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THE GOLDEN SECTION, FIBONACCI NUMBERS, MATHEMATICS OF HARMONY AND "GOLDEN" SCIENTIFIC REVOLUTION

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Abstract. The Publishing House "World Scientific" recently published two fundamental books: Alexey Stakhov "The Mathematics of Harmony" (2009) and Alexey Stakhov and Samuil Aranson "The "Golden" Non-Euclidean Geometry" (2016). In the given article the author develops the Mathematics of Harmony and "Golden" Non-Euclidean Geometry as a new interdisciplinary direction of modern science based on the golden section, Fibonacci numbers and their generalizations. The newest discoveries in different fields of modern science based on the Mathematics of Harmony, namely, mathematics (a general theory of hyperbolic functions and a solution to Hilbert's Fourth Problem, algorithmic measurement theory and "golden" number theory), computer science (the "golden" information technology), crystallography (quasi-crystals), chemistry (fullerenes), theoretical physics and cosmology (Fibonacci-Lorentz transformations, the "golden" interpretation of special theory of relativity and "golden" interpretation of the Universe evolution), botany (new geometric theory of phyllotaxis), genetics ("golden" genomatrices) and so on, creates a general picture of the "Golden" Scientific Revolution, which can influence fundamentally on the development of modern science and education.

Keywords: golden section, mathematics of harmony, binomial coefficients, Pascal's triangle, Fibonacci and Lucas hyperbolic functions, Hilbert's Fourth Problem, Fibonacci matrices, "golden" matrices.

> *The article is dedicated to the blessed memory of my father Peter Stakhov, student of historical faculty of Kharkov University and soldier of the famous Kharkov studbat, who died heroically in October 1941, near Moscow.*
>
> *Alexey Stakhov*

Preface

Differentiation of modern science and its division into separate branches do not allow often seeing the overall picture of science and the main trends of scientific development. However, in science there are research objects, which unite disparate scientific facts into a single picture. The *golden section* is one of these scientific objects. The ancient Greeks raised the *golden section* at the level of "aesthetic canon" and "major ratio" of the Universe. For centuries or even millennia, starting from **Pythagoras, Plato, Euclid,** this ratio was the subject of admiration and worship of eminent minds of humanity - in the Renaissance, **Leonardo da Vinci, Luca Pacioli, Johannes Kepler**, in the 19 century - **Zeizing, Lucas, Binet**. In the 20 century, the interest in this unique irrational number increased in the mathematical community, due to the works of Russian mathematician **Nikolay Vorobyov** and the American mathematician **Verner Hoggatt**.

We are developing in this article a new approach to Euclid's *Elements* (*Proclus hypothesis*) with purpose to find there the sources of new mathematical theory – the *Mathematics of Harmony*, based on the golden section and Platonic solids, and then basing on this approach to predict the most important trends and directions of modern science, which can lead to global processes in modern sci-

ence called *"Golden" Scientific Revolution*. The present article is a result of author's research in the field of the golden section, Fibonacci numbers and their applications in modern science [1-37].

In 1996 the lecture *The Golden Section and Modern Harmony Mathematics* (The Seventh International Conference on Fibonacci Numbers and Their Applications. Graz, Austria, July 15-19, 1996) [14] Alexey Stakhov put forward the concept of the Mathematics of Harmony as a new interdisciplinary direction of modern science. It plays an important integrating role for modern science and allows bringing together all scientific disciplines from the general point of view - the *golden section*. The ideas of the article [14] have been continued and generalized in two fundamental books published by World Scientific: Alexey Stakhov *"The Mathematics of Harmony. From Euclid to Contemporary Mathematics and Computer Science"* (2009) [11] and Alexey Stakhov and Samuil Aranson *"The "Golden" Non-Euclidean Geometry"* (2016) [83].

The main objective of this article is to consider modern science from this point of view. By means of collection and generalization of all the scientific facts and theories related to the *golden section*, the author have suddenly opened for himself the global picture of the Universe based on the *golden section*, and saw the main trend of modern science - the resurgence of the interest in the ideas of **Pythagoras, Plato and Euclid** on the numerical harmony of the Universe and the *golden section* what may result in the *"Golden" Scientific Revolution*. This revolution shows itself, first of all, in modern mathematics *("Golden" Fibonacci Goniometry* and *Hilbert's Fourth Problem*), theoretical physics (*Fibonacci-Lorentz transformations* and *"golden" interpretation of the Universe evolution*), and computer science (*«Golden» Information Technology*) and could become the basis for the mathematical education reform based on the ideas of harmony and the *golden section*.

1 Introduction

1.1. *Mathematics. The Loss of Certainty.* What is mathematics? What are its origins and history? What distinguishes mathematics from other sciences? What is the subject of mathematical research today? How does mathematics influence on the development of modern scientific revolution? What is a connection of mathematics and its history with mathematical education? All these questions always were interesting for both mathematicians, and representatives of other sciences. Mathematics was always a sample of scientific strictness. It is often named "Tsarina of Sciences," what is reflection of its special status in science and technology. For this reason, the occurrence of the book *Mathematics. The Loss of Certainty* [39], written by **Morris Kline** (1908-1992), Professor Emeritus of Mathematics Courant Institute of Mathematical Sciences (New York University), became a true shock for mathematicians. The book is devoted to the analysis of the crisis, in which mathematics found itself in the 20-th century as a result of its "illogical development."

Morris Kline's view on the deep connection of mathematics to theoretical natural sciences is expressed in the following words:

"Science had been the life blood and sustenance of mathematics. Mathematicians were willing partners with physicists, astronomers, chemists, and engineers in the scientific enterprise. In fact, during the 17th and 18th centuries and most of the 19th, the distinction between mathematics and theoretical science was rarely noted. And many of the leading mathematicians did far greater work in astronomy, mechanics, hydrodynamics, electricity, magnetism, and elasticity than they did in mathematics proper. Mathematics was simultaneously the queen and the handmaiden of the sciences."

However, according to the opinion of famous mathematicians **Felix Klein, Richard Courant** and many others, starting from 20-th century mathematics began to lose its deep connections with theoretical natural sciences and to concentrate its attention on its inner "pure" problems.

Thus, by following to **Felix Klein, Richard Courant** and other famous mathematicians, **Morris Kline** asserts that **the main reason of the contemporary crisis of mathematics is the severance of the relationship between mathematics and theoretical natural sciences, what is the greatest "strategic mistake" for the 20th century mathematics**.

1.2. *Dirac's Principle of Mathematical Beauty.* By discussing the fact why mathematics needs in theoretical natural sciences, we should address to *Dirac's Principle of Mathematical Beauty.* On May 13, 2006, the eminent Russian mathematician and academician Vladimir Arnold presented a public lecture: *"The complexity of finite sequences of zeros and units, and the geometry of finite functional spaces"* [40] at the Moscow Mathematical Society. Let us consider some of its general ideas. Arnold notes:

1. In my opinion, mathematics is simply a part of physics, that is, it is an experimental science, which discovers for mankind the most important and simple laws of nature.

2. We must begin with a beautiful mathematical theory. Dirac claims: "If this theory is really beautiful, then it necessarily appears as a perfect model of important physical phenomena. It is necessary to search for these phenomena to develop applications of the beautiful mathematical theory and to interpret them as predictions of new laws of physics." Thus, according to Dirac, all new branches of physics, including relativistic and quantum, are developing in this way.

At Moscow University there is a tradition that the distinguished visiting-scientists are requested to write on a blackboard a self-chosen inscription. When Dirac visited Moscow in 1956, he wrote *"A physical law must possess mathematical beauty."* This inscription is the famous *Principle of Mathematical Beauty* that Dirac developed during his scientific life. No other modern physicist has been preoccupied with the concept of beauty more than Dirac.

Thus, according to Dirac, the *Principle of Mathematical Beauty* is the primary criterion for a mathematical theory to be used as a model of physical phenomena. Of course, there is an element of subjectivity in the definition of the "beauty" of mathematics, but the majority of mathematicians agrees that "beauty" in mathematical objects and theories nevertheless exist.

Paul Adrien Maurice Dirac
(1902-1984)

Let's examine some of "beautiful" mathematical objects, which have a direct relation to the theme of this article.

1.3. *Platonic Solids.* We can find the beautiful mathematical objects in Euclid's *Elements*. As is well known, in Book XIII of his *Elements* Euclid presented a theory of 5 regular polyhedrons called *Platonic Solids* (Fig.1). Really, these remarkable geometrical figures got very wide applications in theoretical natural sciences; in particular, in crystallography (*quasi-crystals*), chemistry (*fullerenes*), biology and so on what is brilliant confirmation of *Dirac's Principle of Mathematical Beauty.*
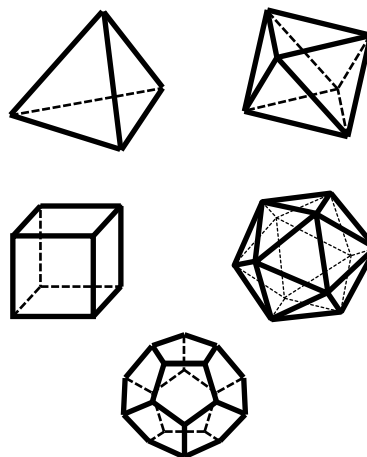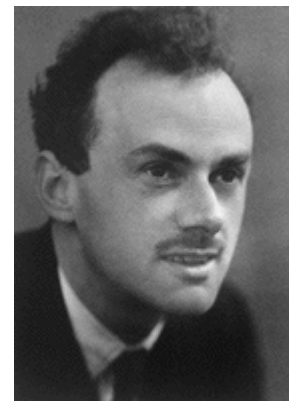
Fig. 1 - Platonic Solids: *tetrahedron, octahedron, cube, icosahedron, dodecahedron*

1.4. *Binomial coefficients, the binomial formula, and Pascal's triangle.*

For the given non-negative integers n and k, there is the following beautiful formula that sets the *binomial coefficients*:

$$C_n^k = \frac{n!}{k!(n-k)!}, \tag{1}$$

where $n! = 1 \times 2 \times 3 \times \ldots \times n$ is a *factorial* of *n.*

One of the most beautiful mathematical formulas, the *binomial formula,* is based upon the *binomial coefficients*:

$$(a+b)^n = a^n + C_n^1 a^{n-1} b + C_n^2 a^{n-2} b^2 + \ldots + C_n^k a^{n-k} b^k + \ldots + C_n^{n-1} a b^{n-1} + b^n. \tag{2}$$

There is a very simple recurrence method for the calculation of the *binomial coefficients* based on their following graceful properties called *Pascal's rule*:

$$C_{n+1}^k = C_n^{n-1} + C_n^k. \tag{3}$$

By using the recurrence relation (3) and taking into consideration that $C_n^0 = C_n^n = 1$ and $C_n^k = C_n^{n-k}$, we can construct the following beautiful table of *binomial coefficients* called *Pascal's triangle* (see Table 1).

Table 1 - Pascal's triangle

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1
1	6		15		20		15		6	1
1	7	21		35		35		21	7	1
1	8	28	56		70		56	28	8	1
1	9	36	84	126		126	84	36	9	1

1.4. *Fibonacci and Lucas numbers.* Let us consider the simplest recurrence relation:

$$F_n = F_{n-1} + F_{n-2}, \tag{4}$$

where $n = 0, \pm1, \pm2, \pm3, \ldots$. This recurrence relation was introduced for the first time by the famous Italian mathematician **Leonardo of Pisa** (nicknamed **Fibonacci**).

For the seeds

$$F_0 = 0 \text{ and } F_1 = 1, \tag{5}$$

the recurrence relation (4) generates a numerical sequence called *Fibonacci numbers* (see Table 2).

In the 19th century the French mathematician **Francois Edouard Anatole Lucas** (1842-1891) introduced the so-called *Lucas numbers* (see Table 2) given by the recurrence relation

$$L_n = L_{n-1} + L_{n-2} \tag{6}$$

with the seeds

$$L_0 = 2 \text{ and } L_1 = 1 \tag{7}$$

Table 2 shows the so-called "extended" Fibonacci and Lucas numbers, which are considered for positive and negative values of the index n. It follows from Table 2 that the "extended" Fibonacci and Lucas numbers build up two infinite numerical sequences, each possessing graceful mathematical properties.

Table 2 - The "extended" Fibonacci and Lucas numbers

n	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55
F_{-n}	0	1	-1	2	-3	5	-8	13	-21	34	-55
L_n	2	1	3	4	7	11	18	29	47	76	123
L_{-n}	2	-1	3	-4	7	-11	18	-29	47	-76	123

As can be seen from Table 2, for the odd indices $n = 2k+1$ the elements F_n and F_{-n} of the *Fibonacci sequence* coincide, that is, $F_{2k+1} = F_{-2k-1}$, and for the even indices $n = 2k$ they are opposite in sign, that is, $F_{2k} = -F_{-2k}$. For the Lucas numbers L_n all is vice versa, that is, $L_{2k} = L_{-2k}; L_{2k+1} = -L_{-2k-1}$.

1.5. *Cassini' formula.* In the 17th century the famous astronomer **Giovanni Domenico Cassini** (1625-1712) deduced the following beautiful formula, which connects three adjacent "extended" Fibonacci numbers in the Fibonacci sequence:

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}. \qquad (8)$$

This wonderful formula evokes a reverent thrill, if we imagine that this formula is valid for any value of *n* (*n* can be any integer within the limits of $-\infty$ to $+\infty$). The alternation of +1 and -1 in the formula (8) within the sequential passage of all "extended" Fibonacci sequence leads to genuine aesthetic enjoyment by its rhythm and beauty.

1.6. **The Golden Section.** *If we take the ratio of two adjacent Fibonacci numbers F_n / F_{n-1} and direct this ratio towards infinity, we come at the following unexpected result:*

$$\lim_{n \to \infty} \frac{F_n}{F_{n-1}} = \Phi = \frac{1+\sqrt{5}}{2}, \qquad (9)$$

where Φ is the famous irrational number, which is the positive root of the algebraic equation:

$$x^2 = x+1. \qquad (10)$$

The number Φ has many beautiful names − *the golden section*, *golden number, golden mean, golden proportion,* and *the divine proportion* (see **Scott Olsen,** page 2 [40]).

The *golden section* or *division of a line segment in extreme and mean ratio* descended to us from Euclid's *Elements* [42]. Over the many centuries the *golden section* has been the subject of enthusiastic worship by outstanding scientists and thinkers including **Pythagoras, Plato, Leonardo da Vinci, Luca Pacioli, Johannes Kepler** and several others.

Note that formula (9) is sometimes called *Kepler's formula* after **Johannes Kepler** (1571-1630) who deduced it for the first time. Many outstanding mathematicians of the past century have proved the uniqueness of the *golden ratio* among other real numbers. In this connection, we should like to draw attention to the brochures of the Russian mathematicians **Alexander Khinchin** (1894-1959) [43] and **Nikolay Vorobyov** (1925-1995) [44]. As it is shown in these works, the unique feature of the *golden ratio* for number theory is the fact that among all irrational numbers **the *golden ratio* is most slowly approximated by rational fractions.** That is, we are talking about the representation of *golden ratio* in the form of a continued fraction as follows:

$$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}} \tag{11}$$

If now we will be approximating the *golden ratio* (11) by rational fractions m/n, which are convergent for Φ, then we come to the numerical sequence consisting of the ratios of the neighboring Fibonacci numbers:

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \ldots \rightarrow \Phi = \frac{1 + \sqrt{5}}{2}.$$

But these ratios represent the famous botanic *Law of phyllotaxis* [45], according to which pine cones, cacti, pineapples, sunflower heads, etc are formed. In other words, Nature uses the unique mathematical feature of the *golden ratio* in its remarkable constructions! This means that the *golden ratio* is not "mathematical fiction" because this unique irrational number exists in Nature! That is, the Fibonacci numbers are a brilliant embodiment of *Dirac's Principle of Mathematical Beauty*.

