow g = h, \tag{35}$$

where g,h can be on any place. This means that the mapping μ_n is one-to-one in each variable. If g,h are on the same i-th place on both sides, the polyadic system is called i-`cancellative`.

The left and right cancellativity are 1-cancellativity and n-cancellativity respectively. A right and left cancellative n-ary semigroup is cancellative (with respect to the same subset).

A polyadic system is called (uniquely) i-solvable, if for all polyads $\boldsymbol{t}$, $\boldsymbol{u}$ and element h, one can (uniquely) resolve the equation (with respect to h) for the fundamental operation

$$\mu_n\left[\boldsymbol{u}, h, \boldsymbol{t}\right] = g \tag{36}$$

where h can be on any i-th place.

A polyadic system which is uniquely i-solvable for *all* places i is called a n-ary (or polyadic) quasigroup.

It follows, that, if (36) uniquely i-solvable for all places, than

$$\boldsymbol{\mu}_n^{\ell_\mu}\left[\boldsymbol{u}, h, \boldsymbol{t}\right] = g \tag{37}$$

can be (uniquely) resolved with respect to h being on any place.

An associative polyadic quasigroup is called a n-ary (or polyadic) group.

The above definition is the most general one, but it is overdetermined. Much work on polyadic groups was done [64] to minimize the set of axioms (solvability not in all places [5, 65], decreasing or increasing number of unknowns in determining equations [66]) or made it in terms of additionally defined objects (various analogs of identity and sequences [67]), as well as using not the total associativity, but various partial ones [57, 68, 69].

In a polyadic group the only solution of (36) is called a querelement of g and denoted by $\bar{g}$ [3], such that

$$\mu_n\left[\boldsymbol{h}, \bar{g}\right] = g, \tag{38}$$

where $\bar{g}$ can be on any place. So, any idempotent g coincides with its querelement $\bar{g} = g$. It follows from (38) and (21), that the polyad

$$\boldsymbol{n}_g = \left(g^{n-2}\bar{g}\right) \tag{39}$$

is neutral for any element of a polyadic group, where $\bar{g}$ can be on any place. If this i-th place is important, then we write $\boldsymbol{n}_{g;i}$. The number of relations in (38) can be reduced from n (number of possible places) to only 2 (when g is on the first and last places [3, 70], or other 2 places). In a polyadic group the Dörnte relations

$$\mu_n\left[g, \boldsymbol{n}_{h;i}\right] = \mu_n\left[\boldsymbol{n}_{h;j}, g\right] = g \tag{40}$$

hold valid for any allowable i, j. In the case of a binary group the relations (40) become $g \cdot h \cdot h^{-1} = h \cdot h^{-1} \cdot g = g$.

The relation (38) can be treated as a definitions of the unary queroperation

$$\bar{\mu}_1\left[g\right] = \bar{g}. \tag{41}$$

A polyadic group is a universal algebra

$$\mathrm{G}_n^{grp} = \left\langle G \mid \mu_n, \bar{\mu}_1 \mid \text{associativity, Dörnte relations} \right\rangle, \tag{42}$$

where μ_n is n-ary associative operation and $\bar{\mu}_1$ is the queroperation.

A straightforward generalization of the queroperation concept and corresponding definitions can be made by substituting in the above formulas (38)–(41) the n-ary multiplication μ_n by the iterating multiplication $\boldsymbol{\mu}_n^{\ell_\mu}$ (3) (cf. [71] for $\ell_\mu = 2$).

Let us define the querpower k of g recursively

$$\bar{g}^{\langle\langle k\rangle\rangle} = \overline{\left(\bar{g}^{\langle\langle k-1\rangle\rangle}\right)}, \tag{43}$$

where $\bar{g}^{\langle\langle 0\rangle\rangle} = g$, $\bar{g}^{\langle\langle 1\rangle\rangle} = \bar{g}$, or as the k composition $\bar{\mu}_1^{\circ k} = \overbrace{\bar{\mu}_1 \circ \bar{\mu}_1 \circ \ldots \circ \bar{\mu}_1}^{k}$ of the queroperation (41).

For instance [66], $\bar{\mu}_1^{\circ 2} = \boldsymbol{\mu}_n^{n-3}$, such that for any ternary group $\bar{\mu}_1^{\circ 2} = \mathrm{id}$, i.e. one has $\bar{\bar{g}} = g$. Using the queroperation in polyadic groups we can define the negative polyadic power of an element g by the following recursive relation

$$\mu_n\left[g^{\langle\ell_\mu - 1\rangle}, g^{n-2}, g^{\langle-\ell_\mu\rangle}\right] = g, \tag{44}$$

or (after use of (17)) as a solution of the equation

$$\boldsymbol{\mu}_n^{\ell_\mu}\left[g^{\ell_\mu(n-1)}, g^{\langle-\ell_\mu\rangle}\right] = g. \tag{45}$$

It is known that the querpower and the polyadic power are mutually connected [72]. Here, we reformulate this connection using the so called Heine numbers [73] or q-deformed numbers [74]

$$\left[\left[k\right]\right]_q = \frac{q^k - 1}{q - 1}, \tag{46}$$

which have the "nondeformed" limit $q \to 1$ as $[k]_q \to k$. Then

$$\bar{g}^{\langle\langle k\rangle\rangle} = g^{\left\langle -[[k]]_{2-n}\right\rangle}, \tag{47}$$

which can be treated as follows: the querpower coincides with the negative polyadic deformed power with the "deformation" parameter q which is equal to the "deviation" $(2-n)$ from the binary group.

HOMOMORPHISMS OF POLYADIC SYSTEMS

Let $\mathrm{G}_n = \langle G; \mu_n\rangle$ and $\mathrm{G}'_{n'} = \langle G'; \mu'_{n'}\rangle$ be two polyadic systems of any kind (quasigroup, semigroup, group, etc.). If they have the multiplications of the same arity $n = n'$, then one can define the mappings from G_n to G'_n. Usually such polyadic systems are similar, and we call mappings between them the equiary mappings.

Let us take $n+1$ mappings $\varphi_i^{GG'} : G \to G'$, $i = 1,\dots,n+1$. An ordered system of mappings $\left\{\varphi_i^{GG'}\right\}$ is called a homotopy from G_n to G'_n, if

$$\varphi_{n+1}^{GG'}\left(\mu_n\left[g_1,\dots,g_n\right]\right) = \mu'_n\left[\varphi_1^{GG'}\left(g_1\right),\dots,\varphi_n^{GG'}\left(g_n\right)\right], \qquad g_i \in G. \tag{48}$$

In general, one should add to this definition the "mapping" of the multiplications

$$\mu_n \overset{\psi_{nn'}^{(\mu\mu')}}{\mapsto} \mu'_{n'}. \tag{49}$$

In such a way, the homotopy can be defined as the extended system of mappings $\left\{\varphi_i^{GG'};\psi_{nn}^{(\mu\mu')}\right\}$.

The existence of the additional "mapping" $\psi_{nn}^{(\mu\mu')}$ acting on the second component of $\langle G; \mu_n\rangle$ is tacitly implied. We will write/mention the "mappings" $\psi_{nn'}^{(\mu\mu')}$ manifestly, e.g.,

$$\mathrm{G}_n \overset{\left\{\varphi_i^{GG'};\psi_{nn}^{(\mu\mu')}\right\}}{\rightrightarrows} \mathrm{G}'_{n'}, \tag{50}$$

as needed only. If all the components $\varphi_i^{GG'}$ of a homotopy are bijections, it is called an isotopy. In case of polyadic quasigroups [56] all mappings $\varphi_i^{GG'}$ are usually taken as permutations of the same set $G = G'$. If the multiplications are also coincide $\mu_n = \mu'_n$, then $\left\{\varphi_i^{GG}; \mathrm{id}\right\}$ is called an autotopy of the polyadic system G_n. Various properties of the homotopy in universal algebras were studied, e.g. in [75, 76].

A homomorphism from G_n to G'_n is given, if there exists a mapping $\varphi^{GG'} : G \to G'$ satisfying

$$\varphi^{GG'}\left(\mu_n\left[g_1,\dots,g_n\right]\right) = \mu'_n\left[\varphi^{GG'}\left(g_1\right),\dots,\varphi^{GG'}\left(g_n\right)\right], \qquad g_i \in G. \tag{51}$$

Usually the homomorphism is denoted by the same one letter $\varphi^{GG'}$, while it would be more consistently to use for its notation the extended pair of mappings $\left\{\varphi^{GG'};\psi_{nn}^{(\mu\mu')}\right\}$. We will use both notations on a par.

We, first, mention some small subset of known generalizations of the homomorphism (for bibliography till 1982 see, e.g., [77]) and then introduce a concrete construction for an analogous mapping which can change arity of the multiplication (fundamental operation) without introducing additional (term) operations. A general approach to mappings between free algebraic systems was initiated in [78], where the so-called basic mapping formulas for generators were introduced, and its generalization to many-sorted algebras was given in [79]. In [80] it was shown that the construction of all homomorphisms between similar polyadic systems can be reduced to some homomorphisms between corresponding mono-unary algebras [81]. The notion of the n-ary homomorphism as a sequence of n consequent homomorphisms φ_i, $i = 1,\dots,n$, of n similar polyadic systems

$$\overbrace{\mathrm{G}_n \overset{\varphi_1}{\to} \mathrm{G}'_n \overset{\varphi_2}{\to} \dots \overset{\varphi_{n-1}}{\to} \mathrm{G}''_n \overset{\varphi_n}{\to} \mathrm{G}'''_n}^{n} \tag{52}$$

(generalizing the Post n-adic substitutions [5]) was introduced in [82], and studied in [83, 84].

The above constructions do not change arity of polyadic systems, because they are based on the corresponding diagram which is a definition of an equiary mapping. To change arity one has to:

1) add another equiary diagram with additional operations using the same formula (51), both do not change arity;

2) use one modified (and not equiary) diagram and the underlying formula (51) by themselves, which will allow us to change arity without introducing additional operations.

The first way leads the concept of the `weak homomorphism` which was introduced in [85–87] for non-indexed algebras and in [88] for indexed algebras, then developed in [89] for Boolean and Post algebras, in [90] for coalgebras and F-algebras [91] (see also [92]). To define the weak homomorphism in our notation we should incorporate into the polyadic systems $\langle G;\mu_n\rangle$ and $\langle G';\mu'_{n'}\rangle$ the following additional term operations of opposite arity $\nu_{n'}: G^{\times n'} \to G$ and $\nu'_n: G'^{\times n} \to G'$ and consider `two` equiary mappings between $\langle G;\mu_n,\nu_{n'}\rangle$ and $\langle G';\mu'_{n'},\nu'_n\rangle$.

A `weak homomorphism` from $\langle G;\mu_n,\nu_{n'}\rangle$ to $\langle G',\mu'_{n'},\nu'_n\rangle$ is given, if there exists a mapping $\varphi^{GG'}: G \to G'$ satisfying `two` relations simultaneously

$$\varphi^{GG'}\left(\mu_n\left[g_1,\ldots,g_n\right]\right)=\nu'_n\left[\varphi^{GG'}\left(g_1\right),\ldots,\varphi^{GG'}\left(g_n\right)\right], \tag{53}$$

$$\varphi^{GG'}\left(\nu_{n'}\left[g_1,\ldots,g_{n'}\right]\right)=\mu'_{n'}\left[\varphi^{GG'}\left(g_1\right),\ldots,\varphi^{GG'}\left(g_{n'}\right)\right]. \tag{54}$$

If only one of the relations (53) or (54) takes place, such a mapping is called a `semi-weak homomorphism` [93]. If $\varphi^{GG'}$ is bijective, then it defines a `weak isomorphism`. Any weak epimorphism can be decomposed into a homomorphism and a weak isomorphism [94], therefore the study of weak homomorphisms reduces to weak isomorphisms (see also [95–97]).

MULTIPLACE MAPPINGS OF POLYADIC SYSTEMS

Let us turn to the second way of changing arity of the multiplication and use only one relation which we modify in some natural manner. First, recall that in any set G there always exists the additional distinguished mapping, viz. the identity id_G. We use the multiplication μ_n with its combination of id_G. We define an (`$\ell_{\mathrm{id}}$-intact`) `id-product` for the polyadic system $\langle G;\mu_n\rangle$ as

$$\mu_n^{(\ell_{\mathrm{id}})}=\mu_n\times\left(\mathrm{id}_G\right)^{\times\ell_{\mathrm{id}}}, \tag{55}$$

$$\mu_n^{(\ell_{\mathrm{id}})}: G^{\times(n+\ell_{\mathrm{id}})}\to G^{\times(1+\ell_{\mathrm{id}})}. \tag{56}$$

To indicate the exact i-th place of μ_n in r.h.s. of (55), we write $\mu_n^{(\ell_{\mathrm{id}})}(i)$, as needed. Here we use the id-product to generalize the homomorphism and consider mappings between polyadic systems of different arity. It follows from (56) that, if the image of the id-product is G alone, than $\ell_{\mathrm{id}}=0$. Let us introduce a `multiplace mapping` $\Phi_k^{(n,n')}$ acting as follows

$$\Phi_k^{(n,n')}: G^{\times k}\to G'. \tag{57}$$

We are allowed to take only one upper $\Phi_k^{(n,n')}$, because of one G' in the upper right corner. Since we already know that the lower right corner is exactly $G'^{\times n'}$ (as a pre-image of one multiplication $\mu'_{n'}$), the lower horizontal arrow should be a product of n' multiplace mappings $\Phi_k^{(n,n')}$. So we can write a definition of a multiplace analog of the homomorphism which changes arity of the multiplication using one relation.

A k-`place heteromorphism` from G_n to $\mathrm{G}'_{n'}$ is given, if there exists a k-place mapping $\Phi_k^{(n,n')}$ (57) such that the corresponding defining equation (a modification of (51)) depends of the place i of μ_n in (55). For $i=1$ it can read as

$$\Phi_k^{(n,n')}\begin{pmatrix}\mu_n\left[g_1,\ldots,g_n\right]\\ g_{n+1}\\ \vdots\\ g_{n+\ell_{\mathrm{id}}}\end{pmatrix}=\mu'_{n'}\left[\Phi_k^{(n,n')}\begin{pmatrix}g_1\\ \vdots\\ g_k\end{pmatrix},\ldots,\Phi_k^{(n,n')}\begin{pmatrix}g_{k(n'-1)}\\ \vdots\\ g_{kn'}\end{pmatrix}\right]. \tag{58}$$

This notion is motivated by [98, 99], where mappings between objects from different categories were considered and called chimera morphisms. See, also, [100].

In the particular case $n=3$, $n'=2$, $k=2$, $\ell_{\mathrm{id}}=1$ we have

$$\Phi_2^{(3,2)}\begin{pmatrix}\mu_3\left[g_1,g_2,g_3\right]\\ g_4\end{pmatrix}=\mu'_2\left[\Phi_2^{(3,2)}\begin{pmatrix}g_1\\ g_2\end{pmatrix},\Phi_2^{(3,2)}\begin{pmatrix}g_3\\ g_4\end{pmatrix}\right]. \tag{59}$$

This formula was used in the construction of the bi-element representations of ternary groups [19]. Consider the example.

Let $G = M_2^{adiag}(\mathbb{K})$, a set of antidiagonal 2×2 matrices over the field $\mathbb{K}$ and $G' = \mathbb{K}$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{H}$. The ternary multiplication μ_3 is a product of 3 matrices. Obviously, μ_3 is nonderived. For the elements $g_i = \begin{pmatrix} 0 & a_i \\ b_i & 0 \end{pmatrix}$, $i = 1, 2$, we construct a 2-place mapping $G \times G \to G'$ as

$$\Phi_2^{(3,2)} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = a_1 a_2 b_1 b_2, \tag{60}$$

which satisfies (59). We can introduce a standard 1-place mapping by $\varphi(g_i) = a_i b_i$. It is important to note, that $\varphi(g_i)$ satisfies (51) for a commutative field $\mathbb{K}$ only ($= \mathbb{R}, \mathbb{C}$) becoming a homomorphism, and in this case we can have the relation between the heteromorhism $\Phi_2^{(3,2)}$ and the standard homomorphism

$$\Phi_2^{(3,2)} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \varphi(g_1) \cdot \varphi(g_2), \tag{61}$$

where the product $(\cdot)$ in l.h.s. is taken in $\mathbb{K}$, such that (51) and (59) coincide. For noncommutative field $\mathbb{K}$ ($= \mathbb{Q}$ or $\mathbb{H}$) we can define the heteromorphism (60) only.

A heteromorphism is called `derived`, if it can be expressed through ordinary (1-place) homomorphism. So, in the above example the heteromorphism is derived (by formula (61)) for a commutative field $\mathbb{K}$ and nonderived for a noncommutative one.

For arbitrary n a slightly modified construction (59) with still binary final arity, defined by $n' = 2$, $k = n-1$, $\ell_{\mathrm{id}} = n-2$,

$$\Phi_{n-1}^{(n,2)} \begin{pmatrix} \mu_n[g_1, \dots, g_{n-1}, h_1] \\ h_2 \\ \vdots \\ h_{n-1} \end{pmatrix} = \mu_2' \left[\Phi_{n-1}^{(n,2)} \begin{pmatrix} g_1 \\ \vdots \\ g_{n-1} \end{pmatrix}, \Phi_{n-1}^{(n,2)} \begin{pmatrix} h_1 \\ \vdots \\ h_{n-1} \end{pmatrix} \right]. \tag{62}$$

was used in [53] to study representations of n-ary groups. However, no new results compared with [19] (other than changing 3 to n in some formulas) were obtained. This reflects the fact that a major role play the final arity n' and number of n-ary multiplications in l.h.s. of (59) and (62). In the above cases, the latter number was one, but can make it arbitrary below n.

A heteromorphism is called a `$\ell_\mu$-ple heteromorphism`, if it contains ℓ_μ multiplications in the argument of $\Phi_k^{(n,n')}$ in its defining relation. According this definition the mapping defined by (58) is the 1_μ-ple heteromorphism. So by analogy with (55)–(56) we define a `$\ell_\mu$-ple $\ell_{\mathrm{id}}$-intact id-product` for the polyadic system $\langle G; \mu_n \rangle$ as

$$\mu_n^{(\ell_\mu, \ell_{\mathrm{id}})} = (\mu_n)^{\times \ell_\mu} \times (\mathrm{id}_G)^{\times \ell_{\mathrm{id}}}, \tag{63}$$

$$\mu_n^{(\ell_\mu, \ell_{\mathrm{id}})} : G^{\times(n\ell_\mu + \ell_{\mathrm{id}})} \to G^{\times(\ell_\mu + \ell_{\mathrm{id}})}. \tag{64}$$

A `$\ell_\mu$-ple $k$-place heteromorphism` from G_n to $\mathrm{G}'_{n'}$ is given, if there exists a k-place mapping $\Phi_k^{(n,n')}$ (57). The corresponding `main heteromorphism equation` is

$$\Phi_k^{(n,n')} \begin{pmatrix} \left.\begin{matrix} \mu_n[g_1, \dots, g_n], \\ \vdots \\ \mu_n\left[g_{n(\ell_\mu - 1)}, \dots, g_{n\ell_\mu}\right] \end{matrix}\right\} \ell_\mu \\ \left.\begin{matrix} g_{n\ell_\mu + 1}, \\ \vdots \\ g_{n\ell_\mu + \ell_{\mathrm{id}}} \end{matrix}\right\} \ell_{\mathrm{id}} \end{pmatrix} = \mu'_{n'} \left[\Phi_k^{(n,n')} \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix}, \dots, \Phi_k^{(n,n')} \begin{pmatrix} g_{k(n'-1)} \\ \vdots \\ g_{kn'} \end{pmatrix} \right]. \tag{65}$$

Obviously, we can consider various permutations of the multiplications in both sides, as further additional demands (associativity, commutativity, etc.), which will be considered below. The system of equation connecting initial and final arities is

$$kn' = n\ell_\mu + \ell_{\mathrm{id}}, \tag{66}$$

$$k = \ell_\mu + \ell_{\mathrm{id}}. \tag{67}$$

Excluding ℓ_μ or ℓ_{id}, we obtain two `arity changing formulas`, respectively

$$n' = n - \frac{n-1}{k} \ell_{\mathrm{id}}, \tag{68}$$

$$n' = \frac{n-1}{k} \ell_\mu + 1, \tag{69}$$

where $\frac{n-1}{k}\ell_{\mathrm{id}} \geq 1$ and $\frac{n-1}{k}\ell_{\mu} \geq 1$ are integer. The following inequalities hold valid

$$1 \leq \ell_\mu \leq k, \tag{70}$$
$$0 \leq \ell_{\mathrm{id}} \leq k-1, \tag{71}$$
$$\ell_\mu \leq k \leq (n-1)\,\ell_\mu, \tag{72}$$
$$2 \leq n' \leq n, \tag{73}$$

which are important for further classification of heteromorphisms. The main statement follows from (73):

The heteromorphism $\Phi_k^{(n,n')}$ defined by the relation (65) decreases arity of the multiplication.

Another important observation is the fact that only the id-product (63) with $\ell_{\mathrm{id}} \neq 0$ leads to change of the arity. In the extreme case, when k approaches its minimum, $k = k_{\min} = \ell_\mu$, the final arity approaches its maximum $n'_{\max} = n$, and the id-rpoduct becomes a product of ℓ_μ initial multiplications μ_n without id's, since now $\ell_{\mathrm{id}} = 0$ in (65). Therefore, we call a heteromorphism defined by (65) with $\ell_{\mathrm{id}} = 0$ `a` $k\,(=\ell_\mu)$`-place homomorphism`. The ordinary homomorphism (48) corresponds to $k = \ell_\mu = 1$, and so it is really the 1-place homomorphism. An opposite extreme case, when the final arity approaches its minimum $n'_{\min} = 2$ (the final operation is binary), corresponds to the maximal value of k, that is $k = k_{\max} = (n-1)\,\ell_\mu$. The number of id's now is $\ell_{\mathrm{id}} = (n-2)\,\ell_\mu \geq 0$, which vanishes, when the initial operation is binary as well. This is the case of the ordinary homomorphism (48) for both binary operations $n' = n = 2$ and $k = \ell_\mu = 1$. We conclude that:

Any polyadic system can be mapped into a binary system by means of the special k-place ℓ_μ-ple heteromorphism $\Phi_k^{(n,n')}$, where $k = (n-1)\,\ell_\mu$ (we call it a `binarizing heteromorphism`) which is defined by (65) with $\ell_{\mathrm{id}} = (n-2)\,\ell_\mu$.

In the relation with the Gluskin-Hosszú theorem [7] (any n-ary group can be constructed by the special binary group and its homomorphism) our statement can be treated as an opposite one: *any n-ary system (not only a group) can be mapped into a binary system, but now using another construction, that is, the binarizing heteromorphism.*

The case of 1-ple binarizing heteromorphism ($\ell_\mu = 1$) corresponds to the formula (62). Further requirements (associativity, commutativity, etc.) will give additional relations between multiplications and $\Phi_k^{(n,n')}$, and fix the exact structure of (65). Thus, we arrive to the following

Classification of ℓ_μ-ple heteromorphisms:

1. $n' = n'_{\max} = n \Longrightarrow \Phi_k^{(n,n)}$ *is the ℓ_μ-place or* `multiplace homomorphism`*, i.e.,*

$$k = k_{\min} = \ell_\mu. \tag{74}$$

2. $2 < n' < n \Longrightarrow \Phi_k^{(n,n')}$ *is the* `intermediate heteromorphism` *with*

$$k = \frac{n-1}{n'-1}\ell_\mu. \tag{75}$$

In this case the number of intact elements is proportional to the number of multiplications

$$\ell_{\mathrm{id}} = \frac{n-n'}{n'-1}\ell_\mu. \tag{76}$$

3. $n' = n'_{\min} = 2 \Longrightarrow \Phi_k^{(n,2)}$ *is the $(n-1)\,\ell_\mu$-place (multiplace)* `binarizing heteromorphism`*, i.e.,*

$$k = k_{\max} = (n-1)\,\ell_\mu. \tag{77}$$

In the extreme (first and third) cases there are no restrictions on the initial arity n, while in the intermediate case n is "quantized" due to the fact that fractions in (68) and (69) should be integer. Thus, we have established a general structure and classification of heteromorphisms defined by (65). The next important issue is preservation of special properties (associativity, commutativity, etc.), while passing from μ_n to $\mu'_{n'}$, which will further restrict the concrete shape of the main relation (65) for each choice of the `heteromorphism parameters`: arities n, n', places k, number of intacts ℓ_{id} and multiplications ℓ_μ.

ASSOCIATIVITY AND HETEROMORPHISMS

The most important property of the heteromorphism, which is needed for its next applications in the representation theory, is associativity of the final operation $\mu'_{n'}$, when the initial operation μ_n is associative. In other words, here we consider the concrete form of semigroup heteromorphisms. In general, this is a complicated task, because it is not clear from (65), which permutation in l.h.s. should be taken to get an associative product in its r.h.s. for each set of the heteromorphism parameters. Straightforward checking of the associativity of the final operation $\mu'_{n'}$ for each permutation in l.h.s. of (65) is almost impossible, especially for higher n. To solve this difficulty we introduce the concept of the associative polyadic quiver and special rules to construct the associative final operation $\mu'_{n'}$.

A `polyadic quiver of product` is the set of elements from G_n (presented as several copies of some matrix of the elements glued together) and arrows, such that the elements along arrows form n-ary product μ_n. For instance, for the multiplication $\mu_4[g_1,h_2,g_2,u_1]$ the 4-adic quiver is denoted by $\{g_1 \to h_2 \to g_2 \to u_1\}$. Here the elements from G_n are arbitrary and have no connection with heteromorphisms.

Next we define `polyadic quivers` which are related to the main heteromorphism equation (65) in the following way: 1) the matrix of elements is the transposed matrix from r.h.s. of (65), such that different letters correspond to their place in $\Phi_k^{(n,n')}$ and the low index of an element is related to its position in the $\mu'_{n'}$ product; 2) the number of polyadic quivers is ℓ_μ, which corresponds to ℓ_μ multiplications in the l.h.s. of (65); 3) the heteromorphism parameters (n, n', k, ℓ_{id} and ℓ_μ) are not arbitrary, but satisfy the arity changing formulas (68)-(69). In this way, a polyadic quiver makes clear visualization of the main heteromorphism equation (65); 4) The intact elements will be placed after semicolon.

For example, the polyadic quiver $\{g_1 \to h_2 \to g_2 \to u_1; h_1, u_2\}$ corresponds to the heteromorphism with $n=4$, $n'=2$, $k=3$, $\ell_{\mathrm{id}}=2$ and $\ell_\mu=1$ is

$$\Phi_3^{(4,2)}\begin{pmatrix}\mu_4[g_1,h_2,g_2,u_1]\\ h_1\\ u_2\end{pmatrix} = \mu'_2\left[\Phi_3^{(4,2)}\begin{pmatrix}g_1\\ h_1\\ u_1\end{pmatrix}, \Phi_3^{(4,2)}\begin{pmatrix}g_2\\ h_2\\ u_2\end{pmatrix}\right]. \tag{78}$$

As it is seen from (78), the product μ'_2 is not associative, if μ_4 is associative. So, not all polyadic quivers preserve associativity.

An `associative polyadic quiver` is a polyadic quiver which ensures the final associativity of $\mu'_{n'}$ in the main heteromorphism equation (65), when the initial multiplication μ_n is associative.

So, one of the associative polyadic quivers which corresponds to the same heteromorphism parameters as the non-associative quiver (78) is $\{g_1 \to h_2 \to u_1 \to g_2; h_1, u_2\}$ which corresponds to

$$\Phi_3^{(4,2)}\begin{pmatrix}\mu_4[g_1,h_2,u_1,g_2]\\ h_1\\ u_2\end{pmatrix} = \mu'_2\left[\Phi_3^{(4,2)}\begin{pmatrix}g_1\\ h_1\\ u_1\end{pmatrix}, \Phi_3^{(4,2)}\begin{pmatrix}g_2\\ h_2\\ u_2\end{pmatrix}\right]. \tag{79}$$

We propose the classification of associative polyadic quivers and the rules of construction of the corresponding heteromorphism equations, and use for the heteromorphism parameters the classification of ℓ_μ-ple heteromorphisms (75). In other words, we describe the consistent procedure of building the semigroup heteromorphisms.

Let us consider the first class of heteromorphisms (without intact elements $\ell_{\mathrm{id}}=0$ or `intactless`), that is ℓ_μ-place (multiplace) homomorphisms. In the simplest case, associativity can be reached, when all elements in a product are taken from the same row. The number of places k is not fixed by the arity relation (68) and can be arbitrary, while the arrows can have various directions. There are 2^k such combinations which preserve associativity. If the arrows have the same direction, this kind of mapping is also called homomorphism. As an example, for $n=n'=3$, $k=2$, $\ell_\mu=2$ we have

$$\Phi_2^{(3,3)}\begin{pmatrix}\mu_3[g_1,g_2,g_3]\\ \mu_3[h_1,h_2,h_3]\end{pmatrix} = \mu'_3\left[\Phi_2^{(3,3)}\begin{pmatrix}g_1\\ h_1\end{pmatrix}, \Phi_2^{(3,3)}\begin{pmatrix}g_2\\ h_2\end{pmatrix}, \Phi_2^{(3,3)}\begin{pmatrix}g_3\\ h_3\end{pmatrix}\right]. \tag{80}$$

Note that the analogous quiver with opposite arrow directions is

$$\Phi_2^{(3,3)}\begin{pmatrix}\mu_3[g_1,g_2,g_3]\\ \mu_3[h_3,h_2,h_1]\end{pmatrix} = \mu'_3\left[\Phi_2^{(3,3)}\begin{pmatrix}g_1\\ h_1\end{pmatrix}, \Phi_2^{(3,3)}\begin{pmatrix}g_2\\ h_2\end{pmatrix}, \Phi_2^{(3,3)}\begin{pmatrix}g_3\\ h_3\end{pmatrix}\right]. \tag{81}$$

The latter mapping was used in constructing the middle representations of ternary groups [19].

The important class of intactless heteromorphisms (with $\ell_{\mathrm{id}}=0$) preserving associativity can be constructed using analogy with the Post substitutions [5], and therefore we call it the `Post-like associative quiver`.

The number of places k is now fixed by $k = n-1$, while $n' = n$ and $\ell_\mu = k = n-1$. An example of the Post-like associative quiver with the same heteromorphisms parameters as in (80)-(81) is

$$\Phi_2^{(3,3)} \begin{pmatrix} \mu_3 [g_1, h_2, g_3] \\ \mu_3 [h_1, g_2, h_3] \end{pmatrix} = \mu_3' \left[\Phi_2^{(3,3)} \begin{pmatrix} g_1 \\ h_1 \end{pmatrix}, \Phi_2^{(3,3)} \begin{pmatrix} g_2 \\ h_2 \end{pmatrix}, \Phi_2^{(3,3)} \begin{pmatrix} g_3 \\ h_3 \end{pmatrix} \right]. \tag{82}$$

This construction appeared in the study of ternary semigroups of morphisms [101–103]. Its n-ary generalization was used in consideration of polyadic operations on Cartesian powers [104], polyadic analogs of the Cayley and Birkhoff theorems [84, 105] and special representations of n-groups [106, 107] (where the n-group with the multiplication μ_2' was called the `diagonal n-group`). Consider the example.

Let Λ be the Grassmann algebra consisting of even and odd parts $\Lambda = \Lambda_{\bar 0} \oplus \Lambda_{\bar 1}$ (see e.g., [108]). The odd part can be considered as a ternary semigroup $\mathrm{G}_3^{(\bar 1)} = \langle \Lambda_{\bar 1}, \mu_3 \rangle$, its multiplication $\mu_3 : \Lambda_{\bar 1} \times \Lambda_{\bar 1} \times \Lambda_{\bar 1} \to \Lambda_{\bar 1}$ is defined by $\mu_3 [\alpha, \beta, \gamma] = \alpha \cdot \beta \cdot \gamma$, where $(\cdot)$ is multiplication in Λ and $\alpha, \beta, \gamma \in \Lambda_{\bar 1}$, so $\mathrm{G}_3^{(\bar 1)}$ is nonderived and contains no unity. The even part can be treated as a ternary group $\mathrm{G}_3^{(\bar 0)} = \langle \Lambda_{\bar 0}, \mu_3' \rangle$ with the multiplication $\mu_3' : \Lambda_{\bar 0} \times \Lambda_{\bar 0} \times \Lambda_{\bar 0} \to \Lambda_{\bar 0}$, defined by $\mu_3 [a, b, c] = a \cdot b \cdot c$, where $a, b, c \in \Lambda_{\bar 0}$, thus $\mathrm{G}_3^{(\bar 0)}$ is derived and contains unity. We introduce the heteromorphism $\mathrm{G}_3^{(\bar 1)} \to \mathrm{G}_3^{(\bar 0)}$ as a mapping (2-place homomorphism) $\Phi_2^{(3,3)} : \Lambda_{\bar 1} \times \Lambda_{\bar 1} \to \Lambda_{\bar 0}$ by the formula

$$\Phi_2^{(3,3)} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \cdot \beta, \tag{83}$$

where $\alpha, \beta \in \Lambda_{\bar 1}$. It is seen that $\Phi_2^{(3,3)}$ defined by (83) satisfies the Post-like heteromorphism equation (82), but not the "vertical" one (80), due to anticommutativity of odd elements from $\Lambda_{\bar 1}$. In other words, $\mathrm{G}_3^{(\bar 0)}$ can be treated as a nontrivial example of the "diagonal" semigroup of $\mathrm{G}_3^{(\bar 1)}$ (according to the notation of [106, 107]).

Note that for $k \geq 3$ there exists additional (to the above) associative quivers having the same heteromorphisms parameters. For instance, when $n' = n = 4$ and $k = 3$ we have the Post-like associative quiver

$$\Phi_3^{(4,4)} \begin{pmatrix} \mu_4 [g_1, h_2, u_3, g_4] \\ \mu_4 [h_1, u_2, g_3, h_4] \\ \mu_4 [u_1, g_2, h_3, u_4] \end{pmatrix} = \mu_4' \left[\Phi_3^{(4,4)} \begin{pmatrix} g_1 \\ h_1 \\ u_1 \end{pmatrix}, \Phi_3^{(4,4)} \begin{pmatrix} g_2 \\ h_2 \\ u_2 \end{pmatrix}, \Phi_3^{(4,4)} \begin{pmatrix} g_1 \\ h_1 \\ u_1 \end{pmatrix}, \Phi_3^{(4,4)} \begin{pmatrix} g_2 \\ h_2 \\ u_2 \end{pmatrix} \right]. \tag{84}$$

Also, we have one intermediate non-Post associative quiver

$$\Phi_3^{(4,4)} \begin{pmatrix} \mu_4 [g_1, u_2, h_3, g_4] \\ \mu_4 [h_1, g_2, u_3, h_4] \\ \mu_4 [u_1, h_2, g_3, u_4] \end{pmatrix} = \mu_4' \left[\Phi_3^{(4,4)} \begin{pmatrix} g_1 \\ h_1 \\ u_1 \end{pmatrix}, \Phi_3^{(4,4)} \begin{pmatrix} g_2 \\ h_2 \\ u_2 \end{pmatrix}, \Phi_3^{(4,4)} \begin{pmatrix} g_1 \\ h_1 \\ u_1 \end{pmatrix}, \Phi_3^{(4,4)} \begin{pmatrix} g_2 \\ h_2 \\ u_2 \end{pmatrix} \right]. \tag{85}$$

The next type of heteromorphisms (intermediate) is described by the equations (66)-(76), it contains intact elements ($\ell_{\mathrm{id}} \geq 1$) and changes (decreases) arity $n' < n$. For each fixed k the arities are not arbitrary. There are many other possibilities (using permutations and different variants of quivers) to obtain an associative final product $\mu_{n'}'$ corresponding the same heteromorphism parameters. The above examples are sufficient to understand the rules of their construction for each concrete case.

MULTIPLACE REPRESENTATIONS OF POLYADIC SYSTEMS

The representation theory (see e.g. [109]) deals with mappings from abstract algebraic systems into linear systems, such that, e.g. linear operators in vector spaces, or into general (semi)groups of transformations of some set. In our notation, this means that in the mapping of polyadic systems (50) the final multiplication $\mu_{n'}'$ is a linear map. This leads to some restriction on the final polyadic structure $\mathrm{G}_{n'}'$, which are considered below.

Let V be a vector space over a field $\mathbb{K}$ (usually algebraically closed) and $\mathrm{End}\, V$ be a set of linear endomorphisms of V, which is in fact a binary group. In the standard way, a linear representation of a binary semigroup $\mathrm{G}_2 = \langle G; \mu_2 \rangle$ is a (1-place) map $\Pi_1 : \mathrm{G}_2 \to \mathrm{End}\, V$, such that Π_1 is a homomorphism

$$\Pi_1 (\mu_2 [g, h]) = \Pi_1 (g) * \Pi_1 (h), \tag{86}$$

where $g, h \in G$ and $(*)$ is the binary multiplication in $\mathrm{End}\, V$ (usually, it is a (semi)group with multiplication as composition of operators or product of matrices, if a basis is chosen). If G_2 is a binary group with the unity e, then we have the additional condition

$$\Pi_1 (e) = \mathrm{id}_V. \tag{87}$$

We generalize these known formulas to corresponding polyadic systems along with the introduced above heteromorphism concept. Our general idea is to use the heteromorphism equation (65) instead of the standard homomorphism equation (86), such that the arity of representation will be different from the arity of the initial polyadic system $n' \neq n$.

Consider the structure of the final n'-ary multiplication $\mu'_{n'}$ in (65), taking into account that the final polyadic system $\mathrm{G}'_{n'}$ should be constructed from $\mathrm{End}\, V$. The most natural and physically applicable way is to consider the binary $\mathrm{End}\, V$ and to put $\mathrm{G}'_{n'} = \mathrm{der}_{n'} (\mathrm{End}\, V)$, as it was proposed for the ternary case in [19]. In this way $\mathrm{G}'_{n'}$ becomes a derived n'-ary (semi)group of endomorphisms of V with the multiplication $\mu'_{n'} : (\mathrm{End}\, V)^{\times n'} \to \mathrm{End}\, V$, where

$$\mu'_{n'} [v_1, \ldots, v_{n'}] = v_1 * \ldots * v_{n'}, \quad v_i \in \mathrm{End}\, V. \tag{88}$$

Because the multiplication $\mu'_{n'}$ (88) is derived and therefore, associative by definition, we consider the associative initial polyadic systems (semigroups and groups) and the associativity preserving mappings, that are the special heteromorphisms constructed in the previous section.

Let $\mathrm{G}_n = \langle G; \mu_n \rangle$ be an associative n-ary polyadic system. By analogy with (57), we introduce the following k-place mapping

$$\Pi_k^{(n,n')} : G^{\times k} \to \mathrm{End}\, V. \tag{89}$$

`A multiplace representation` of an associative polyadic system G_n in a vector space V is given, if there exists a k-place mapping (89) which satisfies the (associativity preserving) heteromorphism equation (65), that is

$$\Pi_k^{(n,n')} \left(\begin{array}{c} \left. \begin{array}{c} \mu_n [g_1, \ldots, g_n], \\ \vdots \\ \mu_n \left[g_{n(\ell_\mu - 1)}, \ldots, g_{n\ell_\mu} \right] \end{array} \right\} \ell_\mu \\ \left. \begin{array}{c} g_{n\ell_\mu + 1}, \\ \vdots \\ g_{n\ell_\mu + \ell_{\mathrm{id}}} \end{array} \right\} \ell_{\mathrm{id}} \end{array} \right) = \overbrace{\Pi_k^{(n,n')} \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix} * \ldots * \Pi_k^{(n,n')} \begin{pmatrix} g_{k(n'-1)} \\ \vdots \\ g_{kn'} \end{pmatrix}}^{n'}, \tag{90}$$

where $\mu_n^{(\ell_\mu, \ell_{\mathrm{id}})}$ is given by (63), ℓ_μ and ℓ_{id} are numbers of multiplications and intact elements in the l.h.s. of (90), respectively.

The exact permutation in the l.h.s. of (90) is given by the associative quiver presented in the previous section. The representation parameters (n, n', k, ℓ_μ and ℓ_{id}) in (90) are the same as the heteromorphism parameters, and they satisfy the same arity changing formulas (68) and (69). Therefore, a general classification of multiplace representations can be done by analogy with one of the heteromorphisms (74)–(77) as follows:

1. The `hom-like multiplace representation` which is a multiplace homomorphism with $n' = n'_{\max} = n$, without intact elements $l_{\mathrm{id}} = l_{\mathrm{id}}^{(\min)} = 0$, and minimal number of places

$$k = k_{\min} = \ell_\mu. \tag{91}$$

2. The `intact element multiplace representation` which is the intermediate heteromorphism with $2 < n' < n$ and the number of intact elements

$$l_{\mathrm{id}} = \frac{n - n'}{n' - 1} \ell_\mu. \tag{92}$$

3. The `binary multiplace representation` which is a binarizing heteromorphism (77) with $n' = n'_{\min} = 2$, maximal number of intact elements $l_{\mathrm{id}}^{(\max)} = (n-2)\, \ell_\mu$ and maximal number of places

$$k = k_{\max} = (n-1)\, \ell_\mu. \tag{93}$$

The multiplace representations for n-ary semigroups have no additional defining relations, as compared with (90). In case of n-ary groups, we need an analog of the “normalizing” relation (87). If the n-ary group has the unity e, then one can put

$$\Pi_k^{(n,n')} \left(\left. \begin{array}{c} e \\ \vdots \\ e \end{array} \right\} k \right) = \mathrm{id}_V . \tag{94}$$

If there is no unity at all, one can “normalize” the multiplace representation, using analogy with (87) in the form

$$\Pi_1 \left(h^{-1} * h \right) = \mathrm{id}_V, \tag{95}$$

as follows

$$\Pi_k^{(n,n')}\left(\begin{array}{c}\left.\begin{array}{c}\bar{h}\\ \vdots\\ \bar{h}\end{array}\right\}\ell_\mu\\ \left.\begin{array}{c}h\\ \vdots\\ h\end{array}\right\}\ell_{\mathrm{id}}\end{array}\right)=\mathrm{id}_V, \tag{96}$$

for all $h\in\mathrm{G}_n$, where $\bar{h}$ is the querelement of h. The latter ones can be placed on any places in l.h.s. of (96) due to the Dörnte identities. Also, the multiplications in l.h.s. of (90) can change their place due to the same reason.

A general form of multiplace representations can be found by applying the Dörnte identities to each n-ary product in l.h.s. of (90). Than, using (96) we schematically have

$$\Pi_k^{(n,n')}\left(\begin{array}{c}g_1\\ \vdots\\ g_k\end{array}\right)=\Pi_k^{(n,n')}\left(\begin{array}{c}t_1\\ \vdots\\ t_{\ell_\mu}\\ \left.\begin{array}{c}g\\ \vdots\\ g\end{array}\right\}\ell_{\mathrm{id}}\end{array}\right), \tag{97}$$

where g is an arbitrary fixed element of the n-ary group and

$$t_a=\mu_n\left[g_{a1},\ldots,g_{an-1},\bar{g}\right],\quad a=1,\ldots,\ell_\mu. \tag{98}$$

This is the special shape of some multiplace representations, while the concrete formula should be obtained in each case separately. Nevertheless, some conclusions can be done from (97). First, the equivalent classes on which $\Pi_k^{(n,n')}$ is constant are determined by fixing $\ell_\mu+1$ elements, i.e. by the surface $t_a=const$, $g=const$. Second, some k-place representations of n-ary group can be reduced to ℓ_μ-place representations of its retract. In the case $\ell_\mu=1$, multiplace representations of n-ary group derived from a binary group correspond to ordinary representations of the latter (see [19, 53]).

The above formulas describe various properties of multiplace representations, but they give no idea, how to build representations for concrete polyadic systems. The most common method of representation construction is using the concept of group action on a set. Below we extend this concept to the multiplace case, as it was done above for homomorphisms and representations.

MULTIACTIONS AND G-SPACES

Let $\mathrm{G}_n=\langle G;\mu_n\rangle$ be a polyadic system and X be a set. A (left) 1-place action of G_n on X is the external binary operation $\boldsymbol{\rho}_1^{(n)}:G\times\mathsf{X}\to\mathsf{X}$ such that it is consistent with the multiplication μ_n, i.e. composition of the binary operations $\boldsymbol{\rho}_1\{g|\mathsf{x}\}$ gives the n-ary product, that is,

$$\boldsymbol{\rho}_1^{(n)}\{\mu_n[g_1,\ldots g_n]|\mathsf{x}\}=\boldsymbol{\rho}_1^{(n)}\left\{g_1|\boldsymbol{\rho}_1^{(n)}\left\{g_2|\ldots|\boldsymbol{\rho}_1^{(n)}\{g_n|\mathsf{x}\}\right\}\ldots\right\},\quad g_1,\ldots,g_n\in G,\ \mathsf{x}\in\mathsf{X}. \tag{99}$$

If the polyadic system is n-ary group, then in addition to (99) it is implied the there exist such $e_x\in G$ (which may or may not coincide with the unity of G_n) that $\boldsymbol{\rho}_1^{(n)}\{e_x|\mathsf{x}\}=\mathsf{x}$ for all $\mathsf{x}\in\mathsf{X}$, and the mapping $\mathsf{x}\mapsto\boldsymbol{\rho}_1^{(n)}\{e_x|\mathsf{x}\}$ is a bijection of X. The right 1-place action of G_n on X is defined in the symmetric way, and therefore we will consider below only one of them. Obviously, that we cannot compose $\boldsymbol{\rho}_1^{(n)}$ and $\boldsymbol{\rho}_1^{(n')}$ with $n\neq n'$. Usually X is called a G-set or G-space depending of its properties (see, e.g., [110]).

The application of the 1-place action defined by (99) to the representation theory of n-ary groups gave mostly repetitions of the ordinary (binary) group representation results (except mentioning trivial b-derived ternary groups) [20]. Also, it is obviously seen that the construction (99) with the binary external operation $\boldsymbol{\rho}_1$ cannot be applied for studying the most important regular representations of polyadic systems, when the X coincides with G_n itself and the action arises from translations.

Here we introduce the multiplace concept of action for polyadic systems which is consistent with heteromorphisms and multiplace representations. Then we show, how it naturally appears when $\mathsf{X}=\mathrm{G}_n$ and apply it to construct examples of representations including the regular ones.

For a polyadic system $\mathrm{G}_n=\langle G;\mu_n\rangle$ and a set X we introduce an external polyadic operation

$$\boldsymbol{\rho}_k:G^{\times k}\times\mathsf{X}\to\mathsf{X}, \tag{100}$$

which is called a (left) k-`place action` or `multiaction`. To generalize the 1-action composition (99), we use the analogy with multiplication laws of the heteromorphisms (65) and the multiplace representations (90) and propose (schematically)

$$\boldsymbol{\rho}_k^{(n)} \left\{ \begin{array}{c} \left.\begin{array}{c} \mu_n [g_1, \ldots, g_n], \\ \vdots \\ \mu_n \left[g_{n(\ell_\mu - 1)}, \ldots, g_{n\ell_\mu}\right] \end{array}\right\} \ell_\mu \\ \left.\begin{array}{c} g_{n\ell_\mu+1}, \\ \vdots \\ g_{n\ell_\mu+\ell_{\mathrm{id}}} \end{array}\right\} \ell_{\mathrm{id}} \end{array} \middle| \, \mathsf{x} \right\} = \boldsymbol{\rho}_k^{(n)} \overbrace{\left\{ \begin{array}{c} g_1 \\ \vdots \\ g_k \end{array} \middle| \ldots \middle| \rho_k^{(n)} \left\{ \begin{array}{c} g_{k(n'-1)} \\ \vdots \\ g_{kn'} \end{array} \middle| \, \mathsf{x} \right\} \ldots \right\}}^{n'}. \tag{101}$$

The connection between all parameters here is the same as in the arity changing formulas (68)–(69). Composition of mappings is associative, therefore in concrete cases we can use the associative quiver technique, as it is described in the previous sections. If G_n is n-ary group, then we should add to (101) the "normalizing" relations analogous with (94) or (96). So, if there is a unity $e \in \mathrm{G}_n$, then

$$\boldsymbol{\rho}_k^{(n)} \left\{ \begin{array}{c} e \\ \vdots \\ e \end{array} \middle| \, \mathsf{x} \right\} = \mathsf{x}, \quad \text{for all } \mathsf{x} \in \mathsf{X}. \tag{102}$$

In terms of the querelement the normalization has the form

$$\boldsymbol{\rho}_k^{(n)} \left\{ \begin{array}{c} \left.\begin{array}{c} \bar{h} \\ \vdots \\ \bar{h} \end{array}\right\} \ell_\mu \\ \left.\begin{array}{c} h \\ \vdots \\ h \end{array}\right\} \ell_{\mathrm{id}} \end{array} \middle| \, \mathsf{x} \right\} = \mathsf{x}, \quad \text{for all } \mathsf{x} \in \mathsf{X} \text{ and for all } h \in \mathrm{G}_n. \tag{103}$$

The multiaction $\rho_k^{(n)}$ is `transitive`, if any two points x and y in X can be "connected" by $\rho_k^{(n)}$, i.e. there exist $g_1, \ldots, g_k \in G_n$ such that

$$\rho_k^{(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ g_k \end{array} \middle| \, \mathsf{x} \right\} = \mathsf{y}. \tag{104}$$

If $g_1, \ldots, g_k$ are unique, then $\rho_k^{(n)}$ is `sharply transitive`. The subset of X, in which any points are connected by (104) with fixed $g_1, \ldots, g_k$ can be called the `multiorbit` of X. If there is only one multiorbit, then we call X the `heterogenous` G-space (by analogy with the homogeneous one). By analogy with the (ordinary) 1-place actions, we define a G-equivariant map Ψ between two G-sets X and Y by (in our notation)

$$\Psi \left(\rho_k^{(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ g_k \end{array} \middle| \, \mathsf{x} \right\} \right) = \boldsymbol{\rho}_k^{(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ g_k \end{array} \middle| \, \Psi(\mathsf{x}) \right\} \in \mathsf{Y}, \tag{105}$$

which makes G-space into a category (for details, see, e.g., [110]). In the particular case, when X is a vector space over $\mathbb{K}$, the multiaction (100) can be called a `multi-`G-module which satisfies (102) and the additional (linearity) conditions

$$\rho_k^{(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ g_k \end{array} \middle| \, a\mathsf{x} + b\mathsf{y} \right\} = a\boldsymbol{\rho}_k^{(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ g_k \end{array} \middle| \, \mathsf{x} \right\} + b\boldsymbol{\rho}_k^{(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ g_k \end{array} \middle| \, \mathsf{y} \right\}, \tag{106}$$

where $a, b \in \mathbb{K}$. Then, comparing (90) and (101) we can define a multiplace representation as a multi-G-module by the following formula

$$\Pi_k^{(n,n')} \left(\begin{array}{c} g_1 \\ \vdots \\ g_k \end{array} \right) (\mathsf{x}) = \boldsymbol{\rho}_k^{(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ g_k \end{array} \middle| \, \mathsf{x} \right\}. \tag{107}$$

In a similar way, one can generalize to the polyadic systems many other notions of the group action theory [109].

REGULAR MULTIACTIONS

The most important role in the study of polyadic systems plays the case, when $\mathsf{X} = \mathrm{G}_n$, and the multiaction coincides with the n-ary analog of translations [111], so called i-translations [56]. In the binary case ordinary translations lead to the regular representations [109], therefore we call such action a `regular multiaction` $\boldsymbol{\rho}_k^{reg(n)}$. In this connection, the analog of the Cayley theorem for n-ary groups was obtained in [105,112]. Now we show in examples, how the regular multiactions can arise from i-translations.

Let G_3 *be a ternary semigroup,* $k=2$*, and* $\mathsf{X} = \mathrm{G}_3$*, then 2-place (left) action can be defined as*

$$\boldsymbol{\rho}_2^{reg(3)} \left\{ \begin{array}{c} g \\ h \end{array} \middle| u \right\} \stackrel{def}{=} \mu_3 \left[g,h,u \right]. \tag{108}$$

This gives the following composition law for two regular multiactions

$$\boldsymbol{\rho}_2^{reg(3)} \left\{ \begin{array}{c} g_1 \\ h_1 \end{array} \middle| \boldsymbol{\rho}_2^{reg(3)} \left\{ \begin{array}{c} g_2 \\ h_2 \end{array} \middle| u \right\} \right\} = \mu_3 \left[g_1,h_1,\mu_3 \left[g_2,h_2,u \right] \right]$$

$$= \mu_3 \left[\mu_3 \left[g_1,h_1,g_2 \right],h_2,u \right] = \boldsymbol{\rho}_2^{reg(3)} \left\{ \begin{array}{c} \mu_3 \left[g_1,h_1,g_2 \right] \\ h_2 \end{array} \middle| u \right\}. \tag{109}$$

*Thus, using the regular 2-action (*108*) we, in fact, derived the associative quiver corresponding to (*59*).*

The formula (108) can be simultaneously treated as 2-translation [56]. In this way, the following `left regular multiaction`

$$\boldsymbol{\rho}_k^{reg(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ g_k \end{array} \middle| h \right\} \stackrel{def}{=} \mu_n \left[g_1,\ldots,g_k,h \right], \tag{110}$$

corresponds to (62), where in r.h.s. there is the i-translation with $i=n$. The `right regular multiaction` corresponds to the i-translation with $i=1$. The binary composition of the left regular multiactions corresponds to (62). In general, the value of i fixes the minimal final arity n'_{reg} which differs for even and odd values of the initial arity n.

It follows from (110) that for regular multiactions the number of places is fixed

$$k_{reg} = n-1, \tag{111}$$

and the arity changing formulas (68)–(69) become

$$n'_{reg} = n - \ell_{\mathrm{id}} \tag{112}$$

$$n'_{reg} = \ell_\mu + 1. \tag{113}$$

From (112)–(113) we conclude that for any n a regular multiaction having one multiplication $\ell_\mu = 1$ is binarizing and has $n-2$ intact elements. For $n=3$ see (109). Also, it follows from (112) that for regular multiactions the number of intact elements gives exactly the difference between initial and final arities.

If the initial arity is odd, then there exist a special `middle regular multiaction` generated by the i-translation with $i = (n+1)\,/\,2$. For $n=3$ the corresponding associative quiver is (81) and such 2-actions were used in [19] to construct middle representations of ternary groups, which did not change arity ($n'=n$). Here we give a more complicated example of middle regular multiaction, which can contain intact elements and therefore can change arity.

Let us consider 5-ary semigroup and the following middle 4-action

$$\boldsymbol{\rho}_4^{reg(5)} \left\{ \begin{array}{c} g \\ h \\ u \\ v \end{array} \middle| s \right\} = \mu_5 \left[g,h, \overset{\overset{i=3}{\downarrow}}{s}, u,v \right]. \tag{114}$$

*Using (*113*) we observe that there are two possibilities for the number of multiplications* $\ell_\mu = 2,4$*. The last case* $\ell_\mu = 4$ *is similar to the vertical associative quiver (*81*), but with more complicated l.h.s., that is*

$$\boldsymbol{\rho}_4^{reg(5)} \left\{ \begin{array}{c} \mu_5 \left[g_1,h_1,g_2,h_2,g_3 \right] \\ \mu_5 \left[h_3,g_4,h_4,g_5,h_5 \right] \\ \mu_5 \left[u_5,v_5,u_4,v_4,u_3 \right] \\ \mu_5 \left[v_3,u_2,v_2,u_1,v_1 \right] \end{array} \middle| s \right\} =$$

$$\boldsymbol{\rho}_4^{reg(5)} \left\{ \begin{array}{c} g_1 \\ h_1 \\ u_1 \\ v_1 \end{array} \middle| \boldsymbol{\rho}_4^{reg(5)} \left\{ \begin{array}{c} g_2 \\ h_2 \\ u_2 \\ v_2 \end{array} \middle| \boldsymbol{\rho}_4^{reg(5)} \left\{ \begin{array}{c} g_3 \\ h_3 \\ u_3 \\ v_3 \end{array} \middle| \boldsymbol{\rho}_4^{reg(5)} \left\{ \begin{array}{c} g_4 \\ h_4 \\ u_4 \\ v_4 \end{array} \middle| \boldsymbol{\rho}_4^{reg(5)} \left\{ \begin{array}{c} g_5 \\ h_5 \\ u_5 \\ v_5 \end{array} \middle| s \right\} \right\} \right\} \right\} \right\}. \tag{115}$$

Now we have an additional case with two intact elements ℓ_{id} and two multiplications $\ell_\mu = 2$ as

$$\boldsymbol{\rho}_4^{reg(5)} \left\{ \begin{array}{c} \mu_5 [g_1, h_1, g_2, h_2, g_3] \\ h_3 \\ \mu_5 [h_3, v_3, u_2, v_2, u_1] \\ v_1 \end{array} \middle| s \right\} = \boldsymbol{\rho}_4^{reg(5)} \left\{ \begin{array}{c} g_1 \\ h_1 \\ u_1 \\ v_1 \end{array} \middle| \boldsymbol{\rho}_4^{reg(5)} \left\{ \begin{array}{c} g_2 \\ h_2 \\ u_2 \\ v_2 \end{array} \middle| \boldsymbol{\rho}_4^{reg(5)} \left\{ \begin{array}{c} g_3 \\ h_3 \\ u_3 \\ v_3 \end{array} \middle| s \right\} \right\} \right\}, \tag{116}$$

with changing arity from $n = 5$ to $n'_{reg} = 3$. In addition to (116) we have 3 more possible regular multiactions due to associativity of μ_5, when the multiplication brackets in the sequences of 6 elements in the first two rows and the second two ones can be shifted independently.

For $n > 3$, in addition to left, right and middle multiactions, there exist intermediate cases. First observe that the i-translations with $i = 2$ and $i = n - 1$ immediately fix the final arity $n'_{reg} = n$. Therefore, the composition of multiactions will be similar to (115), but with some permutations in l.h.s.

Now we consider some multiplace analogs of regular representations of binary groups [109]. The straightforward generalization is considering the introduced regular multiactions (110) in the r.h.s. of (107). Let G_n be a finite polyadic associative system and $\mathbb{K}\mathrm{G}_n$ be a vector space spanned by G_n (some properties of n-ary group rings were considered in [113, 114]). This means that any element of $\mathbb{K}\mathrm{G}_n$ can be uniquely presented in the form $w = \sum_l a_l \cdot h_l$, $a_l \in \mathbb{K}$, $h_l \in G$. Then, using (110) and (107) we define the `i-regular k-place representation` by

$$\Pi_k^{reg(i)} \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix} (w) = \sum_l a_l \cdot \mu_{k+1} [g_1 \dots g_{i-1} h_l g_{i+1} \dots g_k] . \tag{117}$$

Comparing (110) and (117) one can conclude that all general properties of multiplace regular representations are similar to those of the regular multiactions. If $i = 1$ or $i = k$, the multiplace representation is called `right` or `left regular representation` correspondingly. If k is even, the representation with $i = k/2 + 1$ is called a `middle regular representation`. The case $k = 2$ was considered in [19] for ternary groups.

MULTIPLACE REPRESENTATIONS OF TERNARY GROUPS

Let us consider the case $n = 3$, $k = 2$ in more details paying attention on its special peculiarities, which corresponds to the 2-place (bi-element) representations of ternary groups [19]. Let V be a vector space over $\mathbb{K}$ and $\operatorname{End} V$ be a set of linear endomorphisms of V. From now on we denote the ternary multiplication by the square bracket only, as follows $\mu_3 [g_1, g_2, g_3] \equiv [g_1 g_2 g_3]$, and use the "horizontal" notation $\Pi \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \equiv \Pi (g_1, g_2)$.

A `left representation` of a ternary group $G, [\,])$ in V is a map $\Pi^L : G \times G \to \operatorname{End} V$ such that

$$\Pi^L (g_1, g_2) \circ \Pi^L (g_3, g_4) = \Pi^L ([g_1 g_2 g_3], g_4), \tag{118}$$

$$\Pi^L (g, \bar{g}) = \mathrm{id}_V, \tag{119}$$

where $g, g_1, g_2, g_3, g_4 \in G$.

Replacing in (119) g by $\bar{g}$ we obtain $\Pi^L (\bar{g}, g) = id_V$, which means that in fact (119) has the form $\Pi^L (\bar{g}, g) = \Pi^L (g, \bar{g}) = \mathrm{id}_V$, $\forall g \in G$. Note that the axioms considered in the above definition are the natural ones satisfied by left multiplications $g \mapsto [abg]$. For all $g_1, g_2, g_3, g_4 \in G$ we have

$$\Pi^L ([g_1 g_2 g_3], g_4) = \Pi^L (g_1, [g_2 g_3 g_4]) .$$

For all $g, h, u \in G$ we have

$$\Pi^L (g, h) = \Pi^L ([g u \bar{u}], h) = \Pi^L (g, u) \circ \Pi^L (\bar{u}, h) \tag{120}$$

and

$$\Pi^L (g, u) \circ \Pi^L (\bar{u}, \bar{g}) = \Pi^L (\bar{u}, \bar{g}) \circ \Pi^L (g, u) = \mathrm{id}_V, \tag{121}$$

and therefore every $\Pi^L (g, u)$ is invertible and $\left(\Pi^L (g, u)\right)^{-1} = \Pi^L (\bar{u}, \bar{g})$. This means that any left representation gives a representation of a ternary group by a binary group [19]. If the ternary group is medial, then

$$\Pi^L (g_1, g_2) \circ \Pi^L (g_3, g_4) = \Pi^L (g_3, g_4) \circ \Pi^L (g_1, g_2) ,$$

i.e. obtained group is commutative. If the ternary group $\langle G, [\,] \rangle$ is commutative, then also $\Pi^L (g, h) = \Pi^L (h, g)$, because

$$\Pi^L (g, h) = \Pi^L (g, h) \circ \Pi^L (g, \bar{g}) = \Pi^L ([g\, h\, g], \bar{g}) = \Pi^L ([h\, g\, g], \bar{g}) = \Pi^L (h, g) \circ \Pi^L (g, \bar{g}) = \Pi^L (h, g) .$$

In the case of the commutative and idempotent ternary group any its left representation is idempotent and $\left(\Pi^L(g,h)\right)^{-1} = \Pi^L(g,h)$, so that commutative and idempotent ternary groups are represented by Boolean groups.

Let $\langle G, [\,]\rangle = \mathrm{der}\,(G,\odot)$ be a ternary group derived from a binary group $\langle G,\odot\rangle$, then there is one-to-one correspondence between representations of $(G,\odot)$ and left representations of $(G,[\,])$.

Indeed, because $(G,[\,]) = \mathrm{der}\,(G,\odot)$, then $g\odot h = [geh]$ and $\overline{e} = e$, where e is unity of the binary group $(G,\odot)$. If $\pi \in \mathrm{Rep}\,(G,\odot)$, then (as it is not difficult to see) $\Pi^L(g,h) = \pi(g)\circ\pi(h)$ is a left representation of $\langle G,[\,]\rangle$. Conversely, if Π^L is a left representation of $\langle G,[\,]\rangle$ then $\pi(g) = \Pi^L(g,e)$ is a representation of $(G,\odot)$. Moreover, in this case $\Pi^L(g,h) = \pi(g)\circ\pi(h)$, because we have

$$\Pi^L(g,h) = \Pi^L(g,[ehe]) = \Pi^L([geh],e) = \Pi^L(g,e)\circ\Pi^L(h,e) = \pi(g)\circ\pi(h).$$

Let $(G,[\,])$ be a ternary group and $(G\times G,*)$ be a semigroup used to the construction of left representations. According to Post [5] one says that two pairs (a,b), (c,d) of elements of G are equivalent, if there exists an element $g\in G$ such that $[abg]=[cdg]$. Using a covering group we can see that if this equation holds for some $g\in G$, then it holds also for all $g\in G$. This means that

$$\Pi^L(a,b) = \Pi^L(c,d) \Longleftrightarrow (a,b)\sim(c,d),$$

i.e.

$$\Pi^L(a,b) = \Pi^L(c,d) \Longleftrightarrow [abg]=[cdg]$$

for some $g\in G$. Indeed, if $[abg]=[cdg]$ holds for some $g\in G$, then

$$\begin{aligned}\Pi^L(a,b) &= \Pi^L(a,b)\circ\Pi^L(g,\overline{g}) = \Pi^L([abg],\overline{g})\\ &= \Pi^L([cdg],\overline{g}) = \Pi^L(c,d)\circ\Pi^L(g,\overline{g}) = \Pi^L(c,d).\end{aligned}$$

By analogy we can define

A `right representation` of a ternary group $(G,[\,])$ in V is a map $\Pi^R: G\times G\to \mathrm{End}\,V$ such that

$$\Pi^R(g_3,g_4)\circ\Pi^R(g_1,g_2) = \Pi^R(g_1,[g_2g_3g_4]), \tag{122}$$

$$\Pi^R(g,\overline{g}) = \mathrm{id}_V, \tag{123}$$

where $g,g_1,g_2,g_3,g_4\in G$.

From (122)-(123) it follows that

$$\Pi^R(g,h) = \Pi^R(g,[u\,\overline{u}\,h]) = \Pi^R(\overline{u},h)\circ\Pi^R(g,u). \tag{124}$$

It is easy to check that $\Pi^R(g,h) = \Pi^L\left(\overline{h},\overline{g}\right) = \left(\Pi^L(g,h)\right)^{-1}$. So it is enough to consider only left representations (as in binary case). Consider the following example of group algebra ternary generalization [19].

Let G be a ternary group and $\mathbb{K}G$ be a vector space spanned by G, which means that any element of $\mathbb{K}G$ can be uniquely presented in the form $t=\sum_{i=1}^n k_ih_i$, $k_i\in\mathbb{K}$, $h_i\in G$, $n\in\mathbb{N}$ (we do not assume that G has finite rank). Then left and right regular representations are defined by

$$\Pi^L_{reg}(g_1,g_2)\,t = \sum_{i=1}^n k_i\,[g_1g_2h_i], \tag{125}$$

$$\Pi^R_{reg}(g_1,g_2)\,t = \sum_{i=1}^n k_i\,[h_ig_1g_2]. \tag{126}$$

Let us construct the middle representations as follows.

A `middle representation` of a ternary group $\langle G,[\,]\rangle$ in V is a map $\Pi^M: G\times G\to\mathrm{End}\,V$ such that

$$\Pi^M(g_3,h_3)\circ\Pi^M(g_2,h_2)\circ\Pi^M(g_1,h_1) = \Pi^M([g_3g_2g_1],[h_1h_2h_3]), \tag{127}$$

$$\Pi^M(g,h)\circ\Pi^M\left(\overline{g},\overline{h}\right) = \Pi^M\left(\overline{g},\overline{h}\right)\circ\Pi^M(g,h) = \mathrm{id}_V \tag{128}$$

It is seen that a middle representation is a ternary group homomorphism $\Pi^M: G\times G^{op}\to\mathrm{der}\,\mathrm{End}\,V$. Note that instead of (128) one can use $\Pi^M\left(g,\overline{h}\right)\circ\Pi^M(\overline{g},h) = id_V$ after changing g to $\overline{g}$ and taking into account that $g=\overline{\overline{g}}$. In the case idempotents elements g and h we have $\Pi^M(g,h)\circ\Pi^M(g,h)=id_V$, which means that the matrices Π^M are Boolean. Thus all middle representation matrices of idempotent ternary groups are Boolean. The composition $\Pi^M(g_1,h_1)\circ\Pi^M(g_2,h_2)$ is not a middle representation, but the following proposition holds.

Let Π^M is a middle representation of a ternary group $\langle G,[\,]\rangle$, then, if $\Pi_u^L(g,h)=\Pi^M(g,u)\circ\Pi^M(h,\overline{u})$ is a left representation of $\langle G,[\,]\rangle$, then $\Pi_u^L(g,h)\circ\Pi_{u'}^L(g',h')=\Pi_{u'}^L([ghu'],h')$, and, if $\Pi_u^R(g,h)=\Pi^M(u,h)\circ\Pi^M(\overline{u},g)$ is a right representation of $\langle G,[\,]\rangle$, then $\Pi_u^R(g,h)\circ\Pi_{u'}^R(g',h')=\Pi_u^R(g,[hg'h'])$. In particular, Π_u^L (Π_u^R) is a family of left (right) representations.

If a middle representation Π^M of a ternary group $\langle G,[\,]\rangle$ satisfies $\Pi^M(g,\overline{g})=\mathrm{id}_V$ for all $g\in G$, then it is a left and right representation and $\Pi^M(g,h)=\Pi^M(h,g)$ for all $g,h\in G$. Note that in general $\Pi_{reg}^M(g,\overline{g})\neq\mathrm{id}$. For regular representations we have the following commutation relations

$$\Pi_{reg}^L(g_1,h_1)\circ\Pi_{reg}^R(g_2,h_2)=\Pi_{reg}^R(g_2,h_2)\circ\Pi_{reg}^L(g_1,h_1).$$

Let $\langle G,[\,]\rangle$ be a ternary group and let $\left\langle G\times G,[\,]'\right\rangle$ be a ternary group used to the construction of the middle representation. In $\langle G,[\,]\rangle$, and in the consequence in $\left\langle G\times G,[\,]'\right\rangle$, we define the relation

$$(a,b)\sim(c,d)\Longleftrightarrow[aub]=[cud]$$

for all $u\in G$. It is not difficult to see that this relation is a congruence in a ternary group $\left\langle G\times G,[\,]'\right\rangle$. For regular representations $\Pi_{reg}^M(a,b)=\Pi_{reg}^M(c,d)$ if $(a,b)\sim(c,d)$. We have the following relation

$$a\approx a'\Longleftrightarrow a=[\overline{g}a'g]\ \text{ for some }\ g\in G$$

or equivalently

$$a\approx a'\Longleftrightarrow a'=[ga\overline{g}]\ \text{ for some }\ g\in G.$$

It is not difficult to see that it is an equivalence relation on $\langle G,[\,]\rangle$, moreover, if $\langle G,[\,]\rangle$ is medial, then this relation is a congruence.

Let $\left\langle G\times G,[\,]'\right\rangle$ be a ternary group used for a construction of middle representations, then

$$\begin{aligned}(a,b)\approx(a',b)\Longleftrightarrow a'=[ga\overline{g}]\ &\text{ and}\\ b'=[hb\overline{b}]\ &\text{ for some }\ (g,h)\in G\times G\end{aligned}$$

is an equivalence relation on $\left\langle G\times G,[\,]'\right\rangle$. Moreover, if $(G,[\,])$ is medial, then this relation is a congruence. Unfortunately, it is a weak relation. In a ternary group $\mathbb{Z}_3$, where $[ghu]=(g-h+u)\,(\mathrm{mod}\,3)$ we have only one class, i.e. all elements are equivalent. In $\mathbb{Z}_4$ with the operation $[ghu]=(g+h+u+1)\,(\mathrm{mod}\ 4)$ we have $a\approx a'\Longleftrightarrow a=a'$. But for this relation holds the following statement. If $(a,b)\approx(a',b')$, then

$$\mathrm{tr}\,\Pi^M(a,b)=\mathrm{tr}\,\Pi^M(a',b').$$

We have $\mathrm{tr}(AB)=\mathrm{tr}(BA)$ for all $A,B\in\mathrm{End}V$, and

$$\begin{aligned}\mathrm{tr}\,\Pi^M(a,b)&=\mathrm{tr}\,\Pi^M([ga'\overline{g}],[hb'\overline{h}]\,)=\mathrm{tr}\left(\Pi^M(g,\overline{h})\circ\Pi^M(a',b')\circ\Pi^M(\overline{g},h)\right)\\&=\mathrm{tr}\left(\Pi^M(g,\overline{h})\circ\Pi^M(\overline{g},h)\circ\Pi^M(a',b')\right)=\mathrm{tr}\left(id_V\circ\Pi^M(a'b')\right)=\mathrm{tr}\,\Pi^M(a',b')\end{aligned}$$

In our derived case the connection with the standard group representations is given by the following. Let $(G,\odot)$ be a binary group, and the ternary derived group as $\langle G,[\,]\rangle=\mathrm{der}\,(G,\odot)$. There is one-to-one correspondence between a pair of commuting binary groups representations and a middle ternary derived group representation. Indeed, let $\pi,\rho\in Rep\,(G,\odot)$, $\pi(g)\circ\rho(h)=\rho(h)\circ\pi(g)$ and $\Pi^L\in Rep\,(G,[\,])$. We take

$$\Pi^M(g,h)=\pi(g)\circ\rho\left(h^{-1}\right),\quad\pi(g)=\Pi^M(g,e),\quad\rho(g)=\Pi^M(e,\overline{g}).$$

Then using (127) we prove the needed representation laws.

Let $\langle G,[\,]\rangle$ be a fixed ternary group, $\left\langle G\times G,[\,]'\right\rangle$ a corresponding ternary group used in the construction of middle representations, $\left((G\times G)^*,\circledast\right)$ a covering group of $\left\langle G\times G,[\,]'\right\rangle$, $(G\times G,\diamond)=\mathrm{ret}_{(a,b)}(G\times G,\langle\,\rangle)$. If $\Pi^M(a,b)$ is a middle representation of $\langle G,[\,]\rangle$, then π defined by

$$\pi(g,h,0)=\Pi^M(g,h),\quad\pi(g,h,1)=\Pi^M(g,h)\circ\Pi^M(a,b)$$

is a representation of the covering group [5]. Moreover

$$\rho(g,h)=\Pi^M(g,h)\circ\Pi^M(a,b)=\pi(g,h,1)$$

is a representation of the above retract induced by (a,b). Indeed, $(\overline{a},\overline{b})$ is the identity of this retract and $\rho(\overline{a},\overline{b})=\Pi^M(\overline{a},\overline{b})\circ\Pi^M(a,b)=\mathrm{id}_V$. Similarly

$$\begin{aligned}\rho\left((g,h)\diamond(u,u)\right)&=\rho\left(\langle(g,h),(a,b),(u,u)\rangle\right)=\rho\left([gau],[ubh]\right)=\Pi^M\left([gau],[ubh]\right)\circ\Pi^M(a,b)\\&=\Pi^M(g,h)\circ\Pi^M(a,b)\circ\Pi^M(u,u)\circ\Pi^M(a,b)=\rho(g,h)\circ\rho(u,u)\end{aligned}$$

But $\tau(g) = (g, \overline{g})$ is an embedding of $(G, [\,])$ into $\langle G \times G, [\,]' \rangle$. Hence μ defined by $\mu(g, 0) = \Pi^M(g, \overline{g})$ and $\mu(g, 1) = \Pi^M(g, \overline{g}) \circ \Pi^M(a, \overline{a})$ is a representation of a covering group G^* for $(G, [\,])$ (see the Post theorem [5] for $a = c$). On the other hand, $\beta(g) = \Pi^M(g, \overline{g}) \circ \Pi^M(a, \overline{a})$ is a representation of a binary retract $(G, \cdot) = \mathrm{ret}_a(G, [\,])$. That β can induce some middle representation of $(G, [\,])$ (by Gluskin-Hosszú theorem [7]).

Note that in a ternary group of the quaternions $\langle \mathbb{K}, [\,] \rangle$ (with norm 1), where $[ghu] = ghu(-1) = -ghu$ and gh is a multiplication of quaternions (-1 is a central element) we have $\overline{1} = -1$, $\overline{-1} = 1$ and $\overline{g} = g$ for others. In $\langle \mathbb{K} \times \mathbb{K}, [\,]' \rangle$ we have $(a, b) \sim (-a, -b)$ and $(a, -b) \sim (-a, b)$, which gives 32 two-elements equivalence classes. The embedding $\tau(g) = (g, \overline{g})$ suggest that $\Pi^M(i, i) = \pi(i) \neq \pi(-i) = \Pi^M(-i, -i)$. Generally $\Pi^M(a, b) \neq \Pi^M(-a, -b)$ and $\Pi^M(a, -b) \neq \Pi^M(-a, b)$.

The relation $(a, b) \sim (c, d) \iff [abg] = [cdg]$ for all $g \in G$ is a congruence on $(G \times G, *)$. Note that this relation can be defined as "for some g". Indeed, using a covering group we can see that if $[abg] = [cdg]$ holds for some g then holds also for all g. Thus $\pi^L(a, b) = \Pi^L(c, d) \iff (a, b) \sim (c, d)$. Indeed

$$\begin{aligned} \Pi^L(a, b) &= \Pi^L(a, b) \circ \Pi^L(g, \overline{g}) = \Pi^L([a\ b\ g], \overline{g}) \\ &= \Pi^L([c\ d\ g], \overline{g}) = \Pi^L(c, d) \circ \Pi^L(g, \overline{g}) = \Pi^L(c, d). \end{aligned}$$

We conclude, that every left representation of a commutative group $\langle G, [\,] \rangle$ is a middle representation. Indeed,

$$\Pi^L(g, h) \circ \Pi^L(\overline{g}, \overline{h}) = \Pi^L([g\ h\ \overline{g}], \overline{h}) = \Pi^L([g\ \overline{g}\ h], \overline{h}) = \Pi^L(h, \overline{h}) = \mathrm{id}_V$$

and

$$\begin{aligned} \Pi^L(g_1, g_2) \circ \Pi^L(g_3, g_4) \circ \Pi^L(g_5, g_6) &= \Pi^L([[g_1 g_2 g_3] g_4 g_5], g_6) = \Pi^L([[g_1 g_3 g_2] g_4 g_5], g_6) = \Pi^L([g_1 g_3 [g_2 g_4 g_5]], g_6) \\ &= \Pi^L([g_1 g_3 [g_5 g_4 g_2]], g_6) = \Pi^L([g_1 g_3 g_5], [g_4 g_2 g_6]) = \Pi^L([g_1 g_3 g_5], [g_6 g_4 g_2]). \end{aligned}$$

Note that the converse holds only for the special kind of middle representations such that $\Pi^M(g, \overline{g}) = \mathrm{id}_V$. Therefore,

There is one-one-correspondence between left representations of $\langle G, [\,] \rangle$ and binary representations of the retract $\mathrm{ret}_a(G, [\,])$.

Indeed, let $\Pi^L(g, a)$ is given, then define $\rho(g) = \Pi^L(g, a)$ is such representation of the retract which can be directly shown. Conversely, assume that $\rho(g)$ is a representation of the retract $\mathrm{ret}_a(G, [\ ])$. Define $\Pi^L(g, h) = \rho(g) \circ \rho(\overline{h})^{-1}$, then $\Pi^L(g, h) \circ \Pi^L(u, u) = \rho(g) \circ \rho(\overline{h})^{-1} \circ \rho(u) \circ \rho(\overline{u})^{-1} = \rho(g \circledast (\overline{h})^{-1} \circ \circledast u) \circ \rho(\overline{u})^{-1} = \rho([\,[\,g\,a\,[\,\overline{a}\,h\,\overline{a}\,]\,]\,a\,u\,]) \circ \rho(\overline{u})^{-1} = \rho([\,g\,h\,g\,]) \circ \rho(\overline{u})^{-1} = \Pi^L([\,g\,h\,u\,], u)$,

MATRIX REPRESENTATIONS OF TERNARY GROUPS

Here we give several examples of matrix representations for concrete ternary groups. Let $G = \mathbb{Z}_3 \ni \{0, 1, 2\}$ and the ternary multiplication is $[ghu] = g - h + u$. Then $[ghu] = [uhg]$ and $\overline{0} = 0$, $\overline{1} = 1$, $\overline{2} = 2$, therefore $(G, [\,])$ is an idempotent medial ternary group. Thus $\Pi^L(g, h) = \Pi^R(h, g)$ and

$$\Pi^L(a, b) = \Pi^L(c, d) \iff (a - b) = (c - d) \bmod 3. \tag{129}$$

The calculations give the left regular representation in the manifest matrix form

$$\Pi^L_{reg}(0,0) = \Pi^L_{reg}(2,2) = \Pi^L_{reg}(1,1) = \Pi^R_{reg}(0,0) = \Pi^R_{reg}(2,2) = \Pi^R_{reg}(1,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [1] \oplus [1] \oplus [1], \tag{130}$$

$$\begin{aligned} \Pi^L_{reg}(2,0) = \Pi^L_{reg}(1,2) = \Pi^L_{reg}(0,1) = \Pi^R_{reg}(2,1) = \Pi^R_{reg}(1,0) = \Pi^R_{reg}(0,2) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\ = [1] \oplus \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = [1] \oplus \left[-\frac{1}{2} + \frac{1}{2} i\sqrt{3}\right] \oplus \left[-\frac{1}{2} - \frac{1}{2} i\sqrt{3}\right], \end{aligned} \tag{131}$$

$$\begin{aligned} \Pi^L_{reg}(2,1) = \Pi^L_{reg}(1,0) = \Pi^L_{reg}(0,2) = \Pi^R_{reg}(2,0) = \Pi^R_{reg}(1,2) = \Pi^R_{reg}(0,1) &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ = [1] \oplus \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} = [1] \oplus \left[-\frac{1}{2} - \frac{1}{2} i\sqrt{3}\right] \oplus \left[-\frac{1}{2} + \frac{1}{2} i\sqrt{3}\right]. \end{aligned} \tag{132}$$

Consider the middle representation construction. The middle regular representations is defined by

$$\Pi^M_{reg}(g_1,g_2)\,t=\sum_{i=1}^{n}k_i\,[g_1h_ig_2]\,.$$

For regular representations we have

$$\Pi^M_{reg}(g_1,h_1)\circ\Pi^R_{reg}(g_2,h_2)=\Pi^R_{reg}(h_2,h_1)\circ\Pi^M_{reg}(g_1,g_2)\,, \tag{133}$$

$$\Pi^M_{reg}(g_1,h_1)\circ\Pi^L_{reg}(g_2,h_2)=\Pi^L_{reg}(g_1,g_2)\circ\Pi^M_{reg}(h_2,h_1)\,. \tag{134}$$

For the middle regular representation matrices we obtain

$$\Pi^M_{reg}(0,0)=\Pi^M_{reg}(1,2)=\Pi^M_{reg}(2,1)=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},$$

$$\Pi^M_{reg}(0,1)=\Pi^M_{reg}(1,0)=\Pi^M_{reg}(2,2)=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},$$

$$\Pi^M_{reg}(0,2)=\Pi^M_{reg}(2,0)=\Pi^M_{reg}(1,1)=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}.$$

The above representation Π^M_{reg} of $\langle\mathbb{Z}_3,[\,]\rangle$ is equivalent to the orthogonal direct sum of two irreducible representations

$$\Pi^M_{reg}(0,0)=\Pi^M_{reg}(1,2)=\Pi^M_{reg}(2,1)=[1]\oplus\begin{bmatrix}-1&0\\0&1\end{bmatrix},$$

$$\Pi^M_{reg}(0,1)=\Pi^M_{reg}(1,0)=\Pi^M_{reg}(2,2)=[1]\oplus\begin{bmatrix}\frac{1}{2}&-\frac{\sqrt{3}}{2}\\-\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{bmatrix},$$

$$\Pi^M_{reg}(0,2)=\Pi^M_{reg}(2,0)=\Pi^M_{reg}(1,1)=[1]\oplus\begin{bmatrix}\frac{1}{2}&\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{bmatrix},$$

i.e. one-dimensional trivial [1] and two-dimensional irreducible. Note, that in this example $\Pi^M(g,\overline{g})=\Pi^M(g,g)\neq\mathrm{id}_V$, but $\Pi^M(g,h)\circ\Pi^M(g,h)=\mathrm{id}_V$, and so Π^M are of the second degree.

Consider a more complicated example of left representations. Let $G=\mathbb{Z}_4\ni\{0,1,2,3\}$ and the ternary multiplication is

$$[ghu]=(g+h+u+1)\,\mathrm{mod}\,4. \tag{135}$$

We have the multiplication table

$$[g,h,0]=\begin{pmatrix}1&2&3&0\\2&3&0&1\\3&0&1&2\\0&1&2&3\end{pmatrix}\qquad[g,h,1]=\begin{pmatrix}2&3&0&1\\3&0&1&2\\0&1&2&3\\1&2&3&0\end{pmatrix}$$

$$[g,h,2]=\begin{pmatrix}3&0&1&2\\0&1&2&3\\1&2&3&0\\2&3&0&1\end{pmatrix}\qquad[g,h,3]=\begin{pmatrix}0&1&2&3\\1&2&3&0\\2&3&0&1\\3&0&1&2\end{pmatrix}$$

Then the skew elements are $\overline{0}=3,\ \overline{1}=2,\ \overline{2}=1,\ \overline{3}=0$, therefore $(G,[\,])$ is an (non-idempotent) commutative ternary group. The left representation is defined by expansion $\Pi^L_{reg}(g_1,g_2)\,t=\sum_{i=1}^{n}k_i\,[g_1g_2h_i]$, which means that (see the general formula (117))

$$\Pi^L_{reg}(g,h)\,|u>=|\,[ghu]>.$$

Analogously, for right and middle representations

$$\Pi^R_{reg}(g,h)\,|u>=|\,[ugh]>,\quad\Pi^M_{reg}(g,h)\,|u>=|\,[guh]>.$$

Therefore $|[ghu]>=|[ugh]>=|[guh]>$ and

$$\Pi^L_{reg}(g,h)=\Pi^R_{reg}(g,h)\,|u>=\Pi^M_{reg}(g,h)\,|u>,$$

so $\Pi^L_{reg}(g,h)=\Pi^R_{reg}(g,h)=\Pi^M_{reg}(g,h)$. Thus it is sufficient to consider the left representation only.

In this case the equivalence is $\Pi^L(a,b)=\Pi^L(c,d)\Longleftrightarrow(a+b)=(c+d)\bmod 4$, and we obtain the following classes

$$\Pi^L_{reg}(0,0)=\Pi^L_{reg}(1,3)=\Pi^L_{reg}(2,2)=\Pi^L_{reg}(3,1)=\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}=[1]\oplus[-1]\oplus[-i]\oplus[i],$$

$$\Pi^L_{reg}(0,1)=\Pi^L_{reg}(1,0)=\Pi^L_{reg}(2,3)=\Pi^L_{reg}(3,2)=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}=[1]\oplus[-1]\oplus[-1]\oplus[-1],$$

$$\Pi^L_{reg}(0,2)=\Pi^L_{reg}(1,1)=\Pi^L_{reg}(2,0)=\Pi^L_{reg}(3,3)=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}=[1]\oplus[-1]\oplus[i]\oplus[-i],$$

$$\Pi^L_{reg}(0,3)=\Pi^L_{reg}(1,2)=\Pi^L_{reg}(2,1)=\Pi^L_{reg}(3,0)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=[1]\oplus[-1]\oplus[1]\oplus[1].$$

It is seen that, due to the fact that the ternary operation (135) is commutative, there are only one-dimensional irreducible left representations.

Let us "algebralize" the above regular representations in the following way. From (118) we have for the left representation

$$\Pi^L_{reg}(i,j)\circ\Pi^L_{reg}(k,l)=\Pi^L_{reg}(i,[jkl]),\tag{136}$$

where $[jkl]=j-k+l,\ \ i,j,k,l\in\mathbb{Z}_3$. Denote $\gamma^L_i=\Pi^L_{reg}(0,i),\ \ i\in\mathbb{Z}_3$, then we obtain the algebra with the relations

$$\gamma^L_i\gamma^L_j=\gamma^L_{i+j}.\tag{137}$$

Conversely, any matrix representation of $\gamma_i\gamma_j=\gamma_{i+j}$ leads to the left representation by $\Pi^L(i,j)=\gamma_{j-i}$. In the case of the middle regular representation we introduce $\gamma^M_{k+l}=\Pi^M_{reg}(k,l),\ \ k,l\in\mathbb{Z}_3$, then we obtain

$$\gamma^M_i\gamma^M_j\gamma^M_k=\gamma^M_{[ijk]},\quad i,j,k\in\mathbb{Z}_3.\tag{138}$$

In some sense (138) can be treated as a `ternary analog of the Clifford algebra`. As before, any matrix representation of (138) gives the middle representation $\Pi^M(k,l)=\gamma_{k+l}$.

TERNARY ALGEBRAS AND HOPF ALGEBRAS

Let us consider associative ternary algebras [2,115]. One can introduce autodistributivity property $[[xyz]\,ab]=[[xab]\,[yab]\,[zab]]$ (see [72]). If we take 2 ternary operations $\{\ ,\ ,\ \}$ and $[\ ,\ ,\ ]$, then distributivity is $\{[xyz]\,ab\}=[\{xab\}\,\{yab\}\,\{zab\}]$. If $(+)$ is a binary operation (addition), then `left linearity` is

$$[(x+z)\,,a,b]=[xab]+[zab].\tag{139}$$

By analogy one can define central (middle) and right linearity. `Linearity` is defined, when left, middle and right linearity hold valid simultaneously.

An `associative ternary algebra` is a triple $\left(A,\mu_3,\eta^{(3)}\right)$, where A is a linear space over a field $\mathbb{K}$, μ_3 is a linear map $A\otimes A\otimes A\to A$ called `ternary multiplication` $\mu_3(a\otimes b\otimes c)=[abc]$ which is ternary associative $[[abc]\,de]=[a\,[bcd]\,e]=[ab\,[cde]]$ or

$$\mu_3\circ(\mu_3\otimes\mathrm{id}\otimes\mathrm{id})=\mu_3\circ(\mathrm{id}\otimes\mu_3\otimes\mathrm{id})=\mu_3\circ(\mathrm{id}\otimes\mathrm{id}\otimes\mu_3).\tag{140}$$

There are two types [45] of ternary unit maps $\eta^{(3)}:\mathbb{K}\to A$:

1) One `strong` unit map

$$\mu_3\circ\left(\eta^{(3)}\otimes\eta^{(3)}\otimes\mathrm{id}\right)=\mu_3\circ\left(\eta^{(3)}\otimes\mathrm{id}\otimes\eta^{(3)}\right)=\mu_3\circ\left(\mathrm{id}\otimes\eta^{(3)}\otimes\eta^{(3)}\right)=\mathrm{id};\tag{141}$$

2) Two `sequential` units $\eta_1^{(3)}$ and $\eta_2^{(3)}$ satisfying

$$\mu_3 \circ \left(\eta_1^{(3)} \otimes \eta_2^{(3)} \otimes \mathrm{id}\right) = \mu_3 \circ \left(\eta_1^{(3)} \otimes \mathrm{id} \otimes \eta_2^{(3)}\right) = \mu_3 \circ \left(\mathrm{id} \otimes \eta_1^{(3)} \otimes \eta_2^{(3)}\right) = \mathrm{id}; \tag{142}$$

In first case the ternary analog of the binary relation $\eta^{(2)}(x) = x1$, where $x \in \mathbb{K}$, $1 \in A$, is

$$\eta^{(3)}(x) = [x, 1, 1] = [1, 1, x] = [1, x, 1]. \tag{143}$$

Let (A, μ_A, η_A), (B, μ_B, η_B) and (C, μ_C, η_C) be ternary algebras, then the `ternary tensor product` space $A \otimes B \otimes C$ is naturally endowed with the structure of an algebra. The multiplication $\mu_{A\otimes B\otimes C}$ on $A \otimes B \otimes C$ reads

$$[(a_1 \otimes b_1 \otimes c_1)(a_2 \otimes b_2 \otimes c_2)(a_3 \otimes b_3 \otimes c_3)] = [a_1 a_2 a_3] \otimes [b_1 b_2 b_3] \otimes [c_1 c_2 c_3], \tag{144}$$

and so the set of ternary algebras is closed under taking ternary tensor products. A `ternary algebra map` (homomorphism) is a linear map between ternary algebras $f : A \to B$ which respects the ternary algebra structure

$$f([xyz]) = [f(x), f(y), f(z)], \tag{145}$$

$$f(1_A) = 1_B. \tag{146}$$

Let C be a linear space over a field $\mathbb{K}$.

A `ternary comultiplication` $\Delta^{(3)}$ is a linear map over a field $\mathbb{K}$ such that

$$\Delta_3 : C \to C \otimes C \otimes C. \tag{147}$$

In the standard Sweedler notations [37] $\Delta_3(a) = \sum_{i=1}^{n} a_i' \otimes a_i'' \otimes a_i''' = a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$. Consider different possible types of ternary coassociativity [45, 46].

1. A `standard` ternary coassociativity

$$(\Delta_3 \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta_3 = (\mathrm{id} \otimes \Delta_3 \otimes \mathrm{id}) \circ \Delta_3 = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta_3) \circ \Delta_3, \tag{148}$$

2. A `nonstandard` ternary Σ-coassociativity (Gluskin-type positional operatives)

$$(\Delta_3 \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta_3 = (\mathrm{id} \otimes (\sigma \circ \Delta_3) \otimes \mathrm{id}) \circ \Delta_3,$$

   where $\sigma \circ \Delta_3(a) = \Delta_3(a) = a_{(\sigma(1))} \otimes a_{(\sigma(2))} \otimes a_{(\sigma(3))}$ and $\sigma \in \Sigma \subset S_3$.

3. A `permutational` ternary coassociativity

$$(\Delta_3 \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta_3 = \pi \circ (\mathrm{id} \otimes \Delta_3 \otimes \mathrm{id}) \circ \Delta_3,$$

   where $\pi \in \Pi \subset S_5$.

A `ternary comediality` is

$$(\Delta_3 \otimes \Delta_3 \otimes \Delta_3) \circ \Delta_3 = \sigma_{medial} \circ (\Delta_3 \otimes \Delta_3 \otimes \Delta_3) \circ \Delta_3,$$

where $\sigma_{medial} = \binom{123456789}{147258369} \in S_9$. A `ternary counit` is defined as a map $\varepsilon^{(3)} : C \to \mathbb{K}$. In general, $\varepsilon^{(3)} \neq \varepsilon^{(2)}$ satisfying one of the conditions below. If Δ_3 is derived, then maybe $\varepsilon^{(3)} = \varepsilon^{(2)}$, but another counits may exist. There are two types of ternary counits:

1. Standard (`strong`) ternary counit

$$(\varepsilon^{(3)} \otimes \varepsilon^{(3)} \otimes \mathrm{id}) \circ \Delta_3 = (\varepsilon^{(3)} \otimes \mathrm{id} \otimes \varepsilon^{(3)}) \circ \Delta_3 = (\mathrm{id} \otimes \varepsilon^{(3)} \otimes \varepsilon^{(3)}) \circ \Delta_3 = \mathrm{id}, \tag{149}$$

2. Two `sequential` (polyadic) counits $\varepsilon_1^{(3)}$ and $\varepsilon_2^{(3)}$

$$(\varepsilon_1^{(3)} \otimes \varepsilon_2^{(3)} \otimes \mathrm{id}) \circ \Delta = (\varepsilon_1^{(3)} \otimes \mathrm{id} \otimes \varepsilon_2^{(3)}) \circ \Delta = (\mathrm{id} \otimes \varepsilon_1^{(3)} \otimes \varepsilon_2^{(3)}) \circ \Delta = \mathrm{id}, \tag{150}$$

Below we will consider only the first standard type of associativity (148). The σ-`cocommutativity` is defined as $\sigma \circ \Delta_3 = \Delta_3$.

A `ternary coalgebra` is a triple $\left(C, \Delta_3, \varepsilon^{(3)}\right)$, where C is a linear space and Δ_3 is a ternary comultiplication (147) which is coassociative in one of the above senses and $\varepsilon^{(3)}$ is one of the above counits.

Let $\left(A, \mu^{(3)}\right)$ be a ternary algebra and (C, Δ_3) be a ternary coalgebra and $f, g, h \in \mathrm{Hom}_{\mathbb{K}}(C, A)$. `Ternary convolution product` is

$$[f, g, h]_* = \mu^{(3)} \circ (f \otimes g \otimes h) \circ \Delta_3 \tag{151}$$

or in the Sweedler notation $[f, g, h]_*(a) = \left[f\left(a_{(1)}\right) g\left(a_{(2)}\right) h\left(a_{(3)}\right)\right]$.

A ternary coalgebra is called `derived`, if there exists a binary (usual, see e.g. [37]) coalgebra $\Delta_2 : C \to C \otimes C$ such that

$$\Delta_{3,\mathrm{der}} = (\mathrm{id} \otimes \Delta_2) \otimes \Delta_2. \tag{152}$$

A ternary bialgebra B is $\left(B, \mu^{(3)}, \eta^{(3)}, \Delta_3, \varepsilon^{(3)}\right)$ for which $\left(B, \mu^{(3)}, \eta^{(3)}\right)$ is a ternary algebra and $\left(B, \Delta_3, \varepsilon^{(3)}\right)$ is a ternary coalgebra and they are compatible

$$\Delta_3 \circ \mu^{(3)} = \mu^{(3)} \circ \Delta_3 \tag{153}$$

One can distinguish four kinds of ternary bialgebras with respect to a "being derived" property:

1. A `Δ-derived` ternary bialgebra
$$\Delta_3 = \Delta_{3,\mathrm{der}} = (\mathrm{id} \otimes \Delta_2) \circ \Delta_2 \tag{154}$$
2. A `μ-derived` ternary bialgebra
$$\mu^{(3)}_{der} = \mu^{(3)}_{der} = \mu^{(2)} \circ \left(\mu^{(2)} \otimes \mathrm{id}\right) \tag{155}$$
3. A `derived` ternary bialgebra is simultaneously μ-derived and Δ-derived ternary bialgebra.
4. A `non-derived` ternary bialgebra which does not satisfy (154) and (155).

Possible types of ternary antipodes can be defined using analogy with binary coalgebras.

A `skew ternary antipod` is

$$\mu^{(3)} \circ (S^{(3)}_{skew} \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta_3 = \mu^{(3)} \circ (\mathrm{id} \otimes S^{(3)}_{skew} \otimes \mathrm{id}) \circ \Delta_3 = \mu^{(3)} \circ (\mathrm{id} \otimes \mathrm{id} \otimes S^{(3)}_{skew}) \circ \Delta_3 = \mathrm{id}\,. \tag{156}$$

If only one equality from (156) is satisfied, the corresponding skew antipod is called `left`, `middle` or `right`.

`Strong ternary antipod` is

$$\left(\mu^{(2)} \otimes \mathrm{id}\right) \circ (\mathrm{id} \otimes S^{(3)}_{strong} \otimes \mathrm{id}) \circ \Delta_3 = 1 \otimes \mathrm{id},\ \left(\mathrm{id} \otimes \mu^{(2)}\right) \circ (\mathrm{id} \otimes \mathrm{id} \otimes S^{(3)}_{strong}) \circ \Delta_3 = \mathrm{id} \otimes 1,$$

where 1 is a unit of algebra.

If in a ternary coalgebra the relation

$$\Delta_3 \circ S = \tau_{13} \circ (S \otimes S \otimes S) \circ \Delta_3 \tag{157}$$

hold valid, where $\tau_{13} = \binom{123}{321}$, then it is called `skew-involutive`.

A ternary Hopf algebra $\left(H, \mu^{(3)}, \eta^{(3)}, \Delta_3, \varepsilon^{(3)}, S^{(3)}\right)$ is a ternary bialgebra with a ternary antipod $S^{(3)}$ of the corresponding above type .

Let us consider concrete constructions of ternary comultiplications, bialgebras and Hopf algebras. A `ternary group-like element` can be defined by $\Delta_3(g) = g \otimes g \otimes g$, and for 3 such elements we have

$$\Delta_3([g_1 g_2 g_3]) = \Delta_3(g_1) \Delta_3(g_2) \Delta_3(g_3)\,. \tag{158}$$

But an analog of the binary primitive element (satisfying $\Delta^{(2)}(x) = x \otimes 1 + 1 \otimes x$) cannot be chosen simply as $\Delta_3(x) = x \otimes e \otimes e + e \otimes x \otimes e + e \otimes e \otimes x$, since the algebra structure is not preserved. Nevertheless, if we introduce `two` idempotent units e_1, e_2 satisfying "semiorthogonality" $[e_1 e_1 e_2] = 0$, $[e_2 e_2 e_1] = 0$, then

$$\Delta_3(x) = x \otimes e_1 \otimes e_2 + e_2 \otimes x \otimes e_1 + e_1 \otimes e_2 \otimes x \tag{159}$$

and now $\Delta_3([x_1 x_2 x_3]) = [\Delta_3(x_1) \Delta_3(x_2) \Delta_3(x_3)]$. Using (159) $\varepsilon(x) = 0$, $\varepsilon(e_{1,2}) = 1$, and $S^{(3)}(x) = -x$, $S^{(3)}(e_{1,2}) = e_{1,2}$, one can construct a ternary universal enveloping algebra in full analogy with the binary case (see e.g. [39]).

One of the most important examples of noncocommutative Hopf algebras is the well known Sweedler Hopf algebra [37] which in the binary case has two generators x and y satisfying

$$\mu^{(2)}(x,x)=1, \tag{160}$$
$$\mu^{(2)}(y,y)=0, \tag{161}$$
$$\sigma_{+}^{(2)}(xy)=-\sigma_{-}^{(2)}(xy). \tag{162}$$

It has the following comultiplication

$$\Delta_2(x)=x\otimes x, \tag{163}$$
$$\Delta_2(y)=y\otimes x+1\otimes y, \tag{164}$$

counit $\varepsilon^{(2)}(x)=1$, $\varepsilon^{(2)}(y)=0$, and antipod $S^{(2)}(x)=x$, $S^{(2)}(y)=-y$, which respect the algebra structure. In the derived case a `ternary Sweedler algebra` is generated also by two generators x and y obeying

$$\mu^{(3)}(x,e,x)=\mu^{(3)}(e,x,x)=\mu^{(3)}(x,x,e)=e, \tag{165}$$
$$\sigma_{+}^{(3)}([yey])=0, \tag{166}$$
$$\sigma_{+}^{(3)}([xey])=-\sigma_{-}^{(3)}([xey]). \tag{167}$$

The derived Hopf algebra structure is given by

$$\Delta_3(x)=x\otimes x\otimes x, \tag{168}$$
$$\Delta_3(y)=y\otimes x\otimes x+e\otimes y\otimes x+e\otimes e\otimes y, \tag{169}$$
$$\varepsilon^{(3)}(x)=\varepsilon^{(2)}(x)=1, \tag{170}$$
$$\varepsilon^{(3)}(y)=\varepsilon^{(2)}(y)=0, \tag{171}$$
$$S^{(3)}(x)=S^{(2)}(x)=x, \tag{172}$$
$$S^{(3)}(y)=S^{(2)}(y)=-y, \tag{173}$$

and it can be checked that (168)-(170) are algebra maps, while (172) is antialgebra maps. To obtain a non-derived ternary Sweedler example we have the possibilities: 1) one "even" generator x, two "odd" generators $y_{1,2}$ and one ternary unit e; 2) two "even" generators $x_{1,2}$, one "odd" generator y and two ternary units $e_{1,2}$. In the first case the ternary algebra structure is (no summation, $i=1,2$)

$$[xxx]=e, \tag{174}$$
$$[y_iy_iy_i]=0, \tag{175}$$
$$\sigma_{+}^{(3)}([y_ixy_i])=\sigma_{+}^{(3)}([xy_ix])=0, \tag{176}$$
$$[xey_i]=-[xy_ie],$$
$$[exy_i]=-[y_ixe], \tag{177}$$
$$[ey_ix]=-[y_iex], \tag{178}$$
$$\sigma_{+}^{(3)}([y_1xy_2])=-\sigma_{-}^{(3)}([y_1xy_2]). \tag{179}$$

The corresponding ternary Hopf algebra structure is

$$\begin{gathered}\Delta_3(x)=x\otimes x\otimes x,\ \Delta_3(y_{1,2})=y_{1,2}\otimes x\otimes x+e_{1,2}\otimes y_{2,1}\otimes x+e_{1,2}\otimes e_{2,1}\otimes y_{2,1},\\ \varepsilon^{(3)}(x)=1,\ \ \varepsilon^{(3)}(y_i)=0,\ \ S^{(3)}(x)=x,\quad S^{(3)}(y_i)=-y_i.\end{gathered} \tag{180}$$

In the second case we have for algebra structure

$$\begin{gathered}[x_ix_jx_k]=\delta_{ij}\delta_{ik}\delta_{jk}e_i,\quad [yyy]=0,\ \ \sigma_{+}^{(3)}([yx_iy])=0,\ \sigma_{+}^{(3)}([x_iyx_i])=0,\\ \sigma_{+}^{(3)}([y_1xy_2])=0,\quad \sigma_{-}^{(3)}([y_1xy_2])=0,\end{gathered} \tag{181}$$

and the ternary Hopf algebra structure is

$$\Delta_3(x_i) = x_i \otimes x_i \otimes x_i,$$
$$\Delta_3(y) = y \otimes x_1 \otimes x_1 + e_1 \otimes y \otimes x_2 + e_1 \otimes e_2 \otimes y, \tag{182}$$
$$\varepsilon^{(3)}(x_i) = 1, \tag{183}$$
$$\varepsilon^{(3)}(y) = 0, \tag{184}$$
$$S^{(3)}(x_i) = x_i, \tag{185}$$
$$S^{(3)}(y) = -y. \tag{186}$$

TERNARY QUANTUM GROUPS

A ternary commutator can be obtained in different ways [116]. We will consider a simplest version called a Nambu bracket (see e.g. [2,30]). Let us introduce two maps $\omega_{\pm}^{(3)} : A \otimes A \otimes A \to A \otimes A \otimes A$ by

$$\omega_{+}^{(3)}(a \otimes b \otimes c) = a \otimes b \otimes c + b \otimes c \otimes a + c \otimes a \otimes b, \tag{187}$$
$$\omega_{-}^{(3)}(a \otimes b \otimes c) = b \otimes a \otimes c + c \otimes b \otimes a + a \otimes c \otimes b. \tag{188}$$

Thus, obviously $\mu^{(3)} \circ \omega_{\pm}^{(3)} = \sigma_{\pm}^{(3)} \circ \mu^{(3)}$, where $\sigma_{\pm}^{(3)} \in S_3$ denotes sum of terms having even and odd permutations respectively. In the binary case $\omega_{+}^{(2)} = \mathrm{id} \otimes \mathrm{id}$ and $\omega_{-}^{(2)} = \tau$ is the twist operator $\tau : a \otimes b \to b \otimes a$, while $\mu^{(2)} \circ \omega_{-}^{(2)}$ is permutation $\sigma_{-}^{(2)}(ab) = ba$. So the Nambu product is $\omega_{N}^{(3)} = \omega_{+}^{(3)} - \omega_{-}^{(3)}$, and the ternary commutator is $[,,]_N = \sigma_{N}^{(3)} = \sigma_{+}^{(3)} - \sigma_{-}^{(3)}$, or [30]

$$[a, b, c]_N = [abc] + [bca] + [cab] - [cba] - [acb] - [bac] \tag{189}$$

An `abelian ternary algebra` is defined by vanishing of Nambu bracket $[a, b, c]_N = 0$ or `ternary commutation` relation $\sigma_{+}^{(3)} = \sigma_{-}^{(3)}$. By analogy with the binary case a `deformed ternary algebra` can be defined by

$$\sigma_{+}^{(3)} = q\sigma_{-}^{(3)} \text{ or } [abc] + [bca] + [cab] = q([cba] + [acb] + [bac]), \tag{190}$$

where multiplication by q is treated as an external operation.

Let us consider a ternary analog of the Woronowicz example of a bialgebra construction, which in the binary case has two generators satisfying $xy = qyx$ (or $\sigma_{+}^{(2)}(xy) = q\sigma_{-}^{(2)}(xy)$), then the following coproducts

$$\Delta_2(x) = x \otimes x \tag{191}$$
$$\Delta_2(y) = y \otimes x + 1 \otimes y \tag{192}$$

are algebra maps. In the derived ternary case using (190) we have

$$\sigma_{+}^{(3)}([xey]) = q\sigma_{-}^{(3)}([xey]), \tag{193}$$

where e is the ternary unit and ternary coproducts are

$$\Delta_3(e) = e \otimes e \otimes e, \tag{194}$$
$$\Delta_3(x) = x \otimes x \otimes x, \tag{195}$$
$$\Delta_3(y) = y \otimes x \otimes x + e \otimes y \otimes x + e \otimes e \otimes y, \tag{196}$$

which are `ternary algebra maps`, i.e. they satisfy

$$\sigma_{+}^{(3)}([\Delta_3(x)\Delta_3(e)\Delta_3(y)]) = q\sigma_{-}^{(3)}([\Delta_3(x)\Delta_3(e)\Delta_3(y)]). \tag{197}$$

Let us consider the group $G = SL(n, \mathbb{K})$. Then the algebra generated by $a_j^i \in SL(n, \mathbb{K})$ can be endowed by the structure of ternary Hopf algebra (see e.g. [117] for binary case) by choosing the ternary coproduct, counit and antipod as (here summation is implied)

$$\Delta_3(a_j^i) = a_k^i \otimes a_l^k \otimes a_j^l, \quad \varepsilon(a_j^i) = \delta_j^i, \quad S^{(3)}(a_j^i) = (a^{-1})_j^i. \tag{198}$$

This antipod is a skew one since from (156) it follows

$$\mu^{(3)} \circ (S^{(3)} \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta_3(a_j^i) = S^{(3)}(a_k^i) a_l^k a_j^l = (a^{-1})_k^i a_l^k a_j^l = \delta_l^i a_j^l = a_j^i. \tag{199}$$

This ternary Hopf algebra is derived since for $\Delta^{(2)} = a^i_j \otimes a^j_k$ we have

$$\Delta_3 = \left(\mathrm{id} \otimes \Delta^{(2)}\right) \otimes \Delta^{(2)} \left(a^i_j\right) = \left(\mathrm{id} \otimes \Delta^{(2)}\right) \left(a^i_k \otimes a^k_j\right) = a^i_k \otimes \Delta^{(2)} \left(a^k_j\right) = a^i_k \otimes a^k_l \otimes a^l_j. \tag{200}$$

In the most important case $n = 2$ we can obtain the manifest action of the ternary coproduct Δ_3 on components. Possible non-derived matrix representations of the ternary product can be done only by four-rank $n \times n \times n \times n$ twice covariant and twice contravariant tensors $\left\{a^{ij}_{kl}\right\}$. Among all products the non-derived ones are only the following $a^{oi}_{jk} b^{jl}_{oo} c^{ko}_{il}$ and $a^{ij}_{ok} b^{ol}_{io} c^{ko}_{il}$ (where o is any index). So using e.g. the first choice we can define the non-derived Hopf algebra structure by

$$\Delta_3 \left(a^{ij}_{kl}\right) = a^{i\mu}_{v\rho} \otimes a^{v\sigma}_{kl} \otimes a^{\rho j}_{\mu\sigma}, \tag{201}$$

$$\varepsilon \left(a^{ij}_{kl}\right) = \frac{1}{2} \left(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k\right), \tag{202}$$

and the skew antipod $s^{ij}_{kl} = S^{(3)} \left(a^{ij}_{kl}\right)$ which is a solution of the equation $s^{i\mu}_{v\rho} a^{v\sigma}_{kl} = \delta^i_\rho \delta^\mu_k \delta^\sigma_l$.

Next consider ternary dual pair $k(G)$ (push-forward) and $\mathcal{F}(G)$ (pull-back) which are related by $k^*(G) \cong \mathcal{F}(G)$ (see e.g. [118]). Here $k(G) = \mathrm{span}(G)$ is a ternary group algebra (G has a ternary product $[\,]_G$ or $\mu^{(3)}_G$) over a field k. If $u \in k(G)$ ($u = u^i x_i, x_i \in G$), then

$$[uvw]_k = u^i v^j w^l \left[x_i x_j x_l\right]_G \tag{203}$$

is associative, and so $(k(G), [\,]_k)$ becomes a ternary algebra. Define a ternary coproduct $\Delta_3 : k(G) \to k(G) \otimes k(G) \otimes k(G)$ by

$$\Delta_3 (u) = u^i x_i \otimes x_i \otimes x_i \tag{204}$$

(derived and associative), then $\Delta_3 ([uvw]_k) = [\Delta_3 (u) \Delta_3 (v) \Delta_3 (w)]_k$, and $k(G)$ is a ternary bialgebra. If we define a ternary antipod by $S^{(3)}_k = u^i \bar{x}_i$, where $\bar{x}_i$ is a skew element of x_i, then $k(G)$ becomes a ternary Hopf algebra.

In the dual case of functions $\mathcal{F}(G) : \{\varphi : G \to k\}$ a ternary product $[\,]_\mathcal{F}$ or $\mu^{(3)}_\mathcal{F}$ (derived and associative) acts on $\psi(x, y, z)$ as

$$\left(\mu^{(3)}_\mathcal{F} \psi\right)(x) = \psi(x, x, x), \tag{205}$$

and so $\mathcal{F}(G)$ is a ternary algebra. Let $\mathcal{F}(G) \otimes \mathcal{F}(G) \otimes \mathcal{F}(G) \cong \mathcal{F}(G \times G \times G)$, then we define a ternary coproduct $\Delta_3 : \mathcal{F}(G) \to \mathcal{F}(G) \otimes \mathcal{F}(G) \otimes \mathcal{F}(G)$ as

$$(\Delta_3 \varphi)(x, y, z) = \varphi([xyz]_\mathcal{F}), \tag{206}$$

which is derive and associative. Thus we can obtain $\Delta_3 ([\varphi_1 \varphi_2 \varphi_3]_\mathcal{F}) = [\Delta_3 (\varphi_1) \Delta_3 (\varphi_2) \Delta_3 (\varphi_3)]_\mathcal{F}$, and therefore $\mathcal{F}(G)$ is a ternary bialgebra. If we define a ternary antipod by

$$S^{(3)}_\mathcal{F} (\varphi) = \varphi(\bar{x}), \tag{207}$$

where $\bar{x}$ is a skew element of x, then $\mathcal{F}(G)$ becomes a ternary Hopf algebra.

Let us introduce a ternary analog of R-matrix. For a ternary Hopf algebra H we consider a linear map $R^{(3)} : H \otimes H \otimes H \to H \otimes H \otimes H$.

A ternary Hopf algebra $\left(H, \mu^{(3)}, \eta^{(3)}, \Delta_3, \varepsilon^{(3)}, S^{(3)}\right)$ is called `quasifiveangular`[4] if it satisfies

$$(\Delta_3 \otimes \mathrm{id} \otimes \mathrm{id}) = R^{(3)}_{145} R^{(3)}_{245} R^{(3)}_{345}, \tag{208}$$

$$(\mathrm{id} \otimes \Delta_3 \otimes \mathrm{id}) = R^{(3)}_{125} R^{(3)}_{145} R^{(3)}_{135}, \tag{209}$$

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta_3) = R^{(3)}_{125} R^{(3)}_{124} R^{(3)}_{123}, \tag{210}$$

where as usual index of R denotes action component positions.

Using the standard procedure (see e.g. [39,119,120]) we obtain set of abstract `ternary quantum Yang-Baxter equations`, one of which has the form

$$R^{(3)}_{243} R^{(3)}_{342} R^{(3)}_{125} R^{(3)}_{145} R^{(3)}_{135} = R^{(3)}_{123} R^{(3)}_{132} R^{(3)}_{145} R^{(3)}_{245} R^{(3)}_{345}, \tag{211}$$

[4]The reason of such notation is clear from (211).

and others can be obtained by corresponding permutations. The classical ternary Yang-Baxter equations for one parameter family of solutions $R(t)$ can be obtained by the expansion

$$R^{(3)}(t) = e \otimes e \otimes e + rt + \mathcal{O}\left(t^2\right), \tag{212}$$

where r is a ternary classical R-matrix, then e.g. for (211) we have

$$\begin{aligned} &r_{342}r_{125}r_{145}r_{135} + r_{243}r_{125}r_{145}r_{135} + r_{243}r_{342}r_{145}r_{135} + r_{243}r_{342}r_{125}r_{135} + r_{243}r_{342}r_{125}r_{145} \\ &= r_{132}r_{145}r_{245}r_{345} + r_{123}r_{145}r_{245}r_{345} + r_{123}r_{132}r_{245}r_{345} + r_{123}r_{132}r_{145}r_{345} + r_{123}r_{132}r_{145}r_{245}. \end{aligned}$$

For three ternary Hopf algebras $\left(H_{A,B,C}, \mu^{(3)}_{A,B,C}, \eta^{(3)}_{A,B,C}, \Delta^{(3)}_{A,B,C}, \varepsilon^{(3)}_{A,B,C}, S^{(3)}_{A,B,C}\right)$ we can introduce a non-degenerate ternary "pairing" (see e.g. [119] for binary case) $\langle\, ,\, ,\, \rangle^{(3)} : H_A \times H_B \times H_C \to \mathbb{K}$, trilinear over $\mathbb{K}$, satisfying

$$\begin{aligned} \left\langle \eta^{(3)}_A(a), b, c\right\rangle^{(3)} &= \left\langle a, \varepsilon^{(3)}_B(b), c\right\rangle^{(3)}, \quad \left\langle a, \eta^{(3)}_B(b), c\right\rangle^{(3)} = \left\langle \varepsilon^{(3)}_A(a), b, c\right\rangle^{(3)}, \\ \left\langle b, \eta^{(3)}_B(b), c\right\rangle^{(3)} &= \left\langle a, b, \varepsilon^{(3)}_C(c)\right\rangle^{(3)}, \quad \left\langle a, b, \eta^{(3)}_C(c)\right\rangle^{(3)} = \left\langle a, \varepsilon^{(3)}_B(b), c\right\rangle^{(3)}, \\ \left\langle a, b, \eta^{(3)}_C(c)\right\rangle^{(3)} &= \left\langle \varepsilon^{(3)}_A(a), b, c\right\rangle^{(3)}, \quad \left\langle \eta^{(3)}_A(a), b, c\right\rangle^{(3)} = \left\langle a, b, \varepsilon^{(3)}_C(c)\right\rangle^{(3)}, \end{aligned}$$

$$\begin{aligned} \left\langle \mu^{(3)}_A(a_1 \otimes a_2 \otimes a_3), b, c\right\rangle^{(3)} &= \left\langle a_1 \otimes a_2 \otimes a_3, \Delta^{(3)}_B(b), c\right\rangle^{(3)}, \\ \left\langle \Delta^{(3)}_A(a), b_1 \otimes b_2 \otimes b_3, c\right\rangle^{(3)} &= \left\langle a, \mu^{(3)}_B(b_1 \otimes b_2 \otimes b_3), c\right\rangle^{(3)}, \\ \left\langle a, \mu^{(3)}_B(b_1 \otimes b_2 \otimes b_3), c\right\rangle^{(3)} &= \left\langle a, b_1 \otimes b_2 \otimes b_3, \Delta^{(3)}_C(c)\right\rangle^{(3)}, \\ \left\langle a, \Delta^{(3)}_B(b), c_1 \otimes c_2 \otimes c_3\right\rangle^{(3)} &= \left\langle a, b, \mu^{(3)}_C(c_1 \otimes c_2 \otimes c_3)\right\rangle^{(3)}, \\ \left\langle a, b, \mu^{(3)}_C(c_1 \otimes c_2 \otimes c_3)\right\rangle^{(3)} &= \left\langle \Delta^{(3)}_A(a), b, c_1 \otimes c_2 \otimes c_3\right\rangle^{(3)}, \\ \left\langle a_1 \otimes a_2 \otimes a_3, b, \Delta^{(3)}_C(c)\right\rangle^{(3)} &= \left\langle \mu^{(3)}_A(a_1 \otimes a_2 \otimes a_3), b, c\right\rangle^{(3)}, \end{aligned}$$

$$\left\langle S^{(3)}_A(a), b, c\right\rangle^{(3)} = \left\langle a, S^{(3)}_B(b), c\right\rangle^{(3)} = \left\langle a, b, S^{(3)}_C(c)\right\rangle^{(3)},$$

where $a, a_i \in H_A,\ b, b_i \in H_B$. The ternary "paring" between $H_A \otimes H_A \otimes H_A$ and $H_B \otimes H_B \otimes H_B$ is given by $\langle a_1 \otimes a_2 \otimes a_3, b_1 \otimes b_2 \otimes b_3\rangle^{(3)} = \langle a_1, b_1\rangle^{(3)} \langle a_2, b_2\rangle^{(3)} \langle a_3, b_3\rangle^{(3)}$. These constructions can naturally lead to ternary generalization of duality concept and quantum double which are the key ingredients in the theory of quantum groups [39, 120, 121].

CONCLUSIONS

In this paper we presented a review of polyadic systems and their representations, ternary algebras and Hopf algebras. We classified general polyadic systems and considered their homomorphisms and the multiplace generalizations, paying attention on their associativity. We defined multiplace representations and multiactions, gave examples of matrix representations for some ternary groups. We defined and investigated ternary algebras and Hopf algebras, and give some examples. Then we considered some ternary generalizations of quantum groups and the Yang-Baxter equation.

REFERENCES

1. Kerner R. Ternary algebraic structures and their applications in physics // Paris, 2000. - 15 p. (Preprint Univ. P. & M. Curie).
2. de Azcarraga J. A., Izquierdo J. M. n-ary algebras: a review with applications // J. Phys. - 2010. - Vol. A43. - P. 293001.
3. Dörnte W. Unterschungen über einen verallgemeinerten Gruppenbegriff // Math. Z. - 1929. - Vol. 29. - P. 1–19.
4. Prüfer H. Theorie der abelshen Gruppen I. Grundeigenschaften // Math. Z. - 1924. - Vol. 20. - P. 165–187.
5. Post E. L. Polyadic groups // Trans. Amer. Math. Soc. - 1940. - Vol. 48. - P. 208–350.
6. Hosszú L. M. On the explicit form of n-group operation // Publ. Math. Debrecen. - 1963. - Vol. 10. - P. 88–92.

7. Gluskin L. M. Position operatives // Math. Sbornik. - 1965. - Vol. 68(110). - № 3. - P. 444–472.
8. Dudek W. A., Głazek K., Gleichgewicht B. A note on the axioms of n-groups // Coll. Math. Soc. J. Bolyai. 29. Universal Algebra. - Esztergom (Hungary). , 1977. - P. 195–202.
9. Rusakov S. A definition of an n-ary group // Dokl. Akad. Nauk BSSR. - 1979. - Vol. 23. - P. 965–967.
10. Weyl H. Classical Groups, Their Invariants and Representations. - Princeton: Princeton Univ. Press, 1946.
11. Fulton W., Harris J. Representation Theory: a First Course. - N. Y.: Springer, 1991.
12. Cornwell J. F. Group Theory in Physics: An Introduction. - London: Academic Press, 1997. - 349 p.
13. Curtis C. W., Reiner I. Representation theory of finite groups and associative algebras. - Providence: AMS, 1962.
14. Collins M. J. Representations And Characters Of Finite Groups. - Cambridge: Cambridge University Press, 1990. - 242 p.
15. Kapranov M., Gelfand I. M., Zelevinskii A. Discriminants, Resultants and Multidimensional Determinants. - Berlin: Birkhäuser, 1994. - 234 p.
16. Sokolov N. P. Introduction to the Theory of Multidimensional Matrices. - Kiev: Naukova Dumka, 1972. - 175 p.
17. Kawamura Y. Cubic matrices, generalized spin algebra and uncertainty relation // Progr. Theor. Phys. - 2003. - Vol. 110. - P. 579–587.
18. Rausch de Traubenberg M. Cubic extentions of the poincare algebra // Phys. Atom. Nucl. - 2008. - Vol. 71. - P. 1102–1108.
19. Borowiec A., Dudek W., Duplij S. Bi-element representations of ternary groups // Comm. Algebra. - 2006. - Vol. 34. - № 5. - P. 1651–1670.
20. Dudek W. A., Shahryari M. Representation theory of polyadic groups // Algebr. Represent. Theory. - 2012. - Vol. 15. - № 1. - P. 29–51.
21. Lõhmus J., Paal E., Sorgsepp L. Nonassociative Algebras in Physics. - Palm Harbor: Hadronic Press, 1994. - 271 p.
22. Lounesto P., Ablamowicz R. Clifford Algebras: Applications To Mathematics, Physics, And Engineering. - Birkhäuser, 2004. - 626 p.
23. Georgi H. Lie algebras in particle physics. - New York: Perseus Books, 1999. - 320 p.
24. Abramov V. $\mathbb{Z}_3$-graded analogues of Clifford algebras and algebra of $\mathbb{Z}_3$-graded symmetries // Algebras Groups Geom. - 1995. - Vol. 12. - № 3. - P. 201–221.
25. Abramov V. Ternary generalizations of Grassmann algebra // Proc. Est. Acad. Sci., Phys. Math. - 1996. - Vol. 45. - № 2-3. - P. 152–160.
26. Abramov V., Kerner R., Le Roy B. Hypersymmetry: a $\mathbb{Z}_3$-graded generalization of supersymmetry // J. Math. Phys. - 1997. - Vol. 38. - P. 1650–1669.
27. Filippov V. T. n-Lie algebras // Sib. Math. J. - 1985. - Vol. 26. - P. 879–891.
28. Michor P. W., Vinogradov A. M. n-ary Lie and associative algebras // Rend. Sem. Mat. Univ. Pol. Torino. - 1996. - Vol. 54. - № 4. - P. 373–392.
29. Nambu Y. Generalized Hamiltonian dynamics // Phys. Rev. - 1973. - Vol. 7. - P. 2405–2412.
30. Takhtajan L. On foundation of the generalized Nambu mechanics // Commun. Math. Phys. - 1994. - Vol. 160. - P. 295–315.
31. Bagger J., Lambert N. Comments on multiple M2-branes // - 2008. - Vol. 2. - P. 105.
32. Bagger J., Lambert N. Gauge symmetry and supersymmetry of multiple M2-branes // Phys. Rev. - 2008. - Vol. D77. - P. 065008.
33. Gustavsson A. One-loop corrections to Bagger-Lambert theory // Nucl. Phys. - 2009. - Vol. B807. - P. 315–333.
34. Ho P.-M., Hou R.-C., Matsuo Y., Shiba S. M5-brane in three-form flux and multiple M2-branes // - 2008. - Vol. 08. - P. 014.
35. Low A. M. Worldvolume superalgebra of BLG theory with Nambu-Poisson structure // - 2010. - Vol. 04. - P. 089.
36. Abe E. Hopf Algebras. - Cambridge: Cambridge Univ. Press, 1980. - 221 p.
37. Sweedler M. E. Hopf Algebras. - New York: Benjamin, 1969. - 336 p.
38. Montgomery S. Hopf algebras and their actions on rings. - Providence: AMS, 1993. - 238 p.
39. Kassel C. Quantum Groups. - New York: Springer-Verlag, 1995. - 531 p.
40. Shnider S., Sternberg S. Quantum Groups. - Boston: International Press, 1993. - 371 p.
41. Duplij S., Li F. Regular solutions of quantum Yang-Baxter equation from weak Hopf algebras // Czech. J. Phys. - 2001. - Vol. 51. - № 12. - P. 1306–1311.
42. Duplij S., Sinel'shchikov S. Quantum enveloping algebras with von Neumann regular Cartan-like generators and the Pierce decomposition // Commun. Math. Phys. - 2009. - Vol. 287. - № 1. - P. 769–785.
43. Li F., Duplij S. Weak Hopf algebras and singular solutions of quantum Yang-Baxter equation // Commun. Math. Phys. - 2002. - Vol. 225. - № 1. - P. 191–217.
44. Duplij S., Sinel'shchikov S. Classification of $U_q\left(\mathrm{sl}_2\right)$-module algebra structures on the quantum plane // J. Math. Physics, Analysis, Geometry. - 2010. - Vol. 6. - № 6. - P. 21–46.

45. Duplij S. Ternary Hopf algebras // Symmetry in Nonlinear Mathematical Physics. - Kiev. Institute of Mathematics, 2001. - P. 25-34.
46. Borowiec A., Dudek W., Duplij S. Basic concepts of ternary Hopf algebras // Journal of Kharkov National University, ser. Nuclei, Particles and Fields. - 2001. - Vol. 529. - № 3(15). - P. 21–29.
47. Bergman G. M. An invitation to general algebra and universal constructions. - Berkeley: University of California, 1995. - 358 p.
48. Hausmann B. A., Ore Ø. Theory of quasigroups // Amer. J. Math. - 1937. - Vol. 59. - P. 983–1004.
49. Clifford A. H., Preston G. B. The Algebraic Theory of Semigroups. Vol. 1 - Providence: Amer. Math. Soc., 1961.
50. Brandt H. Über eine Verallgemeinerung des Gruppenbegriffes // Math. Annalen. - 1927. - Vol. 96. - P. 360–367.
51. Bruck R. H. A Survey on Binary Systems. - New York: Springer-Verlag, 1966.
52. Bourbaki N. Elements of Mathematics: Algebra 1. - Springer, 1998.
53. Dudek W. A. Remarks to głazek's results on n-ary groups // Discuss. Math., Gen. Algebra Appl. - 2007. - Vol. 27. - № 2. - P. 199–233.
54. Dudek W. A., Michalski J. On retract of polyadic groups // Demonstratio Math. - 1984. - Vol. 17. - P. 281–301.
55. Thurston H. A. Partly associative operations // J. London Math. Soc. - 1949. - Vol. 24. - P. 260–271.
56. Belousov V. D. n-ary Quasigroups. - Kishinev: Shtintsa, 1972. - 225 p.
57. Sokhatsky F. M. On the associativity of multiplace operations // Quasigroups Relat. Syst. - 1997. - Vol. 4. - P. 51–66.
58. Michalski J. Covering k-groups of n-groups // Archiv. Math. - 1981. - Vol. 17. - № 4. - P. 207–226.
59. Pop M. S., Pop A. On some relations on n-monoids // Carpathian J. Math. - 2004. - Vol. 20. - № 1. - P. 87–94.
60. Čupona G., Trpenovski B. Finitary associative operations with neutral elements // Bull. Soc. Math. Phys. Macedoine. - 1961. - Vol. 12. - P. 15–24.
61. Evans T. Abstract mean values // Duke Math J. - 1963. - Vol. 30. - P. 331–347.
62. Głazek K., Gleichgewicht B. Abelian n-groups // Colloq. Math. Soc. J. Bolyai. Universal Algebra (Esztergom, 1977). - Amsterdam. North-Holland, 1982. - P. 321–329.
63. Stojaković Z., Dudek W. A. On σ-permutable n-groups // Publ. Inst. Math., Nouv. Sér. - 1986. - Vol. 40(54). - P. 49–55.
64. Rusakov S. A. Some Applications of n-ary Group Theory. - Minsk: Belaruskaya navuka, 1998. - 180 p.
65. Celakoski N. On some axiom systems for n-groups // Mat. Bilt. - 1977. - Vol. 1. - P. 5–14.
66. Gal'mak A. M. n-ary groups, Part 1. - Gomel: Gomel University, 2003. - 195 p.
67. Ǔsan J. n-groups in the light of the neutral operations // Math. Moravica. - 2003. - Vol. Special vol.
68. Sokolov E. I. On the theorem of Gluskin-Hosszú on Dörnte groups // Mat. Issled. - 1976. - Vol. 39. - P. 187–189.
69. Yurevych O. V. Criteria for invertibility of elements in associates // Ukr. Math. J. - 2001. - Vol. 53. - № 11. - P. 1895–1905.
70. Timm J. Verbandstheoretische Behandlung n-stelliger Gruppen // Abh. Math. Semin. Univ. Hamb. - 1972. - Vol. 37. - P. 218–224.
71. Dudek W. A. Remarks on n-groups // Demonstratio Math. - 1980. - Vol. 13. - P. 165–181.
72. Dudek W. A. Autodistributive n-groups // Annales Sci. Math. Polonae, Commentationes Math. - 1993. - Vol. 23. - P. 1–11.
73. Heine E. Handbuch der Kugelfunktionan. - Berlin: Reimer, 1878.
74. Kac V., Cheung P. Quantum calculus. - New York: Springer, 2002. - 112 p.
75. Petrescu A. On the homotopy of universal algebras. I. // Rev. Roum. Math. Pures Appl. - 1977. - Vol. 22. - P. 541–551.
76. Halaš R. A note on homotopy in universal algebra // Acta Univ. Palacki. Olomuc., Fac. Rerum Nat., Math. - 1994. - Vol. 33. - № 114. - P. 39–42.
77. Głazek K., Gleichgewicht B. Bibliography of n-groups (polyadic groups) and some group like n-ary systems // Proceedings of the Symposium on n-ary structures. - Skopje. Macedonian Academy of Sciences and Arts, 1982. - P. 253–289.
78. Fujiwara T. On mappings between algebraic systems // Osaka Math. J. - 1959. - Vol. 11. - P. 153–172.
79. Vidal J. C., Tur J. S. A 2-categorial generalization of the concept of institution // Stud. Log. - 2010. - Vol. 95. - № 3. - P. 301–344.
80. Novotný M. Homomorphisms of algebras // Czech. Math. J. - 2002. - Vol. 52. - № 2. - P. 345–364.
81. Novotný M. Mono-unary algebras in the work of Czechoslovak mathematicians // Arch. Math., Brno. - 1990. - Vol. 26. - № 2-3. - P. 155–164.
82. Gal'mak A. M. Generalized morphisms of algebraic systems // Vopr. Algebry. - 1998. - Vol. 12. - P. 36–46.
83. Gal'mak A. M. Generalized morphisms of Abelian m-ary groups // Discuss. Math. - 2001. - Vol. 21. - № 1. - P. 47–55.
84. Gal'mak A. M. n-ary groups, Part 2. - Minsk: Belarus State University, 2007. - 324 p.
85. Goetz A. On weak isomorphisms and weak homomorphisms of abstract algebras // Colloq. Math. - 1966. - Vol. 14. - P. 163–167.
86. Marczewski E. Independence in abstract algebras. Results and problems // Colloq. Math. - 1966. - Vol. 14. - P. 169–188.
87. Głazek K., Michalski J. On weak homomorphisms of general non-indexed algebras // Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. - 1974. - Vol. 22. - P. 651–656.

88. Głazek K. Weak homomorphisms of general algebras and related topics // Math. Semin. Notes, Kobe Univ. - 1980. - Vol. 8. - P. 1–36.
89. Traczyk T. Weak isomorphisms of Boolean and Post algebras // Colloq. Math. - 1965. - Vol. 13. - P. 159–164.
90. Denecke K., Wismath S. L. Universal Algebra and Coalgebra. - Singapore: World Scientific, 2009.
91. Denecke K., Saengsura K. State-based systems and (F_1, F_2)-coalgebras // East-West J. Math.,. - 2008. - Vol. Spec. Vol. - P. 1–31.
92. Chung K. O., Smith J. D. H. Weak homomorphisms and graph coalgebras // Arab. J. Sci. Eng., Sect. C, Theme Issues. - 2008. - Vol. 33. - № 2. - P. 107–121.
93. Kolibiar M. Weak homomorphisms in some classes of algebras // Stud. Sci. Math. Hung. - 1984. - Vol. 19. - P. 413–420.
94. Głazek K., Michalski J. Weak homomorphisms of general algebras // Commentat. Math. - 1977. - Vol. 19 (1976). - P. 211–228.
95. Czákány B. On the equivalence of certain classes of algebraic systems // Acta Sci. Math. (Szeged). - 1962. - Vol. 23. - P. 46–57.
96. Mal'tcev A. I. Free topological algebras // Izv. Akad. Nauk SSSR. Ser. Mat. - 1957. - Vol. 21. - P. 171—-198.
97. Mal'tsev A. I. The structural characteristic of some classes of algebras // Dokl. Akad. Nauk SSSR. - 1958. - Vol. 120. - P. 29–32.
98. Ellerman D. A theory of adjoint functors – with some thoughts about their philosophical significance // What is category theory? - Monza (Milan). Polimetrica, 2006. - P. 127–183.
99. Ellerman D. Adjoints and emergence: applications of a new theory of adjoint functors // Axiomathes. - 2007. - Vol. 17. - № 1. - P. 19–39.
100. csi B. P. On Morita contexts in bicategories // Applied Categorical Structures. - 2011. - P. 1–18.
101. Chronowski A. A ternary semigroup of mappings // Demonstr. Math. - 1994. - Vol. 27. - № 3-4. - P. 781–791.
102. Chronowski A. On ternary semigroups of homomorphisms of ordered sets // Arch. Math. (Brno). - 1994. - Vol. 30. - № 2. - P. 85–95.
103. Chronowski A., Novotný M. Ternary semigroups of morphisms of objects in categories // Archivum Math. (Brno). - 1995. - Vol. 31. - P. 147–153.
104. Gal'mak A. M. Polyadic associative operations on Cartesian powers // Proc. of the Natl. Academy of Sciences of Belarus, Ser. Phys.-Math. Sci. - 2008. - Vol. 3. - P. 28–34.
105. Gal'mak A. M. Polyadic analogs of the Cayley and Birkhoff theorems // Russ. Math. - 2001. - Vol. 45. - № 2. - P. 10–15.
106. Gleichgewicht B., Wanke-Jakubowska M. B., Wanke-Jerie M. E. On representations of cyclic n-groups // Demonstr. Math. - 1983. - Vol. 16. - P. 357–365.
107. Wanke-Jakubowska M. B., Wanke-Jerie M. E. On representations of n-groups // Annales Sci. Math. Polonae, Commentationes Math. - 1984. - Vol. 24. - P. 335–341.
108. Berezin F. A. Introduction to Superanalysis. - Dordrecht: Reidel, 1987. - 421 p.
109. Kirillov A. A. Elements of the Theory of Representations. - Berlin: Springer-Verlag, 1976.
110. Husemöller D., Joachim M., Jurčo B., Schottenloher M. G-spaces, G-bundles, and G-vector bundles // Basic Bundle Theory and K-Cohomology Invariants. - Berlin-Heidelberg. Springer, 2008. - P. 149-161.
111. Mal'tcev A. I. On general theory of algebraic systems // Mat. Sb. - 1954. - Vol. 35. - № 1. - P. 3–20.
112. Gal'mak A. M. Translations of n-ary groups. // Dokl. Akad. Nauk BSSR. - 1986. - Vol. 30. - P. 677–680.
113. Zeković B., Artamonov V. A. A connection between some properties of n-group rings and group rings // Math. Montisnigri. - 1999. - Vol. 15. - P. 151–158.
114. Zeković B., Artamonov V. A. On two problems for n-group rings // Math. Montisnigri. - 2002. - Vol. 15. - P. 79–85.
115. Carlsson R. N-ary algebras // Nagoya Math. J. - 1980. - Vol. 78. - № 1. - P. 45–56.
116. Bremner M., Hentzel I. Identities for generalized lie and jordan products on totally associative triple systems // J. Algebra. - 2000. - Vol. 231. - № 1. - P. 387–405.
117. Madore J. Introduction to Noncommutative Geometry and its Applications. - Cambridge: Cambridge University Press, 1995.
118. Kogorodski L. I., Soibelman Y. S. Algebras of Functions on Quantum Groups. - Providence: AMS, 1998.
119. Chari V., Pressley A. A Guide to Quantum Groups. - Cambridge: Cambridge University Press, 1996.
120. Majid S. Foundations of Quantum Group Theory. - Cambridge: Cambridge University Press, 1995.
121. Drinfeld V. G. Quantum groups // Proceedings of the ICM, Berkeley. - Phode Island. AMS, 1987. - P. 798–820.

Steven Duplij (Stepan Anatolievich Douplii) is a theoretical physicist, a Lead Staff Researcher of the Theory Group, Nuclear Physics Laboratory at V.N. Karazin Kharkov National University, and Doctor of Physical and Mathematical Sciences. He has more than hundred scientific publications, several monographs. The main scientific directions are supersymmetry and supermanifolds, quantum groups and their actions, singular theories and constrained systems, nonlinear and conformal electrodynamics, multigravity and Pauli-Fierz models, ternary and polyadic structures, von Neumann regularity and semigroups, DNA theory and genetic code. Editor-compiler of "Concise Encylopedia of Supersymmetry"(Springer), he has received grants from the Fulbright and Humboldt Foundations.