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ALGEBRAIZATION IN STABILITY PROBLEM FOR STATIONARY WAVES OF THE KLEIN-GORDON EQUATION

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Nonlinear traveling waves of the Klein-Gordon equation with cubic nonlinearity are considered. These waves are described by the nonlinear ordinary differential equation of the second order having the energy integral. Linearized equation for variation obtained for such waves is transformed to the ordinary one using separation of variables. Then so-called algebraization by Ince is used. Namely, a new independent variable associated with the solution under consideration is introduced to the equation in variations. Integral of energy for the stationary waves is used in this transformation. An advantage of this approach is that an analysis of the stability problem does no need to use the specific form of the solution under consideration. As a result of the algebraization, the equation in variations with variable in time coefficients is transformed to equation with singular points. Indices of the singularities are found. Necessary conditions of the waves stability are obtained. Solutions of the variational equation, corresponding to boundaries of the stability/instability regions in the system parameter space, are constructed in power series by the new independent variable. Infinite recurrent systems of linear homogeneous algebraic equations to determine coefficients of the series can be written. Non-trivial solutions of these systems can be obtained if their determinants are equal to zero. These determinants are calculated up to the fifth order inclusively, then relations connecting the system parameters and corresponding to boundaries of the stability/ instability regions in the system parameter place are obtained. Namely, the relation between parameters of anharmonicity and energy of the waves are constructed. Analytical results are illustrated by numerical simulation by using the Runge-Kutta procedure for some chosen parameters of the system. A correspondence of the numerical and analytical results is observed.
KEY WORDS: the Klein-Gordon equation, stationary waves stability, Ince algebraization

The nonlinear Klein-Gordon equation appears in different physical problems, namely, in problems of wave propagation through a region of weak superconductivity so-called Jefferson transition), motion of dislocations in crystals, propagation of waves in ferromagnetic materials, propagation of laser pulses in a two-phase medium, studying surfaces with negative Gaussian curvature, relativistic effects etc. [1-4]. Besides, it can be considered as useful mathematical model to describe a behavior of various types of traveling waves. Among the articles on the stability of nonlinear traveling waves, we highlight the paper [5], where the stability of traveling waves in some general distributed nonlinear system is considered, and the paper [6], where the stability of traveling waves in some nonlinear chain is analyzed. A variety of analytical, numerical and hybrid techniques are used to study travelling waves and their properties in [7]. In the presented paper the Klein-Gordon equation with cubic nonlinearity is used to represent new method for studying the stability of traveling waves. Namely, the so-called Ince algebraization [8] is used. Note that this approach was successfully used earlier in study of the stability of nonlinear normal vibration modes in nonlinear systems with a finite number of degrees of freedom [9-11]. Besides, this procedure is similar to one proposed in the paper [5] for the stability of traveling waves problem, but results on such stability problem for concrete systems are not presented in this publication. The Ince algebraization is based on a choice of the new independent variable, determining the traveling wave under consideration. An advantage of the proposed approach is that we do not need to use a specific form of the solution under consideration in analysis of the stability problem.

The present paper aims at contributing of the Ince algebraization to the problem of nonlinear wave stability. Our task is to use the proposed approach in regard of the equations in variations for the traveling waves of the nonlinear Klein-Gordon equation. Numerical simulation illustrates obtained theoretical results.

THE GENERAL MODEL. STATIONARY TRAVELING WAVES

One considers the Klein-Gordon equation with cubic nonlinearity:

$$\frac{\partial^2 u}{\partial t^2} - {c_0}^2 \frac{\partial^2 u}{\partial x^2} + {\omega_0}^2 u = -qu^3 \tag{1}$$

Stationary traveling waves are presented in the following form:

$$u = \Phi(\varphi): \text{where } \varphi = kx - \omega t \tag{2}$$

where φ is the wave phase. Substituting (2) into equation (1), we obtain the following ordinary differential equation for describing traveling waves:

$$\frac{d^2\Phi}{d\varphi^2}(\omega^2 - {c_0}^2 k^2) + {\omega_0}^2 \Phi + q\Phi^3 = 0 \tag{3}$$

The energy integral here is written as follows:

$$\frac{1}{2}\left(\frac{d\Phi}{d\varphi}\right)^2(\omega^2 - {c_0}^2k^2) + {\omega_0}^2\frac{\boldsymbol{\Phi^2}}{2} + q\frac{\boldsymbol{\Phi^4}}{\boldsymbol{4}} = h. \tag{4}$$

In addition, from equation (3) we can obtain the following relation, which will be used later in analysis of the traveling wave stability:

$$\frac{d^2\Phi}{d\varphi^2} = \frac{-{\omega_0}^2\Phi - q\boldsymbol{\Phi^3}}{\omega^2 - {c_0}^2k^2}. \tag{5}$$

Besides, from the equation (4) we get the following relation which will be also used later:

$$\left(\frac{d\Phi}{d\varphi}\right)^2 = \frac{2\left(h - {\omega_0}^2\frac{\boldsymbol{\Phi^2}}{2} - q\frac{\boldsymbol{\Phi^4}}{\boldsymbol{4}}\right)}{\omega^2 - {c_0}^2k^2}. \tag{6}$$

EQUATION IN VARIATIONS

To study the stability of stationary waves, we write out, first of all, the linearized equation in variations $V(t,x)$ obtained for the solution (2). One has from the equation (3) the following:

$$\frac{\partial^2 V}{\partial t^2} = {c_0}^2\frac{\partial^2 V}{\partial x^2} - V({\omega_0}^2 + 3q\boldsymbol{\Phi^2}), \tag{7}$$

where the function $\Phi(\varphi)$ is determined by the equation (3).

As the first step, we now introduce the independent variables φ, t instead of the variables *x, t*. The variational equation (7) in the new variables is rewritten as follows:

$$\frac{\partial^2 V}{\partial \varphi^2}(\omega^2 - {c_0}^2k^2) - 2\omega\frac{\partial^2 V}{\partial \varphi \partial t} + \frac{\partial^2 V}{\partial t^2} = -V({\omega_0}^2 + 3q\boldsymbol{\Phi^2}). \tag{8}$$

Then we use the separation of variables as V=e^{st} ž(φ) and the additional transformation:

$$\check{z}(\varphi) = e^{A\varphi}W, \tag{9}$$

where $A = \frac{s\omega}{\omega^2 - {c_0}^2k^2}$. As a result, instead of the equation (8) we get the following ODE in variations:

$$\frac{d^2W}{d\varphi^2}(\omega^2 - {c_0}^2k^2) = -W(B - {\omega_0}^2 - 3q\boldsymbol{\Phi^2}), \tag{10}$$

where $B = \frac{s^2}{{c_0}^2k^2 - \omega^2}$.

Note that since the parameter s^2 is presented in equation (10), in the case of real values of the parameter s, they can be both positive and negative. In view of the transformation (9), this leads to increase of the variations, that is, to instability. Thus, stability can be observed only under the condition that $s^2 < 0$. One has from here that for the stability there should be the following inequalities:

$$B > 0, \text{ if } {c_0}^2k^2 - \omega^2 < 0 \text{ and } B < 0, \text{ if } {c_0}^2k^2 - \omega^2 > 0. \tag{11}$$

Then, as a new independent variable, instead of φ, the variable Φ, determining the traveling wave under consideration, is chosen. Now, after some transformations, the equation of variations can be presented as,

$$2\frac{d^2W}{d\Phi^2}\left(h - {\omega_0}^2\frac{\boldsymbol{\Phi^2}}{2} - q\frac{\boldsymbol{\Phi^4}}{\boldsymbol{4}}\right) - \frac{dW}{d\Phi}({\omega_0}^2\Phi + 3q\boldsymbol{\Phi^3}) + W(B - {\omega_0}^2 - 3q\boldsymbol{\Phi^2}) = 0, \tag{12}$$

whose singular points are obtained when coefficient near the second derivative is equal to zero. One has the following:

$$h - \omega_0^2\frac{\Phi^2}{2} - q\frac{\Phi^4}{4} \equiv (\Phi - \Phi_0)\mathrm{G}(\Phi, \Phi_0)=0, \tag{13}$$

where Φ_0 is a root of this equation.

The transformation to the equation in variations to the form of equation with singular points (12) is the so-called algebraization by Ince of the stability problem which was first presented in [8]. An advantage of this approach is that an analysis of the stability problem does no need to use the specific form of the solution *Φ(φ)*.

CONSTRUCTION OF BOUNDARIES OF THE STABILITY/INSTABILITY REGIONS

It is shown in [8] that boundaries of the stability/instability regions in the parameter space of the variational equation with singular points are determined by solutions presented in the following series:

$$W = z^r(a_0 + a_1 z + \cdots). \tag{14}$$

Here *r* is one of two indices of the variational equation singularity [12], and $z = (\Phi - \Phi_0)$. To determine the indices of the singular point Φ_0, we introduce the series (14) into equation (12). Collecting the terms with the lowest degree of z, we obtain the following equation to determine these indices:

$$r(r-1)(-{\omega_0}^2\Phi_0 - q\Phi_0^3) - r({\omega_0}^2\Phi_0 + 3q\Phi_0^3) = 0. \tag{15}$$

It follows that

$$r_1 = 0 \text{ and } r_2 = -\frac{2q\Phi_0^2}{{\omega_0}^2+q\Phi_0^2}. \tag{16}$$

Substituting now the series (14) corresponding to the zero index to the equation in variations (12) and equating the coefficients with the same degrees by *z*, we get the following infinite recurrent system (17) of linear homogeneous algebraic equations to determine coefficients of the series:

$$z^0\text{: } 4a_2\left[h - \omega_0^2{\Phi_0}^2 - \frac{q}{4}{\Phi_0}^4\right] - a_1\left(\Phi_0\omega_0^2 + q{\Phi_0}^3 - B + \omega_0^2 + 3q{\Phi_0}^2\right) - a_0\left(-B + \omega_0^2 + 3q{\Phi_0}^2 + \frac{q}{4}{\Phi_0}^4\right) = 0$$

$$z^1\text{: } 12a_3\left(h - \frac{\omega_0^2}{2}{\Phi_0}^2 - \frac{q}{4}{\Phi_0}^4\right) - a_2\left(4\Phi_0\omega_0^2 + 4q{\Phi_0}^3 + 2\omega_0^2 + 6q{\Phi_0}^2\right) + a_1\left(B - 2\omega_0^2 - 6q{\Phi_0}^2\right) - 6a_0q\Phi_0 = 0$$

$$z^2\text{: } -15a_3\left(\Phi_0\omega_0^2 + {\Phi_0}^3q\right) + a_2\left(B - 5\omega_0^2 - 12q{\Phi_0}^2\right) - 9a_1q\Phi_0 - 3a_0q = 0, \tag{17}$$

etc.

The system (17) has a non-trivial solution if its determinant is equal to zero. This determinant was calculated up to the fifth order inclusively, and, thus, the relation connecting the system parameters was obtained; as a result, boundaries of the stability/ instability regions in the system parameter space can be constructed. Note that boundaries obtained by calculation of determinants of the fourth and fifth orders are close, so, we did not calculate determinants of the highest order than five.

Substituting the series (14) corresponding to the root r_2 into the equation in variations (12) and equating the coefficients with the same degrees by *z*, we obtain the infinite recurrent system of linear algebraic homogeneous equations for determining the expansion coefficients. Due to bulkiness of these algebraic equations, we present here only two first ones of them by equations (18):

$$z^{r_2-1}\text{: } 2\left[h - \omega_0^2\frac{{\Phi_0}^2}{2} - q\frac{{\Phi_0}^4}{4}\right]\left\{\frac{2q\Phi_0^2}{{\omega_0}^2+q\Phi_0^2}\left[\frac{2q\Phi_0^2}{{\omega_0}^2+q\Phi_0^2} - 1\right] + \frac{2q\Phi_0^2}{{\omega_0}^2+q\Phi_0^2}\right\}a_1 - 2a_0\frac{2q\Phi_0^2}{{\omega_0}^2+q\Phi_0^2}\left[\frac{2q\Phi_0^2}{{\omega_0}^2+q\Phi_0^2} - 1\right]\left[\left(\omega_0^2\Phi_0 + q{\Phi_0}^3\right)\right] - \frac{2q\Phi_0^2}{{\omega_0}^2+q\Phi_0^2}a_0\left(\omega_0^2\Phi_0 + 3q{\Phi_0}^3\right) = 0$$

$$z^{r_2}\text{: } a_0\,6q\Phi_0\left(\omega_0^2\Phi_0 + q{\Phi_0}^3\right)\left[h - \omega_0^2\frac{{\Phi_0}^2}{2} - q\frac{{\Phi_0}^4}{4}\right] + a_0 3q\Phi_0\left(2 + 2\omega_0^2\Phi_0 + 2q{\Phi_0}^3\right)\left[h - \omega_0^2\frac{{\Phi_0}^2}{2} - q\frac{{\Phi_0}^4}{4}\right] + a_0 2\left(\omega_0^2 + 3q{\Phi_0}^2\right)^2\left[h - \omega_0^2\frac{{\Phi_0}^2}{2} - q\frac{{\Phi_0}^4}{4}\right] - 2a_0\left(\omega_0^2 + 3q{\Phi_0}^2\right)^3 + a_0\left[-\omega_0^2\Phi_0 - q{\Phi_0}^3\right]\left(\omega_0^2 + 3q{\Phi_0}^2\right)\left(2 + 2\omega_0^2\Phi_0 + 2q{\Phi_0}^3\right) + a_0\left[-\frac{\omega_0^2}{2} - q\frac{3{\Phi_0}^2}{2}\right]\left(\omega_0^2\Phi_0 + q{\Phi_0}^3\right)\left(2 + 2\omega_0^2\Phi_0 + 2q{\Phi_0}^3\right) + a_1\left[-\omega_0^2\Phi_0 - q{\Phi_0}^3\right]\left(\omega_0^2\Phi_0 + q{\Phi_0}^3\right)\left(2 + 2\omega_0^2\Phi_0 + 2q{\Phi_0}^3\right) + a_1\left[-\frac{\omega_0^2}{2} - q\frac{3{\Phi_0}^2}{2}\right]\left[\left(2 + 4\omega_0^2\Phi_0 + 4q{\Phi_0}^3\right)\left(\omega_0^2 + 3{\Phi_0}^2q\right)\right] + a_2\left(2 + 2\omega_0^2\Phi_0 + 2q{\Phi_0}^3\right)\left(\omega_0^2\Phi_0 + q{\Phi_0}^3\right) + a_1\left[4\omega_0^2 + 12{\Phi_0}^2\right]\left[h - \omega_0^2\frac{{\Phi_0}^2}{2} - q\frac{{\Phi_0}^4}{4}\right] + a_1\left[-\omega_0^2\Phi_0 - q{\Phi_0}^3\right]\left(4 + 4\omega_0^2\Phi_0 + 4q{\Phi_0}^3\right) + 4a_2\left[h - \omega_0^2\frac{{\Phi_0}^2}{2} - q\frac{{\Phi_0}^4}{4}\right] - a_0\left[\omega_0^2 + 3{\Phi_0}^2q\right]\left[\omega_0^2\Phi_0 + q{\Phi_0}^3\right] - a_1\left(\omega_0^2\Phi_0 + q{\Phi_0}^3\right)\left[\omega_0^2\Phi_0 + q{\Phi_0}^3\right] - a_0\left(1 + \omega_0^2\Phi_0 + q{\Phi_0}^3\right)\left[\omega_0^2 + 3{\Phi_0}^2q\right] + a_0\left[B - \omega_0^2 - 3q{\Phi_0}^2\right] = 0. \tag{18}$$

The resulting system has a nontrivial solution if its determinant is equal to zero. This determinant was calculated up to the fourth order inclusive, and, thus, the relation connecting the system parameters was obtained; so, boundaries

of the stability/ instability regions in the system parameter space are constructed. Note that boundaries obtained by calculation of determinants of the fourth and fifth orders are also close in this case.

Boundaries of the stability/ instability regions in some places of the system parameters are presented in Figs. 1,2. Here we fix the traveling wave amplitude, namely, it is assumed that $\Phi_0 = 1$; the frequency $\omega = 1.5$; $c_0 = 1$; k = 0.1. The boundaries in the place of the parameters *B, h* for index r_1 are chosen in Fig. 1, where the parameter B varies in the interval [2.9….3.2]; the system energy *h* varies in the interval [0..0.1]. Here and further the dimensionless parameter of

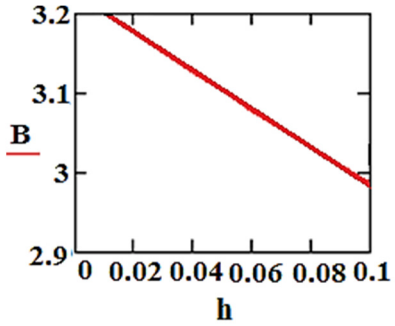


Fig. 1. The boundary between stability/ instability regions in the place (*B, h)* for index r_1.

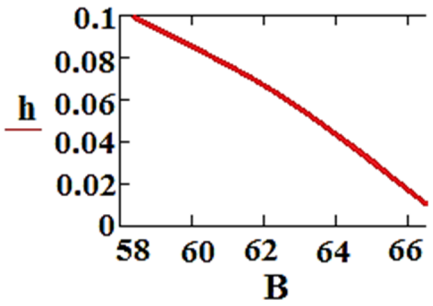


Fig. 2. The boundary between stability/ instability regions in the place (*B, h)* for index r_2.

anharmonicity q is calculated from the equation (13); $c_0{}^2k^2 - \omega^2 = -2.24$. The boundaries in the place of the parameters *B, h* for index r_2 are presented in Fig. 2, where the parameter B varies in the interval [58….66.5]; the system energy *h* varies in the interval [0..0.1]. In Figs. 1,2 regions of stability are situated on the left side of the obtained boundaries. In Fig.3 the system parameters are chosen as $\Phi_0 = 1$; $\omega = 0.5$; $c_0 = 1$; $k = 0.6$. The boundaries in the place of the parameters (*B, h*) for index r_2 are chosen in Fig. 3, where the parameter *B* varies in the interval [-5...0]; the system energy *h* varies in the interval [0…0.1]; the parameter q is calculated from the equation (12); $c_0{}^2k^2 - \omega^2 = 0.11$. Region of stability is disposed above the boundary showed in Fig. 3.

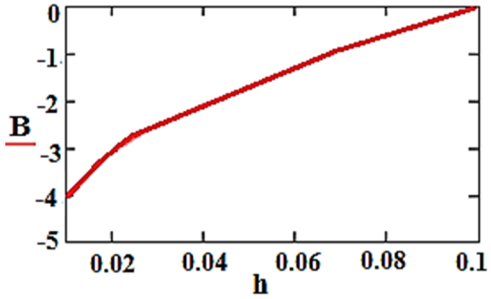
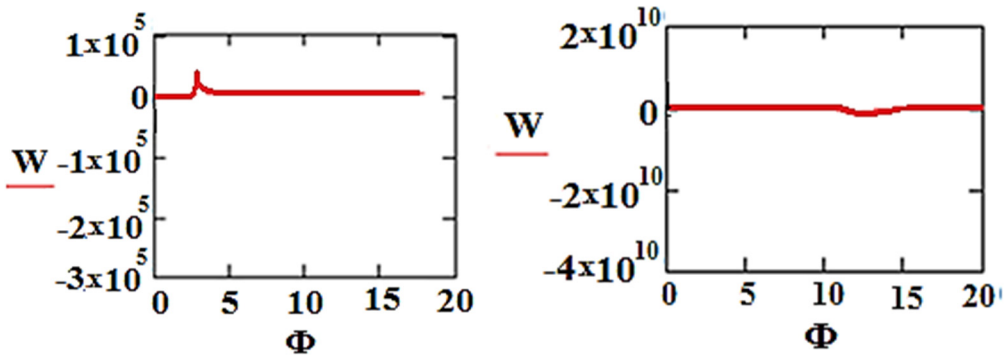


Fig. 3. The boundary between stability/ instability regions in the place (*B, h)* for index r_2.

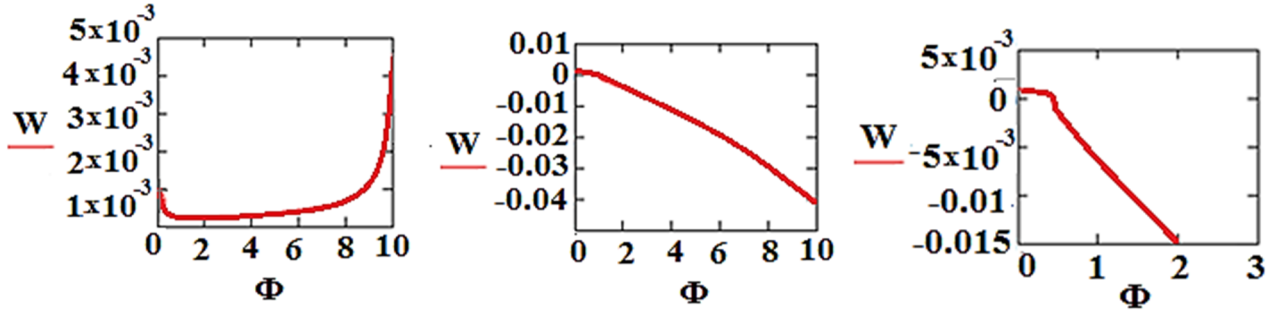
The Runge-Kutta test for the equation in variations (10) shows limited/ unlimited solutions when parameters are chosen from the stability/ instability regions obtained earlier. The same fixed parameters as were used above are used in the calculations. In Fig. 4 the limited solutions of the variational equation are shown. These solutions are chosen in the stability region in the place (*B, h)* presented in Fig.1. Namely, the system energy $h = 0.05$, the dimensionless parameter $B = 3.1$, the dimensionless parameter $q = -4.43$ are used for Fig. 4a. Fig. 4b is obtained for h=0.04, $B = 60$, $q = -4.34$.



a b

Fig. 4. Limited solutions of the equation in variations chosen in region of stability.

Calculations are made for h = 0.05, B = 3.1 (a) and for h=0.04, B = 60 (b). Other parameters correspond to ones used for Fig. 1.

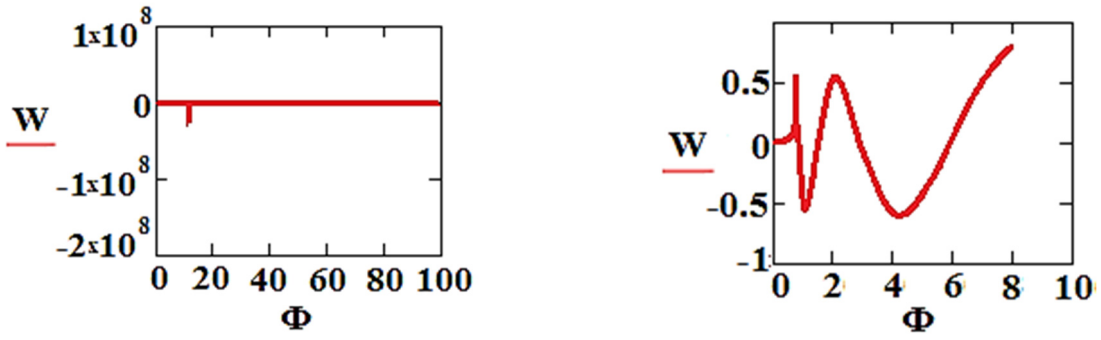


a b c

Fig. 5. Unimited solutions of the equation in variations chosen in regions of instability.
Calculations are made for h=0.08, B = 3.18 (a), for h = 0.06, B = 3.15 (b), and for h = 0.09, B = 66 (c). Other parameters correspond to ones used for Fig. 1,2.

Increasing solutions of the equation in variations are presented in Fig. 5. Namely, the solution presented in Fig. 5a is chosen in the instability region showed in Fig. 1; here h=0.08, B = 3.18, q = -4.18. The solution presented in Fig. 5b is also chosen in this instability region; here h=0.06, B = 3.15, q = -4.26. The solution presented in Fig. 5c is chosen in the instability region showed in Fig. 2 when h=0.09, B = 66, q = -4.1.

Then the limited and unlimited solutions of the variational equation obtained by the Runge-Kutta test are shown in Fig. 6 for regions of stability/ instability presented in Fig. 3. Parameters used in numerical calculations are the same. Namely, the limited solution from the region of stability is presented in Fig. 6a for h=0.01, B = -1, q = - 0.42. The unlimited solution from the region of instability is shown in Fig. 6b for h=0.04, B = -4, q = -0.26.



a b

Fig. 6. Limited solution (a) and unlimited solution (b) of the equation in variations.
Calculations are made for h=0.01, B = -1 (a) and for h=0.04, B = -4 (b). Other parameters correspond to ones used for Fig. 3.

CONCLUSION

We can conclude that the Ince algebraization can be successfully used to analyze stability of nonlinear traveling waves of the Klein-Gordon equation with cubic nonlinearity. Boundaries of the stability/ instability regions in place of the system parameters are obtained by analysis of the linearized equation in variations which is transformed to equation with singular points when the variable connected with solution under consideration is chosen as a new independent variable. Solutions corresponding to these boundaries are constructed in power series. Numerical simulation illustrates this analysis of the traveling wave stability. It seems that the method of algebraization can be used in the stability analysis of other types of nonlinear traveling and standing waves.

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АЛГЕБРАЇЗАЦІЯ В ЗАДАЧІ СТІЙКІСТІ СТАЦІОНАРНИХ ХВИЛЬ РІВНЯННЯ КЛЕЙНА-ГОРДОНА

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Розглянуто нелінійні бігучі хвилі рівняння Клейна-Гордона з кубічною нелінійністю. Ці хвилі описуються звичайним диференціальним рівнянням другого порядку, що має інтеграл енергії. Лінеарізоване рівняння у варіаціях для таких хвиль трансформується у звичайне диференціальне рівняння за допомогою розподілення змінних. Потім використовується так звана алгебраїзація за Айнсом. А саме, у рівняння в варіаціях вводиться нова незалежна змінна, що пов'язана з рішенням, яке розглядається. Під час такої трансформації використовується інтеграл енергії для стаціонарних хвиль. Перевага такого підходу зв'язана з тим, що для аналізу проблеми стійкості не треба використовувати специфічний вигляд рішення, що розглядається. В результаті такої алгебраїзації рівняння у варіаціях зі змінними за часом коефіцієнтами перетворюється у рівняння з особливими точками. Знайдено індекси особливих точок. Отримано необхідні умови стійкості хвиль. Рішення рівнянь у варіаціях, що відповідають межам регіонів стійкості / нестійкості в просторі параметрів системи побудовано у вигляді степеневих рядів за новою незалежною змінною. Можуть бути виписані нескінченні рекурентні системи алгебраїчних рівнянь для розрахунку коефіцієнтів цих рядів. Нетривіальні розв'язки таких систем можуть бути отримані, якщо їх визначники дорівнюють нулю. Ці визначники розраховано до п'ятого порядку включно, а потім отримано зв'язки між параметрами системи і відповідні межі регіонів стійкості/ нестійкості в площині параметрів системи. А саме, встановлено зв'язки між параметрами ангармонізму та енергії хвилі. Аналітичні результати проілюстровано чисельним моделюванням за допомогою процедури Рунге-Кутти. Спостерігається відповідність чисельних та аналітичних результатів.
КЛЮЧОВІ СЛОВА: рівняння Клейна-Гордона, стійкість стаціонарних хвиль, алгебраїзація за Айнсом

АЛГЕБРАИЗАЦИЯ В ЗАДАЧЕ УСТОЙЧИВОСТИ СТАЦИОНАРНЫХ ВОЛН УРАВНЕНИЯ КЛЕЙНА-ГОРДОНА С КУБИЧЕСКОЙ НЕЛИНЕЙНОСТЬЮ

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Рассмотрены нелинейные бегущие волны уравнения Клейна-Гордона с кубической нелинейностью. Эти волны описываются обыкновенным дифференциальным уравнением второго порядка, которое имеет интеграл энергии. Линеаризованное уравнение в вариациях для таких волн преобразуется в обыкновенное дифференциальное уравнение при помощи разделения переменных. Затем используется так называемая алгебраизация по Айнсу. А именно, новая независимая переменная, которая связана с решением, которое рассматривается, вводится в уравнение в вариациях. При этом используется интеграл энергии для стационарных волн. Преимущество такого подхода состоит в том, что для анализа проблемы устойчивости не нужно использование специфической формы решения, которое рассматривается. В результате подобной алгебраизации уравнение в вариациях с переменными по времени коэффициентами преобразуется в уравнение с особыми точками. Найдены индексы особых точек. Решения уравнений в вариациях, которые отвечают границам областей устойчивости/неустойчивости, построены в виде степенных рядов по новой независимой переменной. Могут быть выписаны бесконечные рекуррентные системы алгебраических уравнений для определения коэффициентов этих рядов. Нетривиальные решения таких систем могут быть получены, если их определители равны нулю. Эти определители вычисляются до пятого порядка включительно, затем зависимости между параметрами системы и соответствующие границы областей устойчивости/неустойчивости были получены. А именно, установлены связи между параметрами ангармонизму и енергии волны. Аналитические результаты иллюстрируются численным моделированием при помощи процедуры Рунге-Кутты. Наблюдается соответствие численных и аналитических результатов.
КЛЮЧЕВЫЕ СЛОВА: уравнения Клейна-Гордона, устойчивость стационарных волн, алгебраизация по Айнсу