1.7. *Binet's formulas.* In the *19th* century, French mathematician **Jacques Philippe Marie Binet** (1786-1856) deduced the two magnificent *Binet's formulas*:

$$F_n = \frac{\Phi^n - (-1)^n \Phi^{-n}}{\sqrt{5}}; \; L_n = \Phi^n + (-1)^n \Phi^{-n}. \tag{12}$$

The analysis of *Binet's formulas* (12) gives us a possibility to feel "aesthetic pleasure" and once again to be convinced in the power of human intellect! Really, we know that the Fibonacci and Lucas numbers F_n and L_n always are integers. On the other hand, any power of the *golden ratio* is irrational number. It follows from *Binet's formulas* (12) that the integer numbers F_n and L_n can be represented as the difference or the sum of irrational numbers, the powers of the *golden ratio*!

1.8. *How the golden mean is reflected in modern mathematics and mathematical education?* It is well known the following Kepler's quote, concerning the *golden ratio*:

"Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first, we may compare to a measure of gold; the second we may name a precious stone."

Johannes Kepler (1571-1630)

The above Kepler's statement raises the significance of the *golden ratio* on the level of *Pythagorean Theorem,* one of the most famous theorems of geometry. As a result of the unilateral approach to mathematical education each schoolboy knows *Pythagorean Theorem*, but he has rather vague idea about the *golden ratio,* the second "treasure of geometry." The majority of school textbooks on geometry originate to Euclid's *Elements*. But then we can ask the question: why in the majority of them there is very small description of Euclidean *golden ratio*? There is an impression that "the materialistic pedagogics" has thrown out the *golden ratio* from mathematical education on the dump of

the "doubtful scientific concepts" together with astrology and others esoteric sciences where the *golden ratio* is widely used. We can consider this sad fact as one of the "strategic mistakes" of modern mathematical education [32,33].

Alexey Losev (1893 - 1988), the Russian prominent philosopher and researcher for the aesthetics of Ancient Greece and Renaissance, expressed his relation to the *golden ratio* and *Plato's cosmology* in the following words (cited from [46]):

"From Plato's point of view, and generally from the point of view of all antique cosmology, the Universe is a certain proportional whole that is subordinated to the law of harmonious division, the golden ratio... Greek system of cosmic proportions is considered sometimes in literature as curious result of unrestrained and preposterous fantasy. Full anti-scientific helplessness sounds in the explanations of those who declare this. However, we can understand the given historical and aesthetical phenomenon only in the connection with integral comprehension of history, that is, by using dialectical-materialistic idea of culture and by searching the answer in peculiarities of the ancient social existence."

We can ask the question: how the *golden ratio* is reflected in contemporary mathematics? Unfortunately, the answer is the following: only in the most impoverished manner. In mathematics, Pythagoras and Plato's ideas are considered sometimes as a "curious result of unrestrained and preposterous fantasy." Therefore, the majority of mathematicians consider the study of the *golden ratio* and its applications as an empty pastime, which is unworthy for serious mathematicians.

Unfortunately, we can also find neglectful relation to the *golden ratio* in contemporary theoretical physics. In 2006 "BINOM" publishing house (Moscow) published the interesting scientific book *Metaphysics: Century XXI* [47]. In the Preface to the book, its compiler and editor Professor **Vladimirov** *(Moscow University)* wrote:

"The third part of this book is devoted to a discussion of numerous examples of the manifestation of the golden ratio in art, biology and our surrounding reality. However, paradoxically, the golden ratio is not reflected enough in contemporary theoretical physics. In order to be convinced of this fact, it is enough to merely browse 10 volumes of Theoretical Physics by Landau and Lifshitz. The time has come to fill this gap in physics, all the more given that the golden ratio is closely connected with metaphysics and 'trinitarity' [the 'triune' nature of things]."

Thus, the neglect of the *golden ratio* and its scanty reflection in modern mathematics, mathematical education and theoretical physics is one more "strategic mistake" of modern mathematics, mathematical education and theoretical physics [32,33].

2. Proclus hypothesis: revolutionary idea in the mathematics history

As is known, the first mathematical knowledge originated in the ancient civilizations (Babylon, Egypt and other countries) and they were used for the solution of two important practical problems: *counting* of things and *measurement* of time and distances [48]. Ultimately, the *problem of counting* led to the first fundamental mathematical notion – *natural numbers*. The *problem of measurement* underlies *geometry* origin and then, after the discovery of *incommensurable line segments*, led to the second fundamental mathematical notion – *irrational numbers*. Natural and irrational numbers are the basic notions of the *Classical Mathematics*, which had originated in the ancient Greek science. When we study the ancient Greek science, we should point out on one more important problem, which had influenced fundamentally on the development of the Greek science, including mathematics. We are talking on the *harmony problem*, which was formulated for the first time by Pythagoras, Plato and other ancient thinkers. The harmony problem was connected closely with the *golden ratio*, which was raised in the ancient Greece to the level of aesthetic canon and main mathematical constant of the Universe.

There is very interesting point of view on Euclid's *Elements* suggested by **Proclus Diadochus** (412-485), the best commentator on Euclid's *Elements* [49]. As it is well-known, the concluding book of Euclid's *Elements*, Book XIII, is devoted to the description of the theory of the *five regular polyhedra* (Fig. 1), which played a predominate role in *Plato's cosmology*. They are well known in modern science under the name *Platonic Solids*.

Proclus had paid special attention to this fact. Usually, the most important data are presented in the final part of a scientific work. Based on this fact, Proclus put forward hypothesis that Euclid created his *Elements* primarily not for the description of the axiomatic approach to geometry (although this is very important), but in order to give a systematic theory of the construction of the 5 Platonic Solids, in passing throwing light on some of the most important achievements of the ancient Greek mathematics. Thus, Proclus' hypothesis allows one to suppose that it was well-known in ancient science that the *Pythagorean Doctrine on the Numerical Harmony of the Cosmos* and *Plato's Cosmology*, based on the regular polyhedra, were embodied in Euclid's *Elements*, the greatest Greek work of mathematics. From this point of view, **we can interpret Euclid's *Elements* as the first attempt to create a Mathematical Theory of Harmony what was the primary idea of the ancient Greek science**. This historical information is primary data for the development of new approach to the history of mathematics, described in [29,33,35,36].

A new approach to the mathematics origins is presented in Fig. 2. We can see that three "key" problems - *counting problem, measurement problem*, and *harmony problem* - underlie mathematics origin. The first two "key" problems resulted in the origin of two fundamental mathematics notions - *natural numbers* and *irrational numbers* that underlie the *Classical Mathematics*. The *harmony problem,* connected with the *division in the extreme and mean ratio* (Proposition II.11 of Euclid's *Elements*), resulted in the origin of the *Harmony Mathematics* **-** a new interdisciplinary direction of contemporary science, which has relation to contemporary mathematics, mathematical education, theoretical physics, and computer science. Such approach had resulted in the conclusion, which is unexpected for many mathematicians.
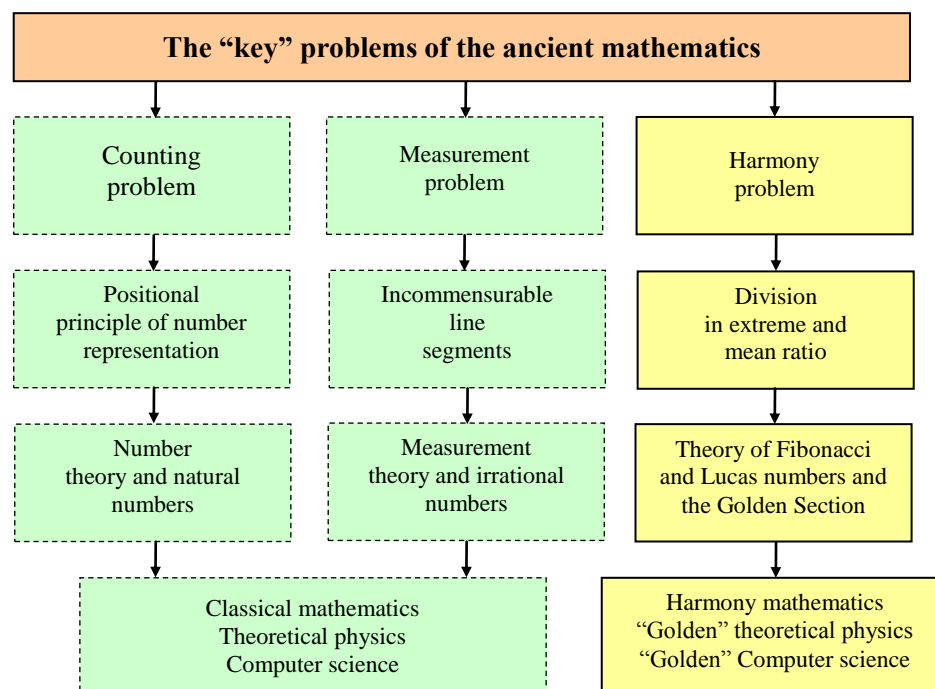

Fig. 2 - Three "key" problems of the ancient mathematics

Prove to be, in parallel with the *Classical Mathematics* one more mathematical direction - the *Harmony Mathematics* - was developing in ancient science. Similarly to the *Classical Mathematics,* the *Harmony Mathematics* takes its origin in Euclid's *Elements.* However, the *Classical Mathematics* accents its attention on "axiomatic approach," while the *Harmony Mathematics* is based on the *golden section* (Theorem II.11) and *Platonic Solids* described in the Book XIII of Euclid's *Elements.* Thus, Euclid's *Elements* is a source of two independent directions in the mathematics development - *Classical Mathematics* and *Harmony Mathematics*.

We affirm that the three greatest mathematical discoveries of the ancient mathematics – *positional principle of number representation, incommensurable line segments,* and *division in extreme and mean ratio (the golden section)* – were those mathematical discoveries, which influenced fundamentally on the mathematics at the stage of its origin. The *positional principle of number representation* (Babylon) became the "key" principle for the development of the concept of *natural numbers* and *number theory.* The *incommensurable line segments* led to the development of the concept of *irrational numbers.* The concepts of *natural numbers* and *irrational numbers* are two great mathematical concepts, which underlie the *Classical Mathematics.* The *division in extreme and mean ratio,* named later *the golden section,* is the third fundamental mathematical discovery, which underlies the *Mathematics of Harmony.*

During many centuries the main forces of mathematicians were directed on the creation of the *Classical Mathematics,* which became *Czarina of Natural Sciences.* However, the forces of many prominent mathematicians - since **Pythagoras, Plato** and **Euclid, Pacioli, Kepler** up to **Lucas, Binet, Vorobyov, Hoggatt** and so on - were directed on the development of the basic concepts and applications of the *Harmony Mathematics.* Unfortunately, these important mathematical directions developed separately one from other. A time came to unite the *Classical Mathematics* and the *Harmony Mathematics.* This unusual union can result in new scientific discoveries in mathematics and natural sciences. The newest discoveries in natural sciences, in particular, *Shechtman's quasicrystals* based on *Plato's icosahedron* (Nobel Prize of 2011) and *fullerenes* based on the *Archimedean truncated icosahedron* (Nobel Prize of 1996) do demand this union. All mathematical theories and directions should be united for one unique purpose to discover and explain Nature's Laws.

A new approach to the mathematics history (see Fig. 2) is very important for school mathematical education. This approach introduces in natural manner the idea of *harmony* and the *golden section* into school mathematical education. This allows to give pupils access to ancient science and to its main achievement – the *harmony idea* – and to tell them on the most important architectural and sculptor works of the ancient arts, based on the *golden section* (*Cheops pyramid, Nefertity, Parthenon, Doriphor, Venus* and so on).

3 The Mathematics of Harmony as a "beautiful" mathematical theory

The Mathematics of Harmony is described in [1-37]. The Mathematics of Harmony suggests new recurrence relations, which generates new numerical sequences and new numerical constants, which can be used for modeling different processes and phenomena of Nature. The most important of them are the following:

3.1. *Generalized Fibonacci p-numbers*. For a given p=0, 1, 2, 3, ... they are given by the following general recurrence relation [1]:

$$F_p(n) = F_p(n-1) + F_p(n-p-1); \quad F_p(0) = 0, F_p(1) = F_p(2) = ... = F_p(p) = 1. \quad (13)$$

Note that the recurrence formula (13) generates an infinite number of different recurrence sequences because every p generates its own recurrence sequences, in particular, the binary sequence 1, 2, 4, 8, 16, … for the case p=0 and classical Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, … for the case p=1.

3.2. *Generalized Lucas p-numbers* are given by the following general recurrence relation:

$$L_p(n) = L_p(n-1) + L_p(n-p-1); \quad L_p(0) = p+1, L_p(1) = L_p(2) = ... = L_p(p) = 1, \quad (14)$$

where p= 0, 1, 2, 3, ... is a given non-negative integer.

Note that the recurrence formula (14) generates an infinite number of different recurrence sequences because every p generates its own recurrence sequences, in particular, the binary sequence 1, 2, 4, 8, 16, … for the case p=0 and classical Lucas numbers 2, 1, 3, 4, 7, 11, 18, … for the case p=1.

3.3. *The golden p-proportions.* It is easy to prove [1] that the ratio of the adjacent Fibonacci and Lucas *p*-numbers aims in limit ($n \to \infty$) for some constant, that is,

$$\lim_{n \to \infty} \frac{F_p(n)}{F_p(n-1)} = \frac{L_p(n)}{L_p(n-1)} = \Phi_p, \tag{15}$$

where Φ_p is a positive root of the following algebraic equation:

$$x^{p+1} = x^p + 1, \tag{16}$$

which for *p=1* is reduced to the algebraic equation (10).

Note that the result (15) is a generalization of *Kepler's formula* (9) for the classical Fibonacci and Lucas numbers (*p=1*).

The positive root of Eq. (16) was named *golden p-proportion* [1]. It is easy to prove [1] that the powers of the *golden p-proportions* are connected between themselves by the following identity:

$$\Phi_p^n = \Phi_p^{n-1} + \Phi_p^{n-p-1} = \Phi_p \times \Phi_p^{n-1}, \tag{17}$$

where $n = 0, \pm 1, \pm 2, \pm 3, \dots$. It follows from (17) that each power of the "golden *p*-proportion" is connected with the preceding powers by the "additive" correlation $\Phi_p^n = \Phi_p^{n-1} + \Phi_p^{n-p-1}$ and by the "multiplicative" correlation $\Phi_p^n = \Phi_p \times \Phi_p^{n-1}$ (similarly to the classical "*golden mean*").

3.4. *Generalized Fibonacci* $\lambda-numbers$. Let $\lambda > 0$ is a given positive real number. Then we can consider the following recurrence relation [48-50]:

$$F_\lambda(n) = \lambda F_\lambda(n-1) + F_\lambda(n-2); \; F_\lambda(0) = 0, F_\lambda(1) = 1. \tag{18}$$

First of all, we note that for the case $\lambda = 1$ the recurrence relation (18) is reduced to the recurrence relation (4), which for the seeds (5) generates the classical *Fibonacci numbers*: 0, 1, 1, 2, 3, 5, 8, 13, …. For other values of λ the recurrence relation (18) generates infinite number of new recurrence numerical sequences. In particular, for the case $\lambda = 2$ the recurrence relation (18) generates the so-called *Pell numbers*: 0, 1, 2, 5, 12, 29, 70, … .

Table 3 shows the four "extended" Fibonacci λ-sequences, corresponding to the cases of $\lambda = 1, 2, 3, 4$.

Table 3 - The "extended" Fibonacci λ-numbers ($\lambda = 1, 2, 3, 4$)

n	0	1	2	3	4	5	6	7	8
$F_1(n)$	0	1	1	2	3	5	8	13	21
$F_1(-n)$	0	1	−1	2	−3	5	−8	13	−21
$F_2(n)$	0	1	2	5	12	29	70	169	408
$F_2(-n)$	0	1	−2	5	−12	29	−70	169	−408
$F_3(n)$	0	1	3	10	33	109	360	1189	3927
$F_3(-n)$	0	1	−3	10	−33	109	−360	1199	−3927
$F_4(n)$	0	1	4	17	72	305	1292	5473	23184
$F_4(-n)$	0	1	−4	17	−72	305	−1292	5473	−23184

3.5. *Generalized Cassini's formula: a unique property of the Fibonacci λ-numbers*. Cassini's formula (8) is one of the most remarkable identities for the classical Fibonacci numbers.

By studying the Fibonacci λ-numbers, **Alexey Stakhov** found in [38] the following unique mathematical property of the Fibonacci λ – *numbers*, which is true for all $\lambda=1,2,3,...$:

$$F_\lambda^2(n) - F_\lambda(n-1)F_\lambda(n+1) = (-1)^{n+1}. \tag{19}$$

As for the case $\lambda=1$ the formula (19) is reduced to the well-known Cassini's formula (8), the formula (19) was named in [38] generalized Cassini's formula, which sounds as follows.

For the given $\lambda=1,2,3,...$ the quadrate of any Fibonacci λ-number $F_\lambda(n)$ are always different from the product of the two adjacent Fibonacci λ-numbers $F_\lambda(n-1)$ and $F_\lambda(n+1)$, which surround the initial Fibonacci λ-number $F_\lambda(n)$, by the number 1; herewith the sign of the difference of 1 depends on the parity of n: if n is even, then the difference of 1 is taken with the sign "minus", otherwise, with the sign "plus".

Until now, we have assumed that only the classic Fibonacci numbers have this unusual property, given by Cassini's formula (8). However, as is shown in [38], a number of such numerical sequences are infinite. All the Fibonacci λ-numbers, generated by the recurrence relation (18) have similar property, given by the generalized Cassini's formula (19)!

As is well known, a study of integer sequences is the area of number theory. The Fibonacci λ-number are integers for the cases $\lambda=1,2,3,...$. Therefore, for many mathematicians in the field of number theory, the existence of the infinite number of the integer sequences, which are given by (18) and satisfy to the generalized Cassini's formula (19), may be a great surprise!

3.6. Metallic means. It follows from (18) the following algebraic equation:

$$x^2 - \lambda x - 1 = 0, \tag{20}$$

which for the case $\lambda=1$ is reduced to (10). A positive root of Eq. (20) produces infinite number of new "harmonic" constants – the *golden λ-proportions,* which are expressed by the following general formula:

$$\Phi_\lambda = \frac{\lambda + \sqrt{4 + \lambda^2}}{2}. \tag{21}$$

According to **Vera W. Spinadel** [50], the *golden λ-proportions* (21) are called also *metallic means* by analogy to the classical *golden mean.*

If we take $\lambda=1,2,3,4$ in (21), then we get the following mathematical constants having, according to **Vera W. Spinadel,** special titles:

$$\Phi_1 = \frac{1 + \sqrt{5}}{2} \ \left(\text{the golden mean, } \lambda=1\right); \ \Phi_2 = 1 + \sqrt{2}\left(\text{the silver mean, } \lambda=2\right);$$
$$\Phi_3 = \frac{3 + \sqrt{13}}{2}\left(\text{the bronze mean, } \lambda=3\right); \Phi_4 = 2 + \sqrt{5}\left(\text{the cooper mean, } \lambda=4\right). \tag{22}$$

Other metallic means ($\lambda \geq 5$) do not have special names:

$$\Phi_5 = \frac{5 + \sqrt{29}}{2}; \ \ \Phi_6 = 3 + 2\sqrt{10}; \ \ \Phi_7 = \frac{7 + 2\sqrt{14}}{2}; \ \ \Phi_8 = 4 + \sqrt{17}.$$

The *metallic means* (21) possess two remarkable properties [49]:

$$\Phi_\lambda = \sqrt{1 + \lambda\sqrt{1 + \lambda\sqrt{1 + \lambda\sqrt{...}}}} \ ; \ \ \Phi_\lambda = \lambda + \cfrac{1}{\lambda + \cfrac{1}{\lambda + \cfrac{1}{\lambda + ...}}}. \tag{23}$$

which are generalizations of similar properties for the classical *golden mean* $\left(\lambda=1\right)$:

$$\Phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{...}}}}; \ \ \Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + ...}}}. \tag{24}$$

4 Introduction into new theory on hyperbolic functions

4.1. *A history of hyperbolic functions and hyperbolic geometry.* Although **Johann Heinrich Lambert** (*1728-1777*), a French mathematician, is often credited with introducing hyperbolic functions, hyperbolic sine and cosine

$$sh(x) = \frac{e^x - e^{-x}}{2}; \quad ch(x) = \frac{e^x + e^{-x}}{2},$$
(25)

it was actually **Vincenzo Riccati** (*1707-1775*), Italian mathematician, who did this in the middle of the 18th century. Riccati found the standard *addition formulas*, similar to trigonometric identities, for hyperbolic functions as well as their derivatives. He revealed the relationship between the hyperbolic functions and the exponential function. For the first time, Riccati used the symbols *sh* and *ch* for the hyperbolic sine and cosine.

In 1826, the Russian mathematician **Nikolay Lobachevski** (*1792-1856*) made revolutionary mathematical discovery. We are talking on the *non-Euclidean geometry*. This *Lobachevski's geometry* is also named *hyperbolic geometry* because all mathematical relations of *Lobachevski's geometry* are based on the hyperbolic functions (25). The first published work on non-Euclidean geometry, Lobachevski's article *About the Geometry Beginnings*, was published in 1829 in *The Kazan Bulletin*. Three years later Hungarian mathematician **Janosh Bolyai** (*1802-1860*) published the article on non-Euclidean geometry, called the *Appendix*. After Gauss' death it was clear that he also had developed geometry similar to those of Lobachevski and Bolyai. A revolutionary significance of hyperbolic geometry consists of the fact that this geometry is beginning of *hyperbolic ideas* in theoretical natural sciences. We recall two remarkable properties of classical hyperbolic functions (25):

$$\textbf{Parity property}: sh(-x) = -sh(x); \quad ch(-x) = ch(x).$$
(26)

$$\textbf{Analog of Phythagoras theorem}: ch^2 x - sh^2 x = 1.$$
(27)

4.2. *Hyperbolic Fibonacci and Lucas functions.* In 1984 **Alexey Stakhov** published the book *Codes of the Golden Proportion* [3]. In this book *Binet's formulas* (12) were represented in the form not used in earlier mathematical literature:

$$F_n = \begin{cases} \dfrac{\Phi^n + \Phi^{-n}}{\sqrt{5}}, n = 2k+1 \\ \dfrac{\Phi^n - \Phi^{-n}}{\sqrt{5}}, n = 2k \end{cases}; \quad L_n = \begin{cases} \Phi^n + \Phi^{-n}, n = 2k \\ \Phi^n - \Phi^{-n}, n = 2k+1 \end{cases},$$
(28)

where $k = 0, \pm 1, \pm 2, \pm 3, \ldots$.

The similarity of *Binet's formulas* (28) to the classical hyperbolic functions (25) is so striking that the formulas (28) can be considered to be a prototype of a new class of hyperbolic functions, based on the *golden ratio*. That is to say, **Alexey Stakhov** in 1984 [3] predicted the appearance of a new class of hyperbolic functions, *hyperbolic Fibonacci and Lucas functions*. The first article on *hyperbolic Fibonacci and Lucas functions* was published by the Ukrainian mathematicians **Alexey Stakhov** and **Ivan Tkachenko** in 1993 [13] (by recommendation of academician **Yuri Mitropolski**). More recently, early of 21-th century, **Alexey Stakhov** and **Boris Rosin** have developed this idea further and introduced in [18] the so-called symmetrical hyperbolic Fibonacci and Lucas functions.

Symmetrical hyperbolic Fibonacci sine and cosine

$$sF(x) = \frac{\Phi^x - \Phi^{-x}}{\sqrt{5}}; \quad cF(x) = \frac{\Phi^x + \Phi^{-x}}{\sqrt{5}}$$
(29)

Symmetrical hyperbolic Lucas sine and cosine

$$sL(x) = \Phi^x - \Phi^{-x}; \quad cL(x) = \Phi^x + \Phi^{-x}. \tag{30}$$

4.3. *Hyperbolic and recursive properties of the symmetrical hyperbolic Fibonacci and Lucas functions*. Note that the symmetrical hyperbolic functions (29), (30), on the one hand, are similar to the classical hyperbolic functions (25), on the other hand, they are generated by *Binet's formulas* (28), given the *extended" Fibonacci and Lucas sequences*. This leads to the following features of the functions (29), (30), which we call **hyperbolic** and **recursive** properties of the functions (29), (30).

Hyperbolic properties

It is proved in [18], that the symmetrical hyperbolic Fibonacci and Lucas functions (29), (30) retain all the known properties of the classical hyperbolic functions (25). In this case, for example, the properties (26), (27) look as follows:

<u>**Parity property**</u> :

$$\begin{cases} sF(-x) = -sF(x); & cF(-x) = cF(x) \\ sL(-x) = -sL(x); & cL(-x) = cL(x) \end{cases} \tag{31}$$

<u>**Analogs of Pythagoras theorem**</u> :

$$\left[cF(x) \right]^2 - \left[sF(x) \right]^2 = \frac{4}{5}; \left[cL(x) \right]^2 - \left[sL(x) \right]^2 = 4 \tag{32}$$

Recursive properties

On the other hand, as it follows from a comparison of *Binet's formulas* (28) with the symmetrical functions (29), (30), the following unique properties connects the "extended" Fibonacci and Lucas sequences (28) with the symmetrical hyperbolical Fibonacci and Lucas functions (29), (30):

$$F_n = \begin{cases} sF(n) \text{ for } n = 2k \\ cF(n) \text{ for } n = 2k+1 \end{cases}; \tag{33}$$

$$L_n = \begin{cases} sL(n) \text{ for } n = 2k+1 \\ cL(n) \text{ for } n = 2k \end{cases}. \tag{34}$$

where $k = 0, \pm 1, \pm 2, \pm 3, ...$.

Note that **the property, given by the formulas (33), (34), is unique because the classical hyperbolic functions (25) do not possess such property**.

The properties (33), (34) gives us the right to name the functions (29), (30) the ***recursive hyperbolic functions.***

Let us consider the examples of the *recursive property* of the functions (29), (30) [18].

Theorem 1. *The following relations, which are similar to the recursive relations for the Fibonacci and Lucas numbers $F_{n+2} = F_{n+1} + F_n$ and $L_{n+2} = L_{n+1} + L_n$, are valid for the recursive hyperbolic Fibonacci and Lucas functions:*

<u>**Recursive relation for the Fibonacci hyperbolic functions**</u> :

$$sF(x+2) = cF(x+1) + sF(x)$$
$$cF(x+2) = sF(x+1) + cF(x)$$

<u>**Recursive relation for the Lucas hyperbolic functions**</u> :   (35)

$$sL(x+2) = cL(x+1) + sL(x)$$
$$cL(x+2) = sL(x+1) + cL(x)$$

Theorem 2 (a generalization of Cassini's formula for continues domain). *The following relations, which are similar to Cassini's formula* $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$, *are valid for the recursive hyperbolic Fibonacci functions:*

Cassini's formula :

$$\left[sF(x) \right]^2 - cF(x+1)cF(x-1) = -1 \qquad (36)$$

$$\left[cF(x) \right]^2 - sF(x+1)sF(x-1) = 1$$

4.4. *Fibonacci and "golden" matrices: a unique class of square matrices*

Fibonacci Q-matrices

It is known that a square matrix *A* is called *non-singular*, if its determinant is not equal to zero, that is

$$\det A \neq 0. \qquad (37)$$

In linear algebra, the non-singular square (n×n)-matrix is called invertible *because every non-singular matrix A has inverse matrix* A^{-1}, *which is connected with the initial* square *matrix A by the following correlation:*

$$AA^{-1} = I_n, \qquad (38)$$

where I_n is the identity (n×n)-matrix.
The *Fibonacci Q-matrix*

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad (39)$$

introduced in [53] is a partial case of the non-singular matrix.
If we raise the *Q*-matrix (39) to the *n*-th power, we obtain:

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}. \qquad (40)$$

It follows from *Cassini formula* (8), that the determinant of the *Q*-matrix (40) is equal:

$$\det Q^n = (-1)^n \qquad (41)$$

Fibonacci G_λ-matrices

Alexey Stakhov introduced in [30] the so-called *Fibonacci G_λ – matrix*:

$$G_\lambda = \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix}, \qquad (42)$$

where $\lambda > 0$ is a given positive real number. It is clear that for the case $\lambda=1$ the Fibonacci G_λ – matrix (42) is reduced to the Fibonacci Q-matrix (39).
The Fibonacci G_λ – matrix (42) is generating matrix for the Fibonacci λ - numbers (18) and has the following properties [30]:

$$G_\lambda^n = \begin{pmatrix} F_\lambda(n+1) & F_\lambda(n) \\ F_\lambda(n) & F_\lambda(n-1) \end{pmatrix} \qquad (43)$$

$$\det G_\lambda^n = (-1)^n. \qquad (44)$$

The "golden" matrices

Alexey Stakhov has introduced in [26] a special class of the square matrices called *"golden" matrices*. Their peculiarity consists of the fact that the *hyperbolic Fibonacci functions* (29) are elements of these matrices. Let us consider the simplest of them [26]:

$$Q_0(x) = \begin{pmatrix} cFs(2x+1) & sFs(2x) \\ sFs(2x) & cFs(2x-1) \end{pmatrix}; \; Q_1(x) = \begin{pmatrix} sFs(2x+2) & cFs(2x+1) \\ cFs(2x+1) & sFs(2x) \end{pmatrix} \tag{45}$$

If we calculate the determinants of the matrices (45), we obtain the following unusual results:

$$\det Q_0(x) = 1; \; \det Q_1(x) = -1. \tag{46}$$

4.5. *Theory of Fibonacci Numbers as a "Degenerate" Case of the Theory of the Recursive Hyperbolic Fibonacci and Lucas Functions.* As follows from Theorems 1 and 2, the two *"continuous"* identities (35), (36) for the recursive hyperbolic Fibonacci and Lucas functions always correspond to one *"discrete"* identity for the "extended" Fibonacci and Lucas numbers. Conversely, we can obtain the *"discrete"* identity for the "extended" Fibonacci and Lucas numbers by using two corresponding *"continuous"* identities for the recursive hyperbolic Fibonacci and Lucas functions (29), (30). As the "extended" Fibonacci and Lucas numbers, according to (33), (34), are the *"discrete"* cases of the recursive hyperbolic Fibonacci and Lucas functions (29), (30), this means that due the introduction of the *recursive hyperbolic Fibonacci and Lucas functions* (29), (30) [18], the classical *"theory of Fibonacci numbers"* [44] as if *"degenerates,"* because this theory is a partial (*"discrete"*) case of the more general (*"continuous"*) theory of the *recursive hyperbolic Fibonacci and Lucas functions* (29), (30). This conclusion is another unexpected result, which follows from the theory of the recursive hyperbolic Fibonacci and Lucas functions [18]. Such approach requires a **revision** of the existed "theory of Fibonacci numbers" [44] from the point of view of the more general (continues) theory of the *recursive hyperbolic Fibonacci and Lucas functions* (29), (30).

5. General theory of the recursive hyperbolic functions

5.1. *Gazale's formulas for Fibonacci and Lucas* λ-*numbers as analog of Binet's formulas.* Based on the *metallic means* (21), **Midchat Gazale** deduced in [51] the following remarkable formula, which allows representing the *Fibonacci λ - numbers* by the *metallic means* (21):

$$F_\lambda(n) = \frac{\Phi_\lambda^n - (-1)^n \Phi_\lambda^{-n}}{\sqrt{4+\lambda^2}}. \tag{47}$$

The formula (47) was named in [30] *Gazale's formula for the Fibonacci λ - numbers* after **Midchat Gazale** [51].

Alexey Stakhov deduced in [30] *Gazale's formula for the Lucas λ - numbers*:

$$L_\lambda(n) = \Phi_\lambda^n + (-1)^n \Phi_\lambda^{-n} \tag{48}$$

Note that for the case $\lambda = 1$ the formulas (47) and (48) are reduced to *Binet's formulas* (12). The formula (48) is analytical representation of new recurrence sequence, *Lucas λ-numbers,* which are given by the recurrence formula:

$$L_\lambda(n) = \lambda L_\lambda(n-1) + L_\lambda(n-2); \; L_\lambda(0) = 2, F_\lambda(1) = \lambda. \tag{49}$$

It is easy to prove [30] that *Gazale's formulas* can be represented in the following form:

$$F_\lambda(n) = \begin{cases} \dfrac{\Phi_\lambda^n - \Phi_\lambda^{-n}}{\sqrt{4+\lambda^2}} & \text{for} \quad n = 2k \\[3mm] \dfrac{\Phi_\lambda^n + \Phi_\lambda^{-n}}{\sqrt{4+\lambda^2}} & \text{for} \quad n = 2k+1 \end{cases} \tag{50}$$

$$L_\lambda(n) = \begin{cases} \Phi_\lambda^n - \Phi_\lambda^{-n} & \text{for} \quad n = 2k+1 \\ \Phi_\lambda^n + \Phi_\lambda^{-n} & \text{for} \quad n = 2k \end{cases} \tag{51}$$

Note that for the case *p=1 Gazale's formulas* (50), (51) are reduced to *Binet's formulas* (28). But *Binet's formulas* (12), (28) are well known in mathematics and belong to the category of outstand-

ing mathematical formulas, like *Euler's formula* and other formulas, which underlie the basis of mathematics. But *Gazale's formulas* (47), (48), (50), (51) are a generalization of *Binet's formulas* (12), (28) and therefore rightfully *Gazale's formulas* (47), (48), (50), (51) can be attributed to the category of outstanding mathematical formulas.

5.2. *Self-similarity and recursion: Gazale's hypothesis.* In mathematics, a **self-similar** object is exactly or approximately similar to a part of itself (i.e. the whole has the same shape as one or more of the parts). Many objects in the real world, such as coastlines, are statistically self-similar: parts of them show the same statistical properties at many scales. **Self-similarity** is a typical property of fractals and underlies botanic phenomenon of *phyllotaxis*. **Recursion** is the process of repeating items in a self-similar way and is brilliant example of reflection of *self-similarity principle* in mathematics. All the recurrent relations (4), (6), (13), (14), (18), (49) are based on the *self-similarity principle.*

Gazale's book *Gnomon. From Pharaohs to Fractals* [51] is devoted to mathematical justification of the *principle of self-similarity.* In this book, **Midhat Gazale** put forward the following hypothesis.

Gazale's hypothesis: *"A key role in the study of self-similarity play a numerical sequences, which I call here the Fibonacci sequence of the order m, where*

$$F_{m,n+2} = F_{m,n} + mF_{m,n+1}.\text{''}$$ (52)

If we compare the recurrence relation for the Fibonacci sequences of the order *m*, given by (52), to the introduced above Fibonacci *λ-numbers*, given by the recurrence relation (18), we come to the unexpected conclusion that the recurrence relations (18) and (52) coincide if we take: $m = \lambda$. This means that **the Fibonacci *λ-numbers* play a key role in the study of the principle of *self-similarity*.**

5.3. *Recursive hyperbolic Fibonacci and Lucas λ-functions.* In 2006 **Alexey Stakhov** has developed in [30] the so-called *hyperbolic Fibonacci and Lucas λ-functions,* which are a generalization of the *symmetrical hyperbolic Fibonacci and Lucas functions* (29), (30):

Hyperbolic Fibonacci λ-sine

$$sF_\lambda(x) = \frac{\Phi_\lambda^x - \Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}} = \frac{1}{\sqrt{4+\lambda^2}}\left[\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x - \left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x}\right]$$ (53)

Hyperbolic Fibonacci λ-cosine

$$cF_\lambda(x) = \frac{\Phi_\lambda^x + \Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}} = \frac{1}{\sqrt{4+\lambda^2}}\left[\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x + \left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x}\right]$$ (54)

Hyperbolic Lucas λ-sine

$$sL_\lambda(x) = \Phi_\lambda^x - \Phi_\lambda^{-x} = \frac{1}{\sqrt{4+\lambda^2}}\left[\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x - \left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x}\right]$$ (55)

Hyperbolic Lucas λ-cosine

$$cL_\lambda(x) = \Phi_\lambda^x + \Phi_\lambda^{-x} = \frac{1}{\sqrt{4+\lambda^2}}\left[\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x + \left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x}\right]$$ (56)

The formulas (53)-(56) give an infinite number of hyperbolic functions because every real number $\lambda > 0$ generates its own class of the hyperbolic functions (53)-(56). In particular, for the case $\lambda = 1$ the hyperbolic functions (53)-(56) are reduced to the *symmetrical hyperbolic Fibonacci and Lucas functions* (29), (30).

By comparing the *"extended" Fibonacci and Lucas* $\lambda - numbers$, represented by *Gazale's formulas* (50), (51) to the functions (53)-(56), it is easy to establish the following simple relationship between them:

$$F_\lambda\ n\ =\begin{cases} sF_\lambda\ n\ , & n=2k \\ cF_\lambda\ n\ , & n=2k+1 \end{cases} ; L_\lambda\ n\ =\begin{cases} cL_\lambda\ n\ , & n=2k \\ sL_\lambda\ n\ , & n=2k+1. \end{cases} \tag{57}$$

For the case *p=1* the formulas (57) are reduced to the relationships (33), (34).

5.4. *Hyperbolic and recursive properties of hyperbolic Fibonacci and Lucas λ-functions.* Let us study the properties of the hyperbolic Fibonacci and Lucas λ-functions (53) – (56).

Hyperbolic properties (example)

As example of hyperbolic properties for the functions (53) – (56) we consider the formulas for **analogs of Pythagoras Theorem**:

$$\left[cF_\lambda(x) \right]^2 - \left[sF_\lambda(x) \right]^2 = \frac{4}{4+\lambda^2} ; \left[cL_\lambda(x) \right]^2 - \left[sL_\lambda(x) \right]^2 = 4 \tag{58}$$

Note that for the case λ=1 the formulas (58) are reduced to the formulas (32).

Recursive properties (examples)

$$F_\lambda(n+2) = \lambda F_\lambda(n+1) + F_\lambda(n) \rightarrow \begin{cases} sF_\lambda(x+2) = \lambda cF_\lambda(x+1) + sF_\lambda(x) \\ cF_\lambda(x+2) = \lambda sF_\lambda(x+1) + cF_\lambda(x) \end{cases} \tag{59}$$

$$F_\lambda^2(n) - F_\lambda(n-1)F_\lambda(n+1) = (-1)^{n+1} \rightarrow \begin{cases} \left[sF_\lambda(x) \right]^2 - cF_\lambda(x+1)cF_\lambda(x-1) = -1 \\ \left[cF_\lambda(x) \right]^2 - sF_\lambda(x+1)sF_\lambda(x-1) = 1 \end{cases} \tag{60}$$

6 Hilbert's Fourth Problem and "Golden" Hyperbolic Geometry: revolution in hyperbolic geometry

6.1. *Hilbert's Fourth Problem.* In the lecture *Mathematical Problems* presented at the *Second International Congress of Mathematicians* (Paris, 1900), **David Hilbert** (*1862 – 1943*) had formulated his famous 23 mathematical problems. These problems determined considerably the development of the 20th century mathematics. This lecture is a unique phenomenon in the mathematics history and in mathematical literature. The Russian translation of Hilbert's lecture and its comments are given in the work [54]. In particular, *Hilbert's Fourth Problem* is formulated in [54] as follows:
 "Whether is possible from the other fruitful point of view to construct geometries, which with the same right can be considered the nearest geometries to the traditional Euclidean geometry".

In particular, **Hilbert** considered that *Lobachevski's geometry* and *Riemannian geometry* are nearest to the *Euclidean geometry*. In mathematical literature *Hilbert's Fourth Problem* is sometimes considered as formulated very vague what makes difficult its final solution. As it is noted in Wikipedia [55], *"the original statement of Hilbert, however, has also been judged too vague to admit a definitive answer."*

In spite of critical attitude of mathematicians to *Hilbert's Fourth Problem*, we should emphasize great importance of this problem for mathematics, particularly for geometry. Without doubts, Hilbert's intuition led him to the conclusion that *Lobachevski's geometry* and *Riemannian geometry*

do not exhaust all possible variants of non-Euclidean geometries. *Hilbert's Fourth Problem* directs attention of mathematicians at finding new non-Euclidean geometries, which are the nearest geometries to the traditional Euclidean geometry.

6.2. *A solution to Hilbert's Fourth Problem based on the Mathematics of Harmony.* As is known, the classical model of *Lobachevski's plane* in *pseudo-spherical coordinates* $(u,v), 0 < u < +\infty, -\infty < v < +\infty$ with the Gaussian curvature $K = -1$ *(Beltrami's interpretation of hyperbolic geometry on pseudo-sphere)* has the following form:

$$\left(ds\right)^2 = \left(du\right)^2 + sh^2\left(u\right)\left(dv\right)^2, \tag{61}$$

where *ds* is an element of length and *sh(u)* is hyperbolic sine.

In connection with *Hilbert's Fourth Problem*, **Alexey Stakhov** and **Samuil Aranson** suggested in [83] an infinite set of models (in dependence on real parameter $\lambda > 0$) of *Lobachevski's plane* at the coordinates $(u,v), 0 < u < +\infty, -\infty < v < +\infty$ of the *Gaussian curvature K= -1*, such that the metric form looks as follows:

$$\left(ds\right)^2 = \ln^2\left(\Phi_\lambda\right)\left(du\right)^2 + \frac{4+\lambda^2}{4}\left[sF_\lambda\left(u\right)\right]^2\left(dv\right)^2, \tag{62}$$

where $\Phi_\lambda = \frac{\lambda + \sqrt{4+\lambda^2}}{2}$ is the *metallic mean* and $sF_\lambda\left(u\right)$ is *hyperbolic Fibonacci λ-sine*.

The partial cases of the metric *λ-forms* of Lobachevski's plane (*golden, silver, bronze, cooper*), corresponding to different values of the parameter λ=1,2,3,4, are given in Table 4.

Table 4 – Metric λ-forms of Lobachevski's plane

Title	λ	Φ_λ	Analitical formula
General form	$\lambda > 0$	$\Phi_\lambda = \frac{\lambda+\sqrt{4+\lambda^2}}{2}$	$\left(ds\right)^2 = \ln^2\left(\Phi_\lambda\right)\left(du\right)^2 + \frac{4+\lambda^2}{4}\left[sF_\lambda\left(u\right)\right]^2\left(dv\right)^2$
"Golden" form	$\lambda = 1$	$\Phi_1 = \frac{1+\sqrt{5}}{2} \approx 1.61803$	$\left(ds\right)^2 = \ln^2\left(\Phi_1\right)\left(du\right)^2 + \frac{5}{4}\left[sFs\left(u\right)\right]^2\left(dv\right)^2$
"Silver" form	$\lambda = 2$	$\Phi_2 = 1 + \sqrt{2} \approx 2.1421$	$\left(ds\right)^2 = \ln^2\left(\Phi_2\right)\left(du\right)^2 + 2\left[sF_2\left(u\right)\right]^2\left(dv\right)^2$
"Bronze" form	$\lambda = 3$	$\Phi_3 = \frac{3+\sqrt{13}}{2} \approx 3.30278$	$\left(ds\right)^2 = \ln^2\left(\Phi_3\right)\left(du\right)^2 + \frac{13}{4}\left[sF_3\left(u\right)\right]^2\left(dv\right)^2$
"Cooper" form	$\lambda = 4$	$\Phi_4 = 2 + \sqrt{5} \approx 4.23607$	$\left(ds\right)^2 = \ln^2\left(\Phi_4\right)\left(du\right)^2 + 5\left[sF_4\left(u\right)\right]^2\left(dv\right)^2$
Classical form	$\lambda_e \approx 2.350402$	$\Phi_{\lambda_e} = e \approx 2.7182$	$\left(ds\right)^2 = \left(du\right)^2 + sh^2\left(u\right)\left(dv\right)^2$

Thus, these considerations result in the conclusion that the *λ-models of Lobachevski's plane* (62), based on the *recursive Fibonacci hyperbolic functions* (53), (54), result in an infinite number of new hyperbolic geometries, having general title *"Golden" Hyperbolic Geometry,* which together with the classical *Lobachevski geometry*, *Riemannian geometry* and *Minkovski geometry "can be considered the nearest geometries to the traditional Euclidean geometry"* (**David Hilbert**).

A new solution to Hilbert's Fourth Problem is brilliant confirmation of effective application of the Mathematics of Harmony to the solution of complicated mathematical problems.

7 New scientific principles based on the Golden Section

7.1. Generalized principle of the golden section. There are some general principles of the division of the whole (the "Unit") into two parts. The most known from them are dichotomy principle, which is based on the trivial identity

$$1 = 2^0 = 2^{-1} + 2^{-1}, \tag{63}$$

and golden section principle based on the identity:

$$1 = \Phi^0 = \Phi^{-1} + \Phi^{-2}, \tag{64}$$

where $\Phi = \left(1 + \sqrt{5}\right)/2$ is the *golden mean*.

It follows from the identity (17) more general principle

$$1 = \Phi_p^0 = \Phi_p^{-1} + \Phi_p^{-p-1}, \tag{65}$$

which is called in [23] the *Generalized Principle of the Golden Section.*

Consider the examples of the use of the Generalized Principle of the Golden Section *(65) in some natural phenomena.*

7.2. *Mathematical theory of biological populations.* As is known, Fibonacci numbers are a result of the solution to *Fibonacci's problem of "rabbit" reproduction*. Let us recall that the *Law of "rabbit" reproduction* boils down to the following rule. Each mature rabbit's pair *A* gives birth to a newborn rabbit pair *B* during one month. The newborn rabbit's pair becomes mature during one month and then in the following month said pair starts to give birth to one rabbit pair each month. Thus, the maturing of the newborn rabbits, that is, their transformation into a mature pair is performed in 1 month. We can model the process of "rabbit reproduction" by using two transitions:

$$A \to AB \tag{66}$$

$$B \to A. \tag{67}$$

Note that the transition (66) simulates the process of the newborn rabbit pair *B* birth and the transition (66) simulates the process of the maturing of the newborn rabbit pair *B*. The transition (66) reflects an *asymmetry* of rabbit reproduction because the mature rabbit pair *A* is transformed into two non-identical pairs, the mature rabbit pair *A* and the newborn rabbit pair *B*.

Note that we should treat "rabbits" in *Fibonacci's problem of "rabbit" reproduction* as some biological objects. For example, as is shown in [11], family tree of honeybees is based strictly on Fibonacci numbers.

Note that *Fibonacci's problem of "rabbit" reproduction* is a primary problem of the *mathematical theory of biological populations* [56].

By using the model of "rabbit reproduction," which is described by the transitions (66) and (67), we can generalize the *problem of "rabbit" reproduction* in the following manner. Let us give a non-negative integer *p≥0* and formulate the "*generalized Fibonacci's problem of "rabbit" reproduction*" for the condition when the transition of newborn rabbits into mature state is realized for *p* month, where *p*=0, 1, 2, 3, … .

It is clear that for the case *p*=1 the generalized variant of the "rabbit reproduction" problem coincides with the classical "rabbit reproduction" problem formulated by Fibonacci in 13th century.

Note that the case *p*=0 corresponds to the "idealized situation," when the rabbits become mature at once after birth. One may model this case by using the transition:

$$A \to AA. \tag{68}$$

It is clear that that the transition (68) reflects *symmetry* of "rabbit reproduction" when the mature rabbit pair *A* turns into two identical mature rabbit pairs *AA*. It is easy to show that for this case the rabbits are reproduced according to the above *dichotomy principle* (63), that is, the amount of rabbits doubles each month: 1, 2, 4, 8, 16, 32, … .

It is easy to prove that for the general case $p \geq 0$ a process of the "rabbit reproduction" is modelled by the recurrence relation (18) generating the *generalized Fibonacci p-numbers*. This means that the *generalized Fibonacci p-numbers* model some general principle of "rabbit reproduction" called the *generalized asymmetry principle of organic nature.*

7.3. *Fibonacci's division of biological cells.* At first appearance the above formulation of the *generalized problem of "rabbit reproduction"* appears to have no real physical sense. However, we should not hurry to such a conclusion! The article [57] is devoted to the application of the *generalized Fibonacci p-numbers* for the simulation of biological cell growth. The article affirms that *"in kinetic analysis of cell growth, the assumption is usually made that cell division yields two daughter*

cells symmetrically. The essence of the semi-conservative replication of chromosomal DNA implies complete identity between daughter cells. Nonetheless, in bacteria, insects, nematodes, and plants, cell division is regularly asymmetric, with spatial and functional differences between the two products of division…. Mechanism of asymmetric division includes cytoplasmic and membrane localization of specific proteins or of messenger RNA, differential methylation of the two strands of DNA in a chromosome, asymmetric segregation of centrioles and mitochondria, and bipolar differences in the spindle apparatus in mitosis." In the models of cell growth based on the Fibonacci 2- and 3-numbers are analyzed [57].

The authors of [57] made the following important conclusion: *"Binary cell division is regularly asymmetric in most species. Growth by asymmetric binary division may be represented by the generalized Fibonacci equation …. Our models, for the first time at the single cell level, provide rational bases for the occurrence of Fibonacci and other recursive phyllotaxis and patterning in biology, founded on the occurrence of regular asymmetry of binary division."*

8 The Mathematics of Harmony: a renaissance of the oldest mathematical theories

8.1. *Algorithmic measurement theory.* The first crisis in the foundations of mathematics was connected with a discovery of *incommensurable line segments.* This discovery turned back mathematics and caused the appearance of *irrational numbers.*

In 19-th century, **Dedekind** and then **Cantor** made an attempt to create a general measurement theory. For this purpose, they introduced the additional axioms into the group of the *continuity axioms.* For instance, let us consider *Cantor's axiom.*

Cantor's continuity axiom (Cantor's principle of nested segments). *If an infinite sequence of segments is given on a straight line A_0B_0, A_1B_1, A_2B_2, …, A_nB_n, …, such that each next segment is nested within the preceding one, and the length of the segments tends to zero, then there exists a unique point, which belongs to all the segments.*

The main result of the mathematical measurement theory that is based on the *continuity axioms* is a proof of the existence and uniqueness of the solution q of the *basic measurement equality*:

$$Q=qV, \tag{69}$$

where V is a measurement unit, Q is a measurable segment, and q is any real number named a *result of measurement.*

However, the *Cantor's axiom* raises the most doubts. According to this axiom, a measurement is a process, which is completed during infinite time. Such idea is a brilliant example of the Cantorian style of mathematical thinking based on the concept of *actual infinity.* However, this concept was subjected to sharp criticism from the side of the representatives of constructive mathematics. The famous Russian mathematician **A.A. Markov** (*1903-1979*) wrote [58]: *"We cannot imagine an endless, that is, never finished process as complete process without rough violence over intellect, which rejects such contradictory fantasies."*

As the concept of *actual infinity* is an internally contradictory notion ("the completed infinity"), this concept cannot be a reasonable basis for the creation of *constructive mathematical measurement theory.* If we reject Cantor's axiom, we can try to construct mathematical measurement theory on the basis of the idea of *potential infinity*, which underlies the *Eudoxus-Archimedes' axiom.* The *constructive approach* to measurement theory led to the creation of the so-called *algorithmic measurement theory* [1].

Algorithmic measurement theory led to new, "optimal" measurement algorithms based of the *generalized Fibonacci p-numbers*. The main outcome of the *algorithmic measurement theory* [1] is that every "optimal" measurement algorithm generates a new positional numeral system. It is proved in [1] that all the known positional numeral systems (binary, decimal, ternary, duodecimal and so on) are generated by the corresponding "optimal" measurement algorithms, which are partial cases of some very general class of the "optimal" measurement algorithms, which generate very unusual positional numeral systems. From these general reasoning's, we can conclude that the

algorithmic measurement theory [1] resulted in *general theory of positional numeral systems*, that is, in new mathematical theory, which is not existed before in mathematics.

The so-called *Fibonacci's measurement algorithm* generates the so-called *Fibonacci p-code*:

$$N = a_n F_p(n) + a_{n-1} F_p(n\text{-}1) + ... + a_i F_p(i) + ... + a_1 F_p(1), \qquad (70)$$

where N is natural number, $a_i \in \{0, 1\}$ is a binary numeral of the i-th digit of the code (70); n is the digit number of the code (70); $F_p(i)$ is the i-th digit weight calculated in accordance with the recurrence relation (18). The abridged notation of the sum (70) has the following form:

$$N = a_n\, a_{n-1}\, ...\, a_i\, ...\, a_1. \qquad (71)$$

Note that the notion of the *Fibonacci p-code* (70) includes an infinite number of different positional "binary" representations of natural numbers because every p produces its own *Fibonacci p-code* (p=0,1,2,3,…). In particular, for the case p=0 the Fibonacci p-code (70) is reduced to the classical binary code:

$$N = a_n 2^{n-1} + a_{n-1} 2^{n-2} + ... + a_i 2^{i-1} + ... + a_1 2^0 \qquad (72)$$

For the case p=1 the Fibonacci p-code (70) is reduced to the following sum:

$$N = a_n F_n + a_{n-1} F_{n-1} + ... + a_i F_i + ... + a_1 F_1. \qquad (73)$$

Note that Fibonacci's representation (73) in the "Fibonacci numbers theory" [53] is called *Zekendorf's sum* after Belgian researcher **Eduardo Zekendorf** (1901-1983). For the case $p = \infty$ all Fibonacci p-numbers in (70) are equal to 1 identically and then the *Fibonacci p-code* (70) is reduced to the sum

$$N = \underbrace{1 + 1 + ... + 1}_{N} \qquad (74)$$

which is known in number theory as *Euclidean definition of natural number.*

Thus, the *Fibonacci p-code* (70) is a wide generalization of the classical *binary code* (72), *Zekendorf's sum* (73) and *Euclidean definition of natural numbers* (74).

8.2. The "golden" number theory. As is known, the first definition of a number was made in the Greek mathematics. We are talking about the *Euclidean definition of natural numbers* (74). In spite of utmost simplicity of the *Euclidean definition* (74), we should note that all number theory begins from the definition (74). This definition underlies many important mathematical concepts, for example, the concept of the *prime* and *composed* numbers, and also the concept of *divisibility* that is one of the major concepts of number theory. Here we would like to note that in mathematics only *natural numbers* have a strong definition (74); all other real numbers do not have such a strong definition.

Within many centuries, mathematicians developed and defined more exactly a concept of *number*. In 17-th century, that is, in period of the creation of new science, in particular, new mathematics, different methods of the "continuous" processes study was developed and the concept of a real number again goes out on the foreground. Most clearly, a new definition of this concept is given by **Isaac Newton** (*1643 – 1727*), one of the founders of mathematical analysis, in his *Arithmetica Universalis* (1707):

"We understand a number not as the set of units, however, as the abstract ratio of one magnitude to another magnitude of the same kind taken for the unit."

This formulation gives us a general definition of numbers, rational and irrational. For example, the binary system

$$A = \sum_i a_i 2^i \qquad (75)$$

is the example of *Newton's definition,* when we chose the number 2 for the unit and represent a number as the sum of the number 2 powers.

Bergman's numeral system

In 1957 the American mathematician **George Bergman** published the article *A number system with an irrational base* [59]. In this article Bergman developed very unusual extension of the notion of the binary positional numeral system. He suggested using the "golden mean" $\Phi = \left(1 + \sqrt{5}\right)/2$ as a base of a special positional numeral system. If we use the sequences Φ^i {$i=0, \pm1, \pm2, \pm3, \dots$} as "digit weights" of the "binary" numeral system, we get the "binary" numeral system with irrational base Φ:

$$A = \sum_i a_i \Phi^i \tag{76}$$

where A is real number, a_i are binary numerals 0 or 1, $i = 0, \pm 1, \pm 2, \pm 3 \dots$, Φ^i is the weight of the *i-th* digit, Φ is the base or radix of the number system (76).

Unfortunately, Bergman's article [59] did not be noticed in that period by mathematicians. Only journalists were surprised by the fact that **George Bergman** made his mathematical discovery in the age of 12 years! In this connection, the Magazine *TIMES* had published the article about mathematical wunderkind of America.

Codes of the Golden p-proportions

Bergman's system (76) allows the following generalization [3]. Consider the set of the following standard line segments:

$$S_p = \left\{\Phi_p^i\right\}, i = 0, \pm1, \pm2, \pm3, \dots \tag{77}$$

where $p \geq 0$ is a given integer, Φ_p is the golden *p*-proportion, a real root of the characteristic equation (20). Remind that the powers of the golden *p*-proportions Φ_p^i are connected between themselves with the mathematical identity (17).

By using the set (77), we can "construct" the following positional representation of real numbers:

$$A = \sum_i a_i \Phi_p^i , \tag{78}$$

Where $a_i \in \{0, 1\}$ is a binary numeral of the *i*-th digit of the positional representation (78), $i = 0, \pm1, \pm2, \pm3, \dots$, Φ_p is a radix of the numeral system (78). We shall name the sums (78) *codes of the golden p-proportion.* Note, that a theory of these codes is described in Stakhov's 1984 book [3].

The formula (78) "generates" an infinite number of different positional numeral systems because every *p* (*p*=0, 1, 2, 3, …) leads to its own numeral system of the kind (78). Note, that for *p*=0 the radix $\Phi_p = \Phi_0 = 2$ and the numeral system (78) is reduced to the classical binary system, the base of modern computers. For the case *p=1* the golden mean $\Phi = \left(1 + \sqrt{5}\right)\!/2$ is the radix of numeral system (78) and, therefore, the numeral system (78) is reduced to *Bergman's system* (76).

Note that for the case $p > 0$ all the radices Φ_p of numeral system (78) are irrationals. This means that the numeral system (78) set a general class of numeral systems with irrational radices. However, for the case $p = 0$ we have the only exception, because for this case the numeral system (78) is reduced to the classical binary system.

The main conclusion from this study is the following. The researchers by **George Bergman** [59] and **Alexey Stakhov** [3] resulted in the discovery of new class of positional numeral systems – *numeral systems with irrational radices,* which can become a basis for new information technology – *"Golden" Information Technology.*

New properties of natural numbers

Let us study the formulas (76) and (78) from number-theoretical point of view. First of all, let us say that the expressions (76) and (78) can be seen as a new (constructive) definition of real num-

bers. It is clear that the sum of (78) specifies an infinite number of such representations because every integer $p \geq 0$ gives its own positional representation in the form (78). Every positional presentation (78) divides all real numbers into two groups, *constructive numbers*, which may be represented as the finite sum of the golden *p*-proportions in the form of (78), and *non-constructive numbers*, which can not be represented in the form of the finite sum (78).

Thus, the definitions (76) and (78) are sources for the new number theory – the *"golden" number theory*. This theory is described for the first time in Stakhov's article [17] (the article [17] was published by recommendation of academician Yuri Mitropolski) and then in the article [79]. Based on this approach, **Alexey Stakhov** has discovered in [17, 79] new properties of natural numbers. Let us consider them for the case of *Bergman's system* (76), which is partial cases of the codes of the golden *p*-proportions (78) for the case *p*=1. Let us represent some natural number *N* in *Bergman's* system:

$$N = \sum_i a_i \Phi^i . \tag{79}$$

It is proved in [17,79] that for arbitrary natural number *N* the sum (79) consists of the finite number of terms, that is, arbitrary natural number *N* is *constructive number* in the system (79). In further we will name the sum (79) Φ-*code of natural number N*. It is proved in [17,79] that this property is valid for all *codes of the golden p-proportions* (78).

The Z-property of natural numbers is based on the following simple reasoning. Let us consider the Φ-*code* of natural number *N* given by the sum (79). It is known [11] the following formula, which connects the golden mean powers $\Phi^i \left(i = 0, \pm 1, \pm 2, \pm 3, ... \right)$ with the Fibonacci and Lucas numbers:

$$\Phi^i = \frac{L_i + F_i \sqrt{5}}{2} . \tag{80}$$

If we substitute Φ^i, given by (80), in the formula (79), then after simple transformation we can write the expression (79) as follows:

$$2N = A + B\sqrt{5} , \tag{81}$$

where

$$A = \sum_i a_i L_i \tag{82}$$

$$B = \sum_i a_i F_i. \tag{83}$$

By studying the "strange" expression (81), we can conclude that the identity (81) can be valid for the arbitrary natural number *N* only if the sum (83) is equal to 0 ("zero"), and the sum (82) is double of *N*, that is,

$$B = \sum_i a_i F_i = 0 \tag{84}$$

$$A = \sum_i a_i L_i = 2N. \tag{85}$$

Let us compare now the sums (84) and (79). Since the binary numerals a_i in these sums coincide, it follows that the expression (84) can be obtained from the expression (79) by simple substitution of every power of the golden mean Φ^i instead the Fibonacci number F_i, where the discrete variable i takes its values from the set $\{0, \pm 1, \pm 2, \pm 3, ...\}$. However, according to (84) the sum (83) is equal to 0 independently of the initial natural number *N* in the expression (79). Thus, we have discovered a new fundamental property of natural numbers, which can be formulated through the following theorem.

Theorem 3 (Z-property of natural numbers). *If we represent an arbitrary natural number N in Bergman's system (79) and then substitute the Fibonacci number F_i instead the power of the golden mean Φ^i in the expression (79), where the discrete variable i takes its values from the set*

{0,±1,±2,±3,…}, then the sum that appear as a result of such a substitution is equal to 0 independently on the initial natural number N, that is, we get the identity (84).

The expression (85) can be formulated as the following theorem.

Theorem 4 (D-property). *If we represent an arbitrary natural number N in Bergman's system (79) and then substitute the Lucas number L_i instead the power of the golden mean Φ^i in the expression (79), where the discrete variable i takes its values from the set {0,±1,±2,±3,…}, then the sum that appears as a result of such a substitution is equal to 2N independently of the initial natural number N, that is, we get the identity (85).*

Thus, Theorems 3 and 4 provide new fundamental properties of natural numbers [17,79]. It is surprising for many mathematicians to find that the new mathematical properties of natural numbers were only discovered at the end of the 20th century, that is, 2½ millennia after the beginning of their theoretical study. The *golden mean* and the *Fibonacci and Lucas numbers* play a fundamental role in this discovery. This discovery connects together two outstanding mathematical concepts of Greek mathematics - *natural numbers* and the *golden section.* This discovery is the next confirmation of the fruitfulness of the constructive approach to the number theory based upon *Bergman's system* (79) and *codes of the golden p-proportions* (78).

9. The "Golden" information technology: a revolution in computer science

9.1. *Fibonacci computers.* The introduced above new positional representations – *Fibonacci p-codes* (70), *Bergman system* (76) and *codes of the golden p-proportions* (78) can be the sources of new computer projects – *Fibonacci computers.* This concept, first described in Stakhov's book [1], is one of the important ideas of modern computer science. The essence of the concept consists of the following. Modern computers are based on the binary system (75), which represents all numbers as the sums of the binary numbers with binary coefficients, 0 and 1. However, the binary system (75) is non-redundant numeral system what does not allow detecting errors, which could appear in computer in the process of its exploitation. In order to eliminate this shortcoming, **Alexey Stakhov** suggested in [1,3] to use the *Fibonacci p-codes* and *codes of the golden p-proportions.*

International recognition of the *Fibonacci Computer concept* began after Stakhov's lecture in Vienna on the joint meeting of the Austrian Computer and Cybernetic Societies in 1976. The very positive reaction to Stakhov's lecture by the Austrian scientists, including Professor **Aigner,** Director of the Mathematics Institute of the Graz Technical University, Professor **Trappel**, President of the Austrian Cybernetic society, Professor **Eier,** Director of the Institute of Data Processing of the Vienna Technical University, and also Professor **Adam** the representative of the Faculty of Statistics and Computer Science of Johannes Kepler Linz University, caused the decision of the Soviet Government to patent Stakhov's inventions in the Fibonacci computer field abroad. The general outcome of the Fibonacci invention patenting surpassed all expectations. 65 foreign patents on various devices for the Fibonacci computer were given by the State Patent Offices of the U.S., Japan, England, France, Germany, Canada, Poland and GDR. These patents testify to the fact that the Fibonacci computer was a world class innovation, because the Western experts could not challenge the Soviet Fibonacci computer inventions. This means, as a result, the Fibonacci patents are the official legal documents, which confirm Soviet priority in this computer direction.

Any expert, who is interested in the Fibonacci computer project, will ask the question: what Fibonacci computer research is done in other countries? Some publications of American scientists on the Fibonacci arithmetic and applications in the Fibonacci computer field are presented in [60-63].

It is important to note the recent applications of the Fibonacci *p*-codes (70) to *digital signal processing* [64,65]. In the Russian science the idea of the use of Fibonacci *p*-numbers for the design of super-fast algorithms for digital signal processing were actively developed by the Professor **Vladimir Chernov**, Doctor in Physics and Mathematics at Samara the Images Processing Institute of the Russian Academy of Science [64]. Also Fibonacci *p*-numbers for the development of super-fast algorithms for digital signal processing are widely used by the research group from the Tampere

International Center for Signal Processing (Finland). As is shown in the book [65], the super fast algorithms for digital signal processing requires a processing of numerical data represented in the Fibonacci *p*-codes (70). This means that for the realization of such super-fast transformations requires for the specialized *Fibonacci signal processors*! This is why the problem of *Fibonacci* processor development is of vital concern today!

9.2. The "golden" ternary mirror-symmetrical arithmetic. In 1958 the ternary *Setun* computer was designed in Moscow University under supervision of **Nikolay Brousentsov.** Its peculiarity was the use of ternary numeral system:

$$A = \sum_i c_i 3^i , \qquad (86)$$

where $c_i \in \left\{1, 0, \bar{1}\right\}$ is a ternary numeral of the *i*-th digit, 3^i is the weight of the *i-th* digit.

Many modern computer experts have come to the conclusion that the ternary computer design principle may bec ome an alternative in the future of computer progress. In this connection, it is important to recall the opinion of well-known Russian scientist, Prof. **Dmitry Pospelov**, on the ternary-symmetrical numeral system (86). In his book [66] he wrote: *"The barriers, which stand in the way of application of ternary-symmetric number systems in computers, are of a technical character. Until now, economical and effective elements with three stable states have not been developed. As soon as such elements will be designed, a majority of computers of the universal kind and many special computers will most likely be re-designed so that they will operate on the ternary-symmetric number system."* Also, American scientist Donald Knuth expressed the opinion [67] that one day the replacement of "flip-flop" by "flip-flap-flop" will occur.

Alexey Stakhov in [16] has developed a new ternary arithmetic, which is original synthesis of the ternary number system (86), used by **Nikolay Brousentsov** (Fig. 3) in the Setun computer, and *Bergman's system* (76). With purpose to explain new ternary representation of numbers, based on the *golden mean,* let us consider infinite sequence of the even powers of the *golden mean*:

$$\left\{\Phi^{2i}\right\}, i = 0, \pm 1, \pm 2, \pm 3, \ldots , \qquad (87)$$

where $\Phi = \left(1 + \sqrt{5}\right)/2$ is the *golden mean*.

It is proved in [16] that we can represent all integers (positive and negative) as the following sum called *ternary Φ-code of integer N*:

$$N = \sum_{i=-\infty}^{+\infty} c_i \left(\Phi^2\right)^i , \qquad (88)$$

where $c_i \in \left\{1, 0, \bar{1}\right\}$ is a ternary numeral of the *i-th* digit, $(\Phi^2)^i$ is the weight of the *i*-th digit of the positional representation (88), and $\Phi^2 = \left(3 + \sqrt{5}\right)/2 \approx 2.618$ is a radix of numeral system (88).

Fig. 3 - Donald Knuth (*born 1938*) and Nikolay Brousentsov (*1925-2014*)

The article *Brousentsov's Ternary Principle, Bergman's Number System and Ternary Mirror-Symmetrical Arithmetic* [16] published in *The Computer Journal* (England) got a high approval of the two outstanding computer specialists - **Donald Knuth** (Fig. 3)**,** Professor-Emeritus of Stanford University and the author of the famous book *The Art of Computer Programming* [67], and **Nikolay Brousentsov,** Professor of Moscow University, a principal designer of the fist ternary Setun computer. And this fact gives a hope that the ternary mirror-symmetrical arithmetic [16] can become a source of new computer projects in the nearest time.

10 The important "golden" discoveries in botany, biology and genetics

10.1. *Bodnar's geometry.* The phyllotaxis phenomenon shows itself in inflorescences and densely packed botanical structures, such as, *pinecones, pineapples, cacti, sunflowers, cauliflowers* and many other structures. As is well known, according to phyllotaxis law the numbers of the left-hand and right-hand spirals on the surface of phyllotaxis objects (Fig. 4) are always the adjacent Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, ... Their ratios

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, ..., \tag{89}$$

are called a *symmetry order* of phyllotaxis objects. The phyllotaxis phenomenon is exciting the best minds of humanity during many centuries (**Johannes Kepler, André Weil, Allan Turing** and others).
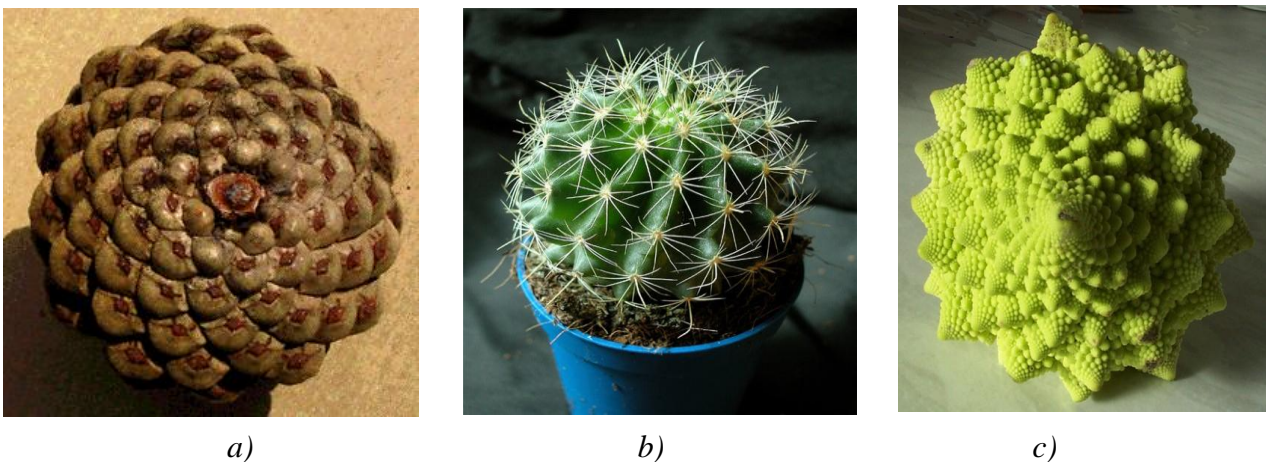

a)   *b)*   *c)*

Fig. 4 - Pyllotaxis structures: - pine cone (*a*); - pineapple (*b*); - Romanesque cauliflower (*c*)

The *puzzle of phyllotaxis* consists of the fact that a majority of bio-forms changes their phyllotaxis orders (95) during their growth. It is known, for example, that sunflower disks that are located on the different levels of the same stalk have different phyllotaxis orders; moreover, the more the age of the disk, the more its phyllotaxis order. This means that during the growth of the phyllotaxis object, a natural modification (an increase) of symmetry happens and this modification of symmetry obeys the law:

$$\frac{2}{1} \rightarrow \frac{3}{2} \rightarrow \frac{5}{3} \rightarrow \frac{8}{5} \rightarrow \frac{13}{8} \rightarrow \frac{21}{13} \rightarrow ... . \tag{90}$$

The law (90) is called *dynamic symmetry.*

Note that the change of orders of phyllotaxis (90) carried out in strict accordance with the *Principle of self-similarity,* that is, all phyllotaxis structures are *self-similar* structures.

Recently the Ukrainian researcher **Oleg Bodnar** had developed very interesting geometric theory of phyllotaxis [45]. He proved that phyllotaxis geometry is a special kind of hyperbolic geometry based on the *"golden" hyperbolic functions* similar to the recursive hyperbolic Fibonacci and Lucas functions (29). Such approach allows explaining geometrically how the "Fibonacci spirals" appear

on the surface of phyllotaxis objects in process of their growth. ***Bodnar's geometry*** [45] **is of fundamental importance because it touches on fundamentals of theoretical natural sciences, in particular, this discovery gives a strict geometrical explanation of the *phyllotaxis law* and *dynamic symmetry* based on *Fibonacci numbers.***

10.2. ***The Golden Section and a heart.*** During many years the Russian biologist **Vladimir Tsvetkov** had fulfilled fundamental scientific researches on the theme *The Golden Section and a Heart* [68,69]. This led to the following conclusions. The *golden mean* is displayed very widely in the work of the heart and all its systems. The main purpose of this work is a creation of stable and energy-optimal system. The mode of the *golden section* brings to maximum economy of energy and building material. The *golden harmony* of the heart activity corresponds to physiological calm of human body. In this state the heart works in economic, "golden" mode. After stopping any physical load, a blood circulation of the body and heart after some time returns back to the "golden" mode as the most economical one. The state of calm is prevailing over the life for even a very active animal. Therefore we can say that the heart and body aim for the *golden harmony* of "opposites"! The availability of the *golden mean* in a wide variety of different heart systems confirms the universality of the *golden mean* for the heart work. **The *golden harmony* is a "sign of quality" of a cardiac system and the heart in the whole.**

10.3. Fibonacci's resonances of genetic code. ***Among the biological concepts* [70] *that are well formalized and have a level of general scientific significance, the genetic code takes special precedence. Discovery of the striking simplicity of the basic principles of the genetic code places it amongst the major modern discoveries of mankind. This simplicity consists of the fact that inheritable information is encoded in the texts from three-lettered words -* triplets *or* codonums *compounded on the basis of the alphabet that consists of the four characters or nitrogen bases:* **A** *(adenine),* **C** *(cytosine),* **G** *(guanine),* **T** *(thiamine). The given system of the genetic information represents a unique and boundless set of diverse living organisms and is called* **genetic code.***

In 1990 **Jean-Claude Perez**, an employee of IBM, made a rather unexpected discovery in the field of the *genetic code*. He discovered the mathematical law that controls the self-organization of bases *A, C, G* and *T* inside of the DNA. He found that the consecutive sets of the DNA nucleotides are organized in frames of remote order called *RESONANCES*. Here, the resonance means a special proportion that divides the DNA sequence according to Fibonacci numbers (1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 …).

The key idea of Perez's discovery, called the *DNA SUPRA-code*, consists of the following. Let us consider some fragment of the genetic code that consists of the *A, C, G* and *T* bases. Suppose that the length of this fragment is equal to some Fibonacci number, for example, 144. If a number of the *T*-bases in the DNA fragment is equal to 55 (Fibonacci number), and a total number of the *C, A* and *G* bases is equal to 89 (Fibonacci number), then this fragment of the genetic code forms is a *RESONANCE*, that is, a proportion between three adjacent Fibonacci numbers (55:89:144). Here it is permissible to consider any combinations of the bases, that is, *C* against *AGT*, *A* against *TCG*, or *G* against *TCA*. The discovery consists of the fact that the arbitrary DNA-chain forms some set of the *RESONANCES*. As a rule, the fragments of the genetic code of the length equal to the Fibonacci number F_n are divided into the subset of the *T*-bases, and the subset of the remaining *A, C, G* bases; here the number of *T*-bases is equal to the Fibonacci number F_{n-2} and the total number of the remaining *A,C, G* bases is equal to the Fibonacci number F_{n-1}, where $F_n=F_{n-1}+F_{n-2}$. If we make a systematic study of all the Fibonacci fragments of the genetic code, we can obtain a set of the resonances that is called the *SUPRA-code of DNA*.

10.4. ***"Golden" genomatrices.*** Recently the Russian researcher **Sergey Petoukhov** made an original discovery in genetics [70]. Petoukhov's discovery [70] shows a fundamental role of the *golden mean* in genetic code. This discovery gives further evidence that the *golden mean* underlies all *Organic Nature*! It is difficult to estimate the full impact of Petoukhov's discovery for the development of modern science. It is clear that this scientific discovery is of revolutionary discovery in this field.

11 The revolutionary "golden" discoveries in crystallography, chemistry, theoretical physics and cosmology

11.1. *Quasi-crystals: revolutionary discovery in crystallography*. According to the *main law of crystallography*, there are strict restrictions imposed on the structure of a crystal. According to classical ideas, the crystal is constructed from one single cell. The identical cells should cover a plane densely without any gaps. As we know, the dense filling of a plane can be carried out by means of *equilateral triangles, squares* and *hexagons*. A dense filling of the plane by means of *pentagons* is impossible, that is, according to the main law of crystallography *pentagonal symmetry* is prohibited for mineral world.

On November 12, 1984 in a small article, published in the authoritative journal *Physical Review Letters*, the experimental proof of the existence of a metal alloy with exclusive physical properties was presented. The Israeli physicist **Dan Shechtman** was one of the authors of this article. A special alloy, discovered by Professor Shechtman in 1982 and called *quasi-crystal*, is the focus of his research. By using methods of electronic diffraction, Shechtman found new metallic alloys having all the symptoms of crystals. Their diffraction pictures were composed from the bright and regularly located points similar to crystals. However, this picture is characterized by the so-called *icosahedral* or *pentagonal* symmetry, strictly prohibited according to geometric reasons. Such unusual alloys are called *quasi-crystals.*

Quasi-crystals are revolutionary discovery in crystallography. The concept of quasi-crystals generalizes and completes the definition of a crystal. **Gratia** wrote in the article [71]: *"A concept of the quasi-crystals is of fundamental interest, because it extends and completes the definition of the crystal. A theory, based on this concept, replaces the traditional idea about the 'structural unit,' repeated periodically, with the key concept of the distant order. This concept resulted in a widening of crystallography and we are only beginning to study the newly uncovered wealth. Its significance in the world of crystals can be put at the same level with the introduction of the irrational numbers to the rational numbers in mathematics."*

What is the practical significance of the discovery of quasi-crystals? Gratia writes in [71] that *"the mechanical strength of the quasi-crystals increased sharply; here the absence of periodicity resulted in slowing down the distribution of dislocations in comparison to the traditional metals. This property is of great practical significance: the use of the "icosahedral" phase allows for light and very stable alloys by means of the inclusion of small-sized fragments of quasi-crystals into the aluminum matrix."*

Note that **Dan Shechtman** published his first article on the quasi-crystals in 1984, that is, exactly 100 years after the publication of Felix Klein's *Lectures on the Icosahedron* in 1884 [72]. This means that this discovery is a worthy gift to the centennial anniversary of Klein's book, in which the famous German mathematician predicted an outstanding role for the icosahedron in future scientific development.

In 2011 **Dan Shechtman** won the Nobel Prize in chemistry for this discovery.

11.2. *Fullerenes: revolutionary discovery in chemistry.* *Fullerenes* are an important modern discovery in chemistry. This discovery was made in 1985, several years after the quasi-crystal discovery. The "fullerene" is named after **Buckminster Fuller** (*1895 -1983*), the American designer, architect, poet, and inventor. Fuller created a large number of inventions, primarily in the fields of design and architecture.

The title of *fullerenes* refers to the carbon molecules C_{60}, C_{70}, C_{76}, and C_{84}. We start from a brief description of the C_{60} molecule. This molecule plays a special role among the *fullerenes.* It is characterized by the greatest symmetry and as a consequence is highly stable. By its shape, the molecule C_{60} (Fig. 4, on the right) has the structure of *Archimedean truncated regular icosahedron* (Fig. 4, on the left).

The atoms of carbon in the molecule C_{60} are located on the spherical surface at the vertices of 20 regular hexagons and 12 regular pentagons; here each hexagon is surrounded by three hexagons and

three pentagons, and each pentagon is surrounded by five hexagons. The most striking property of the C_{60} molecule is its high degree of symmetry. There are 120 symmetry operations that convert the molecule into itself making it the most symmetric molecule.
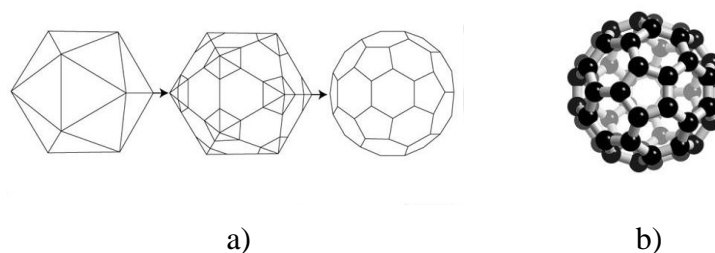

a)   b)

Fig.4 - Archimedean truncated icosahedron (*a*)
and the molecule C_{60} (*b*)

It is not surprising that the shape of the C_{60} molecule has attracted the attention of many artists and mathematicians over the centuries. As mentioned earlier, the *truncated icosahedron* was already known to Archimedes. The oldest known image of the *truncated icosahedron* was found in the *Vatican library*. This picture was from a book by the painter and mathematician **Piero della Francesca**. We can find the truncated icosahedron in Luca Pacioli's *Divina Proportione* (1509). Also **Johannes Kepler** studied the Platonic and Archimedean Solids actually introducing the name *truncated icosahedron* for this shape.

The fullerenes*,* in essence, are "man-made" structures following from fundamental physical research. They were discovered in 1985 by **Robert F. Curl, Harold W. Kroto and Richard E. Smalley**. The researchers named the newly-discovered chemical structure of carbon C_{60} the *buckminsterfullerene* in honor of **Buckminster Fuller**. In 1996 they won the Nobel Prize in chemistry for this discovery.

Fullerenes possess unusual chemical and physical properties. At high pressure the carbon C_{60} becomes firm, like diamond. Its molecules form a crystal structure as though consisting of ideally smooth spheres, freely rotating in a cubic lattice. Owing to this property, C_{60} can be used as firm greasing (dry lubricant). The fullerenes also possess unique magnetic and superconducting properties.

11.3. *Fibonacci's interpretation of Mendeleev's Periodical Table.* Recently the Russian researchers **Shilo** and **Dinkov** have suggested in the work [73] very interesting interpretation of *Mendeleev's Periodical Law* of chemical elements. The essence of this suggestion consists of the following. The Great Russian scientist **Dmitry Mendeleev** suggested the *Periodical Law* 137 years ago. During this time, *Mendeleev's Periodical Law* played a huge role in the development of not only chemistry, but also of physics, biology, geochemistry, mineralogy, petrology, crystallography, and other sciences. In other words, it has stimulated scientific progress in all areas, where chemical elements are the basis of natural or artificial processes. But during this time scientists of different specialties in one or another form expressed dissatisfaction concerning Mendeleev's *Periodical Law*, despite the acclaim of its brilliant fundamental properties.

As it is emphasized in [73], Dmitry Mendeleev suggested a spiral form of the *Periodical System* yet in his first article on this topic. This was his brilliant prediction. Later in his total article *Periodical regularity of chemical elements* Mendeleev wrote: *«In fact, all the distribution of elements is uninterrupted and corresponds, in some degree, to spiral function».* It is asserted in [73] that, in the first days of the *Periodical Law* discovery, Mendeleev had used a dual form of the *Periodic Law*. Now it is clear that all Mendeleev's intuitive and prophetic ideas can be combined in the spatial helical form of the *Periodic Law*.

By studying *Mendleev's Peridical System* from this point of view, **Shilo** and **Dinkov** came in [73] to the important conclusion: *"Thus, the spatial curve (spiral), where chemical elements are placed, are located inside the cone or Lobachevski's pseudo-sphere. The chemical elements are presented of this spiral in discrete points (or «balls»). Projection of the elements on the horizontal plane, that is, on the cone base, presents Fibonacci's spiral, that is, such a spiral, where difference between atomic numbers of any two consecutive chemical elements is equal to Fibonacci numbers."*

Shilo and **Dinkov** pointed in [73] different relations, which determine a connection of the *Periodical System* with the *golden mean* and *Fibonacci numbers*:

1. A ratio of the number of the even mass nuclides of to the number of the even mass nuclides is equal to $2 \times 89 / 2 \times 55 \approx \Phi$, where Φ is the *golden mean.*

2. A ratio the number of the even charge nuclides to the number of the odd charge nuclides is equal to $220/68 \approx \Phi$, where Φ is the *golden mean.*

3. If we arrange in the increase order the 165 even-even nuclides, we get that the well-known "magic" neutron numbers 2, 8, 14, 20, 28, 50, 82, 126 correspond to the following nuclide numbers of our arrangement: 1, 3. 8. 13, 21, 55, 110=2×55, 165=3×55.

It seems that **Shilo** and **Dinkin's** distribution of the chemical elements*,* based on *Fibonacci numbers*, offers great opportunities to predict new properties of chemical elements what plays sometimes a decisive role in their use. And we can agree with the following **Shilo** and **Dinkin**'s assertion: *"If we move in this way, we inevitably will come to a completely new understanding of many processes and phenomena; perhaps, we even will change our ideas on the Universe."*

11.4. *El Nashie's E-infinity theory.* Prominent theoretical physicist and engineering scientist **Mohammed S. El Nashie** is a world leader in the field of the golden mean applications to theoretical physics, in particular, quantum physics [74–76]. El Nashie's discovery of the *golden mean* in the famous physical two-slit experiment-which underlies quantum physics–became a source for many important discoveries in this area, in particular, the *E-infinity theory*. It is also necessary to note the contribution of Slavic researchers to this important area. The book [77], written by the Byelorussian physicist **Vasyl Pertrunenko,** is devoted to the applications of the *golden mean* in quantum physics and astronomy.

11.5. *Fibonacci-Lorentz transformations and the "golden" cosmological interpretation of the Universe evolution.* As is known, *Lorentz's transformations* used in special relativity theory (SRT) are the transformations of the coordinates of the events (*x, y, z, t*) at the transition from one inertial coordinate system (ICS) *K* to another ICS K', which is moving relatively to ICS *K* with a constant velocity *V*.

The transformations were named in honor of Dutch physicist **Hendrik Antoon Lorentz** (*1853-1928*), who introduced them in order to eliminate the contradictions between *Maxwell's electrodynamics* and *Newton's mechanics*. *Lorentz's transformations* were first published in 1904, but at that time their form was not perfect. The French mathematician **Jules Henri Poincaré** (*1854-1912*) brought them to modern form.

In 1908, that is, three years after the promulgation of SRT, the German mathematician **Hermann Minkowski** (*1864-1909*) gave the original geometrical interpretation of *Lorentz's transformations*. In *Minkowski's space*, a geometrical link between two ICS *K* and K' are established with the help of *hyperbolic rotation*, a motion similar to a normal turn of the Cartesian system in *Euclidean space*. However, the coordinates of x' and t' in the ICS K' are connected with the coordinates of *x* and *t* of the ICS *K* by using classical hyperbolic functions. Thus, *Lorentz's transformations* in *Minkowski's geometry* are nothing as the relations of *hyperbolic trigonometry* expressed in physics terms. This means that *Minkowski's geometry* is hyperbolic interpretation of SRT and therefore it is a revolutionary breakthrough in geometric representations of physics, a way out on a qualitatively new level of relations between physics and geometry.

Alexey Stakhov and **Samuil Aranson** put forward in [37] the following hypotheses concerning the "golden" SRT:

1. The first hypothesis concerns the *light velocity in vacuum*. As is well known, the main dispute concerning the SRT, basically, is about the *principle of the constancy of the light velocity in vacuum*. In recent years a lot of scientists in the field of cosmology put forward a hypothesis, which puts doubt the permanence of the light velocity in vacuum - a fundamental physical constant, on which the basic laws of modern physics are based. Thus, **the first hypothesis is that the light velocity in vacuum was changed in process of the Universe evolution.**

2. Another fundamental idea involves with the factor of the *Universe self-organization* in the process of its evolution. According to modern view [82], a few stages of self-organization and degradation can be identified in process of the Universe development: *initial vacuum, the emergence of superstrings, the birth of particles, the separation of matter and radiation, the birth of the Sun, stars, and galaxies, the emergence of civilization, the death of Sun, the death of the Universe.* The main idea of the article [37] is to unite the fact of the *light velocity change* during the Universe evolution with the factor of its *self-organization*, that is, to introduce a dependence of the light velocity in vacuum from some *self-organization parameter ψ,* which does not have dimension and is changing within: $\left(-\infty < \psi < +\infty\right)$. The light velocity in vacuum c is depending on the "self-organization" parameter $\psi \left(-\infty < \psi < +\infty\right)$ and this dependence has the following form:

$$c = c(\psi) = \bar{c}(\psi)c_0. \tag{91}$$

As follows from (91), the *light velocity in vacuum* is a product of the two parameters: c_0 and $\bar{c}(\psi)$. The parameter $c_0 = const$, having dimension [$m.sec^{-1}$], is called *normalizing factor.* It is assumed in [37] that the constant parameter c_0 is equal to *Einstein's light velocity in vacuum* ($2.998 \times 10^8 \, msec^{-1}$) divided by the *golden mean* $\Phi = \left(1 + \sqrt{5}\right)/2 \approx 1,61803$. The dimensionless parameter $\bar{c}(\psi)$ is called *non-singular normalized Fibonacci velocity of light in vacuum.*

3. The *"golden" Fibonacci goniometry* is used for the introduction of the *Fibonacci-Lorentz transformations,* which are a generalization of the classical *Lorentz transformations.* We are talking about the matrix

$$\Omega\left(\psi\right) = \begin{pmatrix} cFs(\psi - 1) & sFs(\psi - 2) \\ sFs(\psi) & cFs(\psi - 1) \end{pmatrix}, \tag{92}$$

whose elements are *symmetric hyperbolic Fibonacci functions* (29). The matrix $\Omega(\psi)$ of the kind (92) is called *non-singular two-dimensional Fibonacci-Lorentz matrix* and the transformation

$$\begin{pmatrix} \xi \\ x_1 \end{pmatrix} = \begin{pmatrix} cFs(\psi - 1) & sFs(\psi - 2) \\ sFs(\psi) & cFs(\psi - 1) \end{pmatrix} \begin{pmatrix} \xi' \\ x_1' \end{pmatrix} \tag{93}$$

is called *non-singular two-dimensional Fibonacci-Lorentz transformations.* The above approach to the SRT led to the new ("golden") cosmological interpretation of the Universe evolution and to the change of the light velocity before, in the moment, and after the bifurcation, called *Big Bang.*

Based on this approach, **Alexey Stakhov** and **Samuil Aranson** have obtained in [37] new cosmological results in the Universe evolution, beginning with the «Big Bang». In particular, they put forward a hypothesis that there are two "bifurcation points" in the Universe evolution. The first one corresponds to the *"Big Bang",* and the second one corresponds to the transition of the Universe from the *Dark Ages* to the *Shining Period,* where light and first stars have arisen. The speed of light immediately after the second "Bifurcation point" is very high, but as far as the evolution of the Universe the speed of light starts to drop and reaches the limit value $C \approx 300000$ km/sec^{-1}.

12 The latest results and publications in the field of Mathematics of Harmony

The ancient Greeks raised the *golden section* at the level of "aesthetic canon" and "major ratio" of the Universe. For centuries or even millennia, starting from **Pythagoras, Plato, Euclid**, this geometric discovery has been the subject of admiration and worship of eminent minds of humanity - in the Renaissance, **Leonardo da Vinci, Luca Pacioli, Johannes Kepler**, in the 19 century -

Zeizing, Lucas, Binet. In 20-th century, the interest in this unique irrational number increased in mathematics, thanks to the works of Russian mathematician **Nikolay Vorobyov** and American mathematician **Verner Hoggatt**. The development of this direction led to the appearance of the *Mathematics of Harmony* [36] as a new interdisciplinary theory of modern science.

The Mathematics of Harmony is now the actively developing mathematical discipline, which expands the scope of its applications and already goes out to the level of mathematical and physical MILLENIUM PROBLEMS [80,81,83]. Let us consider new mathematical results, obtained in the works [49,79,80,81,83]:

1. The article [49] is very important article for the mathematics history. It turns over our ideas about Euclid's *Elements* and the history of mathematics, starting from Euclid. This article is especially important for mathematical education because it brings nearer mathematics to Nature and fine arts and makes mathematics more interesting discipline for learning. **Introduction of the course "Mathematics of Harmony" into the educational programs of schools, colleges and universities can be a revolutionary idea in modern education.**

2. The article [79] is developing the concept of the "golden" number theory, described above. **A discovery of new and unusual properties of natural numbers is a basic mathematical result of the article** [79].

3. The article [80] is devoted to original solution of the most complicated mathematical problem of 20th century, the *Hilbert Fourth Problem.* The *metallic means* by Vera Spinadel [50] and following from them *hyperbolic λ-Fibonacci and Lucas functions* underlie this solution. This solution puts forward a new challenge for theoretical natural sciences, *a search of new hyperbolic worlds of Nature similar to "Bodnar's geometry"* [45]*,* based on the *golden ratio*. **The basic idea of the article [80] is the fact that Hilbert Fourth Problem is MILLENNIUM PROBLEM in Geometry that is still not be understood by modern mathematicians.**

4. The article [81] is very interesting article from the point of view of theoretical physics. The article gives original solution of one of the important physics MILLENNIUM PROBLEMS, the problem of FINE-STRUCTURE CONSTANT, formulated in 2000 by the prominent American physicist **David Gross**. The solution is based on the non-traditional version of the special theory of relativity, which follows from the Fibonacci-Lorentz transformations [37] arising from the theory of the "golden" matrices [26]. On the basis of this approach in the article [81] it is studied changing the FINE-STRUCTURE CONSTANT, depending on the age of the Universe since the Big Bang. **It is shown that the FINE-STRUCTURE CONSTANT in fact is not a physical constant, but is physical dimensionless quantity whose value is dependent on the age of the Universe.**

5. The article [82] is a development of the well-known Stakhov's work on Fibonacci *p*-codes and codes of the golden *p*-proportions [1-5] as new arithmetical and informational foundations of computer science and digital metrology for mission-critical applications. **Designing Fibonacci and "golden" computer and measurement systems for mission-critical applications can be revolutionary idea for future informational technology.**

6. The book "The "Golden" Non-Euclidean Geometry" [83] summarizes the results of the latest applications of the "Mathematics of Harmony", set out in the articles [49,79,80,81,83]. The book [83] is a further development of the book "The Mathematics of Harmony" [11]. The main goal of the book [83] is to show that the "Mathematics of Harmony" [11] is an actively developing direction of contemporary science and that the "Mathematics of Harmony" [11] is indeed a source for new and original scientific ideas and concepts.

Instead Conclusion:
New Challenge for Theoretical Natural Sciences
Based on the Mathematics of Harmony and "Golden" Hyperbolic Geometry

Although the "Mathematics of Harmony" contains in itself a large number of new scientific findings concerning mathematics (*Proclus hypothesis, "golden" number theory and new properties of*

natural numbers, Hilbert's Fourth Problem), computer science (*numeral systems with irrational radices, conception of Fibonacci computers and ternary mirror-symmetrical arithmetic*) and theoretical physics (*Fibonacci-Lorenz transformations and Fibonacci special theory of relativity*), however the main scientific result of the research, described in [80,83], is **a proof of the existence of an infinite number of new hyperbolic functions, *Fibonacci hyperbolic λ-functions*, based on the "metallic proportions".** For the given λ=1,2,3,.. each class of the Fibonacci hyperbolic λ-functions, "generates" new hyperbolic geometry, which leads to the appearance in the "physical world" specific hyperbolic geometries with mathematical properties, based on the "metallic proportions." **The main peculiarity of these new hyperbolic functions, based on "metallic proportions" and Gazale's formulas, is a recursive character of these hyperbolic functions.** This means that the "golden" hyperbolic geometries are also recursive, and they embody in themselves the **Principle of Self-similarity**, which is the basis of **self-organizing systems of Nature.** This is the main difference between the "golden" hyperbolic geometries and classical hyperbolic geometry created by Lobachevski.

The new geometric theory of phyllotaxis, created by Oleg Bodnar [45], is brilliant example of real existing of the "golden" geometry in Nature. Bodnar proved that "the world of phyllotaxis" is a specific "golden hyperbolic world," in which "hyperbolicity" manifests itself in the "Fibonacci spirals" on the surface of "phyllotaxis objects."

Recall that "Bodnar's geometry" [45] is based on the recursive Fibonacci hyperbolic functions:

$$\begin{cases} sFs(x) &= \dfrac{\Phi^x - \Phi^{-x}}{\sqrt{5}} \\ cFs(x) &= \dfrac{\Phi^x + \Phi^{-x}}{\sqrt{5}} \end{cases} \tag{94}$$

which are connected with the "extended" Fibonacci numbers, given by *Binet's formulas* (28), by the formulas (33):

$$F_n = \begin{cases} sF(n) \text{ for } n = 2k \\ cF(n) \text{ for } n = 2k+1 \end{cases}. \tag{95}$$

The property (95) defines a recursive character of the hyperbolic functions (94) and following from them "golden" hyperbolic geometry. Unfortunately, this fundamental property of the new hyperbolic functions, based on the *metallic proportions* Φ_λ, in comparison with the classical hyperbolic functions, based on Euler's number *e* still not understood by many modern mathematicians, excepting mathematicians-thinkers such as Academician **Yuri Mitropolsky**, head of the Ukrainian School of Mathematics, Russian Professor **Samuil Aranson**, one of the leading mathematicians of Russia in the field of topology and geometry, awarded in 2016 by the **GOLD MEDAL of Euro Chamber** for outstanding achievements in the field of science, and also Canadian Professor **M. W. Wong** (*York University*)**,** Editor-in-Chief of the Series on Analysis, Applications and Computation (World Scientific), who recommended the book "The "Golden" Non-Euclidean Geometry" for publication in the World Scientific. Also some philosophers-thinkers such as American Prof. **Scott Olsen,** Russian Prof. **Sergey Abachiev**, and Belarusian philosopher **Eduard Soroko** praised the scientific direction of Alexey Stakhov. We present some excerpts from reviews of famous scientists.

Academician Yuri Mitropolsky [84]:

"One may wonder what place in the general theory of mathematics this work may have. It seems to me-that in the last few centuries-as Nikolay Lobachevsky said, "Mathematicians have turned all their attention to the advanced parts of analytics, and have neglected the origins of Mathematics and are not willing to dig the field that has already been harvested by them and left behind." As a result, this has created a gap between "Elementary Mathematics" - the basis of modern mathematical education - and "Advanced Mathematics." In my opinion, the Mathematics of Harmony developed by Professor Stakhov fills that gap. The Mathematics of Harmony is a huge theoretical contri-

bution to the development of "Elementary Mathematics," and as such should be considered of great importance for mathematical education".

Professor Scott Olsen [85]: *"Now for the past two years I have worked very closely with Prof. Stakhov, editing his book, The Mathematics of Harmony: from Euclid to Contemporary Mathematics and Computer Science, scheduled for publication by World Scientific later this summer in 2009. Some of us are convinced that **his insights in this work are so remarkable, that it may well change not only the way we view the history of mathematics, but the future development of mathematics in its applications to the natural sciences.** In particular, I have found that Professor Stakhov's knowledge of the Golden Section in both its intricacies and ramifications for the natural sciences is the actual state of the art in academia. I know this because I have researched the subject for over 35 years, and in 2006 published the award winning book, The Golden Section: Nature's Greatest Secret".*

A conception of the "Golden" Non-Euclidean Geometries greatly expands the number of possible hyperbolic functions with recursive properties. It is proved in [83] that a number of the recursive hyperbolic geometries is equal infinity, because every $\lambda=1,2,3,...$ "generates" its own "golden" recursive hyperbolic geometry.

In the book [83], the notion of the *normalized distance* between classical Lobachevski's geometry with the base e (*Euler number*) and the "golden" hyperbolic geometries with the bases Φ_λ (*metallic proportions*) has been introduced.

The famous irrational number $\Phi = \dfrac{1+\sqrt{5}}{2} \approx 1.618$ (*the golden ratio*) is the base of the hyperbolic functions (94). The *normalized distance* between *Bodnar's geometry* and *Lobachevski's geometry* is equal $\rho_{12} \approx 0.7336$.

For the first time, the simplest example of the recursive hyperbolic functions has been described in **Stakhov & Rozin's** 2004 article [18] and later **Alexey Stakhov** generalized the result of the article [18] in Stakhov's 2006 article [30]. A detailed study of the *recursive Fibonacci hyperbolic λ-functions* was done in **Stakhov & Aranson's** 2016 article [80] and 2016 book [83]. Basing on the success of *Bodnar's geometry* [45], one can put forward in front to theoretical physics, chemistry, crystallography, botany, biology, and other branches of theoretical natural sciences **the challenge to search new ("harmonic") hyperbolic worlds of Nature, based on other classes of the Fibonacci hyperbolic λ-functions** (53), (54).

However, the *"golden" hyperbolic functions* (94), which underlie the *hyperbolic phyllotaxis world,* are a special case of the *hyperbolic Fibonacci λ-functions* ($\lambda=1$). In this regard, there is every reason to suppose that other types of hyperbolic functions, the *Fibonacci hyperbolic λ-functions*, can be the basis for modeling of new "hyperbolic worlds," which possibly can really exist in Nature. Modern science did not find these special "hyperbolic worlds" until now, because the recursive hyperbolic Fibonacci functions were unknown until early 21th century [18,30]. In this case, perhaps, the next candidate for the new "hyperbolic world" of Nature may be, for example, **"silver" hyperbolic functions**:

$$\begin{cases} sF_2(x) = \dfrac{\Phi_2^x - \Phi_2^{-x}}{\sqrt{8}} = \dfrac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^x - \left(1+\sqrt{2}\right)^{-x}\right] \\[3mm] cF_2(x) = \dfrac{\Phi_2^x + \Phi_2^{-x}}{\sqrt{8}} = \dfrac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^x + \left(1+\sqrt{2}\right)^{-x}\right] \end{cases}, \tag{96}$$

which are connected with *Pell numbers* and are based on the "silver mean" $\Phi_2 = 1+\sqrt{2} \approx 2.41$, called also *Leonardo da Vinci's constant*.

In this regard, we should draw a special attention to the fact that the new hyperbolic geometry, based on the "silver" hyperbolic functions (96), is the closest to Lobachevski's geometry, based of the classical hyperbolic functions with the base $e \approx 2.71$. Its normalized distance to Lobachevski's geometry is equal $\rho_{12} \approx 0.1677$ what is the smallest among all the distances between "golden" hyperbolic λ-forms (62) and Lobachevski's metric form (61). It allows from here the assump-

tion that the "silver" hyperbolic functions (96) and the generated by them "silver" hyperbolic geometry can be soon be found in Nature after *Bodnar's geometry*, based on the "golden" hyperbolic functions (94).

Taking into consideration the above reasoning's, we can conclude that the recursive hyperbolic geometries, based on the *Principle of Self-similarity*, is the new direction in the development of Lobachevski's hyperbolic geometry and therefore searching new hyperbolic worlds of Nature, based on recursive hyperbolic geometries, may lead to new scientific discoveries.

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Золотий перетин, числа Фібоначчі, математика гармонії і "золота" наукова революція.

Анотація. Видавничий дім "World Scientific" нещодавно опублікував дві фундаментальні праці: "Математика гармонії" (2009), автор Олексій Стахов, і "'Золота' неевклідова геометрія" (2016), авторів Олексія Стахова і Самуїла Арансона. У пред-

ставленій статті автор розвиває математику гармонії і 'золоту' неевклідову геометрію як новий меж-дисциплінарний напря-мок сучасної науки, що засноване на золотому перетині, числах Фібоначчі та їх узагальненнях. Новітні відкриття в різних галузях сучасної науки, що засновані на математиці гармонії, а саме, математики (загальної теорії гіперболічних функцій і вирішенні четвертої проблеми Гільберта, алгоритмічної теорії вимірювання і 'золотий' теорії алгоритмів), комп'ютерних науках ('золота' інформаційна технологія), кристалографія (квазікристали), хімія (фулерени), теоретична фізика та космоло-гія (перетворення Фібоначчі-Лоренца, 'золота' інтерпретація спеціальної теорії відносності і 'золота' еволюція Всесвіту), ботаніка (нова геометрична теорія філотаксису), генетика ('золоті' геноматриці) і т.д., створює загальну картину 'золотої' наукової революції, яка може фундаментально впливати на розвиток сучасної науки і освіти.

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Золотое сечение, числа Фибоначчи, математика гармонии и "золотая" научная революция.

Аннотация. Издательский дом "World Scientific" недавно опубликовал две фундаментальные работы: "Математика гармо-нии" (2009), автор Алексей Стахов, и "'Золотая' неевклидова геометрия" (2016), авторов Алексея Стахова и Самуила Аран-сона. В представленной статье автор развивает математику гармонии и 'золотую' неевклидову геометрию как новое меж-дисциплинарное направление современной науки, основанное на золотом сечении, числах Фибоначчи и их обобщениях. Новейшие открытия в разных областях современной науки, основанные на математике гармонии, а именно, математики (общей теории гиперболических функций и решении четвертой проблемы Гильберта, алгоритмической теории измерения и 'золотой' теории алгоритмов), компьютерных науках ('золотая' информационная технология), кристаллография (квазикри-сталлы), химия (фуллерены), теоретическая физика и космология (преобразования Фибоначчи-Лоренца, 'золотая' интерпре-тация специальной теории относительности и 'золотая' эволюция Вселенной), ботаника (новая геометрическая теория фил-лотаксиса), генетика ('золотые' геноматрицы) и т.д., создает общую картину 'золотой' научной революции, которая может фундаментально повлиять на развитие современной науки и образования